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Lie Symmetries and Low-Order Conservation Laws of a Family of Zakharov-Kuznetsov Equations in 2 + 1 Dimensions

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Abstract: In this work, we study a generalised $(2+1)$ equation of the Zakharov–Kuznetsov (ZK)(m,n,k) equation involving three arbitrary functions. From the point of view of the Lie symmetry theory, we have derived all Lie symmetries of this equation depending on the arbitrary functions. Line soliton solutions have also been obtained. Moreover, we study the low-order conservation laws by applying the multiplier method. This family of equations is rich in Lie symmetries and conservation laws. Finally, when the equation is expressed in potential form, it admits a variational structure in the case when two of the arbitrary functions are linear. In addition, the corresponding Hamiltonian formulation is presented.

Keywords: ZK equations; Lie symmetries; conservation laws

1. Introduction

In the context of plasma physics, the Zakharov–Kuznetsov equation (ZK) arises to describe ion-sound waves propagating along the magnetic field [1]. Its formal derivation involves the Euler–Poisson system for uniformly magnetised plasmas [2]. The two-dimensional ZK equation has the form

$$u_t + u\partial_x u + \partial_x \Delta u = 0, \qquad u = u(t,x,y). \tag{1}$$

The ZK equation describes the behaviour of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in a uniform magnetic field. It generalised the well-known Korteweg de Vries (KdV) equation, it is not completely integrable and has a Hamiltonian.

There are several papers in which different generalisations of Equation (1), in two and three dimensions, have been studied. We shall now proceed to show some well-known generalised ZK equations in $(2+1)$-dimensions:

- The modified Zakharov–Kuznetsov equation [3–5]

$$u_t + au^{n/2}u_x + \left(u_{xx} + u_{yy}\right)_x = 0, \tag{2}$$

  where $a \neq 0$ and $n \geq 1$ are arbitrary constants.
- The ZK(m,n,k)

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \tag{3}$$

  where a, b, c are arbitrary constants while m, n and k are positive integers [6–8].
- ZK(m,n,k) equation with generalised evolution and time-dependent coefficients

$$(u^l)_t + a(t)(u^m)_x + b(t)(u^n)_{xxx} + c(t)(u^k)_{yyx} = \alpha(t)u^l, \tag{4}$$

where $a(t), b(t), c(t), \alpha(t)$ are time-dependent coefficients and m, n, k, l are integers [9,10].

In the present paper, we are interested in studying a generalised ZK equation involving arbitrary functions. The simplest generalization would be

$$u_t + u\partial_x f(u) + \partial_x \Delta g(u) = 0. \tag{5}$$

In the previous Equation (5), $\partial_x \Delta g(u)$ represents the dispersion term, with the same dispersion effect for directions x and y. However, in order to get a more general classification, we consider the case in which the dispersion effect could be different in the two directions.

Therefore, we study the following $(2+1)$-dimensional generalised Zakharov–Kuznetsov equation involving three arbitrary functions (gZK)

$$u_t + f(u)_x + g(u)_{xxx} + h(u)_{yyx} = 0, \tag{6}$$

where $f(u)$, $g(u)$ and $h(u)$ are nonzero arbitrary functions.

When functions f, g and h are p-power nonlinearities, it was proved that the equation admits new solitary pattern, solitary wave and singular solitary wave solutions. Moreover, the authors proved a theorem on the convergence of the homotopy analysis method to solve this equation [11].

Due to the physical background of the equation, the objectives of this work are twofold. First, we seek point symmetries of Equation (6). Symmetries of a partial differential equation (PDE) leave invariant the whole space of solutions of the equation and, in that way, symmetries can be used to obtain reductions and exact group-invariant solutions. The Lie method determines all the Lie symmetries that a given PDE admits [12–15]. Moreover, for different cases of the ZK Equation (2)–(4), the Lie method has been shown as a useful tool to get exact solutions, including soliton solutions, cnoidal waves and travelling wave solutions [16–19]. In Section 3, line soliton solutions have been obtained.

Second, we aim to study the conservation laws of the gZK Equation (6) with physical interest. Conservation laws provide basic conserved quantities for all solutions, such as, mass, energy and so on. By using the multiplier method [20–22], we find all local conservation laws admitted by gZK Equation (6). Conservation laws for some special cases of the previous Equations (2)–(4) can be found in [23–25]. Consequently, the study of conservation laws of Equation (6) is also motivated to determine special cases for the arbitrary functions, f, g and h, with extra conservation laws.

Finally, making use of the fact that an equation admits a variational structure if and only if the Frechet derivative of the equation is self-adjoint (i.e., the Helmholtz conditions hold) [15,22], we present the case for the arbitrary functions f, g and h when the potential form of the gZK Equation (6) possesses a variational structure and we write the associated Hamiltonian formulation for the gZK Equation (6). Next, we determine the variational symmetries of the potential equation admitting the variational structure.

2. Lie Symmetries

We apply the Lie method to the two-dimensional gZK Equation (6). Therefore, we consider a one-parameter Lie group of point transformations acting on independent and dependent variables

$$\begin{aligned} \tilde{t} &= t + \varepsilon\tau(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{x} &= x + \varepsilon\xi_1(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{y} &= y + \varepsilon\xi_2(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\eta(t,x,y,u) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{7}$$

where ε is the group parameter. A Lie point symmetry for Equation (6) is a transformation (7) that leaves (6) invariant. From (7) one can obtain the associated vector field which is given by

$$X = \tau(t,x,y,u)\partial_t + \xi_1(t,x,y,u)\partial_x + \xi_2(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u. \tag{8}$$

A generator (8) is a point symmetry of gZK Equation (6) if

$$X^{(3)}\left(u_t + f(u)_x + g(u)_{xxx} + h(u)_{yyx}\right) = 0 \tag{9}$$

when Equation (6) holds. Here, $X^{(3)}$ is the third prolongation of the vector field (8) defined by

$$X^{(3)} = X + \eta^{(1)}_{i_1}\frac{\partial}{\partial u_{i_1}} + \eta^{(2)}_{i_1 i_2}\frac{\partial}{\partial u_{i_1 i_2}} + \eta^{(3)}_{i_1 i_2 i_3}\frac{\partial}{\partial u_{i_1 i_2 i_3}},$$

with coefficients

$$\begin{aligned}
\eta^{(1)}_{i_1} &= D_{i_1}(\eta) - u_t D_{i_1}(\tau) - u_x D_{i_1}(\xi_1) - u_y D_{i_1}(\xi_2),\\
\eta^{(2)}_{i_1 i_2} &= D_{i_2}(\eta^{(1)}_{i_1}) - u_{i_1 t} D_{i_2}(\tau) - u_{i_1 x} D_{i_2}(\xi_1) - u_{i_1 y} D_{i_2}(\xi_2),\\
\eta^{(3)}_{i_1 i_2 i_3} &= D_{i_3}(\eta^{(2)}_{i_1 i_2}) - u_{i_1 i_2 t} D_{i_3}(\tau) - u_{i_1 i_2 x} D_{i_3}(\xi_1) - u_{i_1 i_2 y} D_{i_3}(\xi_2),
\end{aligned}$$

where D is the total derivative operator, $u_i = \frac{\partial u}{\partial x_i}$, $i = 1,2,3$ with $x_1 = t$, $x_2 = x$ and $x_3 = y$, and $i_j = 1,2,3$ for $j = 1,2,3$.

Invariance condition (9) can be expanded and split with respect to the derivatives of u obtaining an overdetermined linear system of equations for the infinitesimals $\tau(t,x,y,u)$, $\xi_1(t,x,y,u)$, $\xi_2(t,x,y,u)$, $\eta(t,x,y,u)$ along with the arbitrary functions $f(u)$, $g(u)$ and $h(u)$. Thus we have the following result:

Theorem 1. *The classification of point symmetries admitted by the* $(2+1)$*-dimensional generalised Zakharov–Kuznetsov Equation (6) is given by the following cases:*

(i) *For arbitrary $f(u)$, $g(u)$ and $h(u)$, the admitted point symmetries are generated by:*

$X_1 = \partial_t$,
time-translation.

$X_2 = \partial_x$,
space-translation.

$X_3 = \partial_y$,
space-translation.

(ii) *Additional point symmetries are admitted by the* $(2+1)$*-dimensional generalised ZK Equation (6) in the following cases:*

- *For arbitrary $g(u)$, $h(u)$, and $f(u) = f_1 u + f_2$, there is an extra generator:*

  $X_4 = 3t\partial_t + (2f_1 t + x)\,\partial_x + y\partial_y$,
  dilation combined with a Galilean boost.

- *For $f(u) = f_1(u+a)^m + f_2 u + f_3$, $g(u) = g_1(u+a)^n + g_2$ and $h(u) = h_1(u+a)^q + h_2$, the additional symmetry is*

  $X_5 = (3m - n - 2)t\partial_t + (2f_2(m-1)t + (m-n)x)\,\partial_x + (m-q)y\partial_y - 2(u+a)\partial_u$,
  shift combined with a scaling and a Galilean boost.

  Moreover:

- *For $m = 1$ and $f_2 = 0$, generator X_4 is also admitted.*
- *For $m = 2$, $n = q = 1$, $a = 0$, the following generator is admitted*

$$X_6 = 2f_1 t\partial_x + \partial_u,$$
Galilean boost

- *For $n = 1$, $m = q = -\dfrac{1}{3}$ and $\dfrac{f_1}{h_1} > 0$, two additional generators are admitted*

$$\begin{aligned} X_7 &= \left(2\cos^2\left(\sqrt{\tfrac{f_1}{h_1}}y\right) - 1\right)\partial_y + 3\sqrt{\tfrac{f_1}{h_1}}\sin\left(2\sqrt{\tfrac{f_1}{h_1}}y\right)(u+a)\partial_u, \\ X_8 &= \sin\left(2\sqrt{\tfrac{f_1}{h_1}}y\right)\partial_y - 3\sqrt{\tfrac{f_1}{h_1}}\left(2\cos^2\left(\sqrt{\tfrac{f_1}{h_1}}y\right) - 1\right)(u+a)\partial_u. \end{aligned}$$

- *For $n = 1$, $m = q = -\dfrac{1}{3}$ and $\dfrac{f_1}{h_1} < 0$, two additional generators are admitted*

$$\begin{aligned} X_9 &= \exp\left(2\sqrt{-\tfrac{f_1}{h_1}}y\right)\left(\partial_y - 3\sqrt{-\tfrac{f_1}{h_1}}(u+a)\partial_u\right), \\ X_{10} &= \exp\left(-2\sqrt{-\tfrac{f_1}{h_1}}y\right)\left(\partial_y + 3\sqrt{-\tfrac{f_1}{h_1}}(u+a)\partial_u\right). \end{aligned}$$

- *For $f(u) = f_1(u+a)^m + f_2u + f_3$, $g(u) = g_1\ln(u+a) + g_2$, and $h(u) = h_1(u+a)^q + h_2$,*

$$X_5|_{n=0} = (3m-2)t\partial_t + (2f_2(m-1)t + mx)\,\partial_x + (m-q)y\partial_y - 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

Moreover, for $m = 1$ and $f_2 = 0$, X_4 is also admitted.

- *For $f(u) = f_1(u+a)^m + f_2u + f_3$, $g(u) = g_1(u+a)^n + g_2$ and $h(u) = h_1\ln(u+a) + h_2$*

$$X_5|_{q=0} = (3m-n-2)t\partial_t + (2f_2(m-1)t + (m-n)x)\,\partial_x + my\partial_y - 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

As before, for $m = 1$ and $f_2 = 0$, X_4 is admitted.

- *For $f(u) = f_1(u+a)^m + f_2u + f_3$, $g(u) = g_1\ln(u+a) + g_2$ and $h(u) = h_1\ln(u+a) + h_2$*

$$X_5|_{n=q=0} = (3m-2)t\partial_t + (2f_2(m-1)t + mx)\,\partial_x + my\partial_y - 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

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- *For $f(u) = f_1\ln(u+a) + f_2u + f_3$, $g(u) = g_1(u+a)^n + g_2$ and $h(u) = h_1(u+a)^q + h_2$*

$$X_5|_{m=0} = (n+2)t\partial_t + (2f_2t + nx)\,\partial_x + qy\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For $f(u) = f_1\ln(u+a) + f_2u + f_3$, $g(u) = g_1\ln(u+a) + g_2$ and $h(u) = h_1(u+a)^q + h_2$*

$$X_5|_{m=n=0} = 2t\partial_t + 2f_2t\partial_x + qy\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1 \ln(u+a) + f_2 u + f_3$, $g(u) = g_1(u+a)^n + g_2$ *and* $h(u) = h_1 \ln(u+a) + h_2$

$$X_5|_{m=q=0} = (n+2)t\partial_t + (2f_2t + nx)\,\partial_x + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1 \ln(u+a) + f_2 u + f_3$, $g(u) = g_1 \ln(u+a) + g_2$ *and* $h(u) = h_1 \ln(u+a) + h_2$

$$X_5|_{m=n=q=0} = t\partial_t + f_2 t\partial_x + (u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1(u+a)\,(\ln(u+a)-1) + f_2 u + f_3$, $g(u) = g_1(u+a)^n + g_2$ *and* $h(u) = h_1(u+a)^q + h_2$

$$X_{11} = (n-1)t\partial_t + (2f_1t + (n-1)x)\,\partial_x + (q-1)y\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1(u+a)\,(\ln(u+a)-1) + f_2 u + f_3$, $g(u) = g_1 \ln(u+a) + g_2$ *and* $h(u) = h_1(u+a)^q + h_2$

$$X_{11}|_{n=0} = -t\partial_t + (2f_1t - x)\,\partial_x + (q-1)y\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1(u+a)\,(\ln(u+a)-1) + f_2 u + f_3$, $g(u) = g_1(u+a)^n + g_2$ *and* $h(u) = h_1 \ln(u+a) + h_2$

$$X_{11}|_{q=0} = (n-1)t\partial_t + (2f_1t + (n-1)x)\,\partial_x - y\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1(u+a)\,(\ln(u+a)-1) + f_2 u + f_3$, $g(u) = g_1 \ln(u+a) + g_2$ *and* $h(u) = h_1 \ln(u+a) + h_2$

$$X_{11}|_{n=q=0} = -t\partial_t + (2f_1t - x)\,\partial_x - y\partial_y + 2(u+a)\partial_u,$$
shift combined with a scaling and a Galilean boost.

- *For* $f(u) = f_1 e^{mu} + f_2 u + f_3$, $g(u) = g_1 e^{nu} + g_2$ *and* $h(u) = h_1 e^{qu} + h_2$

$$X_{12} = (3m-n)t\partial_t + (2mf_2t + (m-n)x)\,\partial_x + (m-q)y\partial_y - 2\partial_u,$$
dilation combined with a Galilean boost.

- *For* $f(u) = f_1 e^{mu} + f_2 u + f_3$, $g(u) = g_1 u + g_2$ *and* $h(u) = h_1 e^{qu} + h_2$

$$X_{12}|_{n=0} = 3mt\partial_t + m\,(2f_2t + x)\,\partial_x + (m-q)y\partial_y - 2\partial_u,$$
dilation combined with a Galilean boost.

- *For* $f(u) = f_1 e^{mu} + f_2 u + f_3$, $g(u) = g_1 e^{nu} + g_2$ *and* $h(u) = h_1 u + h_2$

$$X_{12}|_{q=0} = (3m-n)t\partial_t + (2mf_2t + (m-n)x)\,\partial_x + my\partial_y - 2\partial_u,$$
dilation combined with a Galilean boost.

- *For* $f(u) = f_1 e^{mu} + f_2 u + f_3$, $g(u) = g_1 u + g_2$ *and* $h(u) = h_1 u + h_2$

$$X_{12}|_{n=q=0} = 3mt\partial_t + m\,(2f_2t + x)\,\partial_x + my\partial_y - 2\partial_u,$$
dilation combined with a Galilean boost.

In the above, $f_1 \neq 0$, f_2, f_3, $g_1 \neq 0$, g_2, $h_1 \neq 0$, h_2, a, $m \neq 0$, $n \neq 0$ *and* $q \neq 0$ *are arbitrary constants.*

3. Line Soliton Solution

A line soliton solution

$$u(t,x,y) = w(x + \mu y - \lambda t), \tag{10}$$

is a travelling wave solution of the form $u(t,x,y) = w\,(K \cdot \mathrm{x} - \lambda t)$ with $\mathrm{x} = (x,y)$ and $K = (1,\mu)$. The solution depends on two parameters, where μ represents the direction of propagation of the line soliton, i.e., the inclination of the line soliton in the (x,y)-plane is $\mu = \tan\alpha$, with α the angle from the positive y-axis in counterclockwise direction; whereas $c = \frac{\lambda}{|K|} = \frac{\lambda}{1+\mu^2}$ represents the speed of the wave.

In this section, we will determine line soliton solutions for the $(2+1)$-dimensional generalised Zakharov–Kuznetsov Equation (6). Taking into account the line soliton formulation (10) into Equation (6) we obtain the nonlinear third-order ODE

$$\begin{aligned}&\left(\mu^2 h'(w) + g'(w)\right) w'''(z) + 3\left(\mu^2 h''(w) + g''(w)\right) w'(z)w''(z)\\&\quad + \left(\mu^2 h'''(w) + g'''(w)\right)\left(w'(z)\right)^3 + \left(f'(w) - \lambda\right) w'(z) = 0,\end{aligned} \tag{11}$$

where $z = x + \mu y - \lambda t$. Integrating Equation (11) with respect to z we obtain

$$\Big(\mu^2 h'(w) + g'(w)\Big)\, w''(z) + \Big(\mu^2 h''(w) + g''(w)\Big)\left(w'(z)\right)^2 + f(w) - \lambda w = 0, \tag{12}$$

omitting the constant of integration. We recall that we are interested in a solitary wave solution, therefore we impose w, w', $w'' \longrightarrow 0$ as $z \longrightarrow \pm\infty$. On the other hand, if one considers $f(w) = \lambda w$, $g(w) = -\mu^2 h(w)$, Equation (12) is solved. Thus, it is obvious that

$$u(t,x,y) = a\ sech^m\left(b(x + \mu y - \lambda t)\right), \tag{13}$$

with m, a and b nonzero arbitrary constants, is a smooth solution of Equation (6) for $f(u) = \lambda u$, $g(u) = -\mu^2 h(u)$ satisfying the desired asymptotic conditions.

In [9], the authors proved the existence of solitary wave solutions for Equation (6) with $f(u)$, $g(u)$ and $h(u)$ power functions. As far as we are concerned, line soliton solutions given by (13) have not been previously obtained.

4. Conservation Laws

We apply the direct method of the multipliers in order to obtain conservation laws of the $(2+1)$-dimensional generalised Zakharov–Kuznetsov equation with three arbitrary functions (6).

In two dimensions, a local conservation law is a divergence expression of the form

$$D_t T + D_x X + D_y Y = 0 \tag{14}$$

that holds for the whole set of solutions $u(t,x,y)$.

T is called conserved density, and X, Y are spatial fluxes. The three are functions depending on t, x, y, u and spatial derivatives of u because, by using the equation, time derivatives can be vanished. A conserved current is an expression (T, X, Y).

Every non-trivial local conservation law (14) for Equation (6) can be expressed in its characteristic form

$$D_t T + D_x X + D_y Y = \Big(u_t + f(u)_x + g(u)_{xxx} + h(u)_{yyx}\Big) Q, \tag{15}$$

where

$$Q = \frac{\delta T}{\delta u} \tag{16}$$

is the so called multiplier. Solving the following system of determining equations

$$\frac{\delta}{\delta h}\Big(\left(u_t + f(u)_x + g(u)_{xxx} + h(u)_{yyx}\right)Q\Big) = 0, \tag{17}$$

that holds off the set of solutions of Equation (6), all multipliers Q can be found.

All non-trivial conservation laws arise from multipliers (16). Moreover, as conservation laws of physical importance come from low-order multipliers, we have considered multipliers of the form $Q(t, x, y, u, u_x, u_y, u_{xx}, u_{yy})$. Similarly to Lie symmetries, here the determining Equation (17) for low-order multipliers splits with respect to u_t, u_{xxx}, u_{xyy} and their differential consequences, yielding an overdetermined system of equations for Q and the arbitrary functions which Equation (6) involves.

Once the multipliers are found, the corresponding non-trivial local conservation laws can be obtained by integrating the characteristic Equation (15).

Theorem 2. *All low-order conservation laws of the* $(2+1)$*-dimensional gZK Equation (6) are*

(i) *For arbitrary* $f(u)$, $g(u)$ *and* $h(u)$, *there is one multiplier* $Q_1 = F(u)$ *of which the conservation law is*

$$\begin{aligned} T_1 &= F(y)\,u, \\ X_1 &= F(y)\,g_{uu}{u_x}^2 + \left(\frac{d^2}{dy^2}F(y)\right)h(u) + F(y)\,(g_u u_{xx} + f(u)), \\ Y_1 &= F(y)\,h_u u_{xy} + F(y)\,h_{uu}u_x u_y - \left(\frac{d}{dy}F(y)\right)h_u u_x. \end{aligned} \tag{18}$$

(ii) *Additional conservation laws are admitted in the following cases:*

- *For arbitrary* $h(u)$, $f(u) = f_1\,h(u) + f_2\,u + f_3$, *and* $g(u) = g_1 u + g_2$, *there are seven extra multipliers.*
  - *For the multiplier* $Q_2 = e^{\sqrt{-f_1}y}$,

$$\begin{aligned} T_2 &= e^{\sqrt{-f_1}y}u, \\ X_2 &= e^{\sqrt{-f_1}y}\left(-h_u\sqrt{-f_1}u_y + f_1\,h(u) + f_2\,u + g_1\,u_{xx}\right), \\ Y_2 &= e^{\sqrt{-f_1}y}\left(h_{uu}u_x u_y + h_u u_{xy}\right). \end{aligned} \tag{19}$$

  - *For the multiplier* $Q_3 = e^{-\sqrt{-f_1}y}$,

$$\begin{aligned} T_3 &= e^{-\sqrt{-f_1}y}u, \\ X_3 &= \left(h_u\sqrt{-f_1}u_y + f_1\,h(u) + f_2\,u + g_1\,u_{xx}\right)e^{-\sqrt{-f_1}y}, \\ Y_3 &= \left(h_{uu}u_x u_y + h_u u_{xy}\right)e^{-\sqrt{-f_1}y}. \end{aligned} \tag{20}$$

  - *For the multiplier* $Q_4 = \frac{e^{\sqrt{-f_1}y}\sqrt{g_1}}{\sqrt{f_2}}\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)$,

$$\begin{aligned} T_4 = {} & \frac{e^{\sqrt{-f_1}y}\sqrt{g_1}u}{\sqrt{f_2}}\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right), \\ X_4 = {} & -\frac{e^{\sqrt{-f_1}y}}{\sqrt{f_2}}\left(\left(\sqrt{g_1}\sqrt{-f_1}h_u u_y - \sqrt{g_1}h(u)\,f_1 - {g_1}^{3/2}u_{xx}\right)\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\ & \left. + \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\sqrt{f_2}g_1\,u_x\right), \\ Y_4 = {} & \frac{e^{\sqrt{-f_1}y}}{\sqrt{f_2}}\left(\sqrt{g_1}\left(h_{uu}u_x u_y + h_u u_{xy}\right)\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right) \\ & + \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)h(u)\,\sqrt{f_2}\sqrt{-f_1}\Bigg). \end{aligned} \tag{21}$$

– *For the multiplier* $Q_5 = \frac{\sqrt{g_1}}{\sqrt{f_2}\mathrm{e}^{\sqrt{-f_1}y}} \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)$,

$$\begin{aligned}
T_5 = &\quad \frac{\sqrt{g_1}\mathrm{e}^{-\sqrt{-f_1}y}u}{\sqrt{f_2}} \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right), \\
X_5 = &\quad -\frac{\mathrm{e}^{-\sqrt{-f_1}y}}{\sqrt{f_2}} \left(\left(-\sqrt{g_1}\sqrt{-f_1}h_u u_y - \sqrt{g_1}h\left(u\right)f_1 - {g_1}^{3/2}u_{xx}\right)\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\sqrt{f_2}g_1\, u_x\right), \\
Y_5 = &\quad -\frac{\mathrm{e}^{-\sqrt{-f_1}y}}{\sqrt{f_2}}\left(-\sqrt{g_1}\left(h_{uu}u_x u_y + h_u u_{xy}\right)\sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)h\left(u\right)\sqrt{f_2}\sqrt{-f_1}\right).
\end{aligned} \tag{22}$$

– *For the multiplier* $Q_6 = -\frac{\sqrt{g_1}}{\sqrt{f_2}\mathrm{e}^{\sqrt{-f_1}y}} \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)$,

$$\begin{aligned}
T_6 = &\quad -\frac{\sqrt{g_1}\mathrm{e}^{-\sqrt{-f_1}y}u}{\sqrt{f_2}} \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right), \\
X_6 = &\quad -\frac{\mathrm{e}^{-\sqrt{-f_1}y}}{\sqrt{f_2}} \left(\left(\sqrt{g_1}\sqrt{-f_1}h_u u_y + \sqrt{g_1}h\left(u\right)f_1 + {g_1}^{3/2}u_{xx}\right)\cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + g_1\, \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)u_x\sqrt{f_2}\right), \\
Y_6 = &\quad -\frac{\mathrm{e}^{-\sqrt{-f_1}y}}{\sqrt{f_2}}\left(\sqrt{g_1}\left(h_{uu}u_x u_y + h_u u_{xy}\right)\cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\sqrt{-f_1}h\left(u\right)\sqrt{f_2}\right).
\end{aligned} \tag{23}$$

– *For the multiplier* $Q_7 = -\frac{\mathrm{e}^{\sqrt{-f_1}y}\sqrt{g_1}}{\sqrt{f_2}} \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)$,

$$\begin{aligned}
T_7 = &\quad -\frac{\mathrm{e}^{\sqrt{-f_1}y}\sqrt{g_1}u}{\sqrt{f_2}} \cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right), \\
X_7 = &\quad -\frac{\mathrm{e}^{\sqrt{-f_1}y}}{\sqrt{f_2}} \left(\left(-\sqrt{g_1}\sqrt{-f_1}h_u u_y + \sqrt{g_1}h\left(u\right)f_1 + {g_1}^{3/2}u_{xx}\right)\cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + g_1\, \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)u_x\sqrt{f_2}\right), \\
Y_7 = &\quad \frac{\mathrm{e}^{\sqrt{-f_1}y}}{\sqrt{f_2}}\left(-\sqrt{g_1}\left(h_{uu}u_x u_y + h_u u_{xy}\right)\cos\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\right. \\
&\quad \left. + \sin\left(\frac{\sqrt{f_2}x}{\sqrt{g_1}}\right)\sqrt{-f_1}h\left(u\right)\sqrt{f_2}\right).
\end{aligned} \tag{24}$$

– *For the multiplier* $Q_8 = \frac{\left(\left(\mathrm{e}^{\sqrt{-f_1}y}\right)^2 + 1\right)(tf_2 - x)}{\mathrm{e}^{\sqrt{-f_1}y}f_2}$,

$$\begin{aligned}
T_8 = &\quad \frac{1}{f}_2 u\left(\mathrm{e}^{-\sqrt{-f_1}y} + \mathrm{e}^{\sqrt{-f_1}y}\right)\left(tf_2 - x\right), \\
X_8 = &\quad \frac{1}{f}_2\left(\left(h_u u_y\left(tf_2 - x\right)\sqrt{-f_1} + f_1\,\left(tf_2 - x\right)h\left(u\right)\right.\right. \\
&\quad \left. + {f_2}^2 tu + \left(g_1\, tu_{xx} - xu\right)f_2 - g_1\,\left(xu_{xx} - u_x\right)\right)\mathrm{e}^{-\sqrt{-f_1}y} \\
&\quad + \mathrm{e}^{\sqrt{-f_1}y}\left(-h_u u_y\left(tf_2 - x\right)\sqrt{-f_1} + f_1\,\left(tf_2 - x\right)h\left(u\right)\right. \\
&\quad \left.\left. + {f_2}^2 tu + \left(g_1\, tu_{xx} - xu\right)f_2 - g_1\,\left(xu_{xx} - u_x\right)\right)\right), \\
Y_8 = &\quad \frac{1}{f}_2\left(\left(\sqrt{-f_1}h\left(u\right) + \left(h_{uu}u_x u_y + h_u u_{xy}\right)\left(tf_2 - x\right)\right)\mathrm{e}^{-\sqrt{-f_1}y}\right. \\
&\quad \left. + \left(-\sqrt{-a1}h\left(u\right) + \left(h_{uu}u_x u_y + h_u u_{xy}\right)\left(tf_2 - x\right)\right)\mathrm{e}^{\sqrt{-f_1}y}\right).
\end{aligned} \tag{25}$$

- *For arbitrary $f(u)$, $h(u)$, and $g(u) = g_1 + g_2 h(u)$, there is an additional multiplier $Q_9 = h(u)$, whose associated conservation law is*

$$\begin{aligned} T_9 = & \int h(u)\,\mathrm{d}u, \\ X_9 = & \int h(u) f_u\,\mathrm{d}u + h(u) h_{uu} g_2\, u_x{}^2 + \tfrac{1}{2}\left(-g_2\, u_x{}^2 - u_y{}^2\right) h_u^2 \\ & + h(u) h_u g_2\, u_{xx}, \\ Y_9 = & \; h(u)\left(h_{uu} u_x u_y + h_u u_{xy}\right). \end{aligned} \tag{26}$$

- *For arbitrary $h(u)$, $f(u) = f_1 h(u) + f_2 u + f_3$, and $g(u) = g_1 + g_2 h(u)$, the multiplier is*

$$Q_{10} = -4\,\frac{\sqrt{-g_2}k_1 \mathrm{e}^{k_1 y}}{4\,k_1{}^2 + f_1}\mathrm{e}^{-1/4\,\frac{\sqrt{-g_2}f_1\,f_2\,t}{g_2\,k_1}}\,\mathrm{e}^{\frac{\sqrt{-g_2}k_1\,x}{g_2}}\,\mathrm{e}^{1/4\,\frac{\sqrt{-g_2}f_1\,x}{g_2\,k_1}}\,\mathrm{e}^{-1/4\,\frac{yf_1}{k_1}}\left(\mathrm{e}^{\frac{\sqrt{-g_2}k_1 f_2\,t}{g_2}}\right)^{-1},$$

and its associated conservation law is

$$\begin{aligned} T_{10} = & -4\,\frac{\sqrt{-g_2}k_1 u}{4\,k_1^2+f_1}\mathrm{e}^{1/4\,\frac{-y\left(-4\,k_1{}^2+f_1\right)\sqrt{-g_2}+\left(4\,k_1{}^2+f_1\right)(tf_2-x)}{\sqrt{-g_2}\_c_1}}, \\ X_{10} = & \frac{16}{16\_c_1{}^3+4\,f_1\,k_1}\left(k_1\left(h_u u_x\left(k_1{}^2+f_1/4\right)\sqrt{-g_2} - k_1{}^2 u_x{}^2 g_2\,h_{uu}\right.\right. \\ & + k_1\left(u_y k_1{}^2 - k_1 g_2 u_{xx} - 1/4\,u_y f_1\right) h_u + \left(k_1^2 - f_1/4\right)^2 h(u) \\ & \left. -k_1^2 f_2\,u\right)\sqrt{-g_2}\mathrm{e}^{1/4\,\frac{-y\left(-4\,k_1{}^2+f_1\right)\sqrt{-g_2}+\left(4\,k_1{}^2+f_1\right)(tf_2-x)}{\sqrt{-g_2}k_1}}\Big), \\ Y_{10} = & \frac{4}{4\,k_1^3+f_1\,k_1}\left(-k_1{}^2\left(h_{uu} u_x u_y + h_u u_{xy}\right)\sqrt{-g_2}\right. \\ & \left. + h(u)\left(k_1{}^4 - 1/16\,f_1{}^2\right)\mathrm{e}^{1/4\,\frac{-y\left(-4\,k_1{}^2+f_1\right)\sqrt{-g_2}+\left(4\,k_1{}^2+f_1\right)(tf_2-x)}{\sqrt{-g_2}k_1}}\right). \end{aligned} \tag{27}$$

- *For $f(u) = f_3 + f_2\,u + f_1\,\mathrm{e}^{h_2 u}$, $g(u) = g_2 + g_1\,\mathrm{e}^{h_2 u}$, and $h(u) = h_1\,\mathrm{e}^{h_2 u} + h_3$, there are three multipliers.*

  - *For the multiplier $Q_{11} = \mathrm{e}^{h_2 u}$,*

$$\begin{aligned} T_{11} = & \frac{\mathrm{e}^{h_2 u}}{h_2}, \\ X_{11} = & \frac{1}{2}\frac{\mathrm{e}^{h_2 u}\left(\left(\left(b1\,u_x{}^2 - h_1\,u_y{}^2\right)h_2{}^2 + 2\,g_1\,h_2\,u_{xx} + f_1\right)h_2\,\mathrm{e}^{uh_2} + 2\,f_2\right)}{h_2}, \\ Y_{11} = & \mathrm{e}^{2\,h_2 u} h_1\,h_2\,\left(h_2\,u_x u_y + u_{xy}\right). \end{aligned} \tag{28}$$

  - *For the multiplier $Q_{12} = \frac{tf_1\,h_2\,\mathrm{e}^{h_2 u} + tf_2 - x}{f_1 h_2}$,*

$$\begin{aligned} T_{12} = & \frac{t\mathrm{e}^{h_2 u} f_1 + u(tf_2 - x)}{f_1\,h_2}, \\ X_{12} = & \frac{1}{2f_1 h_2}\left(-th_2 f_1\left(\left(-g_1 u_x{}^2 + h_1 u_y{}^2\right)h_2{}^2 - 2g_1 h_2 u_{xx} - f_1\right)\mathrm{e}^{2h_2 u}\right. \\ & + \left(2g_1 u_x{}^2\,(tf_2 - x)\,h_2{}^2 + 2g_1\,(f_2 t u_{xx} - x u_{xx} + u_x)\,h_2 + 4f_1\,\left(tf_2 - \tfrac{x}{2}\right)\right)\mathrm{e}^{h_2 u} \\ & \left. + 2f_2 u\,(tf_2 - x)\right), \\ Y_{12} = & \frac{h_1}{f_1}\,\mathrm{e}^{h_2 u}\left(tf_1\,c2\,\mathrm{e}^{h_2 u} + tf_2 - x\right)\left(h_2\,u_x u_y + u_{xy}\right). \end{aligned} \tag{29}$$

  - *For the multiplier*

$$\begin{aligned} Q_{13} = & \;4\,\frac{g_1\,h_1\,k_1\mathrm{e}^{k_1 y}}{\sqrt{-g_1\,h_1}\left(4\,k_1{}^2 h_1 + f_1\right)}\mathrm{e}^{\frac{k_1\sqrt{-g_1\,h_1}x}{g_1}}\,\mathrm{e}^{-1/4\,\frac{\sqrt{-g_1\,h_1}f_1\,f_2\,t}{g_1\,h_1\,k_1}}\,\mathrm{e}^{1/4\,\frac{\sqrt{-g_1\,h_1}f_1\,x}{g_1\,h_1\,k_1}}\,\mathrm{e}^{-1/4\,\frac{yf_1}{h_1\,k_1}} \\ & \left(\mathrm{e}^{\frac{k_1\sqrt{-g_1\,h_1}f_2\,t}{g_1}}\right)^{-1}, \end{aligned}$$

$$
\begin{aligned}
T_{13} = \;& 4\,\frac{g_1\,h_1\,k_1 u}{\sqrt{-g_1\,h_1}\left(4\,k_1{}^2 h_1+f_1\right)}\mathrm{e}^{1/4\,\frac{-\left(4\,k_1{}^2 h_1+f_1\right)\left(tf_2-x\right)\sqrt{-g_1\,h_1}-y g_1\,\left(-4\,k_1{}^2 h_1+f_1\right)}{g_1\,h_1\,k_1}}, \\
X_{13} = \;& 16\,\frac{1}{\sqrt{-g_1\,h_1}\left(16\,h_1\,k_1{}^3+4\,f_1\,k_1\right)} g_1\,\left(-k_1\,\left(k_1{}^2 h_1+f_1/4\right)\mathrm{e}^{h_2 u} u_x h_2\,\sqrt{-g_1\,h_1}\right. \\
& +\left(-h_1{}^2 k_1^4-h_1{}^2 h_2\,k_1{}^3 u_y+h_1\,\left(g_1\,h_2\,u_{xx}+g_1\,h_2{}^2 u_x{}^2+f_1/2\right)k_1{}^2\right. \\
& \left.\left.+1/4\,f_1\,h_1\,h_2\,k_1 u_y-1/16\,f_1{}^2\right)\mathrm{e}^{h_2 u}+f_2\,h_1\,k_1^2 u\right) \\
& \mathrm{e}^{1/4\,\frac{-\left(4\,k_1{}^2 h_1+f_1\right)\left(tf_2-x\right)\sqrt{-g_1\,h_1}-y g_1\,\left(-4\,k_1{}^2 h_1+f_1\right)}{g_1\,h_1\,k_1}}, \\
Y_{13} = \;& 16\,\frac{\left(h_1{}^2 k_1{}^4-1/16\,f_1{}^2\right)\sqrt{-g_1\,h_1}+k_1{}^2 g_1\,h_1{}^2 h_2\,\left(h_2\,u_x u_y+u_{xy}\right)}{\sqrt{-g_1\,h_1}\left(16\,h_1\,k_1^3+4\,f_1\,k_1\right)} \\
& \mathrm{e}^{1/4\,\frac{-4\left(tf_2-x\right)\left(k_1{}^2 h_1+f_1/4\right)\sqrt{-g_1\,h_1}+4\,g_1\,\left(k_1{}^2 y h_1+k_1 h_1\,h_2\,u-1/4\,y f_1\right)}{g_1\,h_1\,k_1}}.
\end{aligned}
\tag{30}
$$

- *For arbitrary $f(u)$, $g(u) = g_1 u + g_2$, and $h(u) = h_1 u + h_2$, there are two additional multipliers.*
  - *For the multiplier $Q_{14} = u$,*

$$
\begin{aligned}
T_{14} = \;& 1/2\,u^2, \\
X_{14} = \;& g_1 u\,u_{xx}-1/2\,h_1\,u_y{}^2-1/2\,g_1\,u_x{}^2+\int u\frac{\mathrm{d}}{\mathrm{d}u}f\left(u\right)\,\mathrm{d}u, \\
Y_{14} = \;& h_1 u\,u_{xy}.
\end{aligned}
\tag{31}
$$

  - *For the multiplier $Q_{15} = u_{yy} + \frac{g_1 u_{xx}}{h_1} + \frac{f(u)}{h_1}$,*

$$
\begin{aligned}
T_{15} = \;& 1/2\,\frac{-g_1\,u_x{}^2-h_1\,u_y{}^2+2\,\int f(u)\,\mathrm{d}u}{h_1}, \\
X_{15} = \;& 1/2\,\frac{g_1{}^2 u_{xx}{}^2+2\,g_1 h_1 u_{xx}\,u_{y,y}+h_1{}^2 u_{yy}{}^2+2\,g_1 f(u)\,u_{xx}+2\,h_1 f(u) u_{yy}+2\,g_1\,u_x+(f(u))^2}{h_1}, \\
Y_{15} = \;& u_x u_y.
\end{aligned}
\tag{32}
$$

- *For $f(u) = 1/2\,f_1 u^2 + f_2\,u + a3$, $g(u) = g_1 u + g_2$, and $h(u) = h_1 u + h_2$, there are two extra multipliers.*
  - *For the multiplier $Q_{16} = (-f_1 u - f_2)\,t + x$,*

$$
\begin{aligned}
T_{16} = \;& -1/2\,\left(\left(f_1 u+2\,f_2\right)t-2\,x\right)u, \\
X_{16} = \;& 1/6\,\left(-2\,f_1{}^2 u^3-6\,f_1\,f_2\,u^2+\left(\left(-6\,u_{xx} g_1-6\,h_1\,u_{yy}\right)f_1-6\,f_2{}^2\right)u\right. \\
& \left.+\left(3\,g_1\,u_x{}^2-3\,h_1\,u_y{}^2\right)f_1-6\,f_2\,\left(g_1 u_{xx}+h_1\,u_{yy}\right)\right)t+1/2\,f_1\,x u^2 \\
& +f_2 x u+1/6\,\left(6\,x u_{xx}-6\,u_x\right)g_1+h_1\,x u_{yy}, \\
Y_{16} = \;& h_1\,\left(f_1\,t u_x-1\right)u_y.
\end{aligned}
\tag{33}
$$

  - *For the multiplier $Q_{17} = u_{yy} + \frac{g_1 u_{xx}}{h_1} + 1/2\,\frac{u^2 f_1}{h_1}$,*

$$
\begin{aligned}
T_{17} = \;& 1/6\,\frac{f_1\,u^3-3\,g_1\,u_x{}^2-3\,h_1\,u_y{}^2}{h_1}, \\
X_{17} = \;& \frac{1}{24 h_1}\left(12\,g_1{}^2 u_{xx}{}^2+\left(24\,h_1 u_{xx} u_{yy}+12\,f_1\,u^2 u_{xx}+12\,u_x\left(f_2\,u_x+2\,u_t\right)\right)g_1\right. \\
& \left.+12\,h_1{}^2 u_{yy}{}^2+\left(12\,f_1 u^2\,u_{yy}-12\,f_2\,u_y{}^2\right)h_1+3\,f_1{}^2 u^4+4\,f_1\,f_2\,u^3\right), \\
Y_{17} = \;& \left(f_2\,u_x+u_t\right)u_y.
\end{aligned}
\tag{34}
$$

In the above, $f_1 \neq 0$, f_2, f_3, $g_1 \neq 0$, g_2, $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$ and $k_1 \neq 0$ are arbitrary constants.

5. Variational Structure and Hamiltonian Formulation

The $(2+1)$-dimensional generalised Zakharov–Kuznetsov Equation (6) can be written in potential form

$$
v_{tx} + f(v_x)_x + g(v_x)_{xxx} + h(v_x)_{yyx} = 0,
\tag{35}
$$

in terms of the potential

$$u = v_x. \tag{36}$$

It is straightforward to prove that the potential Equation (35) admits a local Lagrangian structure

$$\frac{\delta L}{\delta v} = 0 \tag{37}$$

if and only if $g(v_x) = g_1 v_x + g_2$ and $h(v_x) = h_1 v_x + h_2$, where $\delta L/\delta v$ is the variational derivative with respect to the variable v

$$\frac{\delta}{\delta v} = \partial_v - D_t\partial_{v_t} - D_x\partial_{v_x} - D_y\partial_{v_y} + D_t^2\partial_{v_{tt}} + D_x^2\partial_{v_{xx}} + D_y^2\partial_{v_{yy}} + D_tD_x\partial_{v_{tx}} + D_tD_y\partial_{v_{ty}} + D_xD_y\partial_{v_{xy}} + \cdots, \tag{38}$$

and where the Lagrangian is given by

$$L = \frac{1}{2}g_1 v_{xx}^2 + \frac{1}{2}h_1 v_{xy}^2 - \frac{1}{2}v_x v_y - \int f(v_x)\, dv_x \tag{39}$$

in terms of the variable v.

For $g(u) = g_1 u + g_2$ and $h(u) = h_1 u + h_2$, the variational structure yields a Hamiltonian formulation for the gZK Equation (6), given by

$$u_t = D_x\Big(\frac{\delta H}{\delta u}\Big) \tag{40}$$

where D_x is the Hamiltonian operator and

$$H = \int_{\mathbb{R}^2} \Big(-\int f(u)\, du - \frac{1}{2}g_1 u_x^2 - \frac{1}{2}h_1 u_y^2 \Big)\, dx\, dy \tag{41}$$

is the Hamiltonian density.

Variational Symmetries of the gZK Potential Equation

The $(2+1)$-dimensional gZK potential Equation (35) with $g(v_x) = g_1 v_x + g_2$ and $h(v_x) = h_1 v_x + h_2$, given by

$$v_{tx} + f(v_x)_x + g_1 v_{xxxx} + h_1 v_{yyxx} = 0, \tag{42}$$

possesses a local Lagrangian structure (37) in terms of the Lagrangian (39), which we use to find the variational symmetries of Equation (42).

A generator

$$Y = \tau(t,x,y,v)\partial_t + \xi_1(t,x,y,v)\partial_x + \xi_2(t,x,y,v)\partial_y + \eta(t,x,y,v)\partial_v, \tag{43}$$

is a Lie point symmetry of the gZK potential Equation (42) if

$$Y^{(4)}\Big(v_{tx} + f(v_x)_x + g_1 v_{xxxx} + h_1 v_{yyxx}\Big) = 0 \tag{44}$$

when Equation (42) holds. Here $Y^{(4)}$ is the fourth prolongation of (43).

The point symmetry (43) has an equivalent characteristic form given by

$$\hat{Y} = P\partial_v, \quad P = \eta - \tau v_t - \xi_1 v_x - \xi_2 v_y \tag{45}$$

in terms of the symmetry characteristic P.

A symmetry (45) will be a variational symmetry if and only if it leaves invariant the Lagrangian density L up to a total divergence,

$$\hat{Y}L = D_t\Psi^t + D_x\Psi^x + D_y\Psi^y, \tag{46}$$

with Ψ^t, Ψ^x, Ψ^y depending on t, x, y, v, and derivatives of v. The invariance condition is equivalent to the equation:

$$\frac{\delta}{\delta v}\left(P\frac{\delta L}{\delta v}\right) = 0, \tag{47}$$

where $\frac{\delta}{\delta v}$ is the variational derivative (38), in terms of the Lagrangian functional L and the characteristic P [15,22]. In order to determine the variational symmetries of the ZK potential Equation (42), we verify which of the symmetries (43) satisfy the variational symmetry condition (47). We are interested in the cases when $f''(v_x) \neq 0$, since the Equation (42) is then nonlinear. Thus, we obtain the following results.

Theorem 3. *The classification of point symmetries admitted by the* $(2+1)$*-dimensional generalised Zakharov–Kuznetsov potential Equation (42) is given by the following cases:*

(i) *For arbitrary* $f(v_x)$, g_1 *and* h_1, *the admitted point symmetries are generated by:*

$$Y_1 = \partial_t,$$
time-translation.

$$Y_2 = \partial_x,$$
space-translation.

$$Y_3 = \partial_y,$$
space-translation.

$$Y_4 = F(t,y)\partial_v.$$

(ii) *Additional point symmetries are admitted by the (2+1)-dimensional generalised ZK potential Equation (6) in the following cases:*

- *For* $f(v_x) = f_1(v_x + a)^m + f_2 v_x + f_3$,

  $$Y_5 = (m-1)3t\partial_t + (m-1)\left(2f_2 t + x\right)\partial_x + (m-1)y\partial_y + ((m-3)v - 2ax)\partial_v,$$
  scaling combined with a Galilean boost.

- *For* $f(v_x) = \frac{1}{6}f_1 v_x^3 + \frac{1}{2}f_2 v_x^2 + f_3 v_x + f_4$,

  $$Y_6 = 3f_1 t\partial_t - \left(f_1(2f_3 t + x) - tf_2^2\right)\partial_x - f_1 y\partial_y + f_2 x\partial_v,$$
  dilation combined with a Galilean boost.

- *For* $f(v_x) = \frac{1}{2}f_1 v_x^2 + f_2 v_x + f_3$,

  $$Y_7 = f_1 t\partial_x + (x - f_2 t)\partial_v,$$
  Galilean boost.

  $$Y_8 = 3t\partial_t + \left(2f_2 t + x\right)\partial_x + y\partial_y - v\partial_v,$$
  scaling combined with a Galilean boost.

- *For* $f(v_x) = f_1 v_x^3 + f_2 v_x + f_3$,

$$Y_9 = 3t\partial_t + (2f_2 t + x)\,\partial_x + y\partial_y,$$
dilation combined with a Galilean boost.

Theorem 4. *The* $(2+1)$*-dimensional gZK potential Equation (42) admits the variational point symmetries spanned by* Y_1, Y_2, Y_3, Y_4 *and* Y_7*. Symmetry* Y_5 *is only variational for* $m = \frac{7}{3}$ *and symmetry* Y_6 *is only variational for* $f_1 = 0$*, which is equivalent to* Y_7*. Finally,* Y_8 *and* Y_9 *are not variational symmetries.*

Noether's theorem states that there is a one-to-one correspondence between conservation laws and variational symmetries for any equation admitting a variational structure [15]. The correspondence is equivalent to the condition:

$$Q = P, \tag{48}$$

where Q is the conservation law multiplier and P is the variational symmetry characteristic. Therefore, for each of the variational symmetries of the gZK potential Equation (42) in Theorem 4, there is a corresponding conservation law associated to the multiplier $Q = P$.

We focus our attention on the variational symmetries and conservation laws of the gZK potential Equation (42) that correspond to local symmetries and local conservation laws of the gZK Equation (6). We observe that the space-translation symmetry Y_2 of the potential Equation (42) correspond to the multiplier $Q = v_x$ of the potential Equation (42), that yields the multiplier $Q_{14} = u$ and conservation law (31) of gZK Equation (6). We also observe that the Galilean boost symmetry Y_7 of the potential Equation (42) corresponds to the multiplier $Q = x - f_2 t - f_1 t v_x$ of the potential Equation (42), that yields the multiplier $Q_{16} = x - f_2 t - f_1 t u$ and conservation law (33) of gZK Equation (6).

6. Conclusions

In this paper, we have considered a generalised ZK equation in $(2+1)$-dimensions depending on three arbitrary functions (6). We have performed a classification of point symmetries admitted by Equation (6) depending on the arbitrary functions $f(u)$, $g(u)$ and $h(u)$. We have shown that if $f(u)$, $g(u)$ and $h(u)$ satisfy certain conditions, Equation (6) admits line soliton solutions. To our knowledge, these solutions have not appeared previously in the literature. We have constructed multipliers of the generalised ZK Equation (6) having dependence on dependent variables, independent variables, and derivatives of dependent variables up to second order. After computing multipliers, we have derived the corresponding low-order local conservation laws. Finally, we have presented the potential form of the gZK Equation (6) and obtained the Lagrangian structure in the case when g and h are linear functions. In addition, we have presented the associated Hamiltonian formulation. Moreover, we have determined the variational symmetries of the potential equation admitting a Lagrangian structure (42).

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Abbreviations

The following abbreviations are used in this manuscript:

ZK	Zakharov–Kuznetsov equation
KdV	Korteweg de Vries
gZK	Generalised Zakharov–Kuznetsov equation with three arbitrary functions
PDE	Partial differential equation

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