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Stochastic orders and co-risk measures under positive dependence

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ABSTRACT

Conditional risk measures (or co-risk measures) and risk contribution measures are increasingly used in actuarial portfolio analysis to evaluate the systemic risk, which is related to the risk that the failure or loss of a component spreads to another component or even to the whole portfolio: while co-risk measures are risk-adjusted versions of measures usually employed to assess isolate risks, risk contribution measures quantify how a stress situation for a component affects another one. In this paper, we provide sufficient conditions under which two random vectors could be compared in terms of CoVaR (conditional value-at-risk), CoES (conditional expected shortfall) and different risk contribution measures. Conditions are given in terms of the increasing convex order, the dispersive order and the excess wealth order of the marginals under some assumptions of positive dependence.

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1. Introduction

Consistency of risk measures with respect to stochastic orderings is an important topic in actuarial theory. Given two risks X and Y with respective distribution functions F and G, the literature on univariate risks contains many results of the form

$$X \prec_P Y \text{ implies } \rho[X] \leq \rho[Y], \text{ for all } \rho \in C, \tag{1}$$

where $\rho[X]$ and $\rho[Y]$ are two risk measures belonging to certain class C and $\prec_P$ denotes the ordering that is induced by some distributional comparison principle P. These results are of interest, for example, in the study of parametric families of risk distributions. Since these families are often ordered under different stochastic orders in terms of their parameters, results of the form (1) provide a way to know how changes in the parameters affect the risk of losses independently of the choice of ρ (whenever it belongs to C). Thus, for example, the well-known result[1]

$$X \leq_{st} Y \text{ implies } \mathbf{VaR}_\alpha[X] \leq \mathbf{VaR}_\alpha[Y], \text{ for all } \alpha \in (0, 1), \tag{2}$$

where $\leq_{st}$ denotes the usual stochastic order and $\mathbf{VaR}_\alpha[X]$ denotes the value at risk of X at level α, tells us, in the context of parametric distributions, how parameter changes causing stochastic increasing of distributions translate into increasing of associated VaRs, independently of the chosen level α. This and many other similar results involving different univariate stochastic orders and risk measures allow to evaluate risks separately, without taking into account how they relate to each other.

In portfolio risk analysis, we cannot ignore interaction among risks. For example, Embrechts et al. (2005), Kaas et al. (2009) and Laeven (2009) prove consistency results for the value-at-risk and expected shortfall of aggregate risk when the marginal distributions of the individual risks are given and some additional information on the dependence structure is available. In this context, risk measures are used not only to evaluate marginal and aggregate risks, but also, more importantly since the recent global financial crisis, to assess the systemic risk, which is related to the risk that the failure or loss of a component i spreads to another component j or even to the whole portfolio. To address this issue, the literature offers, basically, two types of measures: co-risk measures[2] and risk contribution measures. On one hand, co-risk measures can be interpreted as dependence-adjusted versions of measures usually employed to assess isolate risks. The general idea under these measures is to use the conditional distribution of a random variable Y, representing the risk of a particular component (or even the aggregate risk of the portfolio) given that another component, represented by a random variable X, is in stress. Examples are the conditional value at risk (CoVaR) notions introduced by Adrian and Brunnermeier (2016) and Girardi and Ergün (2013), the conditional expected shortfall (CoES) introduced by Mainik and Schaanning (2014) and the marginal expected shortfall (MES) introduced by Acharya et al. (2010). On the other hand, risk contribution measures quantify how a stress situation for a component X affects

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[1] This implication is, in fact, an "if and only if".

[2] Adrian and Brunnermeier (2008) add to existing risk measures the prefix "co", which stands for conditional, comovement, contagion, or contributing.

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another component Y (or the whole portfolio). Examples of risk contribution measures can be found in Adrian and Brunnermeier (2016), Mainik and Schaanning (2014) and Girardi and Ergün (2013). In this paper, we are concerned with the consistency of some co-risk measures and risk contribution measures with respect to different stochastic orderings of the marginals and under different positive dependence assumptions.

The starting point of this work was the observation that some consistency results given by Mainik and Schaanning (2014) for some co-risk measures with respect to stochastic dependence of portfolios can be improved under the assumption that the compared dependence structures are positive (which is a reasonable assumption for insurance portfolios, for example). Then we extended our research to study similar consistency results for risk contribution measures, an issue which has not been addressed before. In this context, we provide conditions, in terms of some positive dependence notions, under which the risk contribution measures agree with the dispersive order (Bickel and Lehmann, 1979) and the excess wealth order (Shaked and Shanthikumar, 1998) of the marginals.

The plan of the paper is as follows. In Section 2 we review some basic definitions and background. In Section 3, given two random vectors (X, Y) and (X', Y') with positively dependent components (this will be formalized later on), we study conditions on the marginals Y and Y' and the respective copulas to order unambiguously (that is, independently of the chosen level of risks) the respective CoESs. In Section 4, we give conditions on Y and Y' (in terms of the dispersive order) and on the dependence structures of the vectors, to make unambiguous comparisons of risk contribution measures based on CoVaRs. In Section 5, we relax the conditions on the marginals (which are given in terms of the weaker excess wealth order) in exchange to make stronger assumptions on the dependence structures of the compared vectors to order risk contribution measures based on CoESs. The results in this section are given in terms of a weaker order (the excess wealth order) and stronger assumptions on the dependence structure of the vectors. Finally, Section 6 contains conclusions.

2. Definitions and background

Let (X, Y) be an absolutely continuous random vector with joint distribution function H and marginal distribution functions F_X and F_Y, where X and Y represent losses or risks. The joint distribution function H can be expressed as

$$H(x, y) = C(F_X(x), F_Y(y)),$$

where C is the copula of (X, Y), that is, the joint distribution function of $(F_X(X), F_Y(Y))$ (see Nelsen, 1999). The copula contains the information about the dependence of the random vector (X, Y) apart from the behavior of the marginal distributions. We denote by $\overline{C}$ the joint tail function for two uniform random variables whose joint distribution function is the copula C, that is

$$\begin{aligned}\overline{C}(u, v) &= P\left[X > F_X^{-1}(u), Y > F_Y^{-1}(v)\right] \\ &= 1 - u - v + C(u, v), \ u, v \in (0, 1).\end{aligned} \tag{3}$$

We first recall the definition of concordance order (Definition 2.8.1 in Nelsen, 1999[alp]).

Definition 1. Given two copulas C and C', we say that C is smaller than C' in the concordance order (and write $C \prec C'$) if $C(u, v) \leq C'(u, v)$ for all $u, v \in (0, 1)$.

One reason for the name is that some measures of concordance, such as Kendall's tau and Spearman's rho, are increasing with respect to "$\prec$". The concordance order, which is sometimes called the "correlation" order (see Dhaene and Goovaerts, 1996, 1997) and the "more PQD" order (see Nelsen, 1999), is a partial order because not every pair of copulas is comparable.[3]

In order to introduce some notions of positive dependence, we first require the definition of the following stochastic orders among random variables (see Chapter 1 in Shaked and Shanthikumar, 2007 for properties and applications).

Definition 2. Let X and Y be two random variables with respective distribution functions F_X and F_Y. Then, X is said to be smaller than Y:
(i) in the stochastic order (denoted by $X \leq_{st} Y$) if $F_X(t) \geq F_Y(t)$, for all t,
(ii) in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f_Y(t)/f_X(t)$ increases in t over the union of the supports of X and Y, where f_X and f_Y are the density functions of X and Y (which are assumed to be absolutely continuous).

The orders $\leq_{lr}$ and $\leq_{st}$ formalize the idea that a random variable is larger in size than another. It is well-known (see, for example, Theorem 1.C.1 in Shaked and Shanthikumar, 2007) that

$$X \leq_{lr} Y \Longrightarrow X \leq_{st} Y. \tag{4}$$

Consequently, the following notions formalize the idea that large values of one component are associated with large values for the other (see Chapter 5 in Barlow and Proschan, 1975, Block et al., 1982 and Joe, 1997 for properties and applications).

Definition 3. (i) The random vector (X, Y) is said to be totally positive of order 2 (denoted TP_2) if $[X|Y = y_1] \leq_{lr} [X|Y = y_2]$, for all $y_1 \leq y_2$ and $[Y|X = x_1] \leq_{lr} [Y|X = x_2]$, for all $x_1 \leq x_2$.
(ii) X is said to be stochastically increasing in Y (denoted as $X \uparrow_{SI} Y$) if $[X|Y = y_1] \leq_{st} [X|Y = y_2]$, for all $y_1 \leq y_2$.
(iii) The random vector (X, Y) is said to be positive dependent through the stochastic order (PDS) if $X \uparrow_{SI} Y$ and $Y \uparrow_{SI} X$.

From (4) it follows that

$$(X, Y) \text{ is } TP_2 \Longrightarrow (X, Y) \text{ is PDS}.$$

It is worth mentioning that (X, Y) is TP_2 (respectively, PDS) if and only if its copula C is TP_2 (respectively, PDS; see Theorem 3.10.19 in Müller and Stoyan (2002) and the discussion on this issue in Cai and Wei (2012)). An example of PDS copula is the Gumbel copula (see Joe, 1997, p. 142), defined as

$$C_\theta(u, v) = \exp\left(-\left((-\log u)^\theta + (-\log v)^\theta\right)^{1/\theta}\right), \ \theta \geq 1,$$

which is often used for modeling strong right tail dependencies. For $\theta = 1$ and $\theta = \infty$ we get, respectively, $C_1(u, v) = uv$ (independence case) and $C_\infty(u, v) = \min\{u, v\}$ (comonotonic case). In addition, $C_\theta(u, v)$ increases in θ, therefore the structure becomes more positively dependent as θ increases (see Fig. 1).

Now we recall the definition of two popular scalar risk measures: the value-at-risk (VaR) and the expected shortfall (ES). The value-at-risk of X at level α is defined by

$$\mathbf{VaR}_\alpha[X] = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}, \ \alpha \in (0, 1)$$

and the corresponding expected shortfall (which is more sensitive to the losses in the tail of the distribution) is given by

$$\mathbf{ES}_\alpha[X] = \frac{1}{1-\alpha}\int_\alpha^1 \mathbf{VaR}_\alpha[X]\, dt, \ \alpha \in (0, 1). \tag{5}$$

The expected shortfall (sometimes called tail-value-at-risk) is a coherent risk measure in the sense of Artzner et al. (1999). If F is

[3] In fact, all the stochastic orders considered in this paper are partial orders.

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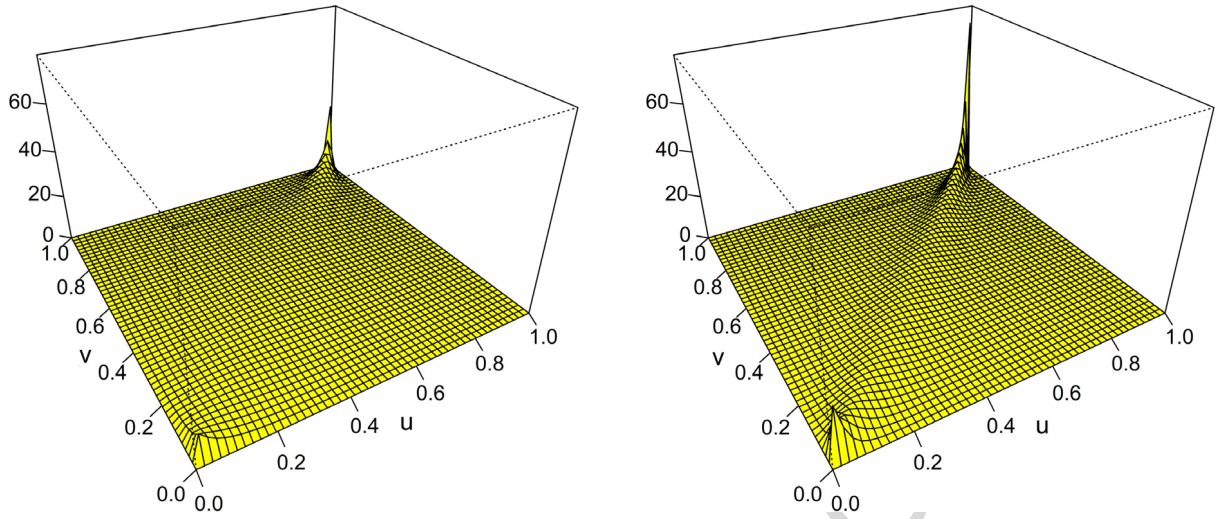


Fig. 1. The left-hand panel shows the density function of a Gumbel copula with $\theta = 3$. The right-hand panel shows the density function of a Gumbel copula with $\theta = 6$. The structure becomes more positively dependent as θ increases.

continuous, it represents the conditional expected loss given that the loss exceeds its VaR:

$$\mathbf{ES}_\alpha[X] = E[X|X > \mathbf{VaR}_\alpha[X]].$$

Following Adrian and Brunnermeier (2016), Girardi and Ergün (2013) or Mainik and Schaanning (2014) among others, we adjust VaR and ES to dependence between X and Y by conditioning the distribution of Y on a stress scenario for X. Among the different kinds of stress scenarios for X, the most common in the literature are $\{X = \mathbf{VaR}_\alpha[X]\}$ (as suggested initially by Adrian and Brunnermeier (2016)) and $\{X \geq \mathbf{VaR}_\alpha[X]\}$ (as modificated by Girardi and Ergün, in a preprint later published in 2013, and successively adopted by Acharya et al. (2010), Klyman (2011) and Mainik and Schaanning (2014), among others[4]). In this paper, we follow the second approach, which presents some favorable theoretical and practical properties, such as: (a) it gives a much more consistent response to dependence than the original stress scenario (Mainik and Schaanning, 2014) and (b) it improves the compatibility of the co-measures with Monte Carlo simulation and non-parametric estimation methods (Girardi and Ergün, 2013).

We first define CoVaR (as considered by Girardi and Ergün, 2013).

Definition 4. For $\alpha, \beta \in (0, 1)$, we set

$$\mathbf{CoVaR}_{\alpha,\beta}[Y|X] = \mathbf{VaR}_\beta[Y|X > \mathbf{VaR}_\alpha[X]].$$

In other words, CoVaR is the value-at-risk at level β of the conditional distribution of Y given that $\{X \geq \mathbf{VaR}_\alpha[X]\}$. Bernardi et al. (2017) provide a directory of CoVaR values for different families of copulas.

The following definition (Mainik and Schaanning, 2014) extends CoVaR to CoES. Under this definition, coherence of ES is inherited by CoES.

Definition 5. For $\alpha, \beta \in (0, 1)$, we set

$$\mathbf{CoES}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 \mathbf{CoVaR}_{\alpha,t}[Y|X]\,dt. \quad (6)$$

For continuous marginals, CoES admits a representation in terms of conditional expectation of Y that is similar to the one holding for the usual expected shortfall, except that CoES incorporates interaction:

$$\mathbf{CoES}_{\alpha,\beta}[Y|X] = E\left[Y|X > \mathbf{VaR}_\alpha[X], Y > \mathbf{CoVaR}_{\alpha,\beta}[Y|X]\right].$$

Risk contribution measures quantify how a stress situation for the component X affects another component Y. One way to measure the risk contribution of X to Y is to compare $\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$, which is the VaR of component Y conditional on component X being in actuarial distress, with $\mathbf{VaR}_\beta[Y]$, which represents the risk of Y in an unstressed state. Another way is to replace the centering term $\mathbf{VaR}_\beta[Y]$ by the conditional VaR of Y given that X is in its benchmark state, which is usually defined in terms of its median[5] (see Adrian and Brunnermeier, 2016, and Mainik and Schaanning, 2014).

Definition 6. For $\alpha, \beta \in (0, 1)$, we set

$$\Delta\mathbf{CoVaR}_{\alpha,\beta}[Y|X] = \mathbf{CoVaR}_{\alpha,\beta}[Y|X] - \mathbf{VaR}_\beta[Y],$$
$$\Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}[Y|X] = \mathbf{CoVaR}_{\alpha,\beta}[Y|X] - \mathbf{CoVaR}_{1/2,\beta}[Y|X].$$

Risk contribution measures can also be defined in terms of CoES. The following definition adapts the previous one to CoES (see Acharya et al., 2010, and Karimalis and Nomikos, 2014).

Definition 7. For $\alpha, \beta \in (0, 1)$, we set

$$\Delta\mathbf{CoES}_{\alpha,\beta}[Y|X] = \mathbf{CoES}_{\alpha,\beta}[Y|X] - \mathbf{ES}_\beta[Y],$$
$$\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X] = \mathbf{CoES}_{\alpha,\beta}[Y|X] - \mathbf{CoES}_{1/2,\beta}[Y|X].$$

Now we require the notion of distortion function, which is very popular in actuarial and finance. A distortion function h is a non-decreasing function from $[0, 1]$ to $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$. If h is continuous, the transformation of the tail function $\overline{F}_Y = 1 - F_Y$ of Y given by $h\left(\overline{F}_Y(x)\right) = h \circ \overline{F}_Y(x)$ defines a new tail function associated to certain random variable Y_h, which is the induced by h distorted random variable of Y. The following lemma will be used throughout the paper (see Theorem 4 in Sordo et al., 2015, for the proof).

Lemma 8. *Let (X, Y) be a random vector with copulas C such that $X \uparrow_{SI} Y$. Then, for $\alpha \in (0, 1)$, the conditional random variable*

[4] In the financial literature, these stress scenarios take the form $\{X \leq \mathbf{VaR}_\alpha[X]\}$.

[5] Another option would be, for example, to define the benchmark state of X in terms of a one-standard deviation event about its mean (as in Girardi and Ergün, 2013).

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$[Y|X \geq VaR_\alpha[X]]$ *is a distorted random variable induced from* Y *by the concave distortion function*

$$h(p) = \frac{\overline{C}(\alpha, 1-p)}{1-\alpha}, \quad p \in [0,1]. \tag{7}$$

Besides the orders $\leq_{lr}$ and $\leq_{st}$, we use in this paper the following stochastic orders.

Definition 9. Let X and Y be two random variables with respective distribution functions F_X and F_Y and tail functions $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$, respectively. Then, X is said to be smaller than Y:
(i) in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\overline{F}_Y(t)/\overline{F}_X(t)$ increases in t,
(ii) in the increasing convex order (denoted by $X \leq_{icx} Y$) if $\int_t^\infty \overline{F}_X(x)dx \leq \int_t^\infty \overline{F}_Y(x)dx$, $\forall t$,
(iii) in the dispersive order (denoted by $X \leq_{disp} Y$) if $F_X^{-1}(p) - F_X^{-1}(q) \leq F_Y^{-1}(p) - F_Y^{-1}(q)$, for all $0 < q < p < 1$,
(iv) in the excess wealth order (denoted by $X \leq_{ew} Y$) if $E\left[\left(X - F_X^{-1}(p)\right)^+\right] \leq E\left[\left(Y - F_Y^{-1}(p)\right)^+\right]$ for all $p \in (0,1)$, where $(x)^+ = \max\{x, 0\}$.

It is well-known (see Shaked and Shanthikumar, 2007) that

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \Longrightarrow X \leq_{st} Y \Longrightarrow X \leq_{icx} Y \tag{8}$$

and

$$X \leq_{disp} Y \Longrightarrow X \leq_{ew} Y$$

but the converses are not true. It is also well-known (see Lemma 2.1 in Sordo and Ramos, 2007) that

$$X \leq_{icx} Y \text{ if and only if } \mathbf{ES}_\alpha[X] \leq \mathbf{ES}_\alpha[Y], \ \forall \alpha \in (0,1), \tag{9}$$

where $\mathbf{ES}_\alpha(X)$ is the expected shortfall given by (5). For applications of the orders (i) and (ii) in risk theory and insurance, see the books by Müller and Stoyan (2002) and Denuit et al. (2005). For applications of the orders (iii) and (iv) (which is sometimes called the right spread order, see Fernández-Ponce et al., 1998) in insurance, see Sordo (2008, 2009), Sordo and Suárez-Llorens (2011), López-Díaz et al. (2012), Belzunce et al. (2016) and the book by Belzunce et al. (2015).

3. Increasing convex order and CoES

Mainik and Schaanning (2014) studied the monotonicity of CoVaR and CoES with respect to copula dependence parameters in the case where the marginals are stochastically ordered (this includes the case of equally distributed marginals). In particular, they proved the following result.

Theorem 10 (*Mainik and Schaanning, 2014*). *Let* (X, Y) *and* (X', Y') *be two random vectors with Gumbel copulas and dependence parameters* θ *and* θ', *respectively. Suppose that* F_X *and* $F_{X'}$ *are continuous. Then,* $Y \leq_{st} Y'$ *and* $\theta \leq \theta'$ *imply* $\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \mathbf{CoES}_{\alpha,\beta}\left[Y'|X'\right]$, *for all* $\alpha, \beta \in (0,1)$.

The motivation for our study comes from the following observation. Comparing co-expected shortfalls of comonotonic vectors (or, equivalently, Gumbels with parameters $\theta = \theta' = \infty$) produces the same result as comparing expected shortfalls of marginals Y and Y' which, according to (9), is consistent with the weaker increasing convex order. This suggests to investigate whether Mainik and Schaanning's result remains true when the assumption $Y \leq_{st} Y'$ is replaced by the weaker $Y \leq_{icx} Y'$. The answer is yes. To prove it, we require Theorem 2.1 in Sordo and Ramos (2007).

Theorem 11 (*Sordo and Ramos, 2007*). *Let* X *and* Y *be two random variables with distribution functions* F_X *and* F_Y, *respectively. Then,* $X \leq_{icx} Y$ *if and only if*

$$\int_0^1 F_X^{-1}(t)\, dA(t) \leq \int_0^1 F_Y^{-1}(t)\, dA(t),$$

for all increasing convex $A : [0,1] \to \mathbf{R}$.

Theorem 12. *Let* (X, Y) *and* (X', Y') *be two random vectors with copulas* C *and* C', *respectively. Assume that either* $X \uparrow_{SI} Y$ *or* $X' \uparrow_{SI} Y'$ *or both hold. Then,* $Y \leq_{icx} Y'$ *and* $C \prec C'$ *imply*

$$\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \mathbf{CoES}_{\alpha,\beta}\left[Y'|X'\right], \ \forall \alpha, \beta \in (0,1). \tag{10}$$

Proof. Suppose $X \uparrow_{SI} Y$ and denote $U = F_X(X)$ and $V = F_Y(Y)$. Using Theorem 3.1 in Mainik and Schaanning (2014), we can write

$$\mathbf{CoES}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 F_Y^{-1}\left(F_{V|U\geq\alpha}^{-1}(p)\right) dp.$$

The change of variable $p = F_{V|U\geq\alpha}(t)$ gives

$$\begin{aligned}\mathbf{CoES}_{\alpha,\beta}[Y|X] &= \frac{1}{1-\beta}\int_{F_{V|U\geq\alpha}^{-1}(\beta)}^1 F_Y^{-1}(t)\, dF_{V|U\geq\alpha}(t) \\ &= \frac{1}{1-\beta}\int_{F_{V|U\geq\alpha}^{-1}(\beta)}^1 F_Y^{-1}(t)\, d\left[1 - \frac{\overline{C}(\alpha,t)}{1-\alpha}\right].\end{aligned}$$

Therefore, we can write

$$\mathbf{CoES}_{\alpha,\beta}[Y|X] = \int_0^1 F_Y^{-1}(t)\, dA_{\alpha,\beta}(t), \tag{11}$$

where $A_{\alpha,\beta}(t)$ is a convex distortion function given by

$$A_{\alpha,\beta}(t) = \max\left\{1 - \frac{\overline{C}(\alpha,t)}{(1-\alpha)(1-\beta)}, 0\right\}. \tag{12}$$

An alternative representation for $A_{\alpha,\beta}(t)$ is

$$A_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } t < F_{V|U\geq\alpha}^{-1}(\beta) \\ 1 - \dfrac{\overline{C}(\alpha,t)}{(1-\alpha)(1-\beta)} & \text{if } t > F_{V|U\geq\alpha}^{-1}(\beta). \end{cases} \tag{13}$$

To see the convexity of $A_{\alpha,\beta}(t)$, note that

$$X \uparrow_{SI} Y \Longrightarrow P[U > \alpha | V = t] \text{ increases in } t.$$

Then, the convexity of $A_{\alpha,\beta}(t)$ follows taking into account that

$$\frac{\partial \overline{C}(\alpha,t)}{\partial t} = -P[U > \alpha | V = t]$$

(see (2.9.1) in Nelsen, 1999). Similarly, it can be shown that

$$\mathbf{CoES}_{\alpha,\beta}\left[Y'|X'\right] = \int_0^1 F_{Y'}^{-1}(t)\, dB_{\alpha,\beta}(t), \tag{14}$$

where $B_{\alpha,\beta}(t)$ is a distortion function given by

$$B_{\alpha,\beta}(t) = \max\left\{1 - \frac{\overline{C'}(\alpha,t)}{(1-\alpha)(1-\beta)}, 0\right\}. \tag{15}$$

On one hand, it follows from Theorem 11 that $Y \leq_{icx} Y'$ implies

$$\int_0^1 F_Y^{-1}(t)\, dA_{\alpha,\beta}(t) \leq \int_0^1 F_{Y'}^{-1}(t)\, dA_{\alpha,\beta}(t). \tag{16}$$

On the other hand, $C \prec C'$ implies $A_{\alpha,\beta}(t) \geq B_{\alpha,\beta}(t)$. Therefore, using that $A_{\alpha,\beta}(0) = B_{\alpha,\beta}(0) = 0$ and $A_{\alpha,\beta}(1) = B_{\alpha,\beta}(1) = 1$, we

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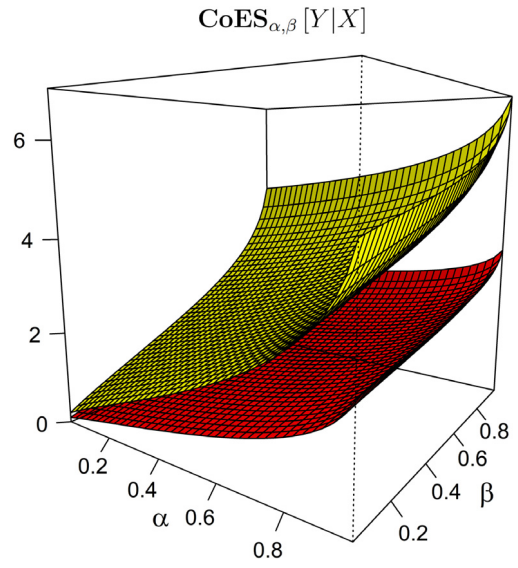


Fig. 2. **CoES**$_{\alpha,\beta}[Y|X]$ as function of α and β. The red surface corresponds to a Gumbel copula C with dependence parameter $\theta = 3$ and normal marginals $X \sim N(1,2)$ and $Y \sim N(0,1)$. The red surface corresponds to a Gumbel copula C' with $\theta = 4$ and marginals $X' \sim N(0,1)$ and $Y' \sim N(0,2)$. Therefore $Y \leq_{icx} Y'$ and $C \prec C'$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

have

$$\int_0^1 F_{Y'}^{-1}(t)\, d\left[A_{\alpha,\beta}(t) - B_{\alpha,\beta}(t)\right] = \int_0^1 \left[B_{\alpha,\beta}(t) - A_{\alpha,\beta}(t)\right] dF_{Y'}^{-1}(t) \leq 0,$$

that is,

$$\int_0^1 F_{Y'}^{-1}(t)\, dA_{\alpha,\beta}(t) \leq \int_0^1 F_{Y'}^{-1}(t)\, dB_{\alpha,\beta}(t). \tag{17}$$

By combining (16) and (17) we get (10). The proof of the case $X' \uparrow_{SI} Y'$ is analogous. ■

Example 13. Let (X,Y) and (X',Y') be two random vectors with Gumbel copulas with dependence parameters $\theta \prec \theta'$ (Gumbel copulas are PDS, in particular $X \uparrow_{SI} Y$). Suppose that $Y \sim N(\mu,\sigma)$ and $Y' \sim N(\mu',\sigma')$, with $\mu = \mu'$ and $\sigma \prec \sigma'$ (which implies $Y \leq_{icx} Y'$, see Table 1.1 in Müller and Stoyan (2002)). It follows from Theorem 12 that **CoES**$_{\alpha,\beta}[Y|X] \leq$ **CoES**$_{\alpha,\beta}\left[Y'|X'\right]$ for all $\alpha,\beta \in (0,1)$. Note that this conclusion does not follow from Mainik and Schaanning's result because $Y \not\leq_{st} Y'$ (two random variables with the same mean are not stochastically ordered unless they have the same distribution, see Theorem 1.A.8 in Shaked and Shanthikumar, 2007). This example is illustrated in Fig. 2.

4. Dispersive order and covar-type risk contribution measures

In this section, given two random vectors (X,Y) and (X',Y'), we give conditions on the marginals Y and Y' and the respective copulas to compare risk contribution measures of the form $\mathbf{\Delta CoVaR}_{\alpha,\beta}[Y|X]$ and $\mathbf{\Delta}^{med}\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$. In order to simplify the proofs of the main results, we first provide the following lemma.

Lemma 14. *Let X and Y be two continuous random variables with distribution functions F_X and F_Y, respectively. Let h be a concave distortion function and let g be another distortion function such that $h(t) \leq g(t)$, for all $t \in [0,1]$. Denote by X_h and Y_g the distorted random variables induced, respectively, from X and Y, by the distortion functions h and g, respectively. If $X \leq_{disp} Y$, then:*
(i) $F_{X_h}^{-1}(p) - F_X^{-1}(p) \leq F_{Y_g}^{-1}(p) - F_Y^{-1}(p)$, *for all* $p \in (0,1)$,
(ii) $F_{X_g}^{-1}(p) - F_{X_h}^{-1}(p) \leq F_{Y_g}^{-1}(p) - F_{Y_h}^{-1}(p)$, *for all* $p \in (0,1)$.

Proof. To prove (i) note that

$$F_{X_h}^{-1}(p) = F_X^{-1}\left(1 - h^{-1}(1-p)\right),$$
$$F_{Y_g}^{-1}(p) = F_Y^{-1}\left(1 - g^{-1}(1-p)\right),$$

for all $p \in (0,1)$. Since h is a concave distortion function, $h(t) \geq t$ for all $t \in [0,1]$ and, consequently, $1 - h^{-1}(1-p) \geq p$ for all $p \in (0,1)$. Therefore, $X \leq_{disp} Y$ implies

$$F_X^{-1}\left(1 - h^{-1}(1-p)\right) - F_X^{-1}(p) \tag{18}$$
$$\leq F_Y^{-1}\left(1 - h^{-1}(1-p)\right) - F_Y^{-1}(p),\ p \in (0,1).$$

Since $0 \leq h(t) \leq g(t) \leq 1$, then

$$0 \leq 1 - h^{-1}(1-p) \leq 1 - g^{-1}(1-p) \leq 1, \quad \text{for all } p \in (0,1), \tag{19}$$

and

$$F_Y^{-1}\left(1 - h^{-1}(1-p)\right) \leq F_Y^{-1}\left(1 - g^{-1}(1-p)\right). \tag{20}$$

Combining (18) and (20) it follows

$$F_X^{-1}\left(1 - h^{-1}(1-p)\right) - F_X^{-1}(p)$$
$$\leq F_Y^{-1}\left(1 - g^{-1}(1-p)\right) - F_Y^{-1}(p),\ p \in (0,1),$$

which is the same as (i). To prove (ii), note that $X \leq_{disp} Y$ and (19) imply

$$F_X^{-1}\left(1 - g^{-1}(1-p)\right) - F_X^{-1}\left(1 - h^{-1}(1-p)\right)$$
$$\leq F_Y^{-1}\left(1 - g^{-1}(1-p)\right) - F_Y^{-1}\left(1 - h^{-1}(1-p)\right),\ p \in (0,1),$$

which is the same as (ii). ■

4.1. Risk contributions of type $\mathbf{\Delta CoVaR}_{\alpha,\beta}[Y|X]$.

Consider two random vectors (X,Y) and (X',Y') with copulas C and C'. In this section we show that if Y is smaller than Y' in the dispersive order, $C \prec C'$ and at least one of the two vectors is positively dependent (this will be formalized later), the risk contribution of X to Y (as measured by $\mathbf{\Delta CoVaR}_{\alpha,\beta}[Y|X]$ is smaller than the risk contribution of X' to Y', for any α and β in $(0,1)$.

Theorem 15. *Let (X,Y) and (X',Y') be two random vectors with copulas C and C', respectively. Assume that either $X \uparrow_{SI} Y$ or $X' \uparrow_{SI} Y'$ or both hold. Then, $Y \leq_{disp} Y'$ and $C \prec C'$ imply*

$$\mathbf{\Delta CoVaR}_{\alpha,\beta}[Y|X] \leq \mathbf{\Delta CoVaR}_{\alpha,\beta}\left[Y'|X'\right],\ \alpha,\beta \in (0,1). \tag{21}$$

Proof. Suppose $X \uparrow_{SI} Y$ (the case $X' \uparrow_{SI} Y'$ is similar). Using the notation of Lemma 14, (21) can be written as

$$F_{Y_{h,\alpha}}^{-1}(\beta) - F_Y^{-1}(\beta) \leq F_{Y'_{g,\alpha}}^{-1}(\beta) - F_{Y'}^{-1}(\beta),\ \alpha,\beta \in (0,1),$$

where $Y_{h,\alpha}$ is the distorted random variable induced from Y by the concave distortion function

$$h_\alpha(p) = \frac{\overline{C}(\alpha, 1-p)}{1-\alpha},\ p \in (0,1) \tag{22}$$

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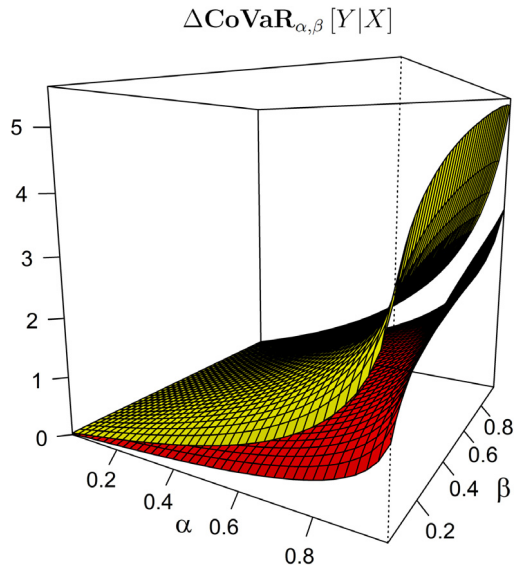


Fig. 3. $\Delta\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$ as function of α and β. The red surface corresponds to a Gaussian copula C with dependence parameter $\rho = 0.8$ and marginals normal $X \sim N(1, 2)$ and Weibull $Y \sim W(1, 1)$. The yellow one corresponds to a Gaussian copula C' with dependence parameter $\rho = 0.9$ and marginals $X' \sim N(1, 2)$ nd $Y' \sim W(1.5, 1)$. Here $Y \leq_{disp} Y'$ and $C \prec C'$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and $Y'_{g,\alpha}$ is the distorted random variable induced from Y' by the distortion function

$$g_\alpha(p) = \frac{\overline{C}'(\alpha, 1-p)}{1-\alpha},\ p \in (0,1).$$

Taking into account that the assumption $C \prec C'$ implies $h_\alpha(p) \leq g_\alpha(p)$ for all $\alpha \in (0,1)$ and $p \in (0,1)$, the result follows by direct application of Lemma 14(i). ■

Example 16. Let (X, Y) and (X', Y') be two random vectors with Gaussian copulas (see Section 4.3.3 in Denuit et al., 2005) with dependence parameters $0 \leq \rho < \rho'$ (which implies $C_\rho \prec C_{\rho'}$). C_ρ and $C_{\rho'}$ are TP_2, in particular $X \uparrow_{SI} Y$. Let $Y \sim W(a, b)$ and $Y' \sim W(a', b')$ be two Weibull distributions with $a \leq a'$ and $b = b'$ (which implies $Y \leq_{disp} Y'$). It follows from Theorem 15 that $\Delta\mathbf{CoVaR}_{\alpha,\beta}[Y|X] \leq \Delta\mathbf{CoVaR}_{\alpha,\beta}[Y'|X']$, for all $\alpha, \beta \in (0,1)$. This example is illustrated in Fig. 3.

Roughly speaking, heavier tailed distributions produce higher VaRs. Intuitively, therefore, given two positively dependent risks X and Y such that (at least) Y is heavy-tailed, the difference $\mathbf{CoVaR}_{\alpha,\beta}[Y|X] - \mathbf{VaR}_\beta[Y]$ is expected to increase as β increases. One way to formalize the notion "heavy-tailed" distributions is to consider the family of DFR risks. The risk Y is said to have decreasing failure rate (or to be DFR) if its distribution function F_Y is logconvex. Examples of DFR distributions include gamma and Weibull distributions with shape parameters less than one and Pareto distribution, which are heavy tailed distributions with applications in insurance. According to the above intuition, we characterize this class of distributions in terms of contribution risk measures of type $\Delta\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$.

Theorem 17. *A random variable Y is DFR if and only if $\Delta\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$ is increasing in β for each fixed α and for each X such that $X \uparrow_{SI} Y$.*

Proof. First we prove the sufficient condition. Let X be a random variable such that $X \uparrow_{SI} Y$. For each fixed α, we know from Lemma 8 that $[Y|X \geq \mathbf{VaR}_\alpha(X)]$ is a distorted random variable induced from Y by the concave distortion function h given in (7). Then, from Corollary 7 of Sordo et al. (2015) it follows that $Y \leq_{hr} [Y|X \geq \mathbf{VaR}_\alpha(X)]$, $\forall \alpha \in (0,1)$, which implies, by applying Theorem 3.B.20(a) in Shaked and Shanthikumar (2007), that $Y \leq_{disp} [Y|X \geq \mathbf{VaR}_\alpha(X)]$. This means that for $0 < \beta_1 < \beta_2 < 1$ we have

$$\mathbf{VaR}_{\beta_2}[Y] - \mathbf{VaR}_{\beta_1}[Y]$$

$$\leq \mathbf{VaR}_{\beta_2}[Y|X \geq \mathbf{VaR}_\alpha[X]] - \mathbf{VaR}_{\beta_1}[Y|X \geq \mathbf{VaR}_\alpha[X]]$$

or, equivalently,

$$\Delta\mathbf{CoVaR}_{\alpha,\beta_1}[Y|X] \leq \Delta\mathbf{CoVaR}_{\alpha,\beta_2}[Y|X],$$

which proves the sufficient condition. In order to prove the necessary condition, we take a random variable X comonotonic with Y (in particular, $X \uparrow_{SI} Y$). Denote by (X^c, Y^c) the corresponding random vector. Given $\alpha, \beta \in (0,1)$, a straightforward computation shows that

$$\mathbf{CoVaR}_{\alpha,\beta}\left[Y^c|X^c\right] = \mathbf{VaR}_{\beta(1-\alpha)+\alpha}[Y] = \mathbf{VaR}_\beta\left[Y|Y > F_Y^{-1}(\alpha)\right].$$

Using this equality and the assumptions, we have that

$$\Delta\mathbf{CoVaR}_{\alpha,\beta}\left[Y^c|X^c\right] = \mathbf{VaR}_\beta\left[Y|Y > F_Y^{-1}(\alpha)\right] - \mathbf{VaR}_\beta[Y]$$

is increasing in β for each fixed α. This means (see Definition 9(iii)) that $Y \leq_{disp} [Y|Y \geq \mathbf{VaR}_\alpha[Y]]$, $\forall \alpha \in (0,1)$, and it follows from Lemma 2.1 in Pellerey and Shaked (1997) that Y is DFR. ■

Example 18. Let (X, Y) be a random vector with a Gumbel copula (which is PDS) such that Y follows a Pareto distribution (which is DFR). Theorem 17 tells us that stress scenarios for X affect more to Y when Y is also stressed. The example is illustrated in Fig. 4.

Combining Theorems 15 and 17, we obtain the following result.

Corollary 19. *Let (X, Y) and (X', Y') be two random vectors with copulas C and C', respectively. Assume that either $X \uparrow_{SI} Y$ or $X' \uparrow_{SI} Y'$ or both hold. Assume also that either Y or Y' or both are DFR. For $\alpha \in (0,1)$, $Y \leq_{disp} Y'$ and $C \prec C'$ imply*

$$\Delta\mathbf{CoVaR}_{\alpha,\beta_1}[Y|X] \leq \Delta\mathbf{CoVaR}_{\alpha,\beta_2}\left[Y'|X'\right],\ 0 < \beta_1 \leq \beta_2 < 1.$$

4.2. Risk contributions of type $\Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}[Y|X]$

In this section, given two random vectors (X, Y) and (X', Y') with the same TP_2 copula C, we show that $Y \leq_{disp} Y'$ ensures $\Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}[Y|X] \leq \Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}\left[Y'|X'\right]$, for any $\beta \in (0,1)$ and $1/2 \leq \alpha \leq 1$. The assumption TP_2 is stronger than $X \uparrow_{SI} Y$ or $X' \uparrow_{SI} Y'$.

Theorem 20. *Let (X, Y) and (X', Y') be two random vectors with the same TP_2 copula C. Then, $Y \leq_{disp} Y'$ implies*

$$\Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}[Y|X] \leq \Delta^{med}\mathbf{CoVaR}_{\alpha,\beta}\left[Y'|X'\right],\quad \alpha \in (1/2,1),\ \beta \in (0,1). \tag{23}$$

Proof. Using the notation of Lemma 14, expression (23) can be written as

$$F^{-1}_{Y_{h,\alpha}}(\beta) - F^{-1}_{Y_{h,1/2}}(\beta) \leq F^{-1}_{Y'_{h,\alpha}}(\beta) - F^{-1}_{Y'_{h,1/2}}(\beta),\quad \alpha \in (1/2,1),\ \beta \in (0,1), \tag{24}$$

where $Y_{h,\alpha}$ is the distorted random variable induced from Y by the distortion function (22) and $Y_{h,1/2}$ is the distorted random variable induced from Y by the distortion function

$$h_{1/2}(p) = 2\overline{C}(1/2, 1-p),\ p \in (0,1). \tag{25}$$

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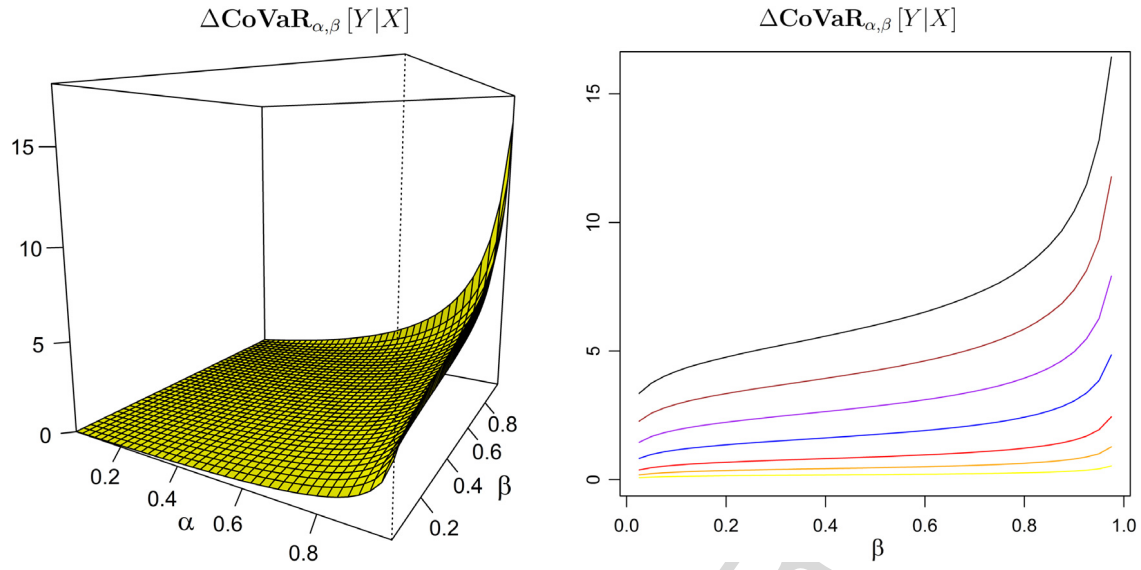


Fig. 4. The left-hand panel shows Δ**CoVaR**$_{\alpha,\beta}[Y|X]$ as function of α and β. The surface corresponds to a Gumbel copula with $\theta = 4$ and marginals normal $X \sim N(0, 2)$ and Pareto $Y \sim P(1, 3)$ (which is DFR). The right 2−dimensional graphic shows different curves for $\alpha = 0.20, 0.40, 0.60, 0.80, 0.90, 0.95, 0.99$. Observe that the contribution of X is larger as Y increases.

Since the copula C is TP_2, it follows from Theorem 2.3(a) of Joe (1997) that

$$\frac{\overline{C}(\alpha, 1-t)}{(1-\alpha)} = P[V > 1-t \,|U > \alpha\,] \text{ is increasing in } \alpha,$$

that is, $h_{1/2}(t) \leq h_\alpha(t)$ for $t \in [0, 1]$ and $1/2 \leq \alpha \leq 1$. Now, (24) follows from Lemma 14(ii). ■

Example 21. Let (X, Y) and (X', Y') be two random vectors with a Gaussian copula with dependence parameter $\rho \geq 0$ (which is TP_2). Let $Y \sim W(a, b)$ and $Y' \sim W(a', b')$ be two Weibull distributions with $a \leq a'$ and $b = b'$ (which implies $Y \leq_{disp} Y'$). It follows from Theorem 20 that Δ^{med}**CoVaR**$_{\alpha,\beta}[Y|X] \leq \Delta^{med}$**CoVaR**$_{\alpha,\beta}[Y'|X']$, for all $\alpha \in (1/2, 1)$ and $\beta \in (0, 1)$. The example is illustrated in Fig. 5.

5. Excess wealth order and CoES-type risk contribution measures

Given a random variable X with distribution function F_X, Sordo (2008) characterizes the dispersive order and the excess wealth order by classes of risk measures of the form

$$\rho_{A,B}(X) = \int_0^1 F_X^{-1}(t)\, dA(t) - \int_0^1 F_X^{-1}(t)\, dB(t), \tag{26}$$

where $A(t)$ and $B(t)$ are two distortion functions.

Theorem 22 (*Sordo, 2008*). *Let X and Y be two random variables with distribution functions F_X and F_Y, respectively. Then, $X \leq_{ew} Y$ if and only if $\rho_{A,B}(X) \leq \rho_{A,B}(Y)$ for all $\rho_{A,B}$ such that $B(t)$ and $AB^{-1}(t)$ are convex.*

5.1. Risk contributions of type ΔCoES$_{\alpha,\beta}[Y|X]$

Our first result shows that, under certain conditions on the copulas of the vectors (X, Y) and (X', Y'), the measures Δ**CoES**$_{\alpha,\beta}[Y|X]$ and Δ**CoES**$_{\alpha,\beta}[Y'|X']$ can be written in the form (26), with A and B satisfying the assumptions of Theorem 22, and they can therefore be ordered for all $\alpha, \beta \in (0, 1)$. The result, which requires the assumption $Y \leq_{ew} Y'$, is especially useful when

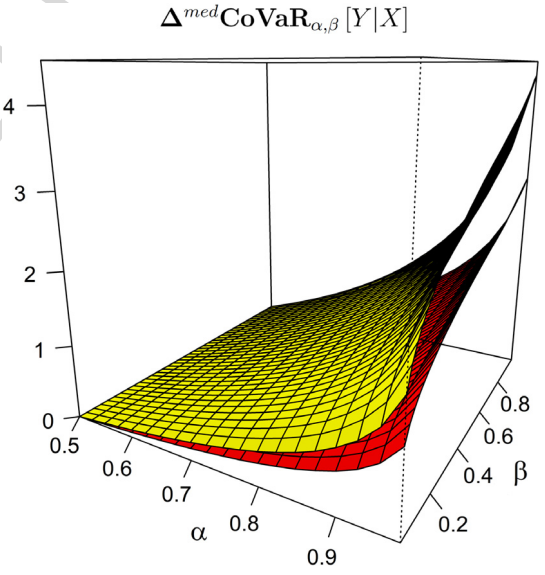


Fig. 5. The graphic plots Δ^{med}**CoVaR**$_{\alpha,\beta}[Y|X]$ as function of α and β. Both surfaces correspond to the same Gaussian copula with dependence parameter $\rho = 0.8$. The red surface has marginals normal $X \sim N(1, 2)$ and Weibull $Y \sim W(1, 1)$. The yellow one has marginals $X' \sim N(1, 2)$ and $Y' \sim W(1.5, 1)$. Therefore $Y \leq_{disp} Y'$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$Y \not\leq_{disp} Y'$ nor $Y \not\geq_{disp} Y'$. In this case, the results in Section 4 cannot be applied to compare risk contributions in terms of Δ**CoVaR**s.

Theorem 23. *Let (X, Y) and (X', Y') be two random vectors with copulas C and C', respectively. Assume that either $X \uparrow_{SI} Y$ or $X' \uparrow_{SI} Y'$ or both hold. Then, $Y \leq_{ew} Y'$ and $C \prec C'$ imply Δ**CoES**$_{\alpha,\beta}[Y|X] \leq \Delta$**CoES**$_{\alpha,\beta}[Y'|X']$, for all $\alpha, \beta \in (0, 1)$.*

Proof. Let $\alpha, \beta \in (0, 1)$ and suppose $X \uparrow_{SI} Y$ (the case $X' \uparrow_{SI} Y'$ is analogous). First note that

$$\mathbf{ES}_\beta[Y] = \int_0^1 F_Y^{-1}(t)\, dH_\beta(t),$$

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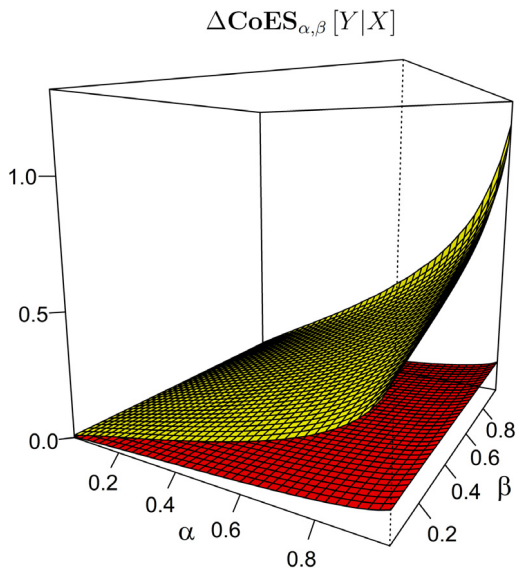


Fig. 6. The graphic plots $\Delta\mathbf{CoES}_{\alpha,\beta}[Y|X]$ as function of α and β. The red surface corresponds to a Gaussian copula C with parameter $\rho = 0.1$ and marginals normal $X \sim N(0,1)$ and Weibull $Y \sim W(1,2)$. The yellow one corresponds to a Gaussian copula C' with $\rho = 0.2$ and marginals $X' \sim N(0,1)$ and $Y' \sim W(1,1)$. Then $C \prec C'$ and $Y \leq_{ew} Y'$, but $Y \not\leq_{disp} Y'$ nor $Y \not\geq_{disp} Y'$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where

$$H_\beta(t) = \max\left\{0, \frac{t-\beta}{1-\beta}\right\} \tag{27}$$

is a convex distortion function (and similarly for $\mathbf{ES}_\beta[Y']$). We use the integral representations for $\mathbf{CoES}_{\alpha,\beta}[Y|X]$ and $\mathbf{CoES}_{\alpha,\beta}[Y'|X']$, given in (11) and (14), in terms of the distortion functions $A_{\alpha,\beta}(t)$ and $B_{\alpha,\beta}(t)$ given in (12) and ((15)). Since $H_\beta^{-1}(t) = (1-\beta)t + \beta$ is lineal and $A_{\alpha,\beta}(t)$ is convex, the composition $A_{\alpha,\beta}H_\beta^{-1}(t)$ is convex. We have the following chain of inequalities

$$\begin{aligned}
&\int_0^1 F_Y^{-1}(t)\,dA_{\alpha,\beta}(t) - \int_0^1 F_Y^{-1}(t)\,dH_\beta(t)\\
&\leq \int_0^1 F_{Y'}^{-1}(t)\,dA_{\alpha,\beta}(t) - \int_0^1 F_{Y'}^{-1}(t)\,dH_\beta(t)\\
&\leq \int_0^1 F_{Y'}^{-1}(t)\,dB_{\alpha,\beta}(t) - \int_0^1 F_{Y'}^{-1}(t)\,dH_\beta(t),
\end{aligned}$$

where the first inequality follows, via Theorem 22, from the assumption $Y \leq_{ew} Y'$ and the second one follows from the fact that $C \prec C'$ implies (17). This proves that $\Delta\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \Delta\mathbf{CoES}_{\alpha,\beta}[Y'|X']$, for all $\alpha, \beta \in (0,1)$. ■

Example 24. Let (X,Y) and (X',Y') be two random vectors with Gaussian copulas with dependence parameters $0 \leq \rho < \rho' \leq 1$, respectively (therefore both are TP_2 and $C_\rho \prec C_{\rho'}$). Suppose that $Y \sim W(a,b)$ and $Y' \sim W(a',b')$, are two Weibull distributions with $b > b'$ and $E[Y] \leq E[Y']$. Then, $Y \leq_{ew} Y'$ but $Y \not\leq_{disp} Y'$ nor $Y \not\geq_{disp} Y'$ (see Example 2.6.24 in Belzunce et al., 2015). In fact, we can find $\alpha, \beta \in (0,1)$ such that (21) is not satisfied. However, it follows from Theorem 23 that $\Delta\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \Delta\mathbf{CoES}_{\alpha,\beta}[Y'|X']$, for all $\alpha, \beta \in (0,1)$. This example is illustrated in Fig. 6.

5.2. Risk contributions of type $\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X]$

In the following result, we provide a sufficient condition for comparing risk contributions of the form $\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X]$. In this case, we consider random vectors with the same copula C, which is required to be TP_2.

Theorem 25. *Let (X,Y) and (X',Y') be two random vectors with the same TP_2 copula C. If $Y \leq_{ew} Y'$ then $\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y'|X']$, for all $0 \leq \beta \leq 1$ and $1/2 \leq \alpha \leq 1$.*

Proof. Let $0 \leq \beta \leq 1$ and denote again $U = F_X(X)$ and $V = F_Y(Y)$. We assume that (U,V) is TP_2. By making $\alpha = 1/2$ in the integral representation for $\mathbf{CoES}_{\alpha,\beta}[Y|X]$ given in (11) we have

$$\mathbf{CoES}_{1/2,\beta}[Y|X] = \int_0^1 F_Y^{-1}(t)\,dA_{1/2,\beta}(t),$$

where $A_{1/2,\beta}(t)$ is a convex distortion function given by

$$A_{1/2,\beta}(t) = \max\left\{1 - \frac{2\overline{C}(1/2,t)}{(1-\beta)}, 0\right\},$$

and we can write

$$\begin{aligned}
\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X] &= \int_0^1 F_Y^{-1}(t)\,dA_{\alpha,\beta}(t)\\
&\quad - \int_0^1 F_Y^{-1}(t)\,dA_{1/2,\beta}(t).
\end{aligned} \tag{28}$$

Similarly, since (X,Y) and (X',Y') have the same copula C, we have

$$\begin{aligned}
\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y'|X'] &= \int_0^1 F_{Y'}^{-1}(t)\,dA_{\alpha,\beta}(t)\\
&\quad - \int_0^1 F_{Y'}^{-1}(t)\,dA_{1/2,\beta}(t).
\end{aligned} \tag{29}$$

Since (U,V) is TP_2, it follows from Theorem 2.3(a) in Joe (1997) that

$$\frac{\overline{C}(\alpha,t)}{(1-\alpha)} = P[V > t\,|U > \alpha] \text{ is increasing in } \alpha,$$

which implies

$$A_{\alpha,\beta}(t) \leq A_{1/2,\beta}(t), \text{ for all } t \in [0,1]. \tag{30}$$

Now we will prove that the continuous function $A_{\alpha,\beta} \circ A_{1/2,\beta}^{-1}(p)$ is convex when $1/2 \leq \alpha \leq 1$. From (13) and (30) it follows that $A_{\alpha,\beta}\left(A_{1/2,\beta}^{-1}(p)\right) = 0$ for $p < A_{1/2,\beta}\left(F_{V|U\geq\alpha}^{-1}(\beta)\right)$ and we only need to prove that $A_{\alpha,\beta}\left(A_{1/2,\beta}^{-1}(p)\right)$ is convex for $p > F_{V|U\geq\alpha}^{-1}(\beta)$. This is equivalent to show that $A'_{\alpha,\beta}(t)/A'_{1/2,\beta}(t)$ is increasing in $t > F_{V|U\geq\alpha}^{-1}(\beta)$ (note that the ratio is well-defined). Using the expression (2.9.1) in Nelsen (1999), this is the same as showing that

$$\frac{P[U > \alpha|V = t]}{P[U > 1/2|V = t]} \text{ increases in } t > F_{V|U\geq\alpha}^{-1}(\beta),$$
$$1/2 \leq \alpha \leq 1. \tag{31}$$

Using that (U,V) is TP_2, it follows, via Definition 3(i) and (8), that $[U|V = v_1] \leq_{hr} [U|V = v_2]$, for $v_1 \leq v_2$, which implies

$$\frac{P[U > \alpha|V = v_2]}{P[U > \alpha|V = v_1]} \text{ increases in } \alpha \text{ for } v_1 \leq v_2. \tag{32}$$

Clearly, (32) implies (31), which proves that $A_{\alpha,\beta} \circ A_{1/2,\beta}^{-1}(p)$ is convex when $1/2 \leq \alpha \leq 1$. Therefore, by applying Theorem 22, from $Y \leq_{ew} Y'$ it follows

$$\begin{aligned}
&\int_0^1 F_Y^{-1}(t)\,dA_{\alpha,\beta}(t) - \int_0^1 F_Y^{-1}(t)\,dA_{1/2,\beta}(t)\\
&\leq \int_0^1 F_{Y'}^{-1}(t)\,dA_{\alpha,\beta}(t) - \int_0^1 F_{Y'}^{-1}(t)\,dA_{1/2,\beta}(t),
\end{aligned}$$

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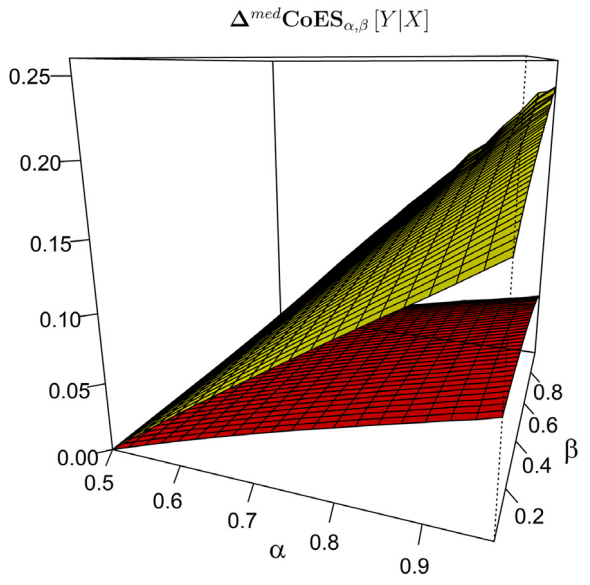


Fig. 7. $\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X]$ as function of α and β. The red surface has a Ali–Mikhail–Haq copula with dependence parameter $\theta = 0.8$ (which is TP_2) and marginals $X \sim N(0,1)$ and $Y \sim W(1,2)$. The yellow one has the same copula and marginals $X' \sim N(0,1)$ and $Y' \sim W(1,1)$. Therefore $Y \leq_{ew} Y'$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

for $0 \leq \beta \leq 1$ and $1/2 \leq \alpha \leq 1$, which is the same, using (28) and (29), as

$$\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \Delta^{med}\mathbf{CoES}_{\alpha,\beta}\left[Y'|X'\right], \quad 0 \leq \beta \leq 1,\ 1/2 \leq \alpha \leq 1. \quad \blacksquare$$

Example 26. Let (X, Y) and (X', Y') be two random vectors with the same Ali–Mikhail–Haq copula (see (3.3.7) in Nelsen, 1999) with dependence parameter $0 \leq \theta \leq 1$ (which is TP_2). As in Example 24 let $Y \sim W(a, b)$ and $Y' \sim W(a', b')$, with $b > b'$ and $E[Y] \leq E[Y']$ (which implies $Y \leq_{ew} Y'$ but $Y \not\leq_{disp} Y'$ nor $Y \not\geq_{disp} Y'$). It follows from Theorem 25 that $\Delta^{med}\mathbf{CoES}_{\alpha,\beta}[Y|X] \leq \Delta^{med}\mathbf{CoES}_{\alpha,\beta}\left[Y'|X'\right]$, for all $\frac{1}{2} \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ (see Fig. 7).

6. Conclusions

In portfolio risk analysis, co-risk measures and risk contribution measures are used to evaluate the systemic risk, which is related to the risk that the loss of a component X spreads to another component Y or even to the whole portfolio. In this paper, we have studied the consistency of these measures with respect to different stochastic orders among the marginals and under different assumptions on the dependence structure of the underlying risks. First, we have improved a result given by Mainik and Schaanning (2014) with respect to stochastic dependence of portfolios under the assumption that at least one of the compared dependence structures is positive (which has been formalized in terms of stochastically increasing risks). Then, we have shown that risk contribution measures based on CoVaR and CoES are consistent with the dispersive order and the excess wealth order, respectively, of the marginals, when the compared dependence structures satisfy certain conditions. We have also provided a new characterization of DFR distributions in terms of a family of risk contribution measures based on CoVaRs. All the results in the paper have been illustrated with examples.

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