# COMPARABILITY, SEPARATIVITY, AND EXCHANGE RINGS.

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**Abstract**

There are several long-standing open problems which ask whether regular rings, and $C^*$-algebras of real rank zero, satisfy certain module cancellation properties. Ara, Goodearl, O'Meara and Pardo recently observed that both types of rings are exchange rings, and showed that separative exchange rings have these good cancellation properties, thus answering the questions affirmatively in the separative case. In this article, we prove that, for any positive integer $s$, exchange rings satisfying $s$-comparability are separative, thus answering the questions affirmatively in the $s$-comparable case.

We also introduce the weaker, more technical, notion of generalized $s$-comparability, and show that this condition still implies separativity for exchange rings. On restricting to directly finite regular rings, we recover results of Ara, O'Meara and Tyukavkin.

# Introduction.

Throughout this article, let $s$ be a positive integer, let $R$ be a ring (associative, with 1), and let $M$ be a monoid (commutative, with operation $+$, and neutral 0).

*Partially supported by the DGICYT (Spain), through the grant PB93-0900, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The results contained in this paper form part of the author's Ph.D.Thesis, written under the supervision of Professor P.Ara.

Recall that the monoid $M$ is said to satisfy *$s$-comparability* if for any $p, q \in M$, either $p$ is a summand of $sq$ (in $M$), or $q$ is a summand of $sp$. Also, $M$ is *separative* if it satisfies the weak cancellation condition that for all $a$, $b$ in $M$, $a + a = a + b = b + b$ only if $a = b$. (In 1956, Hewitt and Zuckerman [11] defined $M$ to be separative if its elements were separated by its characters, and showed that these two conditions were equivalent; see [7, Theorem 5.59]. Hewitt and Zuckerman further showed that separativity was a "local" cancellation condition in the sense that it was equivalent to every archimedean component of $M$ being cancellative; see [7, Theorem 4.13].) We say that $M$ is *strongly separative* if for all $a$, $b$ in $M$, $a+a = a+b$ only if $a = b$.

Let $V(R)$ denote the monoid of isomorphism classes of finitely generated projective right $R$-modules, with the operation induced by the direct sum.

We say that $R$ satisfies *$s$-comparability*, resp. is *separative*, resp. is *strongly separative*, if the monoid $V(R)$ has the corresponding property. The definition of an "exchange" ring will be recalled in Section 2.

Although the foregoing terminology seems natural, it is not used universally. In [8, page 275], a (von Neumann) regular ring $R$ is said to satisfy $s$-comparability if, for each pair of elements $x, y$ of $R$, either $xR$ is isomorphic to a summand of $s(yR)$, or $yR$ is isomorphic to a summand of $s(xR)$; for regular rings this is equivalent to the above usage, by Proposition 2.1(1) of [3]. Also, "strongly separative" for rings is referred to as "has Cancellation of Small Projectives (CSP)" in [1],[3], and, for monoids, as "has Cancellation of Small Elements (CSE)" in [1]; see [1, Lemma 5.5].

There are many open problems concerning cancellation in $V(R)$. For example, if $R$ is regular, or a $C^*$-algebra of real rank zero, it would be interesting to know whether the property of being directly finite is inherited by all matrix rings, whether simplicity or direct finiteness implies stable rank one, and whether the stable rank lies in $\{1, 2, \infty\}$; see [8], [5]. Ara, Goodearl, O'Meara and Pardo [1] proved several cancellation results for the class of separative exchange rings, and observed that regular rings and unital $C^*$-algebras of real rank zero are exchange rings, thus solving the foregoing problems in the separative case. This leaves open the question of whether an exchange ring is separative.

Comparability concepts have proven to be particularly fruitful in the development of the theory of regular rings. The strongest condition, 1-comparability, in the guise of "the comparability axiom", was introduced by Goodearl and Handelman [9], [8, page 80], while $s$-comparability, and slight generalizations theoreof, were used by the same authors [10], [8, Chapter 18] to characterize uniqueness of rank functions on certain regular rings. A regular ring $R$ satisfies *general $s$-comparability* if, for all finitely generated projective right $R$-modules $P$, $Q$, there exists a central idempotent $e$ in $R$ such that $Pe$ is isomorphic to a summand of $s(Qe)$, and $Q(1-e)$ is isomorphic to a summand of $s(P(1-e))$.

General 1-comparability coincides with "general comparability" for regular rings, by [8, Proposition 8.8], and general comparability is important in the theories of operator algebras, Baer rings [12, Theorem 57], and regular right self-injective rings [15], [8, Chapter 9].

Because of the importance of comparability and separativity, it is natural to examine which comparability hypotheses on exchange rings imply separativity. Goodearl and Handelman showed that directly finite regular rings satisfying general 1-comparability have stable rank one [8, Theorem 8.12], so are strongly separative by Theorem 4.14 of [8]. Ara, O'Meara and Tyukavkin [3] showed that a directly finite regular ring satisfying $s$-comparability need not have stable rank one (thus answering [8, Open Problem 4] in the negative), but is necessarily strongly separative. This left open the question of what happens in the case of exchange rings, or even directly infinite regular rings.

The main purpose of this article is to show that an exchange ring is separative if it satisfies $s$-comparability, or even a certain technical generalization thereof called "generalized $s$-comparability", defined and studied in Section 3, and agreeing with the above definition in the case of regular rings. Our techniques, which are different from those of [3], are monoid-theoretic; some of the tools needed were developed in [1] and [4].

In outline the paper is as follows. In Section 1, we recall the necessary definitions, and prove our main monoid-theoretical result, that every conical refinement monoid satisfying $s$-comparability is separative. In Section 2, we deduce that every exchange ring satisfying $s$-comparability is separative, which was proved for directly finite regular rings in [3, Theorem 4.6]. In Section 3, we extend the definition of generalized $s$-comparability to refinement monoids and exchange rings, by considering decompositions of refinement monoids, and we determine the relationship with central idempotents in rings. We then show that the results of the previous two sections remain valid when the hypotheses of $s$-comparability are weakened to generalized $s$-comparability.

# 1 Refinement monoids and $s$-comparability.

In the introduction, we defined the concepts of separativity, strong separativity, and $s$-comparability for $M$, and mentioned their connections with ring theory. We now develop a certain amount of theory about monoids, and postpone to the next section the explanation of its relevance to ring theory.

**Definitions 1.1** We write $M^*$ to denote the set of nonzero elements of $M$.

We say that $M$ is *conical* if, for all $x$, $y$ in $M$, $x+y=0$ only when $x=y=0$.

An element $x$ of $M$ is said to be *directly finite* in $M$, if, for all $y \in M^*$, $x+y \neq x$; otherwise, $x$ is *directly infinite* in $M$. Also $x$ is said to be *stably finite* if $nx$ is directly finite for all positive integers $n$.

For $x \in M$, the *stable rank* of $x$ in $M$, denoted $\mathrm{sr}_M(x)$, is the minimum positive integer $n$ which has the property that, for all $y$, $z$ in $M$ such that $nx + y = x + z$, there exists $w \in M$ such that $nx = x + w$ and $y + w = z$; if no positive integer $n$ has this property, the minimum is understood to be $\infty$. Where there is no ambiguity, we shall write $\mathrm{sr}(x)$ in place of $\mathrm{sr}_M(x)$. This concept was defined in [1, Section 6], and was originally inspired by [17, Theorem 1.3].

Following [18], for example, we say that $M$ is a *refinement monoid* if, for all $a$, $b$, $c$, $d$ in $M$ such that $a+b = c+d$, there exist $w$, $x$, $y$, $z$ in $M$ such that $a = w + x$, $b = y + z$, $c = w + y$ and $d = x + z$. It will usually be convenient to present this situation in the form of a diagram, as follows:

$$\begin{array}{|c|c|c|} \hline & c & d \\ \hline a & w & x \\ \hline b & y & z \\ \hline \end{array}.$$

If $x$, $y$ in $M$, we write $x \leq y$, resp. $x < y$, if there exists $z \in M$, resp. $z \in M^*$, such that $x + z = y$. Here $\leq$ is a translation-invariant pre-order on $M$. Notice that $x < x$ if and only if $x$ is directly infinite.

An element $u$ of $M$ is said to be an *order-unit* for $M$ if, for each $x \in M$, there exists a positive integer $n$ such that $x \leq nu$.

If $u$ is an order-unit of $M$, then we call the pair $(M, u)$ a *monoid with order-unit*, and we say that $(M, u)$ is *directly finite*, resp. *directly infinite*, resp. *stably finite*, if $u$ has the corresponding property.

As in [4], we say that $(M, u)$ satisfies *weak comparability* if, for each $x \in M^*$ such that $x \leq u$, there exists a positive integer $n$ such that for all $y \in M$, $ny \leq u$ only if $y \leq x$.

A subset $S$ of a monoid $M$ is called an *order-ideal*, or simply an *ideal*, if $S$ is a subset of $M$ containing 0, closed under taking sums and summands within $M$; that is, $S$ is a submonoid such that, for all $x \in M$ and $e \in S$, if $x \leq e$ then $x \in S$.

We denote the set of ideals of $M$ by $L(M)$. If $M$ is a refinement monoid then, by [1, Lemma 2.1], $L(M)$ forms a lattice under sum and intersection.

For any $a \in M$, the smallest ideal of $M$ containing $a$ is denoted $M(a)$; thus $M(a) = \{x \in M \mid x \leq na$ for some positive integer $n\}$, and $a$ is an order-unit for $M(a)$. It is clear that an ideal $S$ of $M$ is of this form if and only if there exists an order-unit for $S$.

We say that $M$ is a *simple* monoid if $M$ is nonzero, conical, and every nonzero element is an order-unit, or equivalently, there are exactly two ideals, $M$ and $\{0\}$.

For any ideal $S$ of $M$, the *factor monoid* of $M$ modulo $S$, denoted $M/S$, is the monoid with the sum induced on the set of equivalence classes with respect to the equivalence relation $\sim$ defined by setting $x \sim y$ whenever there exist

$e, f \in S$ such that $x + e = y + f$. The equivalence class of $x$ in $S$ will usually be denoted $[x]$.

We begin with two useful results about the stable rank. The first is a monoid-theoretic analogue of Theorem 1.2 of Warfield's paper [17].

**Lemma 1.2** *Let $M$ be a monoid, $n$ a positive integer, and $x, y, z \in M$. If $x + z = y + z$, and $\mathrm{sr}(z) \le n$, and $nz \le x$, then $x = y$.*

**Proof.** Since $nz \le x$, there exists $w \in M$ such that $x = nz + w$, so $nz + (w + z) = (nz + w) + z = x + z = y + z$. As $\mathrm{sr}(z) \le n$, there exists $v \in M$ such that $nz = z + v$ and $w + z + v = y$. Thus, $x = nz + w = w + z + v = y$. □

**Lemma 1.3** *If $M$ is a conical refinement monoid, and $a$ an element of $M$, then $\mathrm{sr}_M(a) = \mathrm{sr}_{M(a)}(a)$.*

**Proof.** It is clear that $\mathrm{sr}_{M(a)}(a) \le \mathrm{sr}_M(a)$. To prove the reverse inequality, we may assume that $n = \mathrm{sr}_{M(a)}(a)$ is finite. Suppose we have $b, c \in M$ such that $na + b = a + c$. Applying refinement we obtain

$$\begin{array}{c|c|c|} & a & c \\ \hline na & w & y \\ \hline b & x & z \\ \hline \end{array}.$$

Notice that $w$, $x$ and $y$ all lie in $M(a)$. Now $na + x = w + y + x = a + y$ in $M(a)$. Since $n = \mathrm{sr}_{M(a)}(a)$, there exists $e \in M(a)$ such that $na = a + e$ and $x + e = y$. Thus, $b + e = z + x + e = z + y = c$, and we have found $e \in M$ such that $na = a + e$ and $b + e = c$. This shows $\mathrm{sr}_M(a) \le n$, as desired. □

We next record two elementary, but useful, results about ideals.

**Lemma 1.4** *The following hold for any ideal $S$ of $M$.*

(1) *$M/S$ is conical.*

(2) *If $M$ is a refinement monoid, then so are $S$ and $M/S$.*

(3) *If $M$ satisfies s-comparability, then so do $S$ and $M/S$.*

(4) *If $S$ is an ideal in $M$, then there is a natural bijection between the set of ideals of $M/S$ and the set of ideals of $M$ containing $S$.*

(5) *If $J \subseteq K$ are ideals of $M$, then $(M/J)/(K/J) \cong (M/K)$.*

**Proof.** (1) Suppose $x$, $y$ are elements of $M$ such that $[x] + [y] = 0$ in $M/S$. This means that there exist $e, f \in S$ such that $x + y + e = f$. Since $S$ is an ideal, $x, y \in S$, so $[x] = [y] = 0$. It is straightforward to prove (2), (3), (4) and (5). □

**Lemma 1.5** *If $M$ satisfies s-comparability then $L(M)$ is totally ordered by inclusion. If, moreover, $M$ is nonzero, and has an order unit, then $M$ has a unique maximal proper ideal, denoted $\max(M)$.*

**Proof.** Suppose that $J$, $K$ are ideals of $M$ such that $J \not\subseteq K$. Then there exists $x \in J \setminus K$, and for all $y \in K$, $sy \in K$ so $x \not\leq sy$. Hence $y \leq sx$ by $s$-comparability, and $sx \in J$, so $y \in J$. Thus $K \subseteq J$, which shows that $L(M)$ is totally ordered.

If $M$ is nonzero and has an order unit $u$, then the union of all ideals of $M$ not containing $u$ is the unique maximal proper ideal of $M$. □

**Proposition 1.6** *Let $M$ be a conical refinement monoid satisfying $s$-comparability, and let $a$ and $b$ be elements of $M$.*

(1) *If $(s+1)a \leq 2b$ then $a \leq b$.*

(2) *If $M(a) \subset M(b)$, then $a < b$, and, moreover, $na < b$ for each positive integer $n$.*

(3) *If $S$ is an ideal of $M$, and $[a] < [b]$ in $M/S$, then $a < b$.*

(4) *If $a$ is directly finite in $M$, then the image of $a$ is directly finite in any factor monoid of $M$.*

**Proof.** (1) is proved by the same argument used to prove [3, Lemma 2.2].

(2) Suppose $M(a) \subset M(b)$. In particular, $b \notin M(a)$, so $b \not\leq s(s+1)^s a$, so by $s$-comparability, $(s+1)^s a \leq sb \leq 2^s b$. Now, by $s$ applications of (1), $a \leq b$, so $a < b$.

For every positive integer $n$, $M(na) = M(a) \subset M(b)$, so, by the foregoing, $na < b$.

(3) The hypotheses imply there exist $c \in M \setminus S$, and $d, e \in S$ such that $a + c + d = b + e$. Since $M$ is a refinement monoid we have

$$\begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline b & u & v & w \\ \hline e & x & y & z \\ \hline \end{array}.$$

Since $d, e \in S$, we see that $x, y, z, w \in S$. Since $v + y = c \notin S$ we see that $v \notin S$. Thus $M(x) \subseteq S$ and $M(v) \not\subseteq S$. By Lemma 1.5, the ideals of $M$ are totally ordered, so $M(x) \subseteq S \subset M(v)$. By (2), $x < v$, so $a = u + x < u + v \leq u + v + w = b$.

(4) is the case $b = a$ of (3). □

In the foregoing proposition, (1), (2), and (3) are similar to Lemma 2.2, and Propositions 2.3, 2.5, of [1], respectively, while (4) is similar to Corollary 2.7 of [3].

**Proposition 1.7** *If $(M, u)$ is a simple conical refinement monoid with order-unit satisfying $s$-comparability, then the following hold.*

(1) *$M$ satisfies weak comparability.*

(2) *If $u$ is directly finite, then $M$ is cancellative.*

(3) *If $u$ is directly infinite, then $M^*$ is cancellative, and for all $x, y \in M^*$, $x < y$.*

(4) *$M$ is separative.*

**Proof.** (1) Consider any $x \in M^*$ such that $x \leq u$. Since $u$ is an order-unit, there exists a positive integer $k$ such that $u \leq 2^k x$. Let $n = (s+1)^k$. If $y \in M$ such that $ny \leq u$, then $(s+1)^k y = ny \leq 2^k x$, so, by $k$ applications of Proposition 1.6(1), $y \leq x$. This proves weak comparability.

(2), resp. (3), follows immediately from (1) and [14, Corollary 2] (or [4, Proposition 1.4(a)]), resp. [4, Proposition 1.4(b)].

(4) follows immediately from (2) and (3). □

We will not need to use part (4) of the foregoing proposition, but include it out of interest, since it is the "simple" case of the main result of this section. We now come to a crucial step.

**Lemma 1.8** *Let $M$ be a conical refinement monoid satisfying $s$-comparability, and suppose that $a$ is a nonzero element of $M$ such that $(M(a)/\max(M(a)), [a])$ is directly finite. Then $\mathrm{sr}_M(b) \leq 2$ for all $b \in M(a) \setminus \max(M(a))$.*

**Proof.** Since $(M(a)/\max(M(a)), [a])$ is a simple conical refinement monoid with directly finite order-unit, and it satisfies $s$-comparability, $M(a)/\max(M(a))$ is cancellative, by Proposition 1.7(2).

Let $b \in M(a) \setminus \max(M(a))$. Thus $M(a) = M(b)$, so $M(b)/\max(M(b))$ is cancellative. We want to show that $\mathrm{sr}_M(b) \leq 2$, and by Lemma 1.3 it suffices to show that $\mathrm{sr}_{M(b)}(b) \leq 2$. Suppose we have $c$, $d$ in $M(b)$ such that $2b+c = b+d$. We want to find $e \in M(b)$ such that $2b = b+e$ and $c+e = d$. By the refinement condition, we have the following situation:

| | $2b$ | $c$ |
|---|---|---|
| $b$ | $w$ | $x$ |
| $d$ | $y$ | $z$ |

Thus $2b+x = w+y+x = b+y$. In the cancellative monoid $M(b)/\max(M(b))$, $2[b]+[x] = [b]+[y]$, so $[b]+[x] = [y]$. Since $[b]$ is nonzero, we have $[x] < [y]$, so, by Proposition 1.6(3), $x < y$. Hence there exists $e \in M(b)$ such that $x+e = y$. Now $b+e = w+x+e = w+y = 2b$ and $c+e = x+z+e = y+z = d$, as desired. □

We can now prove our main monoid-theoretic result.

**Theorem 1.9** *If $M$ is a conical refinement monoid satisfying $s$-comparability, then $M$ is separative, and, moreover, for all $a$, $b$ in $M$, if $2a = a+b$ then either $M(b) \subset M(a)$ or $a = b$.*

Recall from [1, Proposition 6.4] that all elements in a separative monoid have stable rank 1, 2 or $\infty$.

**Proof.** Let $a$, $b$ be elements of $M$ such that $2a = a+b$. Here $b \in M(a)$, so $M(b) \subseteq M(a)$. Thus we assume that $M(b) = M(a)$, and we will show that

$a = b$. By the refinement condition we have the following situation:

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{|c|}{a} & b \\ \hline a & w & x \\ \hline a & y & z \\ \hline \end{array}.$$

By Lemma 1.5 one of the ideals $M(w)$, $M(y)$ contains the other, and since the two rows of this diagram are interchangeable, we may assume that $M(w) \subseteq M(y)$. If $w = 0$ then $x = a = y$ and $a = y + z = x + z = b$. Thus we may assume that $M(w)$ is nonzero.

We divide the argument into four cases, corresponding to the four possibilities as to whether $M(w) = M(a)$ or not, and whether $(M(w)/\max(M(w)), [w])$ is directly finite or not.

(1) Consider the case where $M(w) \subset M(a)$, so $M(w) \subseteq \max(M(a))$. Since $w+x = a$ and $w+y = a$ do not lie in $\max(M(a))$ we see that $M(w) \subset M(x) = M(y) = M(a) = M(b)$. By Proposition 1.6(2), $w \leq 2w < x, y$.

(1a) Consider the subcase where $(M(w)/\max(M(w)), [w])$ is directly infinite. Here $2[w] < [w]$ in $M(w)/\max(M(w))$ by Proposition 1.7(3), so $2w < w$ in $M(w)$ by Proposition 1.6(3). Thus $2w < w < x, y$ in $M$, so there exist $w'$, $x'$, $y'$ in $M$ such that $x = w + x'$, $y = w + y'$, $w = 2w + w'$. Hence $x = w + x' = 2w + w' + x' = w + w' + x = a + w' = w + y + w' = 2w + w' + y' = w + y' = y$. Hence $a = y + z = x + z = b$.

(1b) Consider the subcase where $(M(w)/\max(M(w)), [w])$ is directly finite. Here $\mathrm{sr}_M(w) \leq 2$ by Lemma 1.8, taking $a$ and $b$ of that lemma to be $w$. Also $x + w = a = y + w$, and $2w \leq x$. By Lemma 1.2, $x = y$. Hence $a = y + z = x + z = b$.

(2) This leaves the case where $M(w) = M(a)$. Here $M(w) = M(y) = M(a) = M(b)$.

(2a) Consider the subcase where $(M(w)/\max(M(w)), [w])$ is directly infinite. Here $2[w] < [w] < [x], [y]$ in $M(w)/\max(M(w))$ by Proposition 1.7(3), so we have $2w < w < x, y$ in $M(w)$ by Proposition 1.6(3). Exactly as in case (1a), we deduce that $a = b$.

(2b) This leaves the subcase where $(M(w)/\max(M(w)), [w])$ is directly finite. Here we argue as in the proof of Theorem 1.7 of [4]. Since $M(w) = M(y)$, there exists a positive integer $n$ such that $w \leq ny$. By the refinement condition there exist $w_1, \ldots, w_n$ in $M$ such that $w = w_1 + \cdots + w_n$ and $w_1 \leq y, \ldots, w_n \leq y$. All the $w_i$ lie in $M(w)$ and their sum lies in $M(w) \setminus \max(M(w))$, so some $w_{i_0}$ lies in $M(w) \setminus \max(M(w))$ which means $M(w_{i_0}) = M(w)$. Let us denote this $w_{i_0}$ by $c$, so $c \leq w$ and $c \leq y$, so $2c \leq w + y = a$. Also $M(c) = M(w) = M(a)$, so there exists a positive integer $m$ such that $a \leq mc$, which means that there exists $d \in M$ such that $a + d = mc$. Hence $a + mc = a + a + d = a + b + d = b + mc$. By Lemma 1.8, taking $a$ and $b$ of that lemma to be $w$ and $c$, respectively, we see that $\mathrm{sr}_M(c) \leq 2$. Thus $a + mc = b + mc$,

$\mathrm{sr}_M(c) \leq 2$, and $2c \leq a$. By $m$ applications of Lemma 1.2, taking $x$, $y$, $z$ of that Lemma to be $a + ic$, $b + ic$, $c$, respectively, for $i = m - 1, \ldots, 0$, we see that $a = b$.

Finally, to see that $M$ is separative, suppose that $2a = a + b = 2b$. Then $M(a) = M(b)$, and, by the foregoing, $a = b$. $\square$

**Remarks 1.10** (1) To see that the refinement hypothesis cannot be deleted in Theorem 1.9, consider the commutative monoid presented on two generators $a$, $b$ with relations saying that $a + a = a + b = b + b$. This monoid is conical and satisfies 2-comparability, but is not separative.

(2) In the terminology introduced by Wehrung [19], $M$ is a *separative positively ordered monoid* if, for all $a$, $b$ in $M$, if $a + a = a + b$ then $b \leq a$; this lies between separativity and strong separativity.

If $M$ is a conical refinement monoid satisfying $s$-comparability, then we can show that $M$ is a separative positively ordered monoid, as follows. Suppose $a + a = a + b$ in $M$. Theorem 1.9 shows that $a = b$, or $M(b) \subset M(a)$, and in the latter case, $b < a$ by Proposition 1.6(2).

Let us record the following consequence of Theorem 1.9.

**Corollary 1.11** *If $(M, u)$ is a conical directly finite refinement monoid satisfying $s$-comparability, then every element of $M$ is directly finite and has stable rank at most 2, and $M$ is strongly separative.*

**Proof.** By Theorem 1.9, $M$ is separative. Let $a$, $b$ be elements of $M$. To see that $a$ is directly finite, we suppose that $a + b = a$ and proceed as in the proof of [1, Theorem 7.1] to show that $b = 0$. Thus, there exists a positive integer $n$ such that $a + c = nu$, so $b + nu = b + a + c = a + c = nu$, and thus $b + u + (n-1)u = u + (n-1)u$. A classical result of Hewitt and Zuckerman [11], reproduced in [1, Lemma 3.1], states that since $b + u$, $u$ are order-units in the separative monoid $M$, the summand $(n-1)u$ can be cancelled, so $b + u = u$. Since $u$ is directly finite, $b = 0$, as desired.

Since $a$ is directly finite in $M(a)$, if $a$ is nonzero then $[a]$ is directly finite in $M(a)/\max(M(a))$, by Proposition 1.6(4). By Lemma 1.8, we see that the stable rank of every element of $M$ is at most two.

Now suppose that $2a = a + b$. We want to show that $a = b$, and we may assume that $a \neq 0$. In $M(a)/\max(M(a))$, $2[a] = [a] + [b]$, we see that $[b] \neq 0$ since $[a]$ is directly finite, so $M(b) = M(a)$. By Theorem 1.9, $a = b$. $\square$

**Remark 1.12** Conical strongly separative refinement monoids need not be directly finite. For example, let $M$ be the commutative monoid presented on two generators $a$, $b$, with a single relation, saying that $a + b = a$. It is not difficult to check that $M$ is a conical refinement monoid satisfying 1-comparability. Then $M(b)$ is a free monoid freely generated by $b$, and $M/M(b)$ is a free monoid freely generated by $[a]$, and these are strongly separative, so by [1, Theorem 5.7], $M$ is strongly separative.

## 2 Exchange rings and $s$-comparability.

Recall that $R$ is a ring, and $V(R)$ is the monoid of isomorphism classes of finitely generated projective right $R$-modules. Thus each element of $V(R)$ is the isomorphism class $[P]$ of a finitely generated projective right $R$-module $P$, which is, of course, unique up to isomorphism.

**Definition 2.1** We say that $R$ is an *exchange* ring if, for every right module $A_R$, and all decompositions

$$A = B \oplus C = \bigoplus_{i \in I} A_i$$

with $B \cong R$ as right $R$-modules, there exist submodules $A_i' \subseteq A_i$ such that

$$A = B \oplus \left( \bigoplus_{i \in I} A_i' \right).$$

For example, all semiregular rings (i.e., rings which modulo the Jacobson radical are regular, and such that idempotents lift modulo the Jacobson radical), all $\pi$-regular rings, and all unital $C^*$-algebras of real rank zero are exchange rings; see [17], [16], [1].

We list the monoid-theoretic aspects of $V(R)$ which allow us to translate Theorem 1.9 into a ring-theoretic result. Clearly, $V(R)$ is conical. If $R$ is an exchange ring then $V(R)$ is a refinement monoid by [1, Proposition 1.1]. By [1, Theorem 6.3], if $R$ is an exchange ring and $[P] \in V(R)$, then $\mathrm{sr}_{V(R)}([P])$ agrees with $\mathrm{sr}(\mathrm{End}_R(P))$, as defined by Bass; see [17].

Combining the above facts with Theorem 1.9 we get our first main result.

**Theorem 2.2** *If $R$ is an exchange ring satisfying $s$-comparability, then $R$ is separative, so has stable rank* 1, 2 *or* $\infty$. $\square$

To see what Corollary 1.11 says about rings, we recall more information about $V(R)$. Clearly $[R]$ is an order-unit in $V(R)$, and $(V(R), [R])$ is directly finite, resp. directly infinite, resp. stably finite, if and only if $R$ has the corresponding property. If $I$ is an ideal of $R$, then the set $V(I, R) = \{[P] \in V(R) \mid P = PI\}$ is an ideal of $V(R)$. By [1, Proposition 2.2], if $R$ is an exchange ring, then $V(R/I) \cong V(R)/V(I, R)$, and, by [1, Theorem 2.3], the map $L(R) \longrightarrow L(V(R))$, $I \mapsto V(I, R)$, is a surjective lattice homomorphism.

Combined with the above, Corollary 1.11 and Proposition 1.6(4) show the following, which was proved for regular rings in [3, Theorem 4.6].

**Theorem 2.3** *If $R$ is a directly finite exchange ring satisfying $s$-comparability, then $R$ is stably finite, strongly separative, and* $\mathrm{sr}(R)$ *is* 1 *or* 2. *Moreover, for every ideal $I$ of $R$, $R/I$ is a directly finite exchange ring satisfying $s$-comparability.* $\square$

**Remarks 2.4** (1) It is not known if all exchange rings are separative, but [2, Example 3.8] gives an example of a regular ring which is not strongly separative.

(2) It follows from Remark 1.10(2) that if $R$ is an exchange ring satisfying $s$-comparability then $V(R)$ is a separative positively ordered monoid; it is not known if this latter property holds for all exchange rings.

(3) Menal and Moncasi gave an example [13, Example 1] of a regular ring which satisfies 1-comparability, is strongly separative and directly infinite.

# 3 Generalized $s$-comparability.

We now recall the decomposability definitions for monoids.

**Definitions 3.1** Let $M$ be a conical monoid. We write $M = M_1 \oplus M_2$ if $M_1$ and $M_2$ are submonoids of $M$ such that each element of $M$ can be written, in a unique way, as the sum of an element of $M_1$ and an element of $M_2$. In this event, we call the expression $M = M_1 \oplus M_2$ a *decomposition*; it is a *trivial* decomposition if either of the monoids is zero. We denote by $\pi_1 : M \to M_1$ the natural projection map, and similarly for $\pi_2$. They are surjective monoid morphisms. Moreover, the ideal ker $(\pi_1)$ is $M_2$, $M/M_2 \cong M_1$.

If $M$ has a nontrivial decomposition, then we say that $M$ is *decomposable*, and it is *indecomposable* if it is nonzero and not decomposable.

**Lemma 3.2** *Let $M$ be a conical monoid, and let $a$ and $b$ be elements of $M$. If in each indecomposable factor of $M$ the images of $a$ and $b$ are equal, then $a = b$.*

**Proof.** Suppose $a \neq b$, and let $\mathcal{C}$ denote the set of those ideals $S$ of $M$ such that $[a] \neq [b]$ in $M/S$. Clearly $\{0\}$ belongs to $\mathcal{C}$. Let $\mathcal{L}$ be a nonempty chain in $\mathcal{C}$, and let $L$ denote the union of the chain $\mathcal{L}$. Then $L$ is an ideal of $M$. If $L$ does not belong to $\mathcal{C}$, then $[a] = [b]$ in $M/L$, and thus there exist elements $e, f \in L$ such that $a+e = b+f$. Hence, there exists $K \in \mathcal{L}$ such that $e, f \in K$, whence $[a] = [b]$ in $M/K$, a contradiction. Thus, $L \in \mathcal{C}$ is an upper bound for $\mathcal{L}$.

By Zorn's Lemma, $\mathcal{C}$ has a maximal element $S$. Since $[a] \neq [b]$ in $M/S$, $M/S$ is not indecomposable, by hypothesis, and is clearly not zero, so it is decomposable. By Lemma 1.4(4) there exist ideals $M_1$, $M_2$ in $M$ properly containing $S$, such that $M/S = M_1/S \oplus M_2/S$. Since $[a] \neq [b]$ in $M/S$, we see that the images of $[a]$ and $[b]$ under the projections to $M_1/S$, or to $M_2/S$, do not agree, and by symmetry, we may assume the former. Thus, the images of $[a]$ and $[b]$ in $(M/S)/(M_2/S)$ do not agree. By Lemma 1.4(5), the images of $a$ and $b$ in $M/M_2$ do not agree. Thus $M_2 \in \mathcal{C}$. This contradicts the maximality of $S$. Thus $a = b$. $\square$

**Corollary 3.3** *A conical refinement monoid $M$ is separative, resp. strongly separative, if and only if every indecomposable factor of $M$ is separative, resp. strongly separative.* $\square$

To define generalized $s$-comparability in the context of monoids, we look at their decompositions.

**Definition 3.4** We say that a conical monoid $M$ satisfies *generalized $s$-comparability* if, for all $x$, $y \in M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $\pi_1(x) \leq s\pi_1(y)$ and $\pi_2(y) \leq s\pi_2(x)$.

Notice that ordinary $s$-comparability corresponds to taking trivial decompositions.

**Theorem 3.5** *Let $M$ be a conical refinement monoid satisfying generalized $s$-comparability.*

(1) *For every ideal $S$ of $M$, $M/S$ is a conical refinement monoid satisfying generalized $s$-comparability.*

(2) *$M$ is separative*

(3) *If every indecomposable factor of $M$ is directly finite, then $M$ is strongly separative.*

**Proof.** (1) By Lemma 1.4(1),(2), it only remains to show that $M/S$ satisfies generalized $s$-comparability. But if $M = M_1 \oplus M_2$ is any decomposition of $M$, then, since $S$ is closed under summands in $M$, we have a corresponding decompostion $S = S_1 \oplus S_2$, so generalized $s$-comparability is easy to check.

(2) By (1), and Theorem 1.9, every indecomposable factor of $M$ is separative, so by Corollary 3.3, $M$ is separative.

(3) By (1), and Corollary 1.11, every indecomposable factor of $M$ is strongly separative, so by Corollary 3.3, $M$ is strongly separative. $\square$

We now want to translate this to exchange rings, and we need to be able to relate indecomposable rings to indecomposable monoids. We write $J(R)$ for the Jacobson radical of $R$.

**Proposition 3.6** *If $R$ is an exchange ring, then $V(R) \cong V(R/J(R))$, and $V(R)$ is indecomposable if and only if $R/J(R)$ is indecomposable.*

**Proof.** Let $J = J(R)$. By [1, Proposition 2.2], $V(R/J) \cong V(R)/V(J,R)$ and, by Nakayama's Lemma, $V(J,R) = 0$. This proves $V(R/J) \cong V(R)$. Results of Bass shows that this condition holds in greater generality. Thus, to prove the second part, we may assume that $R$ is semiprimitive.

Suppose that $R$ is decomposable. Then there is a non-trivial ring decomposition $R = R_1 \times R_2$, and it is well-known that $V(R) = V(R_1) \oplus V(R_2)$, so $V(R)$ is decomposable.

Suppose now that $V(R)$ is decomposable, so $V(R) = M_1 \oplus M_2$ for some nonzero ideals $M_i$ of $V(R)$. By [1, Theorem 2.3], since $R$ is semiprimitive, there exist ideals $I_1, I_2$ of $R$ such that $M_i = V(I_i, R)$, $I_1 \cap I_2 = 0$ and $I_1 + I_2 = R$. Hence, $R$ is decomposable. □

**Example 3.7** The ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field, is an indecomposable semiperfect ring, and hence an exchange ring, but $R/J(R) \cong F \times F$ is decomposable. Observe that $R$ satisfies generalized 1-comparability, but not 1-comparability.

**Definition 3.8** We say that a ring $R$ satisfies *generalized $s$-comparability* if $V(R)$ satisfies generalized $s$-comparability.

Notice that if $R$ is regular, or even a semiprimitve exchange ring, then, by the proof of Proposition 3.6, this usage coincides with the usage given in the introduction.

**Theorem 3.9** *Let $R$ be an exchange ring, and let $\mathcal{I}$ denote the set of those ideals $I$ of $R$ such that $R/I$ is indecomposable and $I \supseteq J(R)$.*

(1) *$R$ is separative if and only if $R/I$ is separative for all $I \in \mathcal{I}$.*

(2) *If $R$ satisfies generalized $s$-comparability, then $R$ is separative, so has stable rank* 1, 2 *or* $\infty$.

(3) *If $R$ satisfies generalized $s$-comparability, and $R/I$ is directly finite for all $I \in \mathcal{I}$, then $R$ is strongly separative.*

**Proof.** By [1, Theorem 2.3] and Proposition 3.7, the indecomposable factors of $V(R)$ are those obtained by factoring out the ideals $V(I,R)$, for $I \in \mathcal{I}$. (1) now follows from Corollary 3.3. (2) and (3) follow from Theorem 3.5 □

**Remarks 3.10** (1) Theorem 3.9(2) generalizes Theorem 2.2, and Theorem 3.9 (3) generalizes Theorem 2.3.

(2) The case $s = 1$ of Theorem 3.9(2) says that any exchange ring satisfying general comparability is separative, and hence any regular ring satisfying general comparability is separative. This was well known for directly finite regular rings (because they are unit-regular), but this generality is new. In particular, right (or left) self-injective regular rings, and right (or left) continuous regular

rings are separative, since both types of rings satisfy general comparability; see [8, Corollary 9.15], [8, Corollary 13.21].

(3) Menal and Moncasi [13, Corollary 7] showed that any regular ring $R$ satisfying general comparability, has stable range 1, 2 or $\infty$. Now Theorem 3.9(2) shows that "regular" can be weakened to "exchange", and "general comparability" can be weakened to "general $s$-comparability".


# Acknowledgments.

I thank Pere Ara, Warren Dicks, Ken Goodearl, Fred Wehrung, and the referee, for their many helpful comments about the preparation of this paper.


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