

FLOW INVARIANTS IN THE CLASSIFICATION OF LEAVITT PATH ALGEBRAS

GENE ABRAMS, ADEL LOULY, ENRIQUE PARDO, AND CHRISTOPHER SMITH

ABSTRACT. We analyze in the context of Leavitt path algebras some graph operations introduced in the context of symbolic dynamics by Williams, Parry and Sullivan, and Franks. We show that these operations induce Morita equivalence of the corresponding Leavitt path algebras. As a consequence we obtain our two main results: the first gives sufficient conditions for which the Leavitt path algebras in a certain class are Morita equivalent, while the second gives sufficient conditions which yield isomorphisms. We discuss a possible approach to establishing whether or not these conditions are also in fact necessary. In the final section we present many additional operations on graphs which preserve Morita equivalence (resp., isomorphism) of the corresponding Leavitt path algebras.

INTRODUCTION

Throughout this article E will denote a row-finite directed graph, and K will denote an arbitrary field. The *Leavitt path algebra of E with coefficients in K* , denoted $L_K(E)$, has received significant attention over the past few years, both from algebraists as well as from analysts working in operator theory. (The precise definition of $L_K(E)$ is given below.) When K is the field \mathbb{C} of complex numbers, the algebra $L_K(E)$ has exhibited surprising similarity to $C^*(E)$, the *graph C^* -algebra of E* . In this context, it is natural to ask whether an analog of the Kirchberg-Phillips Classification Theorem [16, 21] for C^* -algebras holds for various classes of Leavitt path algebras as well. Specifically, the following question was posed in [4]:

The Classification Question for purely infinite simple unital Leavitt path algebras: Let K be a field, and suppose E and F are graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple unital. If $K_0(L_K(E)) \cong K_0(L_K(F))$ via an isomorphism φ having $\varphi([1_{L_K(E)}]) = [1_{L_K(F)}]$, must $L_K(E)$ and $L_K(F)$ be isomorphic?

The Classification Question is answered in the affirmative in [4] for a few specific classes of graphs. We obtain in the current article an affirmative answer for a significantly wider class of graphs. Our approach is as follows. In Section 1 we consider Morita equivalence of Leavitt path algebras. By applying a deep theorem of Franks [14] from the field of symbolic dynamics, we obtain in Theorem 1.25 a sufficient set of conditions on E and F which ensure

2000 *Mathematics Subject Classification.* Primary 16D70, Secondary 46L05.

Key words and phrases. Leavitt path algebra, Morita equivalence, Flow equivalence, K-Theory.

The first author is partially supported by the U.S. National Security Agency under grant number H89230-09-1-0066. The second author was partially supported by a Post-doctoral fellow of the A.E.C.I. (Spain). The third author was partially supported by the DGI and European Regional Development Fund, jointly, through Projects MTM2007-60338 and MTM2008-06201-C02-02, by the Consolider Ingenio “Mathematica” project CSD2006-32 by the MEC, by PAI III grants FQM-298 and P06-FQM-1889 of the Junta de Andalucía, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

that $L_K(E)$ is Morita equivalent to $L_K(F)$. (Ideas from symbolic dynamics were employed in analyzing structures related to Leavitt path algebras in, for instance, [12]; we describe these more fully below.) In Section 2, we exploit these Morita equivalences to obtain sufficient conditions which ensure isomorphism (Theorem 2.7), thereby obtaining the aforementioned partial affirmative answer to the Classification Question.

We complete Section 2 by examining the remaining difficulty in obtaining an affirmative answer to the Classification Question for *all* germane graphs. In Section 3 we extend several results about Morita equivalence and isomorphism to certain classes of graphs E for which $L_K(E)$ is not necessarily purely infinite simple unital, thereby giving more general results than have been previously known about isomorphism and Morita equivalence of Leavitt path algebras.

We briefly recall some graph-theoretic definitions and properties; more complete explanations and descriptions can be found in [1]. A *graph* (synonymously, a *directed graph*) $E = (E^0, E^1, r_E, s_E)$ consists of two sets E^0, E^1 and maps $r_E, s_E : E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. We write s for s_E (resp., r for r_E) if the graph E is clear from context. We emphasize that loops and multiple / parallel edges are allowed. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. All graphs in this paper will be assumed to be row-finite. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*; a vertex w for which $r^{-1}(w)$ is empty is called a *source*.

A *path* μ in a graph E is either a vertex, or a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In the latter case, $s(\mu) := s(e_1)$ is the *source* of μ , $r(\mu) := r(e_n)$ is the *range* of μ , and n is the *length* of μ . If $\mu = v$ is a vertex, we define $s(v) = r(v) = v$, and define the length of v to be 0. An edge f is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(f) = s(e_i)$ and $f \neq e_i$. If μ is a path in E , and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at v* . If $\mu = e_1 \dots e_n$ is a closed path based at $v = s(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*.

The following notation is standard. Let A be a $p \times p$ matrix having non-negative integer entries (i.e., $A = (a_{ij}) \in M_p(\mathbb{Z}^+)$). The graph E_A is defined by setting $(E_A)^0 = \{v_1, v_2, \dots, v_p\}$, and defining $(E_A)^1$ by inserting exactly a_{ij} edges in E_A having source vertex v_i and range vertex v_j . Conversely, if E is a finite graph with vertices $\{v_1, v_2, \dots, v_p\}$, then we define the *incidence matrix* A_E of E by setting $(A_E)_{ij}$ as the number of edges in E having source vertex v_i and range vertex v_j .

Given a graph $E = (E^0, E^1, r, s)$, we define the *transpose graph* E^t to be the graph (E^0, E^1, s, r) with the same vertices as E , but with edges in the opposite direction. Notice that $A_{E^t} = (A_E)^t$, and $E_{A^t} = (E_A)^t$, as implied by the notation.

Our focus in this article is on $L_K(E)$, the Leavitt path algebra of E . We define $L_K(E)$ here, after which we review some important properties and examples.

Definition 0.1. Let E be any row-finite graph, and K any field. The *Leavitt path K -algebra* $L_K(E)$ of E with coefficients in K is the K -algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

- (1) $s(e)e = er(e) = e$ for all $e \in E^1$.
- (2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
- (3) (The ‘‘CK1 relations’’) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.

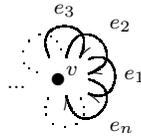
(4) (The “CK2 relations”) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for every vertex $v \in E^0$ for which $s^{-1}(v)$ is nonempty.

When the role of the coefficient field K is not central to the discussion, we will often denote $L_K(E)$ simply by $L(E)$. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

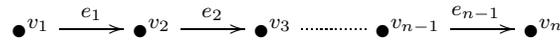
An alternate description of $L_K(E)$ is given in [1], where it is described in terms of a free associative algebra modulo the appropriate relations indicated in Definition 0.1 above. As a consequence, if A is any K -algebra which contains a set of elements satisfying these same relations (we call such a set an E -family), then there is a (unique) K -algebra homomorphism from $L_K(E)$ to A mapping the generators of $L_K(E)$ to their appropriate counterparts in A . We will refer to this conclusion as the *Universal Homomorphism Property* of $L_K(E)$.

If F is a subgraph of E , then F is called *complete* in case $s_F^{-1}(v) = s_E^{-1}(v)$ for every $v \in F^0$ having $s_F^{-1}(v) \neq \emptyset$. In particular, if F is a complete subgraph of E then the Universal Homomorphism Property of $L_K(F)$ yields that there is a K -algebra homomorphism $L_K(F) \rightarrow L_K(E)$ mapping vertices and edges in F with their counterparts in E . This homomorphism is in fact a K -algebra *monomorphism* by [4, Lemma 1.1].

Many well-known algebras arise as the Leavitt path algebra of a row-finite graph. For instance (see e.g. [1, Examples 1.4]), the classical Leavitt algebras L_n for $n \geq 2$ arise as the algebras $L(R_n)$, where R_n is the “rose with n petals” graph



The full $n \times n$ matrix algebra over K arises as the Leavitt path algebra of the oriented n -line graph



while the Laurent polynomial algebra $K[x, x^{-1}]$ arises as the Leavitt path algebra of the “one vertex, one loop” graph



Constructions such as direct sums and the formation of matrix rings produce additional examples of Leavitt path algebras.

We recall now some information and establish notation for unital rings which will be used throughout Sections 1 and 2. We write

$$R \sim_M S$$

to denote that R is Morita equivalent to S . For any ring R we let $\mathcal{V}(R)$ denote the monoid of isomorphism classes of finitely generated projective left R -modules, with operation \oplus . Since $\mathcal{V}(R)$ is conical (i.e., $[P] \oplus [Q] = [0]$ in $\mathcal{V}(R)$ if and only if $[P] = [Q] = [0]$), in fact

$$\mathcal{V}^*(R) = \mathcal{V}(R) \setminus \{[0]\}$$

is a semigroup as well. For any graph E , $\{[L(E)v] \mid v \in E^0\}$ is a set of generators for $\mathcal{V}^*(L(E))$. If $\Phi : R\text{-Mod} \rightarrow S\text{-Mod}$ is a Morita equivalence, then the restriction

$$\Phi_{\mathcal{V}} : \mathcal{V}^*(R) \rightarrow \mathcal{V}^*(S)$$

is an isomorphism of semigroups.

A nonzero idempotent e in a ring R is called *infinite* in case there exist nonzero idempotents f, g for which $e = f + g$, and $Re \cong Rf$ as left R -modules. (That is, e is infinite in case the left ideal Re contains a proper direct summand isomorphic to itself.) A simple unital ring R is called *purely infinite* in case every nonzero left ideal of R contains an infinite idempotent.

By [6, Propositions 2.1 and 2.2], if R is purely infinite simple, then $\mathcal{V}^*(R) = K_0(R)$ (the Grothendieck group of R). In particular, any two elements of $\mathcal{V}^*(R)$ which are equal in $K_0(R)$ are in fact isomorphic as left R -modules. Thus for R, S Morita equivalent purely infinite simple rings, a Morita equivalence $\Phi : R\text{-Mod} \rightarrow S\text{-Mod}$ in fact restricts to an isomorphism

$$\Phi_{\mathcal{V}} : K_0(R) \rightarrow K_0(S).$$

We note that, in general, such an induced isomorphism of K_0 groups need not take $[1_R]$ to $[1_S]$.

Although $L(E)$ can be constructed for any graph E , the Classification Question which is the main subject of this paper pertains to those choices of E for which $L(E)$ is purely infinite simple unital. It is easy to verify that $L(E)$ is unital if and only if E^0 is finite (in which case $\sum_{v \in E^0} v = 1_{L(E)}$), a fact that we will use throughout without explicit mention. Thus for much of the discussion we will assume that E^0 is finite; since for row-finite E the finiteness of E^0 implies the finiteness of E^1 , we simply write E is *finite* in this case. By [1, Theorem 3.11] (and by substituting an equivalent characterization from [9, Lemma 2.8] for one of the conditions therein), we get

Simplicity Theorem. For E finite, $L(E)$ is simple precisely when every cycle of E contains an exit, and there exists a path in E from any vertex to any cycle or sink.

Furthermore, it is shown in [2, Theorem 11] that

Purely Infinite Simplicity Theorem. $L(E)$ is purely infinite simple precisely when $L(E)$ is simple, and E contains a cycle.

Note that, as a consequence, whenever $L(E)$ is purely infinite simple, E does not contain sinks.

1. SUFFICIENT CONDITIONS FOR MORITA EQUIVALENCE BETWEEN PURELY INFINITE SIMPLE UNITAL LEAVITT PATH ALGEBRAS

In this section we establish sufficient conditions on two finite graphs E and F which guarantee that $L(E)$ is Morita equivalent to $L(F)$. In the first step of this process, we build a cache of operations on graphs that preserve Morita equivalence of the associated Leavitt path algebras. Once this arsenal is large enough, the sufficiency result will follow from a well-known theorem of Franks from symbolic dynamics, specifically, from the theory of subshifts of finite type. Our initial goal is to establish enough such Morita-equivalence-preserving operations to allow us to apply Franks' Theorem. With that in mind, we prove only very restrictive versions of the germane properties here, in order to significantly streamline the proofs and arrive at

our main results with maximum haste. (For instance, we present results here only for finite graphs, even though many of these results hold for all row-finite graphs.) For completeness, we provide much more general versions of these properties in Section 3.

Our goal in this section is to establish a Morita equivalence result, i.e., a result which establishes the existence a Morita equivalence between various Leavitt path algebras. However, a specific description of these equivalences, in particular a description of the restriction of these equivalences to the \mathcal{V}^* -semigroups, will be central to our discussion in the subsequent section; we therefore provide such additional information in Propositions 1.4, 1.8, 1.11, and 1.14.

The key lemma which will be used to establish Morita equivalences throughout this section is:

Lemma 1.1. *Suppose R and S are simple unital rings. Let $\pi : R \rightarrow S$ be a nonzero, not-necessarily-identity-preserving ring homomorphism, and let g denote the idempotent $\pi(1_R)$ of S . If $gSg = \pi(R)$, then there exists a Morita equivalence $\Phi : R\text{-Mod} \rightarrow S\text{-Mod}$.*

Moreover, Φ restricts to an isomorphism $\Phi_{\mathcal{V}} : \mathcal{V}^(R) \rightarrow \mathcal{V}^*(S)$ with the property that for any idempotent $e \in R$,*

$$\Phi_{\mathcal{V}}([Re]) = [S\pi(e)].$$

Proof. That π is nonzero, together with the simplicity of R , ensures an isomorphism $R \cong \pi(R) = gSg$ as rings. This gives a Morita equivalence

$$\Pi : R\text{-Mod} \rightarrow gSg\text{-Mod},$$

given on objects by defining, for each left R -module M , $\Pi(M) = M^g$, where $M^g = M$ has gSg -action given by $gsg * m = \pi^{-1}(gsg)m$.

On the other hand, since $g = \pi(1_R) \neq 0$, the simplicity of S ensures that $SgS = S$, from which we conclude that the finitely generated projective left S -module ${}_S Sg$ is a generator of the category of left S -modules. Thus by the well-known result of Morita we get a Morita equivalence

$$\Psi : gSg\text{-Mod} \rightarrow S\text{-Mod}$$

given by defining, for any gSg -module N , $\Psi(N) = Sg \otimes_{gSg} N$.

The composition of these two Morita equivalences gives a Morita equivalence

$$\Phi : R\text{-Mod} \rightarrow S\text{-Mod}.$$

Specifically, for each left R -module M , $\Phi(M) = Sg \otimes_{gSg} M^g$. In particular, Φ restricts to an isomorphism

$$\Phi_{\mathcal{V}} : \mathcal{V}^*(R) \rightarrow \mathcal{V}^*(S).$$

It is tedious and straightforward to show, for each $e = e^2 \in R$, that $Sg \otimes_{gSg} (Re)^g \cong S\pi(e)$ as left S -modules, so the second statement follows as well. \square

Suppose E is a finite graph, let X be any set of distinct vertices of E , and let $e = \sum_{v \in X} v \in L(E)$. It is immediate that every $y \in eL(E)e$ can be written as a K -linear combination of monomials of the form $\mu\nu^*$ for which $s(\mu), s(\nu) \in X$. This observation will be used in the proofs of various results throughout the section without explicit mention.

We now establish the first of the four Morita equivalence results required to achieve Theorem 1.25.

Definition 1.2. Let $E = (E^0, E^1, r, s)$ be a directed graph with at least two vertices, and let $v \in E^0$ be a source. We form the *source elimination graph* $E_{\setminus v}$ of E as follows:

$$\begin{aligned} E_{\setminus v}^0 &= E^0 \setminus \{v\} \\ E_{\setminus v}^1 &= E^1 \setminus s^{-1}(v) \\ s_{E_{\setminus v}} &= s|_{E_{\setminus v}^1} \\ r_{E_{\setminus v}} &= r|_{E_{\setminus v}^1} \end{aligned}$$

Example 1.3. Let E be the graph:



Then the source elimination graph $E_{\setminus v}$ is



It is easy to see that as long as the graph E contains a cycle, repeated source elimination can be used to convert E into a graph with no sources.

Proposition 1.4. *Let E be a finite graph containing at least two vertices such that $L(E)$ is simple, and let $v \in E^0$ be a source. Then $L(E_{\setminus v})$ is Morita equivalent to $L(E)$, via a Morita equivalence*

$$\Phi^{\text{elim}} : L(E_{\setminus v})\text{-Mod} \rightarrow L(E)\text{-Mod}$$

for which $\Phi_{\mathcal{V}}^{\text{elim}}([L(E_{\setminus v})w]) = [L(E)w]$ for all vertices w of $E_{\setminus v}$.

Proof. We begin by noting that, as an easy application of the Simplicity Theorem, $L(E)$ is simple and unital if and only if $L(E_{\setminus v})$ is simple and unital. (The hypothesis that E contains at least two vertices ensures that we are not creating an empty graph by eliminating a single vertex.)

From the definition of $E_{\setminus v}$, it is clear that $E_{\setminus v}$ is a complete subgraph of E . Thus, the K -algebra map defined by the rule

$$\begin{aligned} \pi : L(E_{\setminus v}) &\rightarrow L(E) \\ w &\mapsto w \\ e &\mapsto e \\ e^* &\mapsto e^* \end{aligned}$$

for every $w \in E_{\setminus v}^0$ and every $e \in E_{\setminus v}^1$, is a nonzero ring homomorphism.

We claim that $\pi(L(E_{\setminus v})) = \pi(1_{L(E_{\setminus v})}) L(E) \pi(1_{L(E_{\setminus v})})$. Note that by definition we have $\pi(1_{L(E_{\setminus v})}) = \sum_{w \in E^0, w \neq v} w$. The inclusion $\pi(L(E_{\setminus v})) \subseteq \pi(1_{L(E_{\setminus v})}) L(E) \pi(1_{L(E_{\setminus v})})$ is immediate. For the other direction, it suffices to consider an arbitrary $\mu_1 \mu_2^* \in \pi(1_{L(E_{\setminus v})}) L(E) \pi(1_{L(E_{\setminus v})})$. Then μ_1 and μ_2 are paths in E such that neither has v for its source, and their ranges are equal. But if neither has v for a source, then since v is a source itself, neither path can pass through v at all. Therefore, μ_1 and μ_2 are also paths in $E_{\setminus v}$, such that $\pi(\mu_1 \mu_2^*) = \mu_1 \mu_2^*$. This completes the argument that $\pi(L(E_{\setminus v})) = \pi(1_{L(E_{\setminus v})}) L(E) \pi(1_{L(E_{\setminus v})})$.

Applying Lemma 1.1, we conclude that $L(E_{\setminus v})$ is Morita equivalent to $L(E)$, and that the Morita equivalence restricts to an isomorphism between $\mathcal{V}^*(L(E_{\setminus v}))$ and $\mathcal{V}^*(L(E))$ that maps $[L(E_{\setminus v})w]$ to $[L(E)w]$ for each vertex w of $E_{\setminus v}$. \square

Corollary 1.5. *Let E be a finite graph for which $L(E)$ is purely infinite simple. Then there exists a graph E' which contains no sources, with the property that $L(E)$ is Morita equivalent to $L(E')$ via a Morita equivalence*

$$\Phi^{\text{ELIM}} : L(E')\text{-Mod} \rightarrow L(E)\text{-Mod}$$

for which $\Phi_{\mathcal{V}}^{\text{ELIM}}([L(E')w]) = [L(E)w]$ for all vertices w of E' .

Proof. By continually applying the source elimination procedure described in Definition 1.2, we produce from the finite graph E a new graph E' having no sources. We must show that E' is not the empty graph; that is, we must show that if the source elimination process eventually leads to a graph F with one vertex, then that vertex is not a source. By Proposition 1.4, at each stage of the source elimination process the graph produced in the new stage has Leavitt path algebra Morita equivalent to the Leavitt path algebra of the graph in the previous stage. By [2, Proposition 10], purely infinite simplicity is a Morita invariant. Thus $L(F)$ is purely infinite simple. But a graph F with one vertex for which $L(F)$ is purely infinite simple must contain at least one loop at that vertex (e.g., by the Purely Infinite Simplicity Theorem), so that the vertex is not a source, thus completing the proof. \square

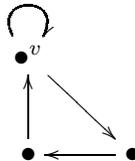
We now build the second of the four indicated Morita equivalence results.

Definitions 1.6. Let $E = (E^0, E^1, r, s)$ be a directed graph, and let $v \in E^0$. Let v^* and f be symbols not in $E^0 \cup E^1$. We form the *expansion graph* E_v from E at v as follows:

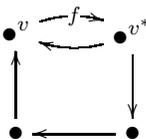
$$\begin{aligned} E_v^0 &= E^0 \cup \{v^*\} \\ E_v^1 &= E^1 \cup \{f\} \\ s_{E_v}(e) &= \begin{cases} v & \text{if } e = f \\ v^* & \text{if } s_E(e) = v \\ s_E(e) & \text{otherwise} \end{cases} \\ r_{E_v}(e) &= \begin{cases} v^* & \text{if } e = f \\ r_E(e) & \text{otherwise} \end{cases} \end{aligned}$$

Conversely, if E and G are graphs, and there exists a vertex v of E for which $E_v = G$, then E is called a *contraction* of G .

Example 1.7. Let E be the graph:



Then the expansion graph E_v is



Proposition 1.8. *Let E be a finite graph such that $L(E)$ is simple, and let $v \in E^0$. Then $L(E)$ is Morita equivalent to $L(E_v)$, via a Morita equivalence*

$$\Phi^{\text{exp}} : L(E)\text{-Mod} \rightarrow L(E_v)\text{-Mod}$$

for which $\Phi_Y^{\text{exp}}([L(E)w]) = [L(E_v)w]$ for all vertices w of E .

Proof. We begin by noting that, as an easy application of the Simplicity Theorem, $L(E)$ is simple and unital if and only if $L(E_v)$ is simple and unital.

For each $w \in E^0$, define $Q_w = w$. For each $e \in s^{-1}(v)$, define $T_e = fe$ and $T_e^* = e^*f^*$. For $e \in E^1$ otherwise, define $T_e = e$ and $T_e^* = e^*$. We claim that $\{Q_w, T_e, T_e^* \mid w \in E^0, e \in E^1\}$ is an E -family in $L(E_v)$. The Q_w 's are mutually orthogonal idempotents because the w 's are. The elements T_e for $e \in E^1$ clearly satisfy $T_e^*T_f = 0$ whenever $e \neq f$. For $e \in E^1$, it is easy to check that $T_e^*T_e = Q_{r(e)}$. Note that $\sum_{e \in s^{-1}(v)} T_e T_e^* = f \left(\sum_{e \in s^{-1}(v^*)} e e^* \right) f^* = f f^* = v = Q_v$. The same property holds immediately for all $w \in E^0$ having $w \neq v$, thereby establishing the claim.

Therefore, by the Universal Homomorphism Property of $L(E)$, there is a K -algebra homomorphism $\pi : L(E) \rightarrow L(E_v)$ that maps $w \mapsto Q_w$, $e \mapsto T_e$, and $e^* \mapsto T_e^*$. Note that π maps w to $Q_w \neq 0$, so π is nonzero. We now claim that $\pi(L(E)) = \pi(1_{L(E)}) L(E_v) \pi(1_{L(E)})$, where $\pi(1_{L(E)}) = \sum_{w \in E^0} w$, viewed as an element of $L(E_v)$. The inclusion $\pi(L(E)) \subseteq \pi(1_{L(E)}) L(E_v) \pi(1_{L(E)})$ is immediate. For the other direction, it suffices to consider arbitrary nonzero terms in $\pi(1_{L(E)}) L(E_v) \pi(1_{L(E)})$ of the form $\mu_1 \mu_2^*$, where μ_1 and μ_2 are paths in E_v , $s(\mu_1), s(\mu_2) \neq v^*$, and $r(\mu_1) = r(\mu_2)$.

Let α be the path in E obtained by removing the edge f from μ_1 any place that it occurs, and similarly let β be the path obtained by removing f from μ_2 . We claim that $\pi(\alpha\beta^*) = \mu_1\mu_2^*$. There are two cases. If $r(\mu_1) \neq v^* \neq r(\mu_2)$, then $\mu_1 = \pi(\alpha)$ and $\mu_2 = \pi(\beta)$, and the result follows. Otherwise, $r(\mu_1) = v^* = r(\mu_2)$. But because μ_1 and μ_2 both begin at a vertex other than v^* , and the only edge entering v^* is f , we must have $\mu_1 = \nu_1 f$ and $\mu_2 = \nu_2 f$, for paths ν_1, ν_2 in E_v , where $r(\nu_1) = v = r(\nu_2)$. But then $\mu_1 \mu_2^* = \nu_1 f f^* \nu_2^* = \nu_1 \nu_2^*$ by the CK2 relation at v , and we are back in the first case again, so $\pi(\alpha\beta^*) = \mu_1 \mu_2^*$, completing the argument.

Applying Lemma 1.1, we conclude that $L(E)$ is Morita equivalent to $L(E_v)$, and that the Morita equivalence restricts to the map given above. \square

If F is a contraction of E (i.e., if there exists a vertex v of F for which $E = F_v$), then we denote by

$$\Phi^{\text{cont}} = (\Phi^{\text{exp}})^{-1}$$

the Morita equivalence from $L(F)\text{-Mod} \rightarrow L(E)\text{-Mod}$. We note that while the Morita equivalence $\Phi^{\text{exp}} : L(E)\text{-Mod} \rightarrow L(E_v)\text{-Mod}$ arises from a ring homomorphism $\pi : L(E) \rightarrow L(E_v)$ as described in Lemma 1.1, the inverse equivalence Φ^{cont} need not in general arise in this way. For instance, if $E = \bullet^v$ is a graph with a single vertex v and no edges, then $E_v = \bullet^v \rightarrow \bullet^{v^*}$, $L(E) \cong K$, $L(E_v) \cong M_2(K)$, and $\pi : L(E) \rightarrow L(E_v)$ is the inclusion map to the upper left corner. But there is no nonzero homomorphism from $L(E_v)$ to $L(E)$.

Our third and fourth Morita equivalence properties require somewhat more cumbersome machinery to build than did the first two. The following definition is borrowed from [10, Section 5].

Definitions 1.9. Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ with $r^{-1}(v) \neq \emptyset$, partition the set $r^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_1^v, \dots, \mathcal{E}_{m(v)}^v$ where $m(v) \geq 1$. (If v is a source then we put $m(v) = 0$.) Let \mathcal{P} denote the resulting partition of E^1 . We form the *in-split graph* $E_r(\mathcal{P})$ from E using the partition \mathcal{P} as follows:

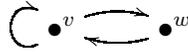
$$\begin{aligned} E_r(\mathcal{P})^0 &= \{v_i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid m(v) = 0\}, \\ E_r(\mathcal{P})^1 &= \{e_j \mid e \in E^1, 1 \leq j \leq m(s(e))\} \cup \{e \mid m(s(e)) = 0\}, \end{aligned}$$

and define $r_{E_r(\mathcal{P})}, s_{E_r(\mathcal{P})} : E_r(\mathcal{P})^1 \rightarrow E_r(\mathcal{P})^0$ by

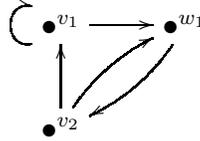
$$\begin{aligned} s_{E_r(\mathcal{P})}(e_j) &= s(e)_j \text{ and } s_{E_r(\mathcal{P})}(e) = s(e) \\ r_{E_r(\mathcal{P})}(e_j) &= r(e)_i \text{ and } r_{E_r(\mathcal{P})}(e) = r(e)_i \text{ where } e \in \mathcal{E}_i^{r(e)}. \end{aligned}$$

Conversely, if E and G are graphs, and there exists a partition \mathcal{P} of E^1 for which $E_r(\mathcal{P}) = G$, then E is called an *in-amalgamation* of G .

Example 1.10. Let E be the graph:



Denote by \mathcal{P} the partition of E^1 that places each edge in its own singleton partition class. Then $E_r(\mathcal{P})$ is:



Proposition 1.11. *Let E be a finite graph with no sources or sinks, such that $L(E)$ is simple. Let \mathcal{P} be a partition of E^1 as in Definitions 1.9, and $E_r(\mathcal{P})$ the in-split graph from E using \mathcal{P} . Then $L(E)$ is Morita equivalent to $L(E_r(\mathcal{P}))$, via a Morita equivalence*

$$\Phi^{\text{ins}} : L(E)\text{-Mod} \rightarrow L(E_r(\mathcal{P}))\text{-Mod}$$

for which $\Phi_Y^{\text{ins}}([L(E)v]) = [L(E_r(\mathcal{P}))v_1]$ for all vertices v of E .

Proof. We begin by noting that, as an easy application of the Simplicity Theorem and a somewhat tedious check, $L(E)$ is simple and unital if and only if $L(E_r(\mathcal{P}))$ is simple and unital. Moreover, E has no sources if and only if $E_r(\mathcal{P})$ has no sources.

For each $v \in E^0$, define $Q_v = v_1$, which exists by the assumption that E contains no sources. For $e \in \mathcal{E}_i^v$, define $T_e = \sum_{f \in s^{-1}(v)} e_1 f_i f_1^*$ and $T_e^* = \sum_{f \in s^{-1}(v)} f_1 f_i^* e_1^*$. The claim is that $\{Q_v, T_e, T_e^* \mid v \in E^0, e \in E^1\}$ is an E -family inside $L(E_r(\mathcal{P}))$. The Q_v 's are mutually orthogonal idempotents because the v_1 's are. It is immediate from the definition above that whenever $v = s(e)$ in E , then $Q_v T_e = T_e$ and $T_e^* Q_v = T_e^*$ in $L(E_r(\mathcal{P}))$, and that whenever $w = r(e)$ in E , $T_e Q_w = T_e$ and $Q_w T_e^* = T_e^*$ in $L(E_r(\mathcal{P}))$. If $e \neq f$, then note that $T_e^* T_f = x e_1^* f_1 y$ for some $x, y \in L(E_r(\mathcal{P}))$, but since $e_1 \neq f_1$, this is zero. Because E and $E_r(\mathcal{P})$ contain no sinks, there is a CK2 relation at every vertex of both graphs. It is now a straightforward matter of computation to check, by applying the CK1 and CK2 relations, that $T_e^* T_e = Q_{r(e)}$, and that $\sum_{e \in s^{-1}(v)} T_e T_e^* = Q_v$.

By the Universal Homomorphism Property, then, there exists a K -algebra homomorphism $\pi : L(E) \rightarrow L(E_r(\mathcal{P}))$ which maps $v \mapsto Q_v$, $e \mapsto T_e$, and $e^* \mapsto T_e^*$. It is easy to verify

that $\pi(v)$ is nonzero for any $v \in E^0$, so π is a nonzero homomorphism. We now claim that $\pi(L(E)) = \pi(1_{L(E)}) L(E_r(\mathcal{P})) \pi(1_{L(E)})$, where $\pi(1_{L(E)}) = \sum_{v \in E^0} v_1$.

The inclusion $\pi(L(E)) \subseteq \pi(1_{L(E)}) L(E_r(\mathcal{P})) \pi(1_{L(E)})$ is immediate. For the opposite inclusion, it suffices to consider arbitrary nonzero terms in $\pi(1_{L(E)}) L(E_r(\mathcal{P})) \pi(1_{L(E)})$ of the form $\mu_1 \mu_2^*$, where μ_1 and μ_2 are finite length paths in $E_r(\mathcal{P})$, and $s(\mu_1) = v_1$ and $s(\mu_2) = w_1$ for some $v, w \in E^0$, and where $r(\mu_1) = r(\mu_2)$.

Let μ be any path in $E_r(\mathcal{P})$ such that $s(\mu) = v_1$ for some $v \in E^0$. Define $r(\mu) = w_k$, where $w \in E^0$ and $1 \leq k \leq m(w)$. We now build a path ν in E , by replacing each v_i in μ with v in ν , and each e_i in μ with e in ν , so that ν is essentially the result of removing subscripts from the edges and vertices of μ . An induction on the length of μ will show that

$$\pi(\nu) = \mu \left(\sum_{f \in s^{-1}(w)} f_k f_1^* \right).$$

If the length of μ is zero, then $\mu = v_1 = w_k$. Applying the CK2 relation at v_1 , we get

$$\pi(v) = v_1 \left(\sum_{f \in s^{-1}(v)} f_1 f_1^* \right).$$

Since $w = v$ and $k = 1$ in this case, this is the result we need. If the length of μ is greater than zero, then $\mu = \mu' e_j$, where $r(\mu') = u_j$, $e \in E^1$, $u \in E^0$, and $1 \leq j \leq m(u)$. We define ν' in the same manner as above, so that from the inductive hypothesis,

$$\pi(\nu) = \pi(\nu') T_e = \mu' \left(\sum_{f \in s^{-1}(u), g \in s^{-1}(w)} f_j f_1^* e_1 g_k g_1^* \right).$$

When $f \neq e$, we have $f_1^* e_1 = 0$ by the CK1 relation, whereas when $f = e$, $f_1^* e_1 = r(e_1)$, which collapses into the adjacent terms. This expression therefore reduces to

$$\pi(\nu) = \mu' e_j \left(\sum_{g \in s^{-1}(w)} g_k g_1^* \right) = \mu \left(\sum_{g \in s^{-1}(w)} g_k g_1^* \right)$$

as desired.

Now, given $\mu_1 \mu_2^* \in \pi(1_{L(E)}) L(E_r(\mathcal{P})) \pi(1_{L(E)})$, we define ν_1 and ν_2 in the manner given above. By a direct computation, it can be verified that $\pi(\nu_1 \nu_2^*) = \mu_1 \mu_2^*$, completing the argument that $\pi(L(E)) = \pi(1_{L(E)}) L(E_r(\mathcal{P})) \pi(1_{L(E)})$.

Applying yet again Lemma 1.1, we conclude that $L(E)$ is Morita equivalent to $L(E_r(\mathcal{P}))$, and that the Morita equivalence restricts to the map above. \square

If F is an in-amalgamation of E (i.e., if there exists a vertex partition \mathcal{P} of F for which $E = F_r(\mathcal{P})$), then we denote by

$$\Phi^{\text{inam}} = (\Phi^{\text{ins}})^{-1}$$

the Morita equivalence from $L(F)\text{-Mod} \rightarrow L(E)\text{-Mod}$.

As a brief remark, we remind the reader that the result established here is not as general as possible. In particular, the hypothesis that E contains no sources or sinks will be weakened in Corollary 3.9. (The difficulties that are avoided by this hypothesis are more notational

than substantial. Nevertheless, the result as stated here is strong enough to serve us for our present goal.)

We now establish the fourth and final tool in our cache. The following definition is borrowed from [10, Section 3].

Definitions 1.12. Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ with $s^{-1}(v) \neq \emptyset$, partition the set $s^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_v^1, \dots, \mathcal{E}_v^{m(v)}$ where $m(v) \geq 1$. (If v is a sink, then we put $m(v) = 0$.) Let \mathcal{P} denote the resulting partition of E^1 . We form the *out-split graph* $E_s(\mathcal{P})$ from E using the partition \mathcal{P} as follows:

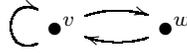
$$\begin{aligned} E_s(\mathcal{P})^0 &= \{v^i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid m(v) = 0\}, \\ E_s(\mathcal{P})^1 &= \{e^j \mid e \in E^1, 1 \leq j \leq m(r(e))\} \cup \{e \mid m(r(e)) = 0\}, \end{aligned}$$

and define $r_{E_s(\mathcal{P})}, s_{E_s(\mathcal{P})} : E_s(\mathcal{P})^1 \rightarrow E_s(\mathcal{P})^0$ for each $e \in \mathcal{E}_{s(e)}^i$ by

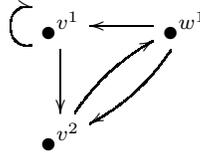
$$\begin{aligned} s_{E_s(\mathcal{P})}(e^j) &= s(e)^i \text{ and } s_{E_s(\mathcal{P})}(e) = s(e)^i \\ r_{E_s(\mathcal{P})}(e^j) &= r(e)^j \text{ and } r_{E_s(\mathcal{P})}(e) = r(e). \end{aligned}$$

Conversely, if E and G are graphs, and there exists a partition \mathcal{P} of E^1 for which $E_s(\mathcal{P}) = G$, then E is called an *out-amalgamation* of G .

Example 1.13. Let E be the graph:



Denote by \mathcal{P} the partition of E^1 that places each edge in its own singleton partition class. Then $E_s(\mathcal{P})$ is:



Our fourth tool follows as a specific case of a result previously established in [4].

Proposition 1.14. *Let E be a finite graph, \mathcal{P} a partition of E^1 as in Definitions 1.12, and $E_s(\mathcal{P})$ the out-split graph from E using \mathcal{P} . Then $L(E)$ is isomorphic to $L(E_s(\mathcal{P}))$. This isomorphism yields a Morita equivalence*

$$\Phi^{\text{outs}} : L(E)\text{-Mod} \rightarrow L(E_s(\mathcal{P}))\text{-Mod}$$

for which $\Phi_{\mathcal{V}}^{\text{outs}}([L(E)v]) = [L(E_s(\mathcal{P})) \sum_{i=1}^{m(v)} v^i]$ for every vertex v of E .

Proof. The indicated isomorphism between $L(E)$ and $L(E_s(\mathcal{P}))$ is established in [4, Theorem 2.8]. Furthermore, the isomorphism given there maps v to $\sum_{i=1}^{m(v)} v^i$, so that the associated Morita equivalence restricts to the desired map. \square

If F is an out-amalgamation of E (i.e., if there exists a vertex partition \mathcal{P} of F for which $E = F_s(\mathcal{P})$), then we denote by

$$\Phi^{\text{outam}} = (\Phi^{\text{outs}})^{-1}$$

the Morita equivalence from $L(F)\text{-Mod} \rightarrow L(E)\text{-Mod}$.

Having built a sufficient arsenal of graph operations, we now proceed toward the first main result of this article. Considerable work has been done in the flow dynamics community regarding the theory of subshifts of finite type; specifically, an explicit description of the flow equivalence relation has been achieved for a large class of such shifts. We refer the interested reader to [19] for a clear, careful introduction to the theory, including the definition of flow equivalence. For our purposes, the following definitions and results will provide all of the connecting information we need.

Definitions 1.15. Let E be a finite (directed) graph. Then E is:

- (1) *irreducible* if given any two vertices v and w in E , there is a path from v to w ([19, Definition 2.2.13]),
- (2) *essential* if there are neither sources nor sinks in E ([19, Definition 2.2.13]), and
- (3) *trivial* if E consists of a single cycle with no other vertices or edges ([14]).

Here is an easily verified observation which will be useful later.

Lemma 1.16. *Let E be a finite graph, let $v \in E^0$, and let \mathcal{P} be a partition of the vertices of E . Then E is essential (resp. nontrivial, resp. irreducible) if and only if $E_s(\mathcal{P})$, $E_r(\mathcal{P})$, and E_v are each essential (resp. nontrivial, resp. irreducible).*

A set of graphs of great interest in the theory of subshifts of finite type are those that are simultaneously irreducible, essential, and nontrivial. The following connecting result is pivotal here.

Lemma 1.17. *Let E be a finite graph. The following are equivalent:*

- (1) E is irreducible, nontrivial, and essential.
- (2) E contains no sources, and $L(E)$ is purely infinite simple.

Proof. Suppose first that E is irreducible, essential, and nontrivial. That E contains no sources is immediate. E also contains no sinks and is finite, so it must contain a cycle. Since E is nontrivial, there must exist some edge or vertex not in any cycle, and either that edge or the path from the cycle to that vertex is an exit to the cycle. Finally, since E is irreducible, there is a path between any two vertices, so there must be a path from any vertex to any cycle.

Conversely, suppose E contains no sources, and that $L(E)$ is purely infinite simple. From the Simplicity Theorem [1, Theorem 3.11], every cycle has an exit, so E is nontrivial. From [9, Lemma 2.8], there is a path from any vertex to any cycle. Since by the Purely Infinite Simplicity Theorem [2, Theorem 11] there is at least one cycle in the graph, there are no sinks. Then E is essential. However, with no sources or sinks in a finite graph, every vertex must belong to a cycle, so there is a path between any two vertices, and E is irreducible. \square

Much of the heavy lifting required to achieve our first goal is provided by deep, fundamental work in flow dynamics. We collect up all the relevant facts in the following few results, then state as Corollary 1.22 the conclusion we need to achieve our goal. (Following Franks, we state some results in the language of matrices. Statements about non-negative integer matrices may be interchanged with statements about directed graphs by exchanging E for its incidence matrix A_E as described in the Introduction.)

Definitions 1.18. We call a graph transformation *standard* if it is one of these six types: in-splitting, in-amalgamation, out-splitting, out-amalgamation, expansion, and contraction.

Analogously, we call a function which transforms a non-negative integer matrix A to a non-negative integer matrix B *standard* if the corresponding graph operation from E_A to E_B is standard.

Definitions 1.19. If E and F are graphs, a *flow equivalence from E to F* is a sequence $E = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n = F$ of graphs and standard graph transformations which starts at E and ends at F . We say that E and F are *flow equivalent* in case there is a flow equivalence from E to F . Analogously, a *flow equivalence between matrices A and B* is defined to be a flow equivalence between the graphs E_A and E_B .

We note that the notion of flow equivalence can be described in topological terms (see e.g. [19]). The definition given here is optimal for our purposes. Specifically, it agrees with the topologically-based definition for essential graphs by an application of [20, Theorem], [25, Corollary 4.4.1], and [19, Corollary 7.15]. Since all graphs under consideration in our main results are essential, this particular definition of flow equivalence will serve us most efficiently.

Theorem 1.20. [14, Theorem] (“*Franks’ Theorem*”) *Suppose that A and B are non-negative irreducible square integer matrices neither of which is in the trivial flow equivalence class. Then the matrices A and B are flow equivalent if and only if:*

$$\det(I_n - A) = \det(I_m - B) \quad \text{and} \quad \mathbb{Z}^n / (I_n - A)\mathbb{Z}^n \cong \mathbb{Z}^m / (I_m - B)\mathbb{Z}^m,$$

where $n \times n$ and $m \times m$ are the sizes of A and B respectively, I_n and I_m are identity matrices, and $\mathbb{Z}^n / (I_n - A)\mathbb{Z}^n$ (resp. $\mathbb{Z}^m / (I_m - B)\mathbb{Z}^m$) denotes the image in \mathbb{Z}^n (resp. \mathbb{Z}^m) of the linear transformation $I_n - A : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ (resp. $I_m - B : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$) induced by matrix multiplication.

Corollary 1.21. *Suppose A and B are irreducible, nontrivial, essential square non-negative integer matrices for which*

$$\det(I_n - A) = \det(I_m - B) \quad \text{and} \quad \mathbb{Z}^n / (I_n - A)\mathbb{Z}^n \cong \mathbb{Z}^m / (I_m - B)\mathbb{Z}^m.$$

Then there exists a sequence of standard transformations which starts with A and ends with B .

As is usual, we denote $\mathbb{Z}^n / (I_n - A)\mathbb{Z}^n$ simply by $\text{coker}(I_n - A)$. By examining the Smith normal form of each matrix, it is easy to show that $\text{coker}(I_n - A) \cong \text{coker}(I_n - A^t)$ for any square matrix A . Furthermore, by a cofactor expansion, it is clear that $\det(I_n - A) = \det(I_n - A^t) = \det(I_n - A)^t$.

If E is a graph for which $L(E)$ is purely infinite simple unital, then by [4, Section 3] there is an isomorphism

$$\text{coker}(I - A_E^t) \rightarrow K_0(L(E)),$$

for which $\overline{x_i} \mapsto [L(E)v_i]$ for each standard basis vector x_i of \mathbb{Z}^n and each vertex v_i of E , $1 \leq i \leq n$. (We note for future use that since $1_{L(E)} = \sum_{v \in E^0} v$ in $L(E)$, this isomorphism takes the element $\sum_{i=1}^n \overline{x_i}$ of $\text{coker}(I - A_E^t)$ to $[L(E)] \in K_0(L(E))$.)

Thus, using Lemma 1.17, we may restate Corollary 1.21 as follows:

Corollary 1.22. *Suppose G and H are finite graphs without sources, for which $L(G)$ and $L(H)$ are purely infinite simple. Suppose*

$$\det(I_n - A_G^t) = \det(I_m - A_H^t) \quad \text{and} \quad K_0(L(G)) \cong K_0(L(H)),$$

where $n = |G^0|$ and $m = |H^0|$. *Then there exists a sequence of standard graph transformations which starts with G and ends with H .*

Our interest here will be in graphs E and F for which $\det(I_n - A_E^t) = \det(I_m - A_F^t)$ and $K_0(L(E)) \cong K_0(L(F))$. The following notation will prove convenient.

Definition 1.23. Let E be a finite graph. The *determinant Franks pair* is the ordered pair

$$\mathcal{F}_{det}(E) = (K_0(L(E)) , \det(I_n - A_E^t))$$

consisting of the abelian group $K_0(L(E))$ and the integer $\det(I_n - A_E^t)$. For finite graphs E, G we write

$$\mathcal{F}_{det}(E) \equiv \mathcal{F}_{det}(G)$$

in case there exists an abelian group isomorphism $K_0(L(E)) \cong K_0(L(G))$, and $\det(I_n - A_E^t) = \det(I_m - A_G^t)$. Clearly \equiv yields an equivalence relation on the set of finite graphs.

We now show that the source elimination process preserves equivalence of the determinant Franks pair.

Lemma 1.24. *Let E be a finite graph for which $L(E)$ is purely infinite simple, and let v be a source in E . Then*

$$\mathcal{F}_{det}(E) \equiv \mathcal{F}_{det}(E_{\setminus v}).$$

Proof. Let $n = |E^0|$. Since v is a source, A_E contains a column of zeros. Then a straightforward determinant computation by cofactors along this column gives $\det(I_n - A_E^t) = \det(I_{n-1} - A_{E_{\setminus v}}^t)$.

Since E satisfies the conditions of the Purely Infinite Simplicity Theorem it is clear by the construction that $E_{\setminus v}$ must as well. But $L(E)$ and $L(E_{\setminus v})$ are Morita equivalent by Proposition 1.4, so that their K_0 groups are necessarily isomorphic. \square

Now we are ready to prove the first of our two main results.

Theorem 1.25. *Let E and F be finite graphs such that $L(E)$ and $L(F)$ are purely infinite simple. Suppose that*

$$\mathcal{F}_{det}(E) \equiv \mathcal{F}_{det}(F);$$

that is, suppose

$$\det(I_n - A_E^t) = \det(I_m - A_F^t) \quad \text{and} \quad K_0(L(E)) \cong K_0(L(F)),$$

where n and m are the number of vertices in E and F , respectively. Then $L(E)$ is Morita equivalent to $L(F)$.

Proof. By Corollary 1.5 there exist graphs E' and F' such that E' and F' contain no sources, and for which $L(E) \sim_M L(E')$ and $L(F) \sim_M L(F')$. By hypothesis, and by applying Lemma 1.24 at each stage of the source elimination process, we have that

$$\det(I - A_{E'}^t) = \det(I - A_E^t) = \det(I - A_F^t) = \det(I - A_{F'}^t),$$

and that

$$K_0(L(E')) \cong K_0(L(E)) \cong K_0(L(F)) \cong K_0(L(F')).$$

Furthermore, $L(E')$ and $L(F')$ are each purely infinite simple unital (either use the Purely Infinite Simplicity Theorem, or apply the fact that purely infinite simplicity is a Morita invariant). So Corollary 1.22 applies, and we conclude that there exists a finite sequence of elementary graph transformations, which starts at E' and ends at F' . By Lemmas 1.16 and 1.17, since E' is purely infinite simple unital with no sources, each time such an operation

is applied the resulting graph is again purely infinite simple unital with no sources. Thus, at each step of the sequence, we may apply the appropriate tool from the cache consisting of Propositions 1.8, 1.11, and 1.14, from which we conclude that each step in the sequence preserves Morita equivalence of the corresponding Leavitt path algebras. Combining these Morita equivalences at each step then yields $L(E') \sim_M L(F')$.

As a result, we have

$$L(E) \sim_M L(E') \sim_M L(F') \sim_M L(F),$$

and the theorem follows. \square

An analysis of objects related to Leavitt path algebras, carried out along somewhat similar lines, is presented in [12]. To wit, Cuntz and Krieger analyze the C^* -algebras \mathcal{O}_A (the now-so-called *Cuntz-Krieger C^* -algebras*); these are C^* -algebras generated by partial isometries which satisfy relations analogous to those in Definition 0.1. The arguments utilized in the C^* -algebra context are based on topological and analytical properties of Markov chains and certain graph operations (e.g., those of [20]). The completely algebraic point of view used in this section, together with the specific construction presented in [14], infuses our approach with a more germane algebraic flavor.

2. SUFFICIENT CONDITIONS FOR ISOMORPHISMS BETWEEN PURELY INFINITE SIMPLE UNITAL LEAVITT PATH ALGEBRAS.

In this section we will use the techniques and results of the previous section to investigate the problem of classifying purely infinite simple unital Leavitt path algebras up to isomorphism. Specifically, in Corollary 2.10 we provide an affirmative answer to the Classification Question for a wide class of graphs. To help establish such a connection we introduce some notation.

Definition 2.1. Let E be a finite graph. The *unitary Franks pair* is the ordered pair

$$\mathcal{F}_{[1]}(E) = (K_0(L(E)) , [1_{L(E)}])$$

consisting of the abelian group $K_0(L(E))$ and the element $[1_{L(E)}]$ of $K_0(L(E))$. For finite graphs E, G we write

$$\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(G)$$

in case there exists an abelian group isomorphism $\varphi : K_0(L(E)) \rightarrow K_0(L(G))$ for which $\varphi([1_{L(E)}]) = [1_{L(G)}]$. Clearly \equiv yields an equivalence relation on the set of finite graphs.

We will show that, in the case of Morita equivalent purely infinite simple Leavitt path algebras over finite graphs, if the unitary Franks pair of their graphs are equivalent, then the algebras are isomorphic. The argument relies on the adaptation to our context of the deep result of Huang [18, Theorem 1.1].

Now suppose E has $L(E)$ purely infinite simple unital, and has no sources. Then by Lemma 1.17 E^t has these same properties. Let

$$E^t = H_0 \xrightarrow{m_1} H_1 \xrightarrow{m_2} H_2 \dots \xrightarrow{m_n} H_n = E^t$$

be a finite sequence of standard graph transformations which starts and ends with E^t . We write $H_i = G_i^t$ (where $G_i = H_i^t$), and so we have a finite sequence of graph transformations

$$E^t = G_0^t \xrightarrow{m_1} G_1^t \xrightarrow{m_2} G_2^t \dots \xrightarrow{m_n} G_n^t = E^t.$$

For any graph G let $\tau_G : G \rightarrow G^t$ be the graph function which is the identity on vertices, but switches the direction of each of the edges. (This is simply the transpose operation on the corresponding incidence matrices.) In particular, any one of the standard graph transformations

$$m : G_i^t \rightarrow G_{i+1}^t$$

yields a graph transformation

$$m' = \tau_{G_{i+1}}^{-1} \circ m \circ \tau_{G_i} : G_i \rightarrow G_{i+1}.$$

Lemma 2.2. *If $m : G_i^t \rightarrow G_{i+1}^t$ is a standard graph transformation, then $m' = \tau_{G_{i+1}}^{-1} \circ m \circ \tau_{G_i} : G_i \rightarrow G_{i+1}$ is also standard.*

Proof. We leave to the reader the straightforward check that

- (1) If m is an expansion (resp. contraction), then m' is an expansion (resp. contraction).
- (2) If m is an in-splitting (resp. out-splitting), then m' is an out-splitting (resp. in-splitting).
- (3) If m is an in-amalgamation (resp. out-amalgamation), then m' is an out-amalgamation (resp. in-amalgamation).

(We note that expansions (resp. contractions) remain expansions (resp. contractions) when passing to the transpose graph, but that the other four standard operations indeed become a different type of standard transformation on the transpose.) \square

As a consequence of Lemma 2.2, if we start with any finite sequence of standard graph transformations

$$E^t = H_0 \xrightarrow{m_1} H_1 \xrightarrow{m_2} H_2 \dots \xrightarrow{m_n} H_n = E^t$$

which starts and ends with E^t , then we get a corresponding finite sequence of standard graph transformations

$$E = G_0 \xrightarrow{m'_1} G_1 \xrightarrow{m'_2} G_2 \dots \xrightarrow{m'_n} G_n = E$$

which starts and ends with E .

By [18, Lemma 3.7], for any graphs E and F , any standard graph transformation $m : E \rightarrow F$ yields the so-called *induced* isomorphism

$$\varphi_m : \text{coker}(I - A_E) \rightarrow \text{coker}(I - A_F).$$

For each of the six types of standard graph transformations, the corresponding induced isomorphism is explicitly described in [18, Lemma 3.7]. As a representative example of these induced isomorphisms, we offer the following description. Suppose $m : E \rightarrow F$ is an in-splitting; that is, $F = E_r(\mathcal{P})$ for some partition \mathcal{P} of the edges of E . By generalizing the

construction of Franks [14, Theorem 1.7] in the natural way, we define matrices

$$R = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where a_{ij} is the number of edges in the j th partition class of E^1 that leave vertex i . The columns of S correspond to vertices of E , and the rows to partition classes of E^1 , where a 1 indicates that a partition class contains edges entering the vertex. With R and S so defined, it is straightforward to show that $A_E = RS$ and $A_F = SR$. By [18, Lemma 3.7], we get $[x] \mapsto [Rx]$ is the induced isomorphism on $\text{coker}(I_n - A_E)$.

As it turns out, the descriptions of the induced isomorphisms coming from expansions and contractions are somewhat different than the descriptions of the induced isomorphisms coming from the other four types of standard graph transformations. However, in each case, an explicit description of the induced isomorphism can be given as above. We leave the details in the other five cases to the interested reader. Here now is the connection between the Morita equivalences of Section 1 and the induced isomorphisms given by Huang.

Proposition 2.3. *Let G_i and G_{i+1} be finite graphs. Suppose G_i has $L(G_i)$ purely infinite simple, and has no sources. Suppose $m_i : G_i^t \rightarrow G_{i+1}^t$ is a standard graph transformation, and let $\varphi_{m_i} : \text{coker}(I - A_{G_i^t}) \rightarrow \text{coker}(I - A_{G_{i+1}^t})$ be the induced isomorphism. Let $m'_i : G_i \rightarrow G_{i+1}$ be the corresponding graph transformation, which, by Lemma 2.2, is also a standard transformation. Let $\Phi^{m'_i} : L(G_i)\text{-Mod} \rightarrow L(G_{i+1})\text{-Mod}$ be the Morita equivalence induced by m'_i as described in Section 1. Then, using the previously described identification between $K_0(L(G_i))$ and $\text{coker}(I - A_{G_i^t})$ (resp. between $K_0(L(G_{i+1}))$ and $\text{coker}(I - A_{G_{i+1}^t})$), we have*

$$\Phi_{\mathcal{Y}}^{m'_i} = \varphi_{m_i}.$$

Proof. Each of the six types of isomomorphisms $\Phi_{\mathcal{Y}}^{m'_i} : K_0(L(G_i)) \rightarrow K_0(L(G_{i+1}))$ have been explicitly described in Section 1. As indicated above, each of the six types of induced isomorphisms $\varphi_{m_i} : \text{coker}(I - A_{G_i^t}) \rightarrow \text{coker}(I - A_{G_{i+1}^t})$ have been explicitly described in [18, Lemma 3.7]. By definition we have $A_{G_i^t} = A_{G_i}^t$ (resp. $A_{G_{i+1}^t} = A_{G_{i+1}}^t$). It is now a tedious but completely straightforward check to verify that, in all six cases, these isomorphisms agree. \square

We are finally in position to adapt the result of Huang to our context. For a ring R , and an automorphism α of $K_0(R)$, we say a Morita equivalence $\Phi : R\text{-Mod} \rightarrow R\text{-Mod}$ restricts to α in case $\Phi_{\mathcal{Y}} = \alpha$.

Proposition 2.4. *Let E be a finite graph for which $L(E)$ is purely infinite simple. Let α be any automorphism of $K_0(L(E))$. Then there exists a Morita equivalence $\Phi : L(E)\text{-Mod} \rightarrow L(E)\text{-Mod}$ which restricts to α .*

Proof. If E contains sources, then Corollary 1.5 guarantees the existence of a Morita equivalence $\Phi^{\text{ELIM}} : L(E')\text{-Mod} \rightarrow L(E)\text{-Mod}$, where E' has no sources. If $\Psi : L(E')\text{-Mod} \rightarrow L(E')\text{-Mod}$ is a Morita equivalence which restricts to the automorphism $(\Phi_{\mathcal{V}}^{\text{ELIM}})^{-1} \circ \alpha \circ \Phi_{\mathcal{V}}^{\text{ELIM}}$ of $K_0(L(E'))$, then $\Phi^{\text{ELIM}} \circ \Psi \circ (\Phi^{\text{ELIM}})^{-1}$ is a Morita equivalence from $L(E)\text{-Mod}$ to $L(E)\text{-Mod}$ which restricts to α . Therefore, it suffices to consider graphs E with no sources.

If $L(E)$ is purely infinite simple, and E has no sources, then E is essential, irreducible, and non-trivial, and hence so is E^t . Since $K_0(L(E))$ is identified with $\text{coker}(I - A_E^t)$, we may view α as an automorphism of $\text{coker}(I - A_E^t) = \text{coker}(I - A_{E^t})$. Therefore, by [18, Theorem 1.1] (details in [17, Theorem 2.15]), there exists a flow equivalence \mathcal{F} from E^t to itself which induces α . Such a flow equivalence can be written as a finite sequence

$$E^t = H_0 \xrightarrow{m_1} H_1 \xrightarrow{m_2} H_2 \dots \xrightarrow{m_n} H_n = E^t$$

of standard graph transformations which starts and ends with E^t . But this then yields a corresponding finite sequence of standard graph transformations

$$E = G_0 \xrightarrow{m'_1} G_1 \xrightarrow{m'_2} G_2 \dots \xrightarrow{m'_n} G_n = E$$

which starts and ends with E , as described in Lemma 2.2. This sequence of standard graph transformations in turn yields a sequence of Morita equivalences (using the results of Section 1) which starts and ends at $L(E)\text{-Mod}$. But by Proposition 2.3, at each stage of the sequence the restriction of the Morita equivalence to the appropriate K_0 group agrees with the induced map coming from the standard graph transformation. If we denote by $\Phi : L(E)\text{-Mod} \rightarrow L(E)\text{-Mod}$ the composition of these Morita equivalences, then Φ restricts to the same automorphism of $K_0(L(E))$ as does \mathcal{F} , namely, the prescribed automorphism α . \square

Here now is the second main result of this article.

Theorem 2.5. *Let E, G be finite graphs such that $L(E), L(G)$ are purely infinite simple unital Leavitt path algebras, and such that $L(E)$ is Morita equivalent to $L(G)$. If*

$$\mathcal{F}_{[1]}(L(E)) \cong \mathcal{F}_{[1]}(L(G)),$$

(i.e., if $K_0(L(E)) \cong K_0(L(G))$ via an isomorphism which sends $[1_{L(E)}]$ to $[1_{L(G)}]$), then there is a ring isomorphism

$$L(E) \cong L(G).$$

Proof. Suppose that $\varphi : K_0(L(E)) \rightarrow K_0(L(G))$ is an isomorphism with $\varphi([1_{L(E)}]) = [1_{L(G)}]$. Since $L(E)$ and $L(G)$ are Morita equivalent by hypothesis, there exists a Morita equivalence

$$\Gamma : L(E)\text{-Mod} \rightarrow L(G)\text{-Mod}.$$

Thus there is an isomorphism $\Gamma_{\mathcal{V}} : K_0(L(E)) \rightarrow K_0(L(G))$.

Now consider the group automorphism

$$\varphi \circ \Gamma_{\mathcal{V}}^{-1} : K_0(L(G)) \rightarrow K_0(L(G)).$$

By Proposition 2.4, there exists a Morita equivalence $\Psi : L(G)\text{-Mod} \rightarrow L(E)\text{-Mod}$ such that

$$\Psi_{\mathcal{V}} = \varphi \circ \Gamma_{\mathcal{V}}^{-1}.$$

Thus, we get a Morita equivalence

$$H := \Psi \circ \Gamma : L(E)\text{-Mod} \rightarrow L(G)\text{-Mod}$$

with

$$H_{\mathcal{V}} = (\Psi \circ \Gamma)_{\mathcal{V}} = \Psi_{\mathcal{V}} \circ \Gamma_{\mathcal{V}} = \varphi \circ \Gamma_{\mathcal{V}}^{-1} \circ \Gamma_{\mathcal{V}} = \varphi.$$

In particular, $H_{\mathcal{V}}([1_{L(E)}]) = \varphi([1_{L(E)}]) = [1_{L(G)}]$. As noted in the Introduction, since $L(E)$ and $L(G)$ are purely infinite simple rings, [6, Corollary 2.2] implies that $[1_{L(E)}] \in K_0(L(E))$ consists of the finitely generated projective left $L(E)$ -modules isomorphic (as left $L(E)$ -modules) to the progenerator ${}_{L(E)}L(E)$, and analogously $[1_{L(G)}] \in K_0(L(G))$ consists of the finitely generated projective left $L(G)$ -modules isomorphic (as left $L(G)$ -modules) to the progenerator ${}_{L(G)}L(G)$. Thus the equation $H_{\mathcal{V}}([1_{L(E)}]) = [1_{L(G)}]$ yields that $H({}_{L(E)}L(E)) \cong {}_{L(G)}L(G)$. Since Morita equivalences preserve endomorphism rings, we get ring isomorphisms

$$L(E) \cong \text{End}_{{}_{L(E)}L(E)}(L(E)) \cong \text{End}_{{}_{L(G)}L(G)}(H({}_{L(E)}L(E))) \cong \text{End}_{{}_{L(G)}L(G)}(L(G)) \cong L(G),$$

and the theorem is established. \square

An easy corollary now gives sufficient, readily computable, and remarkably weak conditions under which two unital purely infinite simple Leavitt path algebras are known to be isomorphic. We combine \mathcal{F}_{det} and $\mathcal{F}_{[1]}$ to obtain these conditions.

Definition 2.6. Let E be a finite graph. We define the *Franks triple* to be the ordered triple

$$\mathcal{F}_3(E) = (K_0(L(E)), [1_{L(E)}], \det(I_n - A_E^t)),$$

consisting of the abelian group $K_0(L(E))$, the element $[1_{L(E)}]$ which represents the order unit of $K_0(L(E))$ containing $1_{L(E)}$, and the integer $\det(I_n - A_E^t)$ (where $n = |E^0|$). For finite graphs E, G we write

$$\mathcal{F}_3(E) \equiv \mathcal{F}_3(G)$$

in case there exists an abelian group isomorphism $\varphi : K_0(L(E)) \rightarrow K_0(L(G))$ for which $\varphi([1_{L(E)}]) = [1_{L(G)}]$, and $\det(I_n - A_E^t) = \det(I_m - A_G^t)$. Clearly \equiv yields an equivalence relation on the set of finite graphs.

When $n = |E^0|$ is clear from context we will often denote the $n \times n$ identity matrix I_n simply by I . We now have the sufficiency result we pursued.

Corollary 2.7. *Let E, G be finite graphs such that $L(E)$ and $L(G)$ are purely infinite simple Leavitt path algebras. If*

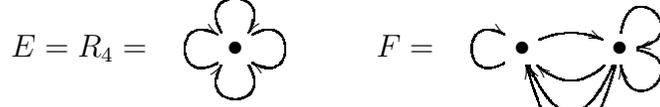
$$\mathcal{F}_3(L(E)) \equiv \mathcal{F}_3(L(G)),$$

(i.e., if $K_0(L(E)) \cong K_0(L(G))$ via an isomorphism which sends $[1_{L(E)}]$ to $[1_{L(G)}]$, and $\det(I - A_E^t) = \det(I - A_G^t)$), then there is a ring isomorphism

$$L(E) \cong L(G).$$

Proof. Since $\mathcal{F}_3(L(E)) \equiv \mathcal{F}_3(L(G))$, we have in particular that $\mathcal{F}_{det}(L(E)) \equiv \mathcal{F}_{det}(L(G))$, so that $L(E)$ and $L(G)$ are Morita equivalent by Theorem 1.25. At the same time, we have $\mathcal{F}_{[1]}(L(E)) \equiv \mathcal{F}_{[1]}(L(G))$, which together with Theorem 2.5 gives the isomorphism we seek. \square

Example 2.8. Let E and F be the graphs



Then

$$A_E = (4) \quad \text{and} \quad A_F = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix},$$

so that

$$I - A_E^t = (-3) \quad \text{and} \quad I - A_F^t = \begin{pmatrix} 0 & -3 \\ -1 & -1 \end{pmatrix}.$$

It is well known (and easy to compute) that $K_0(L(E)) \cong \mathbb{Z}_3$, with $[1_{L(E)}] = 1$ in \mathbb{Z}_3 . Similarly, it is not hard to show that $K_0(L(F)) \cong \text{coker}(I - A_F^t) \cong \mathbb{Z}_3$ as well, and that $[1_{L(F)}] = 1$ in \mathbb{Z}_3 . Clearly $\det(I - A_E^t) = -3 = \det(I - A_F^t)$. Since both graphs yield purely infinite simple unital Leavitt path algebras, we conclude by Corollary 2.7 that the Leavitt path algebras $L(E)$ and $L(F)$ are isomorphic.

It is worth remarking that the proof of Proposition 2.4, and therefore the proof of Theorem 2.5, hinges on results of Huang ([17] and [18]). Huang's results are *constructive*; specifically, the flow equivalence from E^t to itself which induces α (as in the proof of Proposition 2.4) is explicitly described. In [5] we present in detail an algorithmic description of the resulting isomorphism $L(E) \cong L(G)$ of Theorem 2.5.

Corollary 2.7 establishes that equivalence of the Franks triple is a sufficient condition to conclude isomorphism of the corresponding purely infinite simple Leavitt path algebras over finite graphs. For the remainder of this section we consider whether or not the unitary Franks pair (i.e., the pair $(K_0(L(E)), [1_{L(E)}])$ *without* the $\det(I - A_E^t)$ information) precisely classifies these algebras. It is known that the converse is true: namely, that an isomorphism $L(E) \cong L(G)$ implies the equivalence of the unitary Franks pairs $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(G)$ (see e.g. [4, Theorem 5.11]).

It turns out that equivalence of the unitary Franks invariants *almost* guarantees equivalence of the corresponding Franks triples; the only possible difference can be in the sign of the determinant. In particular, we can recast Corollary 2.7 as follows.

Corollary 2.9. *If E and G are finite graphs for which the Leavitt path algebras $L(E)$ and $L(G)$ are purely infinite simple, for which $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(G)$, and for which the integers $\det(I - A_E^t)$ and $\det(I - A_G^t)$ have the same sign, then there is a ring isomorphism $L(E) \cong L(G)$.*

Proof. Since $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(G)$, we have in particular that $\text{coker}(I - A_E^t) \cong \text{coker}(I - A_G^t)$, whence the Smith normal forms of these two matrices are the same. But the Smith normal form of a matrix is achieved by a process which involves multiplication by various matrices, each having determinant 1 or -1 . In particular, this yields $|\det(I - A_E^t)| = |\det(I - A_G^t)|$. So $\det(I - A_E^t)$ and $\det(I - A_G^t)$ having the same sign implies equality of these two integers, whence the result follows from Corollary 2.7. \square

We note that there are classes of graphs for which equivalence of the unitary Franks pair automatically implies equivalence of the corresponding Franks triple, which then in turn implies isomorphism of the corresponding Leavitt path algebras by Corollary 2.7. For instance,

isomorphisms between various sized matrix rings over the Leavitt algebra L_n (see the Introduction) can be recast as isomorphisms between Leavitt path algebras over appropriate graphs (see [4, Section 5]). In this context, one can show that graphs having equivalent unitary Franks pair indeed have identical (negative) $\det(I - A^t)$, so that [4, Theorem 5.9] in fact follows from Corollary 2.7.

Similarly, isomorphisms between purely infinite simple Leavitt path algebras $L(E)$ and $L(G)$, for which neither E nor G have parallel edges, and for which both $|E^0| \leq 3$ and $|G^0| \leq 3$, are established in [4, Section 4]. In this context as well, one can show that graphs having equivalent unitary Franks pair indeed have identical (negative) $\det(I - A^t)$, so that [4, Propositions 4.1 and 4.2] follow from Corollary 2.7 as well.

The previous two paragraphs notwithstanding, the cited isomorphism results from [4] are more than merely special cases of Corollary 2.7, since the isomorphisms of [4] are in fact explicitly constructed.

An immediate, interesting consequence of Corollary 2.7 is the following result along these same lines.

Corollary 2.10. *Let E, G be finite graphs such that $L(E), L(G)$ are purely infinite simple Leavitt path algebras with infinite Grothendieck groups. If $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(G)$, then $L(E) \cong L(G)$. In other words, in this situation, equivalence of the unitary Franks pairs is sufficient to yield isomorphism of the Leavitt path algebras.*

Proof. The condition that $L(E)$ and $L(G)$ have infinite Grothendieck groups implies that $\det(I - A_E^t) = 0 = \det(I - A_G^t)$, and Corollary 2.7 then applies. \square

So, in the case of infinite K_0 -groups, the unitary Franks pair $(K_0(L(E)), [1_{L(E)}])$ is an invariant for classifying the purely infinite simple unital Leavitt path algebras up to isomorphism.

We note that somewhat stronger conclusions may be drawn in Theorem 2.5 and the subsequent three Corollaries, to wit, that there exist K -algebra isomorphisms between the corresponding Leavitt path algebras. The tools required to establish the existence of such K -algebra isomorphisms are more extensive than the tools utilized here. This approach follows an approach similar to one developed by Cuntz, which is described in [23, Theorem 6.5].

For the remainder of this section we investigate whether or not the result of Corollary 2.10 can be generalized to all purely infinite simple unital Leavitt path algebras. Rephrased, we seek to show either

(1) that there exist non-isomorphic purely infinite simple Leavitt path algebras $L(E), L(F)$ over finite graphs E, F such that $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(F)$, for which the signs of $\det(I - A_E^t)$ and $\det(I - A_F^t)$ are unequal, or

(2) that the sign of $\det(I - A_E^t)$ plays no role in guaranteeing the existence of an isomorphism between the purely infinite simple unital Leavitt path algebras $L(E)$ and $L(F)$, for which $\mathcal{F}_{[1]}(E) \equiv \mathcal{F}_{[1]}(F)$.

A key observation related to the analysis of the Classification Question developed by the authors in the present paper and [4] is that the graph operations we have already considered cannot help us in this final step, because all of these graph operations preserve flow equivalence on subshifts of finite type, and thus preserve the sign of $\det(I - A_E^t)$. So it is clear that to attain the final goal of classifying these kinds of Leavitt path algebras using the unitary

Franks pair $(K_0(L(E)), [1_{L(E)}])$ as an invariant requires a completely new set of ideas and strategies.

In the context of Cuntz-Krieger C^* -algebras, the irrelevance of the sign of the determinant in the analogous Classification Question was shown by Rørdam [23]; and in the case of graph C^* -algebras, this irrelevance is a direct consequence of the Kirchberg-Phillips Classification Theorem [16, 21] and the computation of the K-theoretic invariant for such a class of algebras (see e.g. [22]). In this direction, a useful tool is Cuntz's Theorem (presented by Rørdam [23, Theorem 7.2]), whose adaptation to our context gives the possibility of reducing the above situation to a single pair of algebras. We describe the situation, following Cuntz's argument.

Definition 2.11. For any finite graph E having vertices v_1, \dots, v_n , such that $L(E)$ is a purely infinite simple algebra and v_n belongs to a cycle, let E_- be the graph whose incidence matrix and pictorial representation is

$$A_{E_-} = \begin{pmatrix} & & 0 & 0 \\ & A_E & \vdots & \vdots \\ & & 0 & 0 \\ & & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix} \quad E_- = \begin{array}{c} \boxed{E} \\ \vdots \\ \bullet_{v_n} \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$$

Specifically, if $E^0 = \{v_1, \dots, v_n\}$ where v_n belongs to a cycle, then the new graph E_- has $E_-^0 = \{v_1, \dots, v_n, v_{n+1}, v_{n+2}\}$, while E_-^1 is the union of E^1 with six new edges: one from v_n to v_{n+1} ; one from v_{n+1} to each of v_n , v_{n+1} , and v_{n+2} ; and one from v_{n+2} to each of v_{n+1} and v_{n+2} .

It is straightforward to show (using the Purely Infinite Simplicity Theorem) that $L(E)$ is purely infinite simple unital if and only if $L(E_-)$ is purely infinite simple unital. Furthermore, we have

Proposition 2.12. *Let E be a finite graph for which $L(E)$ is purely infinite simple, and let E_- be the graph defined above. Then*

$$K_0(L(E_-)) \cong K_0(L(E)), \quad \text{and} \quad \det(I - A_{E_-}^t) = -\det(I - A_E^t).$$

Proof. For proving $K_0(L(E_-)) \cong K_0(L(E))$, notice that, by [7, Theorem 3.5], the monoid $V(L(E_-))$ is generated by v_1, \dots, v_{n+2} , with relations

$$v_i = \sum_{j=1}^n A_e(i, j)v_j \quad (1 \leq i \leq n-1),$$

together with the three relations

$$v_n = v_{n+1} + \sum_{j=1}^n A_e(i, j)v_j, \quad v_{n+1} = v_n + v_{n+1} + v_{n+2}, \quad \text{and} \quad v_{n+2} = v_{n+1} + v_{n+2}.$$

Since $L(E_-)$ is purely infinite simple we get $K_0(L(E_-)) \cong V(L(E_-))^*$ by [6, Corollary 2.2]. Thus $K_0(L(E_-))$ is the group generated by $[v_1], \dots, [v_{n+2}]$, with relations

$$[v_i] = \sum_{j=1}^n A_e(i, j)[v_j] \quad (1 \leq i \leq n-1),$$

together with the three relations

$$[v_n] = [v_{n+1}] + \sum_{j=1}^n A_e(i, j)[v_j], \quad [v_n] = -[v_{n+2}], \quad \text{and} \quad [v_{n+1}] = [0].$$

In particular, since $[v_{n+1}] = [0]$ we in fact have that $[v_i] = \sum_{j=1}^n A_e(i, j)[v_j]$ for all $1 \leq i \leq n$ (i.e., including n as well). This yields that the relations between generators of $K_0(L(E))$ remain the same when viewed as elements of $K_0(L(E_-))$, so that the inclusion map $K_0(L(E)) \mapsto K_0(L(E_-))$ is a group homomorphism. The equations $[v_{n+1}] = [0]$ and $[v_{n+2}] = -[v_n]$ show that this inclusion is actually a surjection, so that $K_0(L(E_-))$ is isomorphic to $K_0(L(E))$.

(We note that the isomorphism from $K_0(L(E))$ to $K_0(L(E_-))$ given here does not necessarily take $[1_{L(E)}]$ to $[1_{L(E_-)}]$, since in general we need not have $[v_{n+1}] + [v_{n+2}] = [0]$ in $K_0(L(E_-))$.)

With respect to the determinants, the result is an elementary computation. \square

As a specific, important example, consider the Leavitt path algebras $L(\mathbf{2})$ and $L(\mathbf{2}_-)$, where $\mathbf{2}$ and $\mathbf{2}_-$ are the graphs with incidence matrices

$$A_{\mathbf{2}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{2}_-} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Pictorially, these graphs are given by

$$\mathbf{2} = \left(\bullet^{v_1} \rightleftarrows \bullet^{v_2} \right)$$

and

$$\mathbf{2}_- = \left(\bullet^{v_1} \rightleftarrows \bullet^{v_2} \begin{matrix} \circlearrowleft \\ \circlearrowright \end{matrix} \rightleftarrows \bullet^{v_3} \begin{matrix} \circlearrowleft \\ \circlearrowright \end{matrix} \rightleftarrows \bullet^{v_4} \right)$$

Notice that

$$(K_0(L(\mathbf{2})), [1_{L(\mathbf{2})}], \det(I - A_{\mathbf{2}}^t)) = (\{0\}, 0, -1), \quad \text{while} \\ (K_0(L(\mathbf{2}_-)), [1_{L(\mathbf{2}_-)}], \det(I - A_{\mathbf{2}_-}^t)) = (\{0\}, 0, 1).$$

Now consider the standard representations of $L(\mathbf{2})$ and $L(\mathbf{2}_-)$ in $R = \text{End}_K(V)$, where V is a K -vector space of countable dimension with basis $\{v_i\}_{i \geq 1}$. (For a description of this process, see [22, page 8].) Let $\mathbf{u} \in R$ be the endomorphism defined by the rule $\mathbf{u}(v_i) = \delta_{1,i}v_1$. Let \mathcal{E}_2 be the subalgebra of R generated by $L(\mathbf{2})$ and \mathbf{u} , and similarly let \mathcal{E}_{2_-} be the subalgebra of R generated by $L(\mathbf{2}_-)$ and \mathbf{u} .

Hypothesis: *There exists a K -algebra isomorphism $\tau : L(\mathbf{2}) \rightarrow L(\mathbf{2}_-)$ which extends to an isomorphism $T : \mathcal{E}_2 \rightarrow \mathcal{E}_{2_-}$ such that $T(\mathbf{u}) = \mathbf{u}$.*

Using the argument presented in [23, Theorem 7.2], it is long but straightforward to show that

Theorem 2.13. *If the Hypothesis holds, then for any finite graph E such that $L(E)$ is a purely infinite simple Leavitt path algebra, there is a Morita equivalence $L(E) \sim_M L(E_-)$.*

Therefore, as a consequence of Corollary 2.9 and Proposition 2.12 we would then have

Theorem 2.14. *If the Hypothesis holds, then K_0 precisely classifies purely infinite simple unital Leavitt path algebras up to Morita equivalence.*

Proof. Let E, G be finite graphs for which $L(E)$ and $L(G)$ are purely infinite simple and $K_0(L(E)) \cong K_0(L(G))$. By Corollary 2.9, either $\det(I - A_E^t) = \det(I - A_G^t)$, or $\det(I - A_E^t) = -\det(I - A_G^t)$. In the first case, we have $\mathcal{F}_{\det}(L(E)) \equiv \mathcal{F}_{\det}(L(G))$, so Theorem 1.25 gives Morita equivalence. Otherwise, we have $\mathcal{F}_{\det}(L(E_-)) \equiv \mathcal{F}_{\det}(L(G))$, so by Theorems 2.13 and 1.25, we get

$$L(E) \sim_M L(E_-) \sim_M L(G),$$

and the theorem follows. \square

Following the same strategy as before, we now push this Morita equivalence result to yield isomorphisms by applying Theorem 2.5. In order to do so, we will need another graph construction.

Definition 2.15. For any finite graph E having vertices v_1, \dots, v_n , such that $L(E)$ is a purely infinite simple algebra and v_n belongs to a cycle, let E_{1-} be the graph whose incidence matrix and pictorial representation is

$$A_{E_{1-}} = \begin{pmatrix} & & & 0 & 0 & 0 \\ & A_E & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad E_{1-} = \begin{array}{c} \boxed{E} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} .$$

Immediately, we note that $E_- = (E_{1-})_{\setminus v_{n+3}}$. It is again straightforward to show (using the Purely Infinite Simplicity Theorem) that $L(E)$ is purely infinite simple if and only if $L(E_{1-})$ is purely infinite simple. Furthermore, we have

Proposition 2.16. *Let E be a finite graph for which $L(E)$ is purely infinite simple, and let E_{1-} be the graph defined above. Then*

$$\mathcal{F}_{[1]}(L(E_{1-})) \equiv \mathcal{F}_{[1]}(L(E)) \quad \text{and} \quad \det(I - A_{E_{1-}}^t) = -\det(I - A_E^t).$$

Proof. Since $E_- = (E_{1-})_{\setminus v_{n+3}}$, we get from Lemma 1.24 that $\det(I - A_{E_{1-}}^t) = \det(I - A_{E_-}^t) = -\det(I - A_E^t)$. We follow a strategy similar to the one used in the proof of Proposition 2.12 to conclude that the inclusion map is an isomorphism between $K_0(L(E))$ and $K_0(L(E_{1-}))$. Specifically, by again using [7, Theorem 3.5], the monoid $V(L(E_{1-}))$ is generated by v_1, \dots, v_{n+3} with relations $v_i = \sum_{j=1}^n A_e(i, j)v_j$ ($1 \leq i \leq n-1$), together with the four relations

$$v_n = v_{n+1} + \sum_{j=1}^n A_e(i, j)v_j, \quad v_{n+1} = v_n + v_{n+1} + v_{n+2}, \quad v_{n+2} = v_{n+1} + v_{n+2}, \quad v_{n+3} = v_n.$$

Again, we apply [6, Corollary 2.2] to get the isomorphism $K_0(L(E_{1-})) \cong V(L(E_{1-}))^*$, and note that $[v_{n+2}] = -[v_n]$, $[v_{n+3}] = [v_n]$ and $[v_{n+1}] = [0]$, so that $[v_i] = \sum_{j=1}^n A_e(i, j)[v_j]$ ($1 \leq i \leq n$). Hence, the map

$$\begin{aligned} [v_i] &\mapsto [v_i] \quad (1 \leq i \leq n) \\ [v_{n+2}] &\mapsto -[v_n] \\ [v_{n+3}] &\mapsto [v_n] \end{aligned}$$

defines an isomorphism φ from $K_0(L(E_{1-}))$ to $K_0(L(E))$. In addition,

$$\varphi([1_{L(E_{1-})}]) = [1_{L(E)}] + [v_{n+1}] + [v_{n+2}] + [v_{n+3}] = [1_{L(E)}] + [v_n] - [v_n] = [1_{L(E)}],$$

which yields the equivalence $\mathcal{F}_{[1]}(L(E_{1-})) \equiv \mathcal{F}_{[1]}(L(E))$. \square

In particular, for any finite graph E with $L(E)$ purely infinite simple, we now have a construction of another graph E_{1-} which shares its unitary Franks pair, but differs in $\text{sgn}(\det(I - A_E^t))$. We now assume the Hypothesis, and analyze the consequences for isomorphisms.

Proposition 2.17. *If the Hypothesis holds, then for any finite graph E such that $L(E)$ is a purely infinite simple Leavitt path algebra, there is a Morita equivalence $L(E) \sim_M L(E_{1-})$.*

Proof. Applying Theorem 2.13 and Proposition 1.4, we get

$$L(E) \sim_M L(E_-) = L((E_{1-})_{\setminus v_{n+3}}) \sim_M L(E_{1-}).$$

\square

Combining the equivalence of the unitary Franks pair from Proposition 2.16 with the Morita equivalence from Proposition 2.17, we are in position to apply Theorem 2.5 and obtain the following key connecting result, one which allows us to cross the ‘‘determinant gap’’.

Proposition 2.18. *If the Hypothesis holds, then for any finite graph E such that $L(E)$ is a purely infinite simple Leavitt path algebra, there is an isomorphism $L(E) \cong L(E_{1-})$.*

As a consequence, we obtain

Theorem 2.19. *If the Hypothesis holds, then $\mathcal{F}_{[1]}$ precisely classifies purely infinite simple unital Leavitt path algebras up to isomorphism.*

Proof. Let E, G be finite graphs for which $L(E)$ and $L(G)$ are purely infinite simple and $\mathcal{F}_{[1]}(L(E)) \equiv \mathcal{F}_{[1]}(L(G))$. By Corollary 2.9, either $\det(I - A_E^t) = \det(I - A_G^t)$, or $\det(I - A_E^t) = -\det(I - A_G^t)$. In the first case, we have $\mathcal{F}_3(L(E)) \equiv \mathcal{F}_3(L(G))$, so Corollary 2.7 gives the desired isomorphism. Otherwise, we have $\mathcal{F}_3(L(E_{1-})) \equiv \mathcal{F}_3(L(G))$, so by Proposition 2.18 and Corollary 2.7, we get

$$L(E) \cong L(E_{1-}) \cong L(G),$$

and the theorem follows. \square

3. SOME GENERAL ISOMORPHISM AND MORITA EQUIVALENCE RESULTS FOR LEAVITT PATH ALGEBRAS

In Section 1 we presented four results regarding Morita equivalences between Leavitt path algebras. These four specific results were precisely those which we needed to achieve the first main result of this article, Theorem 1.25. In the final section of this article, we present a number of similarly-flavored results which we believe are of interest in their own right. Along the way we will give generalizations of Propositions 1.4, 1.8, and 1.11 to wider classes of graphs.

Information about various topics presented in this section pertaining to Leavitt path algebras (e.g. the \mathbb{Z} -grading on $L(E)$, and the natural action as automorphisms of K^* on $L_K(E)$) can be found in [4]. Information about Morita equivalence for not-necessarily-unital rings can be found in [15].

Here is the indicated generalization of Proposition 1.4.

Proposition 3.1. *Let E be a row-finite graph with no sinks, let $v \in E^0$ be a source, and let $E_{\setminus v}$ be the source elimination graph. Then $L(E)$ and $L(E_{\setminus v})$ are Morita equivalent.*

Proof. By definition of $F = E_{\setminus v}$, it is clear that F is a (complete) subgraph of E . Thus, the K -algebra map defined by the rule

$$\begin{array}{ccc} \phi & L(F) & \longrightarrow & L(E) \\ & w & \mapsto & w \\ & e & \mapsto & e \\ & e^* & \mapsto & e^* \end{array}$$

for every $w \in F^0$ and every $e \in F^1$, is a \mathbb{Z} -graded ring homomorphism such that $\phi(w) \neq 0$ for every $w \in F^0$. Hence ϕ is injective by [4, Lemma 1.1].

Set $F^0 = \{w_i\}_{i \geq 1}$. For each $n \geq 1$ define $e = \sum_{i=1}^n w_i$. Then $\{e_n\}_{n \geq 1}$ is a set of local units for $L(F)$, and since v is a source, $\phi(L(F)) = \bigcup_{n \geq 1} e_n L(E) e_n$. Moreover, as $r(s^{-1}(v)) \subset F^0$, E^0 turns out to be the hereditary saturated closure of F^0 . Hence, we get $L(E) = \bigcup_{n \geq 1} L(E) e_n L(E)$.

Thus, it is not difficult to see that

$$\left(\sum_{n \geq 1} e_n L(E) e_n, \sum_{n \geq 1} L(E) e_n L(E), \sum_{n \geq 1} L(E) e_n, \sum_{n \geq 1} e_n L(E) \right)$$

is a (surjective) Morita context for the rings $L(E)$ and $L(F)$, as desired. \square

The following definition are borrowed from [10, Section 4].

Definitions 3.2. Let $E = (E^0, E^1, r, s)$ be a row-finite graph. A map $d_s : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ such that

- (1) if $w \in E^0$ is not a sink then $d_s(w) = \sup\{d_s(e) \mid s(e) = w\}$, and
- (2) if $d_s(x) = \infty$ for some x , then x is a sink

is called a *Drinen source-vector*. Note that only vertices are allowed to have an infinite d_s -value. From this data we construct a new graph $d_s(E)$ as follows: Let

$$d_s(E)^0 = \{v^i \mid v \in E^0, 0 \leq i \leq d_s(v)\}, \text{ and}$$

$$d_s(E)^1 = E^1 \cup \{f(v)^i \mid 1 \leq i \leq d_s(v)\},$$

and for $e \in E^1$ define $r_{d_s(E)}(e) = r(e)^0$ and $s_{d_s(E)}(e) = s(e)^{d_s(e)}$. For $f(v)^i$ define $s_{d_s(E)}(f(v)^i) = v^{i-1}$ and $r_{d_s(E)}(f(v)^i) = v^i$. The resulting directed graph $d_s(E)$ is called the *out-delayed graph of E* for the Drinen source-vector d_s .

In the out-delayed graph the original vertices correspond to those vertices with superscript 0. Intuitively, the edge $e \in E^1$ is “delayed” from leaving $s(e)^0$ and arriving at $r(e)^0$ by a path of length $d_s(e)$.

Theorem 3.3. *Let K be an infinite field. Let E be a row-finite graph and let $d_s : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ be a Drinen source-vector. Then $L(d_s(E))$ is Morita equivalent to $L(E)$.*

Proof. The argument is essentially the same as in the proof of [10, Theorem 4.2], except for the proof of the injectivity of the map π , and the proof of the Morita equivalence of $L(E)$ and $L(d_s(E))$. We include the whole argument for the sake of completeness.

Given $e \in E^1$ and $v \in E^0$, define $Q_v = v^0$, and define T_e by setting

$$T_e = f(s(e))^1 \dots f(s(e))^{d_s(e)} e \text{ if } d_s(e) \neq 0, \quad \text{and} \quad T_e = e \text{ otherwise.}$$

We claim that $\{T_e, Q_v \mid e \in E^1, v \in E^0\}$ is an E -family in $L(d_s(E))$. The Q_v 's are nonzero mutually orthogonal idempotents since the v^0 's are. The elements T_e for $e \in E^1$ clearly satisfy $T_e^* T_f = 0$ whenever $e \neq f$, because they consist of sums of elements with the same property. For $e \in E^1$ it is routine to check that $T_e^* T_e = Q_{r(e)}$.

If $v \in E^0$ is not a sink, then $d_s(v) < \infty$. If $d_s(v) = 0$, then we certainly have $Q_v = \sum_{\{s(e)=v\}} T_e T_e^*$. Otherwise, for $0 \leq j \leq d_s(v) - 1$ we have

$$(1) \quad v^j = \sum_{\{s(e)=v, d_s(e)=j\}} e e^* + f(v)^{j+1} v^{j+1} (f(v)^{j+1})^*,$$

and since we must have some edges with $s(e) = v$ and $d_s(e) = d_s(v)$ we have

$$(2) \quad v^{d_s(v)} = \sum_{\{s(e)=v, d_s(e)=d_s(v)\}} e e^*.$$

Using (1) recursively and (2) when $j = d_s(v) - 1$ we see that

$$Q_v = v^0 = \sum_{\{s(e)=v, d_s(e)=0\}} T_e T_e^* + \dots + \sum_{\{s(e)=v, d_s(e)=d_s(v)\}} T_e T_e^* = \sum_{\{s(e)=v\}} T_e T_e^*,$$

and this establishes our claim.

So by the Universal Homomorphism Property of $L_K(E)$ there is a K -algebra homomorphism

$$\pi : L_K(E) \rightarrow L_K(d_s(E))$$

which takes e to T_e and v to Q_v .

Let $\alpha^{d_s(E)}$ denote the K -action as automorphisms on $L_K(d_s(E))$ satisfying, for each $t \in K^* = K \setminus \{0\}$,

$$\alpha_t^{d_s(E)}(e) = te, \quad \alpha_t^{d_s(E)}(f(v)^i) = f(v)^i \text{ for } 1 \leq i \leq d_s(v), \quad \text{and} \quad \alpha_t^{d_s(E)}(v^i) = v^i \text{ for } 0 \leq i \leq d_s(v).$$

We now establish the injectivity of π for all fields K . It is straightforward to check that $\pi \circ \tau_t^E = \alpha_t^{d_s(E)} \circ \pi$ for each $t \in K^*$, where τ^E is the standard action of K on $L_K(E)$ given in [4, Definition 1.5]. Since K is infinite, it follows from [4, Theorem 1.8] that π is injective.

Now enumerate the vertices $v^0 \in d_s(E)^0$ by $\{v_i^0 \mid i \geq 1\}$, and define $\{e_n\}_{n \geq 1}$ by $e_n = \sum_{i=1}^n v_i^0$. Then $\{e_n\}_{n \geq 1}$ is an ascending chain of idempotents in $L(d_s(E))$. Let $A = \bigcup_{n \geq 1} e_n L(d_s(E)) e_n$ be the subalgebra of $L(d_s(E))$ with set of local units $\{e_n\}_{n \geq 1}$. The same argument as in [10, Theorem 4.2] shows that $A = \pi(L(E))$, which is isomorphic to $L(E)$ by the previously established injectivity of π . Also, the same argument as in [9, Lemma 2.4] shows that A is Morita equivalent to the ideal

$$I = \bigcup_{n \geq 1} L(d_s(E)) e_n L(d_s(E)) = \sum_{v \in E^0} L(d_s(E)) v^0 L(d_s(E)) = I(\{v^0 \mid v \in E^0\}).$$

Since $d_s(E)^0$ is the hereditary saturated closure of $\{v^0 \mid v \in E^0\}$, $I = L(d_s(E))$ by [9, Lemma 2.1], whence the result holds. \square

Let E be a row-finite graph, and let $v \in E^0$. We define the following Drinen source-vector:

- (1) For every $w \in E^0 \setminus \{v\}$, $d_s(w) = 0$, while $d_s(v) = 1$.
- (2) For every $f \in E^1 \setminus s^{-1}(v)$, $d_s(f) = 0$, while $d_s(e) = 1$ for any $e \in s^{-1}(v)$.

Then it is straightforward to see that

$$E_v = d_s(E).$$

In other words, the out-delay graph $d_s(E)$ related to this particular Drinen source-vector is precisely the expansion graph E_v . With this observation, Theorem 3.3 then immediately yields this more general version of Proposition 1.8

Corollary 3.4. *Let K be an infinite field. Let E be a row-finite graph, and $v \in E^0$. Then $L(E_v)$ is Morita equivalent to $L(E)$.*

We again borrow definitions from [10, Section 4].

Definitions 3.5. Let $E = (E^0, E^1, r, s)$ be a row-finite graph. A map $d_r : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ satisfying

- (1) if w is not a source then $d_r(w) = \sup \{d_r(e) \mid r(e) = w\}$, and
- (2) if $d_r(x) = \infty$ then x is either a source or receives infinitely many edges

is called a *Drinen range-vector*. We construct a new graph $d_r(E)$, called the *in-delayed graph of E* for the Drinen range-vector d_r , as follows:

$$\begin{aligned} d_r(E)^0 &= \{v_i \mid v \in E^0, 0 \leq i \leq d_r(v)\}, \\ d_r(E)^1 &= E^1 \cup \{f(v)_i \mid 1 \leq i \leq d_r(v)\}. \end{aligned}$$

For $e \in d_r(E)^1$ with $e \in E^1$ we define $r_{d_r(E)}(e) = r(e)_{d_r(e)}$ and $s_{d_r(E)}(e) = s(e)_0$. For $f \in d_r(E)^1$ of the form $f = f(v)_i$ we define $s_{d_r(E)}(f(v)_i) = v_i$ and $r_{d_r(E)}(f(v)_i) = v_{i-1}$.

Theorem 3.6. *Let K be an infinite field. Let E be a row-finite graph and let $d_r : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ be a Drinen range-vector. Then $L(d_r(E))$ is Morita equivalent to $L(E)$.*

Proof. The proof is essentially identical to the proof of [10, Theorem 4.5], using arguments analogous to those used in the proof of Theorem 3.3. \square

We now give an additional condition on the previously defined notion of an *in-split graph* (see the notation presented in Definitions 1.9).

Definition 3.7. Let E be a graph, let \mathcal{P} be a partition of E^1 , and let m be as described in Definitions 1.9. \mathcal{P} is called *proper* if for every vertex v which is a sink we have $m(v) = 0$ or $m(v) = 1$. (That is, \mathcal{P} is proper if \mathcal{P} does not in-split at a sink.)

To relate the Leavitt path algebra of a graph to the Leavitt path algebras of its in-splittings we use a variation of the method introduced in [13, Section 4.2]: If $E_r(\mathcal{P})$ is the in-split graph formed from E using the partition \mathcal{P} then we may define a Drinen range-vector $d_{r,\mathcal{P}} : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ by $d_{r,\mathcal{P}}(v) = m(v) - 1$ if $m(v) \geq 1$ and $d_{r,\mathcal{P}}(v) = 0$ otherwise. For $e \in \mathcal{E}_i^{r(e)}$ we put $d_{r,\mathcal{P}}(e) = i - 1$. Hence, if v receives $n \geq 2$ edges then we create an in-delayed graph in which v is given delay of size $m(v) - 1$ and all edges with range v are given a delay one less than their label in the partition of $r^{-1}(v)$. If v is a source or receives only one edge then there is no delay attached to v .

Theorem 3.8. *Let K be an infinite field. Let E be a row-finite graph, \mathcal{P} a partition of E^1 , $E_r(\mathcal{P})$ the in-split graph formed from E using \mathcal{P} and $d_{r,\mathcal{P}} : E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ the Drinen range-vector defined as above. Then $L(E_r(\mathcal{P})) \cong L(d_{r,\mathcal{P}}(E))$ if and only if \mathcal{P} is proper.*

Proof. The proof is analogous to the proof of [10, Theorem 5.3], using the arguments of the proof of Theorem 3.3. \square

Applying Theorem 3.8 and Theorem 3.6, we get the following analog to [10, Corollary 5.4], which in turn gives a generalization of Proposition 1.11. (Note that the hypotheses of Proposition 1.11 include that E contains no sinks, so that every partition of E^1 is vacuously proper.)

Corollary 3.9. *Let K be an infinite field. Let E be a row-finite graph, \mathcal{P} a partition of E^1 and $E_r(\mathcal{P})$ the in-split graph formed from E using \mathcal{P} . Then $L(E_r(\mathcal{P}))$ is Morita equivalent to $L(E)$ if and only if \mathcal{P} is proper.*

Having now finished the work of extending Propositions 1.4, 1.8, and 1.11 to wider classes of graphs, we note here for completeness that, by [4, Theorem 2.8], Proposition 1.14 extends verbatim to row-finite graphs as well.

We conclude this article by analyzing the relationship between the Leavitt path algebra $L(E)$ of a graph E and the Leavitt path algebra $L(E^t)$ of its transpose graph E^t . An easy example shows that in general these two algebras need not be Morita equivalent. For instance, if E is the graph

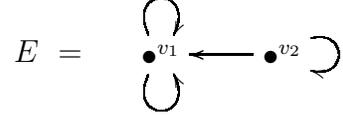
$$E = \bullet \longleftarrow \bullet \longrightarrow \bullet$$

then

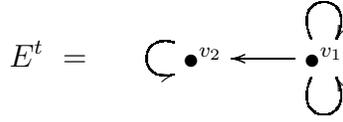
$$E^t = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

By [3, Proposition 3.5] we get that $L_K(E) \cong M_2(K) \oplus M_2(K)$, while $L_K(E^t) \cong M_3(K)$; these two algebras are not Morita equivalent.

Indeed, we can find a finite graph E having neither sinks nor sources for which $L(E)$ and $L(E^t)$ are not Morita equivalent. Specifically, consider the graph E



whose transpose graph E^t is



Then $v_1 \in E^0$ generates the unique proper graded two-sided ideal of $L_K(E)$, and the quotient ring $L_K(E)/\langle v_1 \rangle$ is isomorphic to $K[x, x^{-1}]$. Thus, $L(E)$ has no purely infinite simple unital quotients. Since E contains loops, [8, Theorem 2.8] implies that the stable rank $\text{sr}(L(E))$ equals 2. On the other hand, $v_2 \in (E^t)^0$ generates a proper graded two-sided ideal in $L(E^t)$, whose quotient ring $L(E^t)/\langle v_2 \rangle$ is isomorphic to the Leavitt algebra $L_K(1, 2)$. Thus $\text{sr}(L(E^t)) = \infty$ by [8, Theorem 2.8]. But the stable rank is a Morita invariant for unital Leavitt path algebras of row-finite graphs [8, Remark 3.4(1)], so that $L(E)$ and $L(E^t)$ cannot be Morita equivalent.

However, in contrast to the previous two examples, we get the following consequence of Theorem 2.7.

Proposition 3.10. *If E is a finite graph without sources such that $L(E)$ is a purely infinite simple Leavitt path algebra, then $L(E)$ and $L(E^t)$ are Morita equivalent.*

Proof. There is an isomorphism $\text{coker}(I - A_E^t) \cong \text{coker}(I - A_{E^t}^t) = \text{coker}(I - A_E)$, since the Smith normal forms of $I - A_E^t$ and $I - A_E$ are equal. Furthermore, cofactor expansions clearly yield $\det(I - A_E^t) = \det(I - A_{E^t}^t) = \det(I - A_E)$. Thus we have $\mathcal{F}_{\det}(E) \equiv \mathcal{F}_{\det}(E^t)$. By Lemma 1.17 we have that E is irreducible, essential, and nontrivial. But these three conditions on a graph are easily seen to pass to the transpose graph E^t , so that (again by Lemma 1.17) we have that $L(E^t)$ is purely infinite simple. Thus Theorem 1.25 applies to yield the result. \square

The result of Proposition 3.10 does not extend to isomorphisms, as the following example demonstrates.

Example 3.11. Consider the graph E whose incidence matrix is

$$A_E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then E is a graph with no sources, for which $L(E)$ is purely infinite simple. It is not hard to show (see e.g. [24, pp. 67-68]) that $K_0(L(E)) = \mathbb{Z}_2$, and $[1_E] = [1]$ in \mathbb{Z}_2 . On the other hand, a similarly easy computation yields that $K_0(L(E^t)) = \mathbb{Z}_2$ as well, but $[1_{E^t}] = [0]$ in \mathbb{Z}_2 .

Since, as noted above, an isomorphism between Leavitt path algebras yields an equivalence of the corresponding unitary Franks pairs, we conclude that $L(E) \not\cong L(E^t)$.

ACKNOWLEDGMENTS

The authors thank the referee for a very thorough and careful reading of the original version of this article. Part of this work was done during a visit of the third author to the Department of Mathematics of the University of Colorado at Colorado Springs. The third author thanks both the Department of Mathematics of UCCS and the Department of Mathematics of The Colorado College for their kind hospitality.

REFERENCES

- [1] G. ABRAMS, G. ARANDA PINO, The Leavitt path algebra of a graph, *J. Algebra* **293** (2005), 319–334.
- [2] G. ABRAMS, G. ARANDA PINO, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra*, **207** (2006), 553–563.
- [3] G. ABRAMS, G. ARANDA PINO, M. SILES MOLINA, Finite dimensional Leavitt path algebras, *J. Pure Appl. Algebra* **209**(3) (2007), 753–762.
- [4] G. ABRAMS, P. N. ÁNH, A. LOULY, E. PARDO, The classification question for Leavitt path algebras, *J. Algebra* **320** (2008), 1983–2026.
- [5] G. ABRAMS, C. SMITH, Explicit isomorphisms between purely infinite simple Leavitt path algebras, in preparation.
- [6] P. ARA, K. GOODEARL, E. PARDO, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002), 69–100.
- [7] P. ARA, M.A. MORENO, E. PARDO, Nonstable K-Theory for graph algebras, *Algebr. Represent. Theory* **10** (2007), 157–178.
- [8] P. ARA, E. PARDO, Stable rank of Leavitt path algebras, *Proc. Amer. Math. Soc.* **136**(7) (2008), 2375–2386.
- [9] G. ARANDA PINO, E. PARDO, M. SILES MOLINA, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305** (2006), 912–936.
- [10] T. BATES, D. PASK, Flow equivalence of graph algebras, *Ergodic Theory Dynam. Systems* **24** (2004), 367–382.
- [11] R. BOWEN, J. FRANKS, Homology for zero-dimensional nonwandering sets, *Ann. of Math.* **106** (1977), 73–92.
- [12] J. CUNTZ, W. KRIEGER, A class of C^* -algebras and topological Markov chains, *Invent. Math.* **56** (1980), 251–268.
- [13] D. DRINEN, Flow equivalence and graph groupoid isomorphism, *Dartmouth College* (2001), preprint.
- [14] J. FRANKS, Flow equivalence of subshifts of finite type, *Ergodic Theory Dynam. Systems* **4** (1984), 53–66.
- [15] J.L. GARCÍA, J.J. SIMÓN, Morita equivalence for idempotent rings, *J. Pure Appl. Algebra* **76**(1) (1991), 39–56.
- [16] E. KIRCHBERG, The classification of purely infinite C^* -algebras using Kasparov theory, preprint.
- [17] D. HUANG, Flow equivalence of reducible shifts of finite type, *Ergodic Theory Dynam. Systems* **14** (1994), 695–720.
- [18] D. HUANG, Automorphisms of Bowen-Franks groups of shifts of finite type, *Ergodic Theory Dynam. Systems* **21** (2001), 1113–1137.
- [19] D. LIND, B. MARCUS, “An Introduction to Symbolic Dynamics and Coding”. Cambridge Univ. Press, Cambridge, 1995, Reprinted 1999 (with corrections). ISBN 0-521-55900-6.
- [20] W. PARRY, D. SULLIVAN, A topological invariant for flows on one-dimensional spaces, *Topology* **14** (1975), 297–299.
- [21] N.C. PHILLIPS, A classification theorem for nuclear purely infinite simple C^* -algebras, *Doc. Math.* **5** (2000), 49–114.

- [22] I. RAEBURN, “Graph algebras”. CBMS Regional Conference Series in Mathematics **103**, American Mathematical Society, Providence, 2005. ISBN 0-8218-3660-9
- [23] M. RØRDAM, Classification of Cuntz-Krieger algebras, *K-Theory* **9** (1995), 31–58.
- [24] M. TOMFORDE, Structure of graph C*-algebras and generalizations, in “Graph algebras: bridging the gap between analysis and algebra”, G. Aranda Pino, F. Perera, M. Siles Molina (eds.), Universidad de Málaga Press, 2007. ISBN 978-84-9747-177-0.
- [25] R.F. WILLIAMS, Classification of subshifts of finite type, *Ann. of Math.* **98**(2) (1973), 120–153.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, COLORADO SPRINGS CO 80918 U.S.A.
E-mail address: `abrams@math.uccs.edu`, `cdsmith@gmail.com`

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CÁDIZ, CAMPUS DE PUERTO REAL, 11510 PUERTO REAL (CÁDIZ), SPAIN.
E-mail address: `loully.adel@uca.es`, `enrique.pardo@uca.es`