

REPRESENTING FINITELY GENERATED REFINEMENT MONOIDS AS GRAPH MONOIDS

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ABSTRACT. Graph monoids arise naturally in the study of non-stable K-theory of graph C^* -algebras and Leavitt path algebras. They play also an important role in the current approaches to the realization problem for von Neumann regular rings. In this paper, we characterize when a finitely generated conical refinement monoid can be represented as a graph monoid. The characterization is expressed in terms of the behavior of the structural maps of the associated I -system at the free primes of the monoid.

1. INTRODUCTION

The class of commutative monoids satisfying the Riesz refinement property –refinement monoids for short– has been largely studied over the last decades in connection with various problems, as non-stable K-Theory of rings and C^* -algebras (see e.g. [5, 8, 9, 16, 21]), classification of Boolean algebras (see e.g. [17], [22]), or its own structure theory (see e.g. [13, 14, 25]).

An important invariant in non-stable K-theory is the commutative monoid $\mathcal{V}(R)$ associated to any ring R , consisting of the isomorphism classes of finitely generated projective (left, say) R -modules, with the operation induced from direct sum. If R is a (von Neumann) regular ring or a C^* -algebra with real rank zero (more generally, an exchange ring), then $\mathcal{V}(R)$ is a refinement monoid (e.g., [8, Corollary 1.3, Theorem 7.3]).

The *realization problem* asks which refinement monoids appear as a $\mathcal{V}(R)$ for R in one of the above-mentioned classes. Wehrung [26] constructed a conical refinement monoid of cardinality \aleph_2 which is not isomorphic to $\mathcal{V}(R)$ for any regular ring R , but it is an important open problem, appearing for the first time in [15], to determine whether every countable conical refinement monoid can be realized as $\mathcal{V}(R)$ for some regular R . See [4] for a survey on this problem, and [7] for some recent progress on the problem, with connections with the Atiyah Problem.

An interesting situation in which the answer to the realization problem is affirmative is the following:

Theorem 1.1 ([6, Theorem 4.2, Theorem 4.4]). *Let E be a row-finite graph, let $M(E)$ be its graph monoid, and let K be any field. Then there exists a (non necessarily unital) von Neumann regular K -algebra $Q_K(E)$ such that $\mathcal{V}(Q_K(E)) \cong M(E)$.*

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Thus, an intermediate step that could be helpful to give an answer to the realization problem is to characterize which conical refinement monoids are representable as graph monoids. The first author, Perera and Wehrung gave such a characterization in the concrete case of finitely generated antisymmetric refinement monoids [11, Theorem 5.1]. These monoids are a particular case of primely generated refinement monoids (see e.g. [13]). Recall that an element p in a monoid M is a *prime element* if p is not invertible in M , and, whenever $p \leq a + b$ for $a, b \in M$, then either $p \leq a$ or $p \leq b$ (where $x \leq y$ means that $y = x + z$ for some $z \in M$). The monoid M is *primely generated* if every non-invertible element of M can be written as a sum of prime elements. Primely generated refinement monoids enjoy important cancellation properties, such as separative cancellation and unperforation, as shown by Brookfield in [13, Theorem 4.5, Corollary 5.11(5)]. Moreover, it was shown by Brookfield that any finitely generated refinement monoid is automatically primely generated [13, Corollary 6.8].

In [10], the authors of the present paper showed that any primely generated refinement monoid can be represented, up to isomorphism, as the monoid associated to an I -system (a sort of semilattice of cancellative semigroups defined over a suitable poset I). This result generalizes the representation of two well-known classes of monoids:

- (1) The class of *primitive monoids*, i.e. antisymmetric primely generated refinement monoids, see [22]. These monoids are described by means of a set I endowed with an antisymmetric transitive relation.
- (2) The class of primely generated *regular* conical refinement monoids. These monoids were characterized by Dobbertin in [14] in terms of partial orders of abelian groups.

In the present paper, we will benefit of the picture developed in [10] to state a characterization of monoids representable as graph monoids, in the case of finitely generated conical refinement monoids. The condition, that relies on the behavior of free primes in the representation by I -systems, generalizes [11, Theorem 5.1].

A main device used to obtain realization results for refinement monoids by regular rings or C^* -algebras of real rank zero has been the consideration of algebras associated to graphs. Of particular importance has been the use of graph C^* -algebras (see e.g. [18], [19]) and of Leavitt path algebras (see e.g. [1], [2], [9]). Indeed, the von Neumann regular algebra $Q_K(E)$ appearing in Theorem 1.1 is a specific universal localization of the Leavitt path algebra $L_K(E)$ associated to the row-finite graph E . Some other algebras associated to graphs are also important in further constructions of von Neumann regular rings, as the first author showed in [5]. In that paper, given any finite poset \mathbb{P} , a von Neumann regular algebra $Q_K(\mathbb{P})$ is constructed so that $\mathcal{V}(Q_K(\mathbb{P})) \cong M(\mathbb{P})$, where $M(\mathbb{P})$ is the primitive monoid with associated poset of primes \mathbb{P} and with all primes being free. In the present paper, Leavitt path algebras play an instrumental role, being essential in determining the condition which characterizes finitely generated graph monoids (see Proposition 5.8). It is interesting to observe that, as shown in that Proposition, the obstruction to realize an arbitrary finitely generated conical refinement monoid as a graph monoid is of K -theoretical nature. The solution of this K -theoretical problem was one of the keys to obtain the main result of [5] (see [5, Theorem 3.2]), and will surely play a vital role in the forthcoming approaches to the resolution of the realization problem for finitely generated refinement monoids.

The paper is organized as follows. In Section 2, we recall all the definitions and results that will be necessary to follow the contents of the subsequent sections. In Section 3, we study

the special case of finitely generated *regular* refinement monoids, and we prove that every finitely generated conical regular refinement monoid can be represented as a graph monoid (Theorem 3.6). In the short Section 4, we use the techniques developed in Section 3 to offer an easy presentation of the monoid $M = \mathbb{Z}^+ \cup \{\infty\}$ as a graph monoid. This was done in a somewhat more involved way in [11, Example 6.5]. In Section 5, we prove the main result of the paper, obtaining the characterization of the finitely generated conical refinement monoids which are graph monoids. The result is stated in terms of the theory of I -systems established in [10], and concerns the behavior of the maps associated to the I -system at the free primes (see Theorem 5.6). Finally we use our main result to recover the characterization of finitely generated primitive graph monoids obtained in [11].

2. PRELIMINARIES

In this section, we will recall the definitions and results necessary to follow the contents of the paper in a self-contained way. We divide this section in four parts.

2.1. Basics on commutative monoids. All semigroups and monoids considered in this paper are commutative. We will denote by \mathbb{N} the semigroup of positive integers, and by \mathbb{Z}^+ the monoid of non-negative integers.

Given a commutative monoid M , we set $M^* := M \setminus \{0\}$. We say that M is *conical* if M^* is a semigroup, that is, if, for all x, y in M , $x + y = 0$ only when $x = y = 0$.

Given a monoid M , the antisymmetrization \overline{M} of M is the quotient monoid of M by the congruence given by $x \equiv y$ if and only if $x \leq y$ and $y \leq x$ (see [13, Notation 5.1]). We will denote the class of an element x of M in \overline{M} by \overline{x} .

We say that a monoid M is *separative* provided $2x = 2y = x + y$ always implies $x = y$; there are a number of equivalent formulations of this property, see e.g. [8, Lemma 2.1]. We say that M is a *refinement monoid* if, for all a, b, c, d in M such that $a + b = c + d$, there exist w, x, y, z in M such that $a = w + x$, $b = y + z$, $c = w + y$ and $d = x + z$. It will often be convenient to present this situation in the form of a diagram, as follows:

$$\begin{array}{|c|c|c|} \hline & c & d \\ \hline a & w & x \\ \hline b & y & z \\ \hline \end{array}.$$

A basic example of refinement monoid is the monoid $M(E)$ associated to a countable row-finite graph E [9, Proposition 4.4].

If $x, y \in M$, we write $x \leq y$ if there exists $z \in M$ such that $x + z = y$. Note that \leq is a translation-invariant pre-order on M , called the *algebraic pre-order* of M . All inequalities in commutative monoids will be with respect to this pre-order. An element p in a monoid M is a *prime element* if p is not invertible in M , and, whenever $p \leq a + b$ for $a, b \in M$, then either $p \leq a$ or $p \leq b$. The monoid M is *primely generated* if every non-invertible element of M can be written as a sum of prime elements.

An element $x \in M$ is *regular* if $2x \leq x$. An element $x \in M$ is an *idempotent* if $2x = x$. An element $x \in M$ is *free* if $nx \leq mx$ implies $n \leq m$. Any element of a separative monoid is either free or regular. In particular, this is the case for any primely generated refinement monoid, by [13, Theorem 4.5].

A subset S of a monoid M is called an *order-ideal* if S is a subset of M containing 0, closed under taking sums and summands within M . An order-ideal can also be described as a submonoid I of M , which is hereditary with respect to the canonical pre-order \leq on M : $x \leq y$ and $y \in I$ imply $x \in I$. A non-trivial monoid is said to be *simple* if it has no non-trivial order-ideals.

If $(S_k)_{k \in \Lambda}$ is a family of (commutative) semigroups, $\bigoplus_{k \in \Lambda} S_k$ (resp. $\prod_{k \in \Lambda} S_k$) stands for the coproduct (resp. the product) of the semigroups S_k , $k \in \Lambda$, in the category of commutative semigroups. If the semigroups S_k are subsemigroups of a semigroup S , we will denote by $\sum_{k \in \Lambda} S_k$ the subsemigroup of S generated by $\bigcup_{k \in \Lambda} S_k$. Note that $\sum_{k \in \Lambda} S_k$ is the image of the canonical map $\bigoplus_{k \in \Lambda} S_k \rightarrow S$. We will use the notation $\langle X \rangle$ to denote the semigroup generated by a subset X of a semigroup S .

Given a semigroup M , we will denote by $G(M)$ the Grothendieck group of M . There exists a semigroup homomorphism $\psi_M: M \rightarrow G(M)$ such that for any semigroup homomorphism $\eta: M \rightarrow H$ to a group H there is a unique group homomorphism $\tilde{\eta}: G(M) \rightarrow H$ such that $\tilde{\eta} \circ \psi_M = \eta$. $G(M)$ is abelian and it is generated as a group by $\psi(M)$. If M is already a group then $G(M) = M$. If M is a semigroup of the form $\mathbb{N} \times G$, where G is an abelian group, then $G(M) = \mathbb{Z} \times G$. In this case, we will view G as a subgroup of $\mathbb{Z} \times G$ by means of the identification $g \leftrightarrow (0, g)$.

Let M be a conical commutative monoid, and let $x \in M$ be any element. The archimedean component of M generated by x is the subsemigroup

$$G_M[x] := \{a \in M : a \leq nx \text{ and } x \leq ma \text{ for some } n, m \in \mathbb{N}\}.$$

For any $x \in M$, $G_M[x]$ is a simple semigroup. If M is separative, then $G_M[x]$ is a cancellative semigroup; if moreover x is a regular element, then $G_M[x]$ is an abelian group.

2.2. Primely generated refinement monoids. The structure of primely generated refinement monoids has been recently described in [10]. We recall here some basic facts.

Given a poset (I, \leq) , we say that a subset A of I is a *lower set* if $x \leq y$ in I and $y \in A$ implies $x \in A$. For any $i \in I$, we will denote by $I \downarrow i = \{x \in I : x \leq i\}$ the lower subset generated by i . We will write $x < y$ if $x \leq y$ and $x \neq y$.

The following definition [10, Definition 1.1] is crucial for this work:

Definition 2.1 ([10, Definition 1.1]). Let $I = (I, \leq)$ be a poset. An *I-system*

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$$

is given by the following data:

- (a) A partition $I = I_{free} \sqcup I_{reg}$ (we admit one of the two sets I_{free} or I_{reg} to be empty).
- (b) A family $\{G_i\}_{i \in I}$ of abelian groups. We adopt the following notation:
 - (1) For $i \in I_{reg}$, set $M_i = G_i$, and $\widehat{G}_i = G_i = M_i$.
 - (2) For $i \in I_{free}$, set $M_i = \mathbb{N} \times G_i$, and $\widehat{G}_i = \mathbb{Z} \times G_i$.

Observe that, in any case, \widehat{G}_i is the Grothendieck group of M_i .

- (c) A family of semigroup homomorphisms $\varphi_{ji}: M_i \rightarrow G_j$ for all $i < j$, to whom we associate, for all $i < j$, the unique extension $\widehat{\varphi}_{ji}: \widehat{G}_i \rightarrow G_j$ of φ_{ji} to a group homomorphism from the Grothendieck group of M_i to G_j (we look at these maps as maps from \widehat{G}_i to \widehat{G}_j). We require that the family $\{\varphi_{ji}\}$ satisfies the following conditions:

(1) The assignment

$$\left\{ \begin{array}{l} i \mapsto \widehat{G}_i \\ (i < j) \mapsto \widehat{\varphi}_{ji} \end{array} \right\}$$

defines a functor from the category I to the category of abelian groups (where we set $\widehat{\varphi}_{ii} = \text{id}_{\widehat{G}_i}$ for all $i \in I$).

(2) For each $i \in I_{free}$ we have that the map

$$\bigoplus_{k < i} \varphi_{ik} : \bigoplus_{k < i} M_k \rightarrow G_i$$

is surjective.

We say that an I -system $\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$ is *finitely generated* in case I is a finite poset and all the groups G_i are finitely generated.

To every I -system \mathcal{J} one can associate a primely generated conical refinement monoid $M(\mathcal{J})$, and conversely to any primely generated conical refinement monoid M , we can associate an I -system \mathcal{J} such that $M \cong M(\mathcal{J})$, see Sections 1 and 2 of [10] respectively.

Given a poset I , and an I -system \mathcal{J} , we construct a semilattice of groups based on the partial order of groups (I, \leq, \widehat{G}_i) , by following the model introduced in [14]. Let $A(I)$ be the semilattice (under set-theoretic union) of all the finitely generated lower subsets of I . These are precisely the lower subsets a of I such that the set $\text{Max}(a)$ of maximal elements of a is finite and each element of a is under some of the maximal ones. In case I is finite, and since the intersection of lower subsets of I is again a lower subset, $A(I)$ is a lattice. For any $a \in A(I)$, we define $\widehat{H}_a = \bigoplus_{i \in a} \widehat{G}_i$, and we define $f_a^b (a \subseteq b)$ to be the canonical embedding of \widehat{H}_a into \widehat{H}_b . Given $a \in A(I)$, $i \in a$ and $u \in \widehat{G}_i$, we define $\chi(a, i, u) \in \widehat{H}_a$ by

$$\chi(a, i, u)_j = \begin{cases} u & \text{if } j = i, \\ 0_j & \text{if } j \neq i. \end{cases}$$

Let U_a be the subgroup of \widehat{H}_a generated by the set

$$\{\chi(a, i, u) - \chi(a, j, \widehat{\varphi}_{ji}(u)) : i < j \in \text{Max}(a), u \in \widehat{G}_i\}.$$

Now, for any $a \in A(I)$, set $\widetilde{G}_a = \widehat{H}_a / U_a$, and let $\Phi_a : \widehat{H}_a \rightarrow \widetilde{G}_a$ be the natural onto map. Then, for any $a \subseteq b \in A(I)$ we have that $f_a^b(U_a) \subseteq U_b$, so that there exists a unique homomorphism $\widetilde{f}_a^b : \widetilde{G}_a \rightarrow \widetilde{G}_b$ which makes the diagram

$$\begin{array}{ccc} \widehat{H}_a & \xrightarrow{f_a^b} & \widehat{H}_b \\ \Phi_a \downarrow & & \downarrow \Phi_b \\ \widetilde{G}_a & \xrightarrow{\widetilde{f}_a^b} & \widetilde{G}_b \end{array}$$

commutative. Hence, $(A(I), (\widetilde{G}_a)_{a \in A(I)}, \widetilde{f}_a^b (a \subseteq b))$ is a semilattice of groups. Thus, the set

$$\widetilde{M}(\mathcal{J}) := \bigsqcup_{a \in A(I)} \widetilde{G}_a,$$

endowed with the operation $x + y := \tilde{f}_a^{a \cup b}(x) + \tilde{f}_b^{a \cup b}(y)$ for any $a, b \in A(I)$ and any $x \in \tilde{G}_a, y \in \tilde{G}_b$, is a primely generated regular refinement monoid by [14, Proposition 1]. Note that $\widehat{H}_\emptyset = \tilde{G}_\emptyset = \{0\}$. We refer the reader to [14] for further details on this construction.

Let H_a be the subsemigroup of \widehat{H}_a defined by

$$H_a = \left\{ (z_i)_{i \in a} \in \widehat{H}_a : z_i \in \begin{cases} \mathbb{N} \times G_i & \text{for } i \in \text{Max}(a)_{\text{free}} \\ \{(0, 0_i)\} \cup (\mathbb{N} \times G_i) & \text{for } i \in a_{\text{free}} \setminus \text{Max}(a)_{\text{free}} \end{cases} \right\}.$$

In what follows, whenever $i < j \in I$ with j a free element, $x = (n, g) \in \mathbb{N} \times G_j$ and $y \in M_i$, we will see $x + \varphi_{ji}(y)$ as the element $(n, g + \varphi_{ji}(y)) \in \mathbb{N} \times G_j$. This is coherent with our identification of G_j as the subgroup $\{0\} \times G_j$ of $\widehat{G}_j = \mathbb{Z} \times G_j$.

By [10, Lemma 1.3], we can define a semilattice of semigroups

$$(A(I), (H_a)_{a \in A(I)}, f_a^b(a \subset b)).$$

Now, we construct a monoid associated to it. For this, consider the congruence \sim defined on H_a , for $a \in A(I)$, given by

$$x \sim y \iff x - y \in U_a.$$

Lemma 2.2 ([10, Lemma 1.4]). *Let $a \in A(I)$. The congruence \sim on H_a agrees with the congruence \equiv , generated by the pairs $(x + \chi(a, i, \alpha), x + \chi(a, j, \varphi_{ji}(\alpha)))$, for $x \in H_a$, $i < j \in \text{Max}(a)$ and $\alpha \in M_i$.*

Corollary 2.3 ([10, Corollary 1.5]). *For every $a \in A(I)$, $M_a := H_a / \sim = H_a / \equiv$ is a submonoid of \tilde{G}_a .*

Definition 2.4. Given an I -system $\mathcal{J} = (I, \leq, G_i, \varphi_{ji}(i < j))$, we denote by $M(\mathcal{J})$ the set $\bigsqcup_{a \in A(I)} M_a$. By [10, Lemma 1.3] and Corollary 2.3, $M(\mathcal{J})$ is a submonoid of $\widetilde{M}(\mathcal{J})$.

Observe that Lemma 2.2 gives:

Corollary 2.5 ([10, Corollary 1.6]). *$M(\mathcal{J})$ is the monoid generated by M_i , $i \in I$, with respect to the defining relations*

$$x + y = x + \varphi_{ji}(y), \quad i < j, x \in M_j, y \in M_i.$$

Notation. Assume \mathcal{J} is an I -system. For $i \in I$ and $x \in M_i$ we will denote by $\chi_i(x)$ the element $[\chi(I \downarrow i, i, x)] \in M(\mathcal{J})$. Note that, by Corollary 2.5, $M(\mathcal{J})$ is the monoid generated by $\chi_i(x)$, $i \in I$, $x \in M_i$, with the defining relations

$$\chi_j(x) + \chi_i(y) = \chi_j(x + \varphi_{ji}(y)), \quad i < j, x \in M_j, y \in M_i.$$

Lemma 2.6. *For each $a \in A(I)$, the Grothendieck group of M_a is the group \tilde{G}_a .*

Proof. Since M_a is a submonoid of \tilde{G}_a , we only have to show that every element of \tilde{G}_a can be written as a difference of two elements of M_a . For this, it is enough to show that for each $i \in a$ and each $x \in \widehat{G}_i$, the element $\chi(a, i, x)$ of \widehat{H}_a can be written as a difference of two elements in H_a . If $i \in a_{\text{reg}}$, then we can write

$$\chi(a, i, x) = \left(\chi(a, i, x) + \sum_{j \in a_{\text{free}}} \chi(a, j, (1, e_j)) \right) - \left(\sum_{j \in a_{\text{free}}} \chi(a, j, (1, e_j)) \right) \in H_a - H_a.$$

If $i \in a_{\text{free}}$, then select $n \in \mathbb{N}$ such that $(n, e_i) + x \in M_i$, and write

$$\chi(a, i, x) = \left(\chi(a, i, (n, e_i) + x) + \sum_{j \in a_{\text{free}}} \chi(a, j, (1, e_j)) \right) - \left(\chi(a, i, (n, e_i)) + \sum_{j \in a_{\text{free}}} \chi(a, j, (1, e_j)) \right) \in H_a - H_a.$$

□

Lemma 2.7. *Let $i \in I$ and set $a = I \downarrow i$. Then $M_a = M_i$ and $\tilde{G}_a = \hat{G}_i$.*

Proof. We have a surjective monoid homomorphism $\phi: M_i \rightarrow M_a$ sending $x \in M_i$ to $\chi_i(x)$. Define a monoid homomorphism $\psi: H_a \rightarrow M_i$ by

$$\psi\left(\sum_{j \leq i} \chi(x_j, j, a)\right) = \sum_{j \leq i} \varphi_{ij}(x_j),$$

where $x_i \in M_i$, $x_j \in M_j = G_j$ for $j \in a_{\text{reg}} \setminus \{i\}$ and $x_j \in \{(0, e_j)\} \cup M_j$ for $j \in a_{\text{free}} \setminus \{i\}$. (We are setting here $\varphi_{ii} = \text{Id}_{M_i}$.) Then, ψ clearly factors through the congruence \equiv described in Lemma 2.2, and so induces a monoid homomorphism $\bar{\psi}: M_a \rightarrow M_i$, which is clearly the inverse map of ϕ . A similar proof gives that $\tilde{G}_a = \hat{G}_i$. □

We will denote by $\mathcal{L}(M)$ the lattice of order-ideals of a monoid M and by $\mathcal{L}(I)$ the lattice of lower subsets of a poset I .

Proposition 2.8 ([10, Proposition 1.9]). *Let \mathcal{J} be an I -system. Then there is a lattice isomorphism*

$$\mathcal{L}(I) \cong \mathcal{L}(M(\mathcal{J})).$$

More precisely, given a lower subset J of I , the restricted J -system is

$$\mathcal{J}_J := (J, \leq, (G_i)_{i \in J}, \varphi_{ji}, (i < j \in J)),$$

and the map $J \mapsto M(\mathcal{J}_J)$ defines a lattice isomorphism from $\mathcal{L}(I)$ onto $\mathcal{L}(M(\mathcal{J}))$.

Lemma 2.9. *Let I be a poset and let \mathcal{J} be an I -system. Let J be a finitely generated lower subset of I and let \mathcal{J}_J be the restricted J -system. Then the Grothendieck group of the associated order-ideal $M(\mathcal{J}_J)$ of $M(\mathcal{J})$ is precisely \tilde{G}_J .*

Proof. We have $M(\mathcal{J}_J) = \bigsqcup_{a \in A(J)} M_a$. Let x be an element in M_J , and define a semigroup homomorphism

$$\tau: M(\mathcal{J}_J) \longrightarrow G(M_J)$$

by $\tau(z) = (x + f_a^J(z)) - x$ for $z \in M_a$. Then it is easily seen that τ is the canonical map from $M(\mathcal{J}_J)$ to its Grothendieck group, that is, that for every semigroup homomorphism $\lambda: M(\mathcal{J}_J) \rightarrow G$, where G is a group, there is a unique group homomorphism $\tilde{\lambda}: G(M_J) \rightarrow G$ such that $\lambda = \tilde{\lambda} \circ \tau$. Indeed, given λ as above, $\tilde{\lambda}$ is just the canonical map from the Grothendieck group $G(M_J)$ of M_J to G induced by the restriction of λ to M_J .

Now, we can apply Lemma 2.6 to derive the result. □

Recall that given a poset I , and an element $i \in I$, the *lower cover* of i in I is the set

$$L(I, i) = \{j \in I : j < i \text{ and } [j, i] = \{j, i\}\}.$$

We now consider the special case of a finitely generated regular conical refinement monoid M . Our goal will be to realize M as the graph monoid $M(E)$ for some row-finite directed graph E . In this case the results from [10] reduce to Dobbertin's results [14].

For the rest of this subsection, let M be a finitely generated regular conical refinement monoid. The antisymmetrization \overline{M} of M is then an antisymmetric regular refinement monoid (cf. [13, Theorem 5.2]). The set \mathbb{P} is defined by taking a representative p for each prime $\overline{p} \in \mathbb{P}(\overline{M})$. For each $p \in \mathbb{P}$, the archimedean component M_p of p is a finitely generated abelian group, denoted by G_p . Since we have $\overline{p} = \overline{x}$ for all $x \in G_p$, we may take $p = e_p$, the neutral element of the group G_p , as a canonical representative of \overline{p} , so that

$$\mathbb{P} = \{e \in M : e = 2e \text{ and } e \text{ is prime}\}.$$

With the order induced from M , \mathbb{P} is a finite poset. Note that $e \leq f$ in \mathbb{P} if and only if $f = e + f$. For $e \in \mathbb{P}$, the associated group is $G_e = \{x \in M : e \leq x \leq e\}$, which is precisely the archimedean component $G_M[e]$ of e . Finally if $e \leq f$ in \mathbb{P} , then the induced map $\varphi_{fe}: G_e \rightarrow G_f$ is defined by $\varphi_{f,e}(x) = x + f$ for $x \in G_e$. This structure defines the \mathbb{P} -system \mathcal{J}_M associated to M .

2.3. Graph monoids. Now, we will recall the basic elements about graphs and its monoids that are necessary in the sequel.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s: E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

A vertex $v \in E^0$ is a sink if $s^{-1}(v) = \emptyset$. A graph E is *finite* if E^0 and E^1 are finite sets. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. We will only deal with row-finite graphs in this paper, so **we make the convention that all graphs appearing henceforth are row-finite**; we will make this assumption explicit in the statements of the main results. A *path* μ in a graph E is a sequence of edges $\mu = (\mu_1, \dots, \mu_n)$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the *source* of μ and $r(\mu) := r(\mu_n)$ is the *range* of μ . If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*. We say that a cycle $\mu = (\mu_1, \dots, \mu_n)$ has an *exit* if there is a vertex $v = s(\mu_i)$ and an edge $f \in s^{-1}(v) \setminus \{\mu_i\}$. If $v = s(\mu) = r(\mu)$ and $s(\mu_i) \neq v$ for every $i > 1$, then μ is called a *closed simple path based at v*. For a path μ we denote by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) \mid i = 1, \dots, n\}$. For $n \geq 2$ we define E^n to be the set of paths of length n , and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths.

We define a relation \geq on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A set is *saturated* if every vertex which feeds into H and only into H is again in H , that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by \mathcal{H} (or by \mathcal{H}_E when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0 .

The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the *tree* of v , and it is the smallest hereditary subset of E^0 containing v . We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X . It is shown in [9] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$$\begin{aligned} \Lambda_0(X) &= T(X), \text{ and} \\ \Lambda_n(X) &= \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X), \text{ for } n \geq 1. \end{aligned}$$

We recall here some graph-theoretic constructions which will be of interest. For a hereditary subset of E^0 , the *quotient graph* E/H is defined as

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and the *restriction graph* is

$$E_H = (H, \{e \in E^1 \mid s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Definition 2.10. Given a graph E :

- (1) We say that E is transitive if every two vertices of E^0 are connected through a finite path.
- (2) We say that a nonempty subset S of E^0 is strongly connected if the graph

$$(S, s^{-1}(S) \cap r^{-1}(S), s|_S, r|_S)$$

is transitive. In particular, if F is a subgraph of E , we say that F is strongly connected if so does the subset F^0 of E^0 .

For a row-finite graph E , the *graph monoid* associated to E , denoted by $M(E)$, is the commutative monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \quad \text{for every } v \in E^0 \text{ that emits edges.} \quad (2.1)$$

Let \mathbb{F} be the free commutative monoid on the set E^0 . The nonzero elements of \mathbb{F} can be written in a unique form up to permutation as $\sum_{i=1}^n x_i$, where $x_i \in E^0$. Now we will give a description of the congruence on \mathbb{F} generated by the relations (2.1) on \mathbb{F} . It will be convenient to introduce the following notation. For $x \in E^0$, write

$$\mathbf{r}(x) := \sum_{\{e \in E^1 \mid s(e)=x\}} r(e) \in \mathbb{F}.$$

With this new notation relations (2.1) become $x = \mathbf{r}(x)$ for every $x \in E^0$ that emits edges.

Definition 2.11 ([9, Section 4]). Define a binary relation \rightarrow_1 on $\mathbb{F} \setminus \{0\}$ as follows. Let $\sum_{i=1}^n x_i$ be an element in \mathbb{F} as above and let $j \in \{1, \dots, n\}$ be an index such that x_j emits edges. Then $\sum_{i=1}^n x_i \rightarrow_1 \sum_{i \neq j} x_i + \mathbf{r}(x_j)$. Let \rightarrow be the transitive and reflexive closure of \rightarrow_1 on $\mathbb{F} \setminus \{0\}$, that is, $\alpha \rightarrow \beta$ if and only if there is a finite string $\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \dots \rightarrow_1 \alpha_t = \beta$. Let \sim be the congruence on \mathbb{F} generated by the relation \rightarrow_1 (or, equivalently, by the relation \rightarrow). Namely $\alpha \sim \alpha$ for all $\alpha \in \mathbb{F}$ and, for $\alpha, \beta \neq 0$, we have $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. The number n above will be called the *length* of the string. \square

It is clear that \sim is the congruence on \mathbb{F} generated by relations (2.1), and so $M(E) = \mathbb{F}/\sim$.

Lemma 2.12 ([9, Lemma 4.3]). *Let α and β be nonzero elements in \mathbb{F} . Then $\alpha \sim \beta$ if and only if there is $\gamma \in \mathbb{F}$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$.*

Proposition 2.13 ([9, Proposition 4.4]). *The monoid $M(E)$ associated with any row-finite graph E is a refinement monoid.*

Let E be a graph. For any subset H of E^0 , we will denote by $I(H)$ the order-ideal of $M(E)$ generated by H .

Recall that we denote by $\mathcal{L}(M)$ the lattice of order-ideals of a commutative monoid M . Order-ideals of $M(E)$ correspond to hereditary saturated subsets of E^0 , as follows:

Proposition 2.14 ([9, Proposition 5.2]). *For any row-finite graph E , there is a natural lattice isomorphism from \mathcal{H}_E to $\mathcal{L}(M(E))$ sending $H \in \mathcal{H}_E$ to the order-ideal $I(H)$ generated by H .*

Also, we have:

Lemma 2.15 ([9, Proposition 5.2], [12, Lemma 2.1]). *Let H be a subset of E^0 , with hereditary saturated closure \overline{H} . Then $I(H) = I(\overline{H})$, and $\overline{H} = I(H) \cap E^0$.*

This means that $I(H)$ is generated as a monoid by the set $\overline{H} = I(H) \cap E^0$. We can improve this result, as follows

Lemma 2.16. *Let E be a graph, and let H be a hereditary subset of E^0 . Then, the order-ideal $I(H)$ is generated as a monoid by the set $\{a_v : v \in H\}$.*

Proof. Take $v \in \Lambda_1(H)$. Then $r(s^{-1}(v))$ is a finite subset of H , and so $a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$ belongs to the submonoid generated by $\{a_w : w \in H\}$. By induction, the same happens for a vertex in $\Lambda_n(H)$ for all $n \geq 1$. Since $\overline{H} = \bigcup_{n=0}^{\infty} \Lambda_n(H)$, and $I(H)$ is generated as a monoid by \overline{H} , we obtain the result. \square

Thus, we can conclude:

Lemma 2.17. *Let E be a directed graph and let H be a hereditary subset of E^0 . Then the order-ideal $I(H)$ generated by H is isomorphic to the graph monoid of the restriction graph E_H .*

Proof. Since H is a hereditary subset of E^0 , the map

$$\begin{aligned} \phi : M(E_H) &\rightarrow I(H) \\ a_v &\mapsto a_v \end{aligned}$$

is a well-defined monoid homomorphism. Moreover, ϕ is an onto map by Lemma 2.16.

To show it is injective, let α and β be elements in the free commutative monoid generated by H , and assume that the elements of $M(E)$ represented by α and β agree. By Lemma 2.12, there exists $\gamma \in \mathbb{F}$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$. By the definition of the relation \rightarrow and the fact that H is hereditary, it follows that the vertices appearing in γ belong to H , and that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$ in $M(E_H)$. Hence, α and β represent the same element of $M(E_H)$. \square

Given an order-ideal S of a monoid M we define a congruence \sim_S on M by setting $a \sim_S b$ if and only if there exist $e, f \in S$ such that $a + e = b + f$. Let M/S be the factor monoid obtained from the congruence \sim_S ; see e.g. [8].

Lemma 2.18 ([9, Lemma 6.6]). *Let E be a row-finite graph. For a saturated hereditary subset H of E^0 , consider the order-ideal $I(H)$ associated with H . Then there is a natural monoid isomorphism $M(E)/I(H) \cong M(E/H)$.*

2.4. K-Theory for rings. Here, we recall a few elements on K-Theory for Leavitt path algebras, that will be necessary in the sequel.

For a ring R (with local units), let $M_\infty(R)$ be the directed union of $M_n(R)$ ($n \in \mathbb{N}$), where the transition maps $M_n(R) \rightarrow M_{n+1}(R)$ are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. We define $\mathcal{V}(R)$ to be the set of isomorphism classes (denoted $[P]$) of finitely generated projective left R -modules, and we endow $\mathcal{V}(R)$ with the structure of a commutative monoid by imposing the operation

$$[P] + [Q] := [P \oplus Q]$$

for any isomorphism classes $[P]$ and $[Q]$. Equivalently [23, Chapter 1], $\mathcal{V}(R)$ can be viewed as the set of equivalence classes $\mathcal{V}(e)$ of idempotents e in $M_\infty(R)$ with the operation

$$\mathcal{V}(e) + \mathcal{V}(f) := \mathcal{V}\left(\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right)$$

for idempotents $e, f \in M_\infty(R)$. The group $K_0(R)$ of a ring R with local units is the universal group of $\mathcal{V}(R)$. Recall that, as any universal group of a commutative monoid, the group $K_0(R)$ has a standard structure of partially pre-ordered abelian group. The set of positive elements in $K_0(R)$ is the image of $\mathcal{V}(R)$ under the natural monoid homomorphism $\mathcal{V}(R) \rightarrow K_0(R)$.

Let $E = (E^0, E^1, r, s)$ be a graph, and let K be a field. We define the *Leavitt path algebra* $L_K(E)$ associated with E as the K -algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

- (1) $s(e)e = er(e) = e$ for all $e \in E^1$.
- (2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
- (3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
- (4) $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Note that the relations above imply that $\{ee^* \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L_K(E)$. In general the algebra $L_K(E)$ is not unital, but it has a set of local units given by $\{\sum_{v \in F} v\}$, where F ranges on all finite subsets of E^0 . So, the above facts apply.

Let E be a graph. For any subset H of E^0 , with hereditary saturated closure \overline{H} , we will denote by $\mathfrak{J}(H)$ the ideal of $L_K(E)$ generated by H .

Theorem 2.19 ([9, Theorem 3.5]). *Let E be a row-finite graph. Then there is a natural monoid isomorphism $\mathcal{V}(L_K(E)) \cong M(E)$.*

Moreover, by [9, Theorem 3.5], [9, Theorem 5.2], [9, Lemma 6.6] and [12, Lemma 2.3(1)], we conclude that

Lemma 2.20. *Let E be a row-finite graph, and let H a hereditary subset of E^0 . Then, for any field K we have that*

$$\mathcal{V}(L_K(E))/\mathcal{V}(\mathfrak{J}(H)) \cong \mathcal{V}(L_K(E/H)) \cong \mathcal{V}(L_K(E)/\mathfrak{J}(H)).$$

3. REPRESENTING FINITELY GENERATED REGULAR REFINEMENT MONOIDS

Dobbertin showed in [14] that every finitely generated regular conical refinement monoid can be represented as a partial order of finitely generated abelian groups (see also [10]). Thus, in order to represent a finitely generated regular conical refinement monoid as a graph monoid, it suffices to show that some basic diagrams of abelian groups and homomorphisms

of groups can be represented as graph monoids. In this section, we will prove that this is possible without further restrictions. In order to make clearer the argument, we will divide this task in several steps.

First, we will show that any finitely generated abelian group can be represented using a graph monoid.

Lemma 3.1. *Let H be a finitely generated abelian group. Then, H is representable as the semigroup $M(E)^*$ associated to a (non necessarily finite) row-finite directed graph E .*

Proof. We start by fixing notation. Using the Structure Theorem for Finitely Generated Abelian Groups, we can assume that

$$H = \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s}$$

where $r, s \geq 0$, and $n_i \geq 1$ for all i . We define $N := r + s$.

Now, we will define our graph. The graph E has vertices

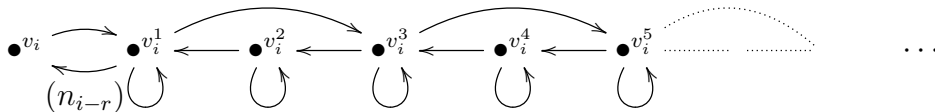
$$E_1^0 = \{v_1, \dots, v_{N+1}, v_i^j \ (r+1 \leq i \leq N, j \geq 1)\}.$$

Instead of fixing what are the edges of E extensively, we will express the relations that these edges define on the graph monoid $M(E_1)$, as follows:

- (1) $v_{N+1} = 2v_{N+1} + \sum_{i=1}^r v_i + \sum_{i=r+1}^N n_{i-r}v_i$
- (2) For every $r+1 \leq i \leq N$:
 - (a) $v_i^1 = n_{i-r}v_i + v_i^1 + v_i^3$
 - (b) for every $j \geq 1$, $v_i^{2j} = v_i^{2j-1} + v_i^{2j}$
 - (c) for every $j \geq 1$, $v_i^{2j+1} = v_i^{2j} + v_i^{2j+1} + v_i^{2j+3}$
- (3) For every $1 \leq i \leq r$, $v_i = 2v_i + \sum_{j=1, j \neq i}^r v_j + \sum_{j=r+1}^N n_{j-r}v_j + v_{N+1}$
- (4) For every $r+1 \leq i \leq N$, $v_i = \sum_{j=1}^r v_j + \sum_{j=r+1, j \neq i}^N n_{j-r}v_j + (n_{i-r} + 1)v_i + v_{N+1} + v_i^1$

Let us carefully explain which are the properties enjoyed by the graph E :

- (a) First, we will show a graphical representation of the part of E described in point (2) above, for any vertex v_i with $r+1 \leq i \leq N$. Consider the subgraph E_i :



Notice that E_i is transitive, and moreover every vertex in E_i^0 is basis point of at least two different simple closed paths. So, the monoid $M(E_i)$ associated to E_i is simple and regular (because in $M(E_i)$ we have $2v_i \leq v_i$ and $2v_i^j \leq v_i^j$ for every $j \geq 1$).

- (b) By relation (1) above, there are paths connecting v_{N+1} with every vertex v_i ($1 \leq i \leq N$), and thus also with every vertex v_i^j ($r+1 \leq i \leq N, j \geq 1$) by relations (2) and (4) above. Moreover, in $M(E)$ we have $2v_{N+1} \leq v_{N+1}$.
- (c) By relation (3) above, for every $1 \leq k \leq r$, there are paths connecting v_k with every vertex v_i ($1 \leq i \leq N+1$), and thus also with every vertex v_i^j ($r+1 \leq i \leq N, j \geq 1$) by relations (2) and (4) above. Moreover, in $M(E)$ we have $2v_k \leq v_k$.

- (d) By relation (4) above, for every $r+1 \leq k \leq N$, there are paths connecting v_k with every vertex v_i ($1 \leq i \leq N+1$), and thus also with every vertex v_i^j ($r+1 \leq i \leq N, j \geq 1$) by relations (2) and (4) above. Moreover, in $M(E)$ we have $2v_k \leq v_k$.

As a consequence:

- (e) The graph E is transitive, and thus $M(E)$ is a simple monoid.
 (f) For every $v \in E_1^0$, $2v \leq v$ in $M(E)$, whence $M(E)$ is regular. Thus, since $M(E)$ is simple, we conclude that $M(E)^*$ is an abelian group.

Now, we will identify this group up to isomorphism. By relation (1) above, the neutral element of $M(E)^*$ is

$$e = v_{N+1} + \sum_{i=1}^r v_i + \sum_{i=r+1}^N n_{i-r} v_i. \quad (3.1)$$

In particular, relations (1) and (3) above simply say that $v_i = v_i + e$ for $i \in \{1, \dots, r, N+1\}$. Now, by using relation (2) above, we have:

- $e = n_{i-r} v_i + v_i^3$ by relation (2a).
- $e = v_i^{2j-1}$ for every $j \geq 1$ by (2b).
- $e = v_i^{2j} + v_i^{2j+3}$ for every $j \geq 1$ by (2c), and then $e = v_i^{2j}$ for every $j \geq 1$ by the previous identity.

As a consequence, $e = v_i^j$ for every $r+1 \leq i \leq N$ and every $j \geq 1$, and thus relation (2a) says that

$$e = n_{i-r} v_i \text{ for every } r+1 \leq i \leq N. \quad (3.2)$$

Hence, $\langle v_{r+1}, \dots, v_N \rangle$ generates a copy of $\mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_s}$ into $M(E)^*$.

Replacing equation (3.2) in the corresponding places of equation (3.1), we obtain

$$e = v_{N+1} + \sum_{i=1}^r v_i. \quad (3.3)$$

Hence, $\langle v_1, \dots, v_r, v_{N+1} \rangle$ generates a copy of \mathbb{Z}^r into $M(E)^*$. Summarizing, the semigroup of nonzero elements of $M(E)$ is isomorphic to H , as desired. \square

Remark 3.2.

- (a) In Lemma 3.1, we are representing the group H as a group generated by vertices. In order to ease operating with this representation, we need to identify the inverses of the vertices, seen as elements of the group. If we follow the notation of Lemma 3.1, $\langle v_{r+1}, \dots, v_N \rangle$ generates a copy of $\mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_s}$ into $M(E)^*$, and $(n_i - 1)v_i$ will be the symmetric of v_i in $M(E)^*$ for $r+1 \leq i \leq N$. Also, $\langle v_1, \dots, v_r, v_{N+1} \rangle$ generates a copy of \mathbb{Z}^r into $M(E)^*$. In this copy, $\{v_1, \dots, v_r\}$ are the free generators of the group, while v_{N+1} help us to express r -tuples of \mathbb{Z}^r with negative entries. To be precise, since $e = v_{N+1} + \sum_{i=1}^r v_i$, for any $1 \leq i \leq r$ the element

$$v_1 + \dots + v_{i-1} + v_{i+1} + \dots + v_r + v_{N+1}$$

will be the symmetric of v_i in $M(E)^*$ for $1 \leq i \leq r$. In order to simplify the notation in the sequel, when we work with a graph monoid $M(E)$, we will denote by v^- the

symmetric of the vertex $v \in E^0$ seen as an element in the archimedean component of $M(E)$ containing v .

- (b) Note that in the isomorphism $M(E)^* \cong H = \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, the vertices v_1, \dots, v_N correspond to the canonical generators of H as an abelian group. We will refer to this fact saying that v_1, \dots, v_N are the canonical generators of H .
- (c) We will work later in Section 5 with *semigroup* generators of a group. Observe that $\{v_1, \dots, v_N, v_{N+1}\}$ is indeed a family of semigroup generators of H .

Let G and H be finitely generated abelian groups, and let $\varphi: H \rightarrow G$ be a group homomorphism. According to [14], there exists a regular refinement monoid $M(\varphi)$ associated to φ . The monoid $M(\varphi)$ has exactly two prime idempotents e and f , with $e + f = f$, $G_M[e] = H$, $G_M[f] = G$, and the map $\varphi_{f,e}: G_M[e] \rightarrow G_M[f]$ given by $\varphi_{f,e}(x) = x + f$ is exactly the map φ . We will show that $M(\varphi)$ is representable as a graph monoid.

Proposition 3.3. *Let $\varphi: H \rightarrow G$ be a homomorphism between finitely generated abelian groups H and G , and let $M(\varphi)$ be the associated regular refinement monoid. Then $M(\varphi)$ is representable as the monoid $M(E)$ of a (non necessarily finite) row-finite directed graph E .*

Proof. We start by fixing notation. Using the Structure Theorem for Finitely Generated Abelian Groups, we can assume that

$$H = \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s}$$

and

$$G = \mathbb{Z}^t \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l},$$

where $r, s, t, l \geq 0$, and $n_i, m_j \geq 1$ for all i, j . We define $N := r + s$ and $M := t + l$. Moreover, if $M < N$, we can add $m_{l+1} = m_{l+2} = \cdots = m_{l+(N-M)} = 1$, so that we are thinking

$$G = \mathbb{Z}^t \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l} \oplus \mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_1.$$

Thus, without loss of generality, we can assume that $N \leq M$. Under this representation of H and G , the homomorphism φ is represented by a block matrix

$$A = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

with entries labeled $a_{i,j} \in \mathbb{Z}$, and without loss of generality we can assume that the entries in blocks $A_{2,l}$ ($l = 1, 2$) satisfy $0 < a_{i,j} \leq m_{i-t}$ for every j .

We will define our graph E in two layers, the first one representing the group H , while the second will represent simultaneously both the group G and the homomorphism φ .

To construct the first layer, we use Lemma 3.1 to build a graph E_1 such that $M(E_1)^* \cong H$, with

$$E_1^0 = \{v_1, \dots, v_{N+1}, v_i^j (r+1 \leq i \leq N, j \geq 1)\},$$

such that v_1, \dots, v_N are the canonical generators of $M(E_1)^* = H$ (see Remark 3.2(b)).

Now, we will construct the layer corresponding to G , and simultaneously the group homomorphism $\varphi: H \rightarrow G$. To this end, we will define a new collection of vertices E_2^0 , jointly with a family of relations stated in similar terms as those used in Lemma 3.1. Nevertheless,

these new relations will involve not only the vertices of E_2^0 , but also those of E_1^0 . The new set of vertices E_2^0 is

$$E_2^0 = \{w_1, \dots, w_{M+1}, w_i^j (t+1 \leq i \leq M, j \geq 1)\}.$$

As in Lemma 3.1, instead of fixing what are the edges emitted by the vertices of E_2^0 extensively, we will express the relations that these edges define on the graph monoid. Let E be the graph defined by taking $E^0 = E_1^0 \sqcup E_2^0$ and where all the relations enjoyed by the vertices are as follows:

- The relations given by the set of edges of the graph E_1 .

- The relations we list below:

$$(1) w_{M+1} = 2w_{M+1} + \sum_{i=1}^t w_i + \sum_{i=t+1}^M m_{i-t} w_i$$

- (2) For every $t+1 \leq i \leq M$:

$$(a) w_i^1 = m_{i-t} w_i + w_i^1 + w_i^3$$

$$(b) \text{ for every } j \geq 1, w_i^{2j} = w_i^{2j-1} + w_i^{2j}$$

$$(c) \text{ for every } j \geq 1, w_i^{2j+1} = w_i^{2j} + w_i^{2j+1} + w_i^{2j+3}$$

- (3) For every $1 \leq i \leq t$,

$$w_i = (a_{i,i} + 2)w_i + \sum_{j=1, j \neq i}^t (a_{j,i} + 1)w_j + \sum_{j=t+1}^M (a_{j,i} + m_{j-t})w_j + w_{M+1} + v_i^-.$$

Notice that whenever $a_{j,i} < 0$ for some $1 \leq j \leq t$, we will replace in the above

relation $a_{j,i}w_j$ by $\sum_{k=1, k \neq j}^t (-a_{j,i})w_k + (-a_{j,i}w_{M+1})$.

- (4) For every $t+1 \leq i \leq M$,

$$w_i = \sum_{j=1}^t (a_{j,i} + 1)w_j + \sum_{j=t+1, j \neq i}^M (a_{j,i} + m_{j-t})w_j + (a_{i,i} + m_{i-t} + 1)w_i + w_{M+1} + w_i^1 + v_i^-.$$

As in the previous relation, whenever $a_{j,i} < 0$ for some $1 \leq j \leq t$, we will replace

in the above relation $a_{j,i}w_j$ by $\sum_{k=1, k \neq j}^t (-a_{j,i})w_k + (-a_{j,i}w_{M+1})$.

If $N < M$, and $N < i \leq M$, then the term v_i^- appearing in (4) (or in (3) if $N < i \leq t$) should be interpreted as the neutral element 0 of the graph monoid. So, for these values of i , the relations are of the form (4) (or (3)) in Lemma 3.1.

Now, consider the graph E , and let us identify the properties enjoyed by the elements of $M(E)$:

- Observe that E_1^0 is a hereditary and saturated subset of E^0 , so that $M(E_1)$ is an order-ideal of $M(E)$, by Lemma 2.17. Moreover $M(E_1)^*$ is the archimedean component of e in $M(E)$ and, by construction $M(E_1)^* \cong H$.
- By similar arguments to those used in Lemma 3.1, every vertex in E_2^0 is a regular element.
- By similar arguments to those used in Lemma 3.1, for any $1 \leq i, j \leq M+1$ there exist paths from w_i to w_j , and also from w_i to w_j^k and from w_j^k to w_i for every $k \geq 1$.

Hence, $M(E_1)^*$ equals $G_{M(E)}[e]$. Also, E_2^0 is strongly connected in E , and thus the vertices in E_2^0 generate an archimedean component $G_{M(E)}[f]$ of $M(E)$ with neutral element f . Moreover, $M(E)$ is a regular refinement monoid.

By using the relations defined above, and arguing as in Lemma 3.1, we have that:

- By relation (1) above,

$$f = \sum_{i=1}^t w_i + w_{M+1} + \sum_{i=t+1}^M m_{i-t} w_i. \quad (3.4)$$

- By relation (2) above, $f = w_i^j$ for every $t+1 \leq i \leq M$ and every $j \geq 1$, while

$$f = m_{i-t} w_i \text{ for every } t+1 \leq i \leq M. \quad (3.5)$$

- By replacing equation (3.5) in equation (3.4), we obtain

$$f = w_1 + \cdots + w_t + w_{M+1} \quad (3.6)$$

Hence, the monoid $\langle w_1, \dots, w_M, w_{M+1} \rangle$ generated by w_1, \dots, w_{M+1} is indeed a group, which is a subgroup of $G_{M(E)}[f]$.

Now, we will state the relation between e and f in $M(E)$. To this end, notice that:

- Since $v_1^- \in M(E_1)^*$, we have that $e \leq v_1^-$.
- By relation (3) above (or (4) if $t = 0$), $v_1^- \leq w_1$, so that $e \leq w_1$.
- By equation (3.6) (or (3.5) if $t = 0$), $w_1 \leq f$.

Hence, $e \leq f$ in $M(E)$. So there are only two archimedean components in $M(E)$, corresponding to the idempotents e and f , and $e \leq f$ in $M(E)$. We want to describe the homomorphism

$$\phi_e^f : \begin{array}{ccc} G_{M(E)}[e] & \rightarrow & G_{M(E)}[f] \\ x & \mapsto & x + f \end{array} .$$

Observing that v_1, \dots, v_N are canonical generators of $G_{M(E)}[e] \cong H$, we are going to compute $\phi_e^f(v_i)$. We will compute $\phi_e^f(v_i)$ for every $1 \leq i \leq N$, by using relations (3) and (4) above. To be precise, recall that $r \leq r+s = N \leq M = t+l$. So, it can occur either $N \leq t$ or $N > t$. In the first case, to determine $\phi_e^f(v_i)$ we will only need relation (3) above. In the second case, we will use relation (3) above to determine $\phi_e^f(v_i)$ for $1 \leq i \leq t$, and relation (4) above to determine $\phi_e^f(v_i)$ for $t+1 \leq i \leq N$. Let us suppose that we are in the second case. Take $i \in \{1, \dots, t\}$, and add v_i on both sides of relation (3), as follows:

$$\begin{aligned} w_i + v_i &= \\ (a_{i,i} + 2)w_i + \sum_{j=1, j \neq i}^t (a_{j,i} + 1)w_j + \sum_{j=t+1}^M (a_{j,i} + m_{j-t})w_j + w_{M+1} + v_i^- + v_i &= \\ w_i + \sum_{j=1}^M a_{j,i}w_j + \left[\sum_{j=1}^t w_j + w_{M+1} + \sum_{j=t+1}^M m_{j-t}w_j \right] + (v_i^- + v_i). \end{aligned}$$

By definition of v_i^- and w_i^- , we have that $v_i^- + v_i = e$ and $w_i^- + w_i = f$. Also, $e + f = f$, and by equation (3.4), $f = \sum_{j=1}^t w_j + w_{M+1} + \sum_{j=t+1}^M m_{j-t}w_j$. Thus, by adding w_i^- on both sides of

the previous identity, we have

$$f + v_i = f + \sum_{j=1}^M a_{j,i} w_j + f + e = \sum_{j=1}^M a_{j,i} w_j + f = \sum_{j=1}^M a_{j,i} w_j$$

because f is the neutral element of the group $G_{M(E)}[f]$. On the other hand, if $i \in \{t + 1, \dots, N\}$, a similar argument shows that adding $v_i + w_i^-$ on both sides of relation (4) gives us

$$v_i + f = \sum_{j=1}^M a_{j,i} w_j.$$

Hence, using (3.5), (3.6) and the above relations, we obtain a monoid homomorphism

$$\gamma: M(\varphi) \rightarrow M(E)$$

extending the canonical isomorphism $H = G_{M(\varphi)}[e] \rightarrow G_{M(E)}[e]$ and sending the canonical generators of $G = G_{M(\varphi)}[f]$ to w_1, \dots, w_M . We are going to define an inverse $\delta: M(E) \rightarrow M(\varphi)$ of the map γ . For this, it is enough to define the map on the vertices of E and to show that the defining relations of $M(E)$ are preserved by this assignment. The images of the vertices in E_1 are dictated by the inverse map of the isomorphism from H onto $G_{M(E)}[e]$. Let $\mathbf{x}_1, \dots, \mathbf{x}_M$ be the canonical generators of G . We define $\delta(w_i) = \mathbf{x}_i$ for $1 \leq i \leq M$, and

$$\delta(w_{M+1}) = -(\mathbf{x}_1 + \dots + \mathbf{x}_t).$$

Finally define $\delta(w_i^j) = f$, where f is the neutral element of G . It is easy to show that all relations (1)-(4) are preserved by δ , so that this assignment gives a well-defined monoid homomorphism δ . It is now clear that δ is the inverse of γ .

Therefore we obtain that γ is an isomorphism from $M(\varphi)$ onto $M(E)$. Observe that γ sends the canonical set of generators of $G = G_{M(\varphi)}[f]$ onto w_1, \dots, w_M , so that the vertices w_1, \dots, w_M are canonical generators of $G_{M(E)}[f]$ and the canonical map ϕ_e^f has associated matrix A with respect to the canonical generators v_1, \dots, v_N of $G_{M(E)}[e]$ and w_1, \dots, w_M of $G_{M(E)}[f]$. \square

Remark 3.4. In the above proof, we could have assumed that $N \leq l$ and use only relations (4) to encode the map φ in the graph monoid $M(E)$. The proof is then a little bit shorter, since we would not need to distinguish different cases when we deal with the computation of the elements $\phi_e^f(v_i)$. This is the approach that we will follow in the proof of the general case (see Proposition 5.12).

The next step is to show a result analogous to Proposition 3.3, which allows us to represent confluent maps of groups. This is precisely what we need to use in the inductive step of the proof of the general result. The proof is very similar to the one of Proposition 3.3, so we will only give a brief sketch of it.

Let H_1, \dots, H_n, G be finitely generated abelian groups, and let $\varphi_i: H_i \rightarrow G$ be group homomorphisms ($1 \leq i \leq n$). Let $M(\varphi)$ be the regular refinement monoid associated to $\varphi := (\varphi_1, \dots, \varphi_n)$, that is, $M(\varphi)$ has exactly $n + 1$ prime idempotents e_1, \dots, e_n and f , with $e_i + f = f$, and $G_M[e_i] = H_i$, $G_M[f] = G$, and the map $\varphi_{f e_i}: G_M[e_i] \rightarrow G_M[f]$ given by $\varphi_{f e_i}(x) = x + f$ is exactly the map φ_i for $1 \leq i \leq n$.

Proposition 3.5. *Let H_1, \dots, H_n, G be finitely generated abelian groups, and let $\varphi_i : H_i \rightarrow G$ be group homomorphisms ($1 \leq i \leq n$). Then, these maps can be represented simultaneously as the monoid $M(E)$ associated to a (non necessarily finite) row-finite direct graph E .*

Proof. We start by fixing notation. Using the Structure Theorem for Finitely Generated Abelian Groups, we can assume that

$$H_i = \mathbb{Z}^{t_i} \oplus \mathbb{Z}_{m_1^i} \oplus \cdots \oplus \mathbb{Z}_{m_{l_i}^i},$$

for $i = 1, \dots, n$, and

$$G = \mathbb{Z}^r \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_s},$$

where $t_i, l_i, r, s \geq 0$, and $m_i^j, p_i \geq 1$ for all i, j . We define $N_i := t_i + l_i$ ($i = 1, 2$) and $M := r + s$; moreover, as in Proposition 3.3, we can assume $\sum_{i=1}^n N_i \leq M$.

Now, we will define our graph E essentially as in Proposition 3.3, defining a first layer corresponding to the groups H_1, \dots, H_n , and a second layer corresponding to the group G and to the different maps φ_i , which are represented by suitable matrices as in Proposition 3.3.

To deal with the first layer we use Lemma 3.1 to build n mutually disconnected strongly connected graphs E_i such that $M(E_i)^* \cong H_i$ for $i = 1, \dots, n$, with corresponding sets of vertices such that the first N_i vertices of graph E_i are a canonical set of generators of $G_{M(E_i)}[e_i] = H_i$.

The final step consists of constructing the layer corresponding to the group G , together with the homomorphisms $\varphi_i : H_i \rightarrow G$. We will do that by adding a new collection of vertices T^0 , jointly with a family of relations, similar to the ones used in Proposition 3.3. These relations will involve not only vertices in T^0 , but also in E_i^0 . The idea is to apply again the procedure described in the proof of Proposition 3.3. Observe that, just as in Proposition 3.3 we obtain an isomorphism $M(\varphi) \rightarrow M(E)$ sending the canonical generating sets of each H_i and of G to the canonical generating sets of vertices spanning the corresponding archimedean components $G_{M(E)}[e_i]$ and $G_{M(E)}[f]$ respectively. \square

We can now obtain the main result of this section.

Theorem 3.6. *If M is a conical finitely generated regular refinement monoid, then there exists a countable row-finite directed graph E such that $M \cong M(E)$.*

Proof. By Dobbertin's result [14], M is of the form $M(\mathcal{J}_M)$, for the I -system \mathcal{J}_M of abelian groups $\{G_M[e] \mid e \in I\}$, where e ranges on the poset I of prime idempotents of M , as explained in Section 2.

Since M is finitely generated, all the groups $G_M[e]$ are finitely generated. We start by fixing a suitable form for these groups, namely we set

$$G_M[e] = \mathbb{Z}^{r_e} \oplus \mathbb{Z}_{n_1^e} \oplus \cdots \oplus \mathbb{Z}_{n_{s_e}^e},$$

where $r_e, s_e \geq 0$, and $n_i^e \geq 1$ for all i . Set $N_e = r_e + s_e$. We can assume that for each $f \in I$, we have $\sum_{i=1}^n N_{e_i} \leq N_f$, where $L(I, f) = \{e_1, \dots, e_n\}$ is the lower cover of f in I .

We can now proceed to show the result by order-induction. So assume that we have a lower subset J of I , and that we have built a countable row-finite graph E_J such that $E_J^0 = \bigsqcup_{e \in J} E_e^0$, where each E_e is a strongly connected graph, with

$$E_e^0 = \{v_1^e, \dots, v_{N_e+1}^e, (v_i^e)^j \mid (r_e + 1 \leq i \leq N_e, j \geq 1)\}.$$

and an isomorphism

$$\gamma_J: M(J) \rightarrow M(E_J),$$

where $M(J)$ is the order-ideal of M generated by J , such that γ_J sends the canonical generators of $G_M[e]$ to $v_1^e, \dots, v_{N_e}^e$ for all $e \in J$.

In case $J \neq I$, let f be a minimal element of $I \setminus J$ and write $J' = J \cup \{f\}$. We will show that the above statement holds for the lower subset J' in place of J . This clearly establishes the result, because I is a finite poset.

Let E_f be the graph associated to $G_M[f]$, as in Lemma 3.1.

There are two cases to consider:

(1) **f is a minimal element of I .** Set $E_{J'} := E_f \sqcup E_J$. Then

$$M(J') = M(\{f\}) \oplus M(J) \cong M(E_f) \oplus M(E_J) = M(E_{J'})$$

in a canonical way, showing the result.

(2) **f is not a minimal element of I .** Let $L(I, f) = \{e_1, \dots, e_n\}$ be the lower cover of f in I . Write $\varphi_i = \varphi_{f, e_i}$ for $i = 1, \dots, n$, and consider the graph E associated to the maps φ_i , $i = 1, \dots, n$, as in Proposition 3.5. The first layer of this graph consists exactly of the disjoint union of the graphs E_{e_i} , for $i = 1, \dots, n$, so it is a subgraph of our graph E_J . Let $E_{J'}$ be the graph with $E_{J'}^0 = E_J^0 \sqcup E_f^0$ and with $E_{J'}^1 = E_J^1 \sqcup s_E^{-1}(E_f)$. The graph $E_{J'}$ is a countable row-finite graph with $E_{J'}^0 = \bigsqcup_{e \in J'} E_e^0$, where each E_e^0 is a strongly connected subset of E^0 , and the sets of edges E_e^1 have the desired form for all $e \in J'$. We have to prove the existence of the isomorphism $\gamma_{J'}$.

Since E_J^0 is a hereditary and saturated subset of $E_{J'}^0$, we get from Lemma 2.17 that the order-ideal of $M(E_{J'})$ generated by E_J^0 is precisely $M(E_J)$. Similarly, the order-ideal of $M(J')$ generated by J is precisely $M(J)$, and $M(J')$ coincides with $M(\mathcal{J}_{J'})$, which is the monoid associated to the partial order of groups $\mathcal{J} := \mathcal{J}_M$ restricted to J' (see e.g. [10, Proposition 1.9]). Define a map

$$\gamma_{J'}: M(J') \rightarrow M(E_{J'})$$

as follows. The map $\gamma_{J'}$ agrees with the isomorphism γ_J when restricted to the order-ideal $M(J)$ of $M(J')$. The map $\gamma_{J'}$ restricted to $G_{M(J')}[f] = G_M[f]$ is just the map obtained by sending the canonical generators of $G_M[f]$ to the canonical generators $v_1^f, \dots, v_{N_f}^f$ of $G_{M(E_{J'})}[f]$. In order to prove that $\gamma_{J'}$ gives a well-defined monoid homomorphism, it suffices by [10, Corollary 1.6] to show that if $e < f$ and $x \in G_M[e]$ then $\gamma_{J'}(x) + \gamma_{J'}(f) = \gamma_{J'}(\varphi_{fe}(x)) + \gamma_{J'}(f)$, that is, $\gamma_{J'}(x) + f = \gamma_{J'}(\varphi_{fe}(x))$. Since $e < f$ and I is finite, there is some i such that $e \leq e_i$. Using the properties of the map defined in Proposition 3.5 and the induction hypothesis, we get

$$\begin{aligned} \gamma_{J'}(\varphi_{fe}(x)) &= \gamma_{J'}(\varphi_{f, e_i}(\varphi_{e_i, e}(x))) = \phi_{e_i}^f(\gamma_{J'}(\varphi_{e_i, e}(x))) \\ &= \phi_{e_i}^f(\gamma_J(\varphi_{e_i, e}(x))) = \phi_{e_i}^f(\phi_e^{e_i}(\gamma_J(x))) \\ &= \phi_e^f(\gamma_{J'}(x)) = f + \gamma_{J'}(x). \end{aligned}$$

This shows that the defining relations of $M(J')$ are preserved and so the map $\gamma_{J'}$ is a well-defined homomorphism from $M(J')$ to $M(E_{J'})$. To build the inverse $\delta_{J'}$ of $\gamma_{J'}$ we follow the idea in the proof of Proposition 3.3. The image by $\delta_{J'}$ of the vertices in E_J is determined by the

inverse of γ_J . Let $\mathbf{x}_1, \dots, \mathbf{x}_{N_f}$ be the canonical generators of $G_M[f]$. We define $\delta_{J'}(v_i^f) = \mathbf{x}_i$ for $1 \leq i \leq N_f$, and

$$\delta_{J'}(v_{N_f+1}^f) = -(\mathbf{x}_1 + \dots + \mathbf{x}_{r_f}).$$

Finally define $\delta_{J'}((v_i^f)^j) = f$. It is easily checked that $\delta_{J'}$ preserves the defining relations of $M(E_{J'})$, and so it gives a well-defined homomorphism from $M(E_{J'})$ to $M(J')$, which is clearly the inverse of $\gamma_{J'}$.

This concludes the proof of the result. \square

4. A SIMPLE EXAMPLE

We are going to represent one of the most simple examples of primitive monoids, using a method similar to the described above.

This is the monoid $M = \mathbb{Z}^+ \cup \{\infty\}$ which appears in [11]. Note that this example can be described as $M = \langle p, a \mid a = 2a, a = a + p \rangle$. Our method is completely different from the method used in [11], and provides a simpler graph than the one given in [11, Example 6.5].

Lemma 4.1. *The monoid $M = \langle p, a \mid a = 2a, a = a + p \rangle$ can be represented by a graph monoid.*

Proof. The lower component is free generated by p , so we introduce a vertex v_0 with a single loop e_0 around it.

Now the second component is regular with trivial associated group, so we use the above method setting $G = \mathbb{Z}_1$, and we consider vertices

$$E^0 = \{v_0, v_1, v_1^j (j \geq 1)\}.$$

Instead of fixing what are the edges of E extensively, we will express the relations that these edges define on the graph monoid $M(E)$, as follows:

- (1) $v_0 = v_0$
- (2) (a) $v_1^1 = v_1 + v_1^1 + v_1^3$
 (b) for every $j \geq 1$, $v_1^{2j} = v_1^{2j-1} + v_1^{2j}$
 (c) for every $j \geq 1$, $v_1^{2j+1} = v_1^{2j} + v_1^{2j+1} + v_1^{2j+3}$
- (3) $v_1 = 2v_1 + v_1^1 + v_0$.

Then one can show as in the previous results that $v_1 = v_1^j = f$ for all $j \geq 1$, so we obtain from (3) that $f = f + v_0$, so the graph monoid $M(E)$ gives the desired monoid M . Note that M is not representable by a graph monoid of a *finite* graph by [11, Theorem 6.1]. \square

5. REPRESENTING FINITELY GENERATED REFINEMENT MONOIDS

In this section, we will obtain our main result (Theorem 5.6), which gives a characterization of the finitely generated conical refinement monoids which are graph monoids. For this, it is fundamental to use the characterization of these monoids obtained in [10], which generalizes the results of Dobbertin [14] for the regular case, and the results of Pierce [22] for the antisymmetric case.

5.1. Assume that M is a finitely generated conical refinement monoid, with poset of primes I . Let p be a free prime and let $L(I, p) = \{q_1, \dots, q_n\}$ be the lower cover of p . Then $M_p = \mathbb{N} \times G_p$

for a finitely generated abelian group G_p . We will assume that q_1, \dots, q_r are free primes and that q_{r+1}, \dots, q_n are regular primes.

Let J_p be the lower subset of I generated by q_1, \dots, q_n . Then, by Lemma 2.9, the Grothendieck group of the order-ideal $M(\mathcal{J}_{J_p})$ associated to J_p is precisely $\tilde{G}_{J_p} = G(M_{J_p})$. There is a surjective semigroup homomorphism

$$\varphi_p: M_{J_p} \rightarrow G_p$$

induced by the maps $\varphi_{p,q}$ for $q < p$ (see Section 2). By the universal property of the Grothendieck group, we get a unique group homomorphism

$$G(\varphi_p): \tilde{G}_{J_p} \rightarrow G_p$$

such that $G(\varphi_p) \circ \iota_{J_p} = \varphi_p$, where $\iota_{J_p}: M_{J_p} \rightarrow G(M_{J_p})$ is the natural map.

Definition 5.2. We define the set $G(M_{J_p})^{++}$ of *strictly positive* elements of $G(M_{J_p})$ as the image in $G(M_{J_p})$ of the natural map $\iota_{J_p}: M_{J_p} \rightarrow G(M_{J_p})$. Note that $G(M_{J_p})^{++}$ is just a subsemigroup of $G(M_{J_p})$, which does not contain the neutral element of $G(M_{J_p})$ in general.

Lemma 5.3. *Every element in $\tilde{G}_{J_p}^{++} = G(M_{J_p})^{++}$ can be represented by an element of the form $\sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i)$ for some $n_i \in \mathbb{N}$, $i = 1, \dots, r$ and $g_i \in G_{q_i}$, $i = 1, \dots, n$.*

Proof. This follows immediately from the description of M_{J_p} given in Section 2, using that $\{q_1, \dots, q_n\} = \text{Max}(J_p)$. \square

Example 5.4. Note that the positive cone $G(M_{J_p})^{++}$ does not coincide in general with the positive cone obtained by considering the image of $M(\mathcal{J}_{J_p})$ in $G(M(\mathcal{J}_{J_p})) = G(M_{J_p})$. For instance consider the graph monoid $M = \langle a, b \mid b = b + 2a \rangle$, and J the poset of primes $\{a, b\}$ of M . Then $\iota(a)$ is in the image of the canonical map $\iota: M \rightarrow G(M)$, but $\iota(a) \notin G(M_{J_b})^{++}$, since $G(M_{J_b}) = \mathbb{Z} \times \mathbb{Z}_2$ and $G(M_{J_b})^{++} = \mathbb{N} \times \mathbb{Z}_2$.

The following definition is handy to express the conditions characterizing finitely generated graph monoids.

Definition 5.5. Let G_1 be an abelian group with a distinguished subsemigroup G_1^{++} of strictly positive elements, and let G_2 be an abelian group. We say that a group homomorphism $f: G_1 \rightarrow G_2$ is an *almost isomorphism* in case f is surjective and the kernel of f is a cyclic subgroup of G_1 generated by an element in G_1^{++} .

We can now state the main result of the paper.

Theorem 5.6. *Let M be the finitely generated conical refinement monoid. Then M is a graph monoid if and only if for each free prime p of M , the map*

$$G(\varphi_p): \tilde{G}_{J_p} \rightarrow G_p$$

is an almost isomorphism.

In particular, we obtain, using the results in [6], the following partial affirmative answer to the realization problem for von Neumann regular rings.

Corollary 5.7. *Let M be a finitely generated conical refinement monoid such that, for all free prime p of M , the map $G(\varphi_p): \tilde{G}_{J_p} \rightarrow G_p$ is an almost isomorphism. Then, there exists a (countable) row-finite graph E such that, for any field K , the von Neumann regular K -algebra $Q_K(E)$ of the quiver E satisfies $\mathcal{V}(Q_K(E)) \cong M$.*

We will split the proof in two parts. First, we prove that the condition is necessary.

Proposition 5.8. *Let M be the finitely generated conical refinement monoid, and let p be a free prime of M . If M is a graph monoid then the map*

$$G(\varphi_p): \widetilde{G}_{J_p} \rightarrow G_p$$

is an almost isomorphism.

Proof. Assume that M is a graph monoid. Let E be a (row-finite) countable graph without sinks such that $M \cong M(E)$. (The condition that E does not have sinks can be assumed because we can add a loop at every sink, without changing the corresponding graph monoid.) Let p be a free prime of M . Let $\mathcal{J} = \mathcal{J}_M$ be the I -system associated to M , so that $M = M(\mathcal{J})$ [10, Theorem 2.7]. The condition in the statement only depends on the restricted ($I \downarrow p$)-system $\mathcal{J}_{I \downarrow p}$. Moreover, by Proposition 2.8, Proposition 2.14 and Lemma 2.17, we get

$$M(\mathcal{J}_{I \downarrow p}) \cong I(H) \cong M(E_H),$$

where H is the hereditary and saturated subset of E^0 corresponding to the order-ideal $M(\mathcal{J}_{I \downarrow p})$ of $M = M(\mathcal{J})$. Therefore $M(\mathcal{J}_{I \downarrow p})$ is a graph monoid and, restricting attention to $I \downarrow p$, we may (and will) assume that p is the largest element of I .

We have

$$\mathcal{V}(L_k(E)) \cong M(E) \cong M$$

for any field k [9, Theorem 3.5]. Fix a field k for the rest of the argument. By Proposition 2.8, the order-ideals of M correspond to the lower subsets of the poset I . Moreover, the order-ideals of M correspond to the graded-ideals of $L_k(E)$ and to the hereditary saturated subsets of E^0 [9, Theorem 5.3].

Since p is the largest element of I , and $p + x = p + \varphi_{p,q}(x)$ for all $x \in M_q$, we get that $K_0(L_k(E)) = G(M) = G(M_p) = \mathbb{Z} \times G_p$, which is denoted by \widehat{G}_p (see Lemmas 2.7 and 2.9). Now, let \mathfrak{J} be the graded ideal of $L_k(E)$ corresponding to the lower subset J_p of I generated by the primes q_1, \dots, q_n in the lower cover of p . Let H be the hereditary and saturated subset of E^0 corresponding to \mathfrak{J} , so that $\mathfrak{J} = \mathfrak{J}(H)$. Since J_p is a maximal lower subset of I , it follows that $\mathfrak{J} = \mathfrak{J}(H)$ is a maximal graded-ideal of $L_k(E)$, and at the same time it gives rise to the maximal order-ideal $S := M(\mathcal{J}_{J_p})$ of M associated to the restricted J_p -system \mathcal{J}_{J_p} . Observe that $M/S \cong \mathbb{Z}^+$. Moreover, by Lemma 2.20, we have

$$\mathcal{V}(L_k(E)/\mathfrak{J}) \cong \mathcal{V}(L_k(E))/\mathcal{V}(\mathfrak{J}) \cong M/S \cong \mathbb{Z}^+.$$

Since $L_k(E)/\mathfrak{J} = L_k(E)/\mathfrak{J}(H) \cong L_k(E/H)$ is a graded-simple Leavitt path algebra, the tricotomy holds for $L_k(E/H)$ (see [2, Proposition 3.1.14]). Since $\mathcal{V}(L_k(E/H)) \cong \mathbb{Z}^+$, $L_k(E/H)$ must be Morita-equivalent to either k or $k[t, t^{-1}]$, the k -algebra of Laurent polynomials. As the graph E is row-finite and does not have sinks, the only possibility is that $L_k(E/H)$ is Morita-equivalent to $k[t, t^{-1}]$. Thus, the graph E/H must have a unique cycle c , without exits, to which all the other vertices of E/H connect. Let E' be the graph obtained from E by removing all vertices in $E^0 \setminus (H \cup c^0)$ and all the edges emitted by them. Then $(E')^0$ is hereditary in E^0 , and its saturation in E is precisely E^0 . Therefore $M(E') = M(E)$ by Lemma 2.17. Hence, replacing E with E' , we can assume that E/H consists exactly of a unique cycle c . Further, let v be a vertex in the cycle c . It is easily shown that replacing

the cycle c with a single loop based at v does not change the monoid $M(E)$. Thus, we can assume that E/H is a single loop, based at v . In particular, we have

$$L_k(E)/\mathfrak{J}(H) \cong L_k(E/H) \cong k[t, t^{-1}].$$

Observe that $K_0(\mathfrak{J}) = G(\mathcal{V}(\mathfrak{J})) = G(M_{J_p}) = \widetilde{G}_{J_p}$ by Lemma 2.9. Hence, the map $K_0(\mathfrak{J}) \rightarrow K_0(L_k(E))$ can be identified with the map $\iota \circ G(\varphi_p)$, where $\iota: G_p \rightarrow \widehat{G}_p = \mathbb{Z} \times G_p$ is the canonical inclusion. On the other hand, we have

$$K_1(L_k(E)/\mathfrak{J}) = K_1(k[t, t^{-1}]) = K_1(k) \oplus K_0(k) \cong k^\times \oplus \mathbb{Z}$$

(see e.g. [24, Theorem III.3.8]). Clearly, we have that the factor k^\times , which is generated by the units of the field k , is contained in the image of the map $K_1(L_k(E)) \rightarrow K_1(L_k(E)/\mathfrak{J})$, while the factor \mathbb{Z} is generated by multiplication by the unit t of $k[t, t^{-1}]$.

Hence, the exact sequence in K_0 and K_1 corresponding to the short exact sequence

$$0 \longrightarrow \mathfrak{J} \longrightarrow L_k(E) \longrightarrow k[t, t^{-1}] \longrightarrow 0$$

gives the exact sequence

$$\mathbb{Z} \xrightarrow{\partial} \widetilde{G}_{J_p} \xrightarrow{\iota \circ G(\varphi_p)} \widehat{G}_p \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (5.1)$$

The map $\partial: \mathbb{Z} \rightarrow \widetilde{G}_{J_p}$ is induced by the connecting homomorphism in algebraic K-theory. Since we can lift the unit t to the von Neumann regular element c in $vL_k(E)v$, we obtain (cf. [20, Proposition 1.3]) that

$$\partial([t]) = [v - cc^*] - [v - c^*c] \in K_0(\mathfrak{J}).$$

Now, $c^*c = v$ and $cc^* + \sum_{e \in s^{-1}(v) \setminus \{c\}} ee^* = v$, so that

$$\partial([t]) = \sum_{e \in s^{-1}(v) \setminus \{c\}} [ee^*] = \sum_{e \in s^{-1}(v) \setminus \{c\}} [r(e)] \in K_0(\mathfrak{J}).$$

Hence, the kernel of the canonical map $G(\varphi_p): \widetilde{G}_{J_p} \rightarrow G_p$ is generated by the element in $\widetilde{G}_{J_p} = G(M_{J_p})$ corresponding to the image of the element $\sum_{e \in s^{-1}(v) \setminus \{c\}} [r(e)]$ under the isomorphism $K_0(\mathfrak{J}) \rightarrow G(M_{J_p})$.

It remains to show that $\sum_{e \in s^{-1}(v) \setminus \{c\}} [r(e)]$ is a strictly positive element of \widetilde{G}_{J_p} . Let $\gamma: M \rightarrow M(E)$ be the isomorphism between M and $M(E)$. Then each $\gamma(q_i)$ is a prime element of the graph monoid $M(E_H)$ (see Lemma 2.17). Consider the tree $T(v)$ of v . Then, $T(v)$ is a hereditary subset of E^0 containing v , and by hypothesis the order-ideal $\mathfrak{J}(T(v))$ generated by $T(v)$ must be $M(E)$. By Lemma 2.16, $\mathfrak{J}(T(v))$ (and so $M(E)$) is generated *as a monoid* by all the elements of the form a_w with $w \in T(v)$. Fix an index $i \in \{1, \dots, n\}$. By the preceding argument, we can write $\gamma(q_i) = \sum_{j=1}^m a_{w_j}$, where $w_j \in T(v)$ for all j . Since $a_{w_j} \leq \gamma(q_i)$ for all j , we see that all $w_j \in H$. Since $\gamma(q_i)$ is prime, we get that $\gamma(q_i) \leq a_{w_j}$ for some j . So, for some vertex $w \in T(v) \cap H$, $\gamma(q_i) \equiv a_w$ (where \equiv denotes the antisymmetric relation generated by \leq). Thus, we can choose an edge $e \in s^{-1}(v) \setminus \{c\}$ such that $r(e)$ connects to w , and so $\gamma(q_i) \leq a_{r(e)}$. This shows that $q_i \leq \gamma^{-1}(a_{r(e)}) \in M(\mathcal{J}_{J_p})$. Therefore, there exists some lower subset J' of J_p such that $\gamma^{-1}(a_{r(e)}) \in M_{J'}$, and since $q_i \leq \gamma^{-1}(a_{r(e)})$, it follows that $q_i \in J'$. Since this holds for every $i = 1, \dots, n$, we conclude that $\gamma^{-1}(\sum_{e \in s^{-1}(v) \setminus \{c\}} a_{r(e)})$ belongs to the maximal component M_{J_p} of $M(\mathcal{J}_{J_p})$. Hence, $\sum_{e \in s^{-1}(v) \setminus \{c\}} [r(e)]$ is the image of

$\gamma^{-1}(\sum_{e \in s^{-1}(v) \setminus \{e\}} a_{r(e)})$ under the canonical map $M_{J_p} \rightarrow G(M_{J_p})$, and so it is strictly positive in $G(M_{J_p})$.

This shows that $G(\varphi_p)$ is an almost isomorphism. \square

To show the converse, we will use a method which is similar to the method employed in the proof of Theorem 3.6.

5.9. In order to establish the correct setting for the induction argument, we need to introduce some terminology and notation.

Since M is finitely generated, all the groups $G_M[p]$, for p a regular prime, are finitely generated. In the case where p is regular, we will assume that p is the neutral element of the group $G_p = M_p = G_M[p]$.

If p is a free prime, then the archimedean component $G_M[p]$ is of the form

$$G_M[p] = \mathbb{N} \times G_p,$$

where G_p is a finitely generated group. Here the free prime p is identified with the element $(1, e_p)$ of $\mathbb{N} \times G_p$.

We start by fixing a suitable form for these groups, namely we set, for $p = e$ a regular prime,

$$G_e = \mathbb{Z}^{r_e} \oplus \mathbb{Z}_{n_1^e} \oplus \cdots \oplus \mathbb{Z}_{n_{s_e}^e},$$

where $r_e, s_e \geq 0$, and $n_i^e \geq 1$ for all i . For $p = e$ a regular prime, set $N_e = r_e + s_e$. We denote by $\mathbf{x}_1, \dots, \mathbf{x}_{N_e}$ the canonical set of group generators of the group $G_e = \mathbb{Z}^{r_e} \oplus \mathbb{Z}_{n_1^e} \oplus \cdots \oplus \mathbb{Z}_{n_{s_e}^e}$. Further, we set $\mathbf{x}_{N_e+1} = -(\mathbf{x}_1 + \cdots + \mathbf{x}_{r_e})$, and we observe that $\mathbf{x}_1, \dots, \mathbf{x}_{N_e}, \mathbf{x}_{N_e+1}$ is a family of semigroup generators for G_e , which we will call the *canonical family of semigroup generators for G_e* .

For p a free prime, we will denote by $\{g_1^p, \dots, g_{N_p}^p\}$ a family of *semigroup generators* of the group G_p , that is, every element of G_p is a finite sum of some of the elements in the family $\{g_1^p, \dots, g_{N_p}^p\}$. Now, we have the following result:

Lemma 5.10. *If p is a free prime, then we can assume that each g_i^p is of the form $\varphi_{p,q}(g)$ for $q < p$, where g is either one of the canonical semigroup generators of G_q if q is a regular prime, or g is q if q is a free prime.*

Proof. We will show the result by (order-)induction. If p is a minimal prime which is free, then G_p is a trivial group, and the statement holds vacuously, taking the empty family of generators for G_p . Assume now that p is a free prime, and that the statement holds for all free primes below p . By [10, Definition 1.1(c2)] the map

$$\bigoplus_{q < p} \varphi_{p,q}: \bigoplus_{q < p} M_q \rightarrow G_p$$

is surjective. Thus, a family of semigroup generators for G_p is obtained by taking $\varphi_{p,\tilde{q}}(h)$, where $\tilde{q} < p$ ranges on the set of regular primes below p , and h ranges on the family of canonical semigroup generators of the group $G_{\tilde{q}}$, together with the family $\{\varphi_{p,q}(q)\} \cup \{\widehat{\varphi}_{p,q}(h)\}$, where $q < p$ ranges on the set of free primes below p , and h ranges on the family of canonical semigroup generators of G_q . By induction hypothesis, applied to the free prime q , each h can be taken of the form $\varphi_{q,q'}(h')$, where either $q' < q$ is a regular prime and h' is a canonical

semigroup generator in $G_{q'}$, or it is the form $\varphi_{q,q'}(q')$, where $q' < q$ is a free prime. In the former case, we get

$$\widehat{\varphi}_{p,q}(h) = \widehat{\varphi}_{p,q}(\varphi_{q,q'}(h')) = \widehat{\varphi}_{p,q}(\widehat{\varphi}_{q,q'}(h')) = \widehat{\varphi}_{p,q'}(h') = \varphi_{p,q'}(h'),$$

and in the latter case we get

$$\widehat{\varphi}_{p,q}(h) = \widehat{\varphi}_{p,q}(\widehat{\varphi}_{q,q'}(q')) = \widehat{\varphi}_{p,q'}(q') = \varphi_{p,q'}(q'),$$

which shows the result. \square

Definition 5.11. Let $p \in I_{\text{free}}$. We say that a family of elements $\{g_1^p, \dots, g_{N_p}^p\}$ of G_p is *the canonical set of semigroup generators* if this family consists of all elements of the form $\varphi_{p,q}(g)$ for $q < p$, where g is either one of the canonical semigroup generators of G_q if q is a regular prime, or g is q if q is a free prime.

We are now ready to prove the converse of Proposition 5.8.

Proposition 5.12. *Let M be a finitely generated conical refinement monoid such that the natural map $G(\varphi_p): \widetilde{G}_{J_p} \rightarrow G_p$ is an almost isomorphism for every free prime. Then there exists a countable row-finite directed graph E such that $M \cong M(E)$.*

Proof. Notice that, thanks to Lemma 5.10, we can always choose a canonical set of semigroup generators of G_p for any free prime $p \in M$. We will also require that $r + \sum_{i=1}^n N_{q_i} \leq s_p$ for each $p \in I_{\text{reg}}$, where $L(I, p) = \{q_1, \dots, q_n\}$ is the lower cover of p , and r is the number of free primes in $L(I, P)$.

We will show the result by order-induction.

Assume that we have a lower subset J of I , and that we have built a countable row-finite graph E_J such that $E_J^0 = \bigsqcup_{q \in J} E_q^0$, where each E_q is a strongly connected graph, with

$$E_q^0 = \{v_1^q, \dots, v_{N_q+1}^q, (v^q)_i^j \ (r_q + 1 \leq i \leq N_q, j \geq 1)\}$$

for a regular prime $q \in J$, and $E_q^0 = \{v^q\}$ if $q \in J$ is a free prime. We also assume that there is an isomorphism

$$\gamma_J: M(J) \rightarrow M(E_J),$$

where $M(J)$ is the order-ideal of M generated by J , such that γ_J sends the canonical semigroup generators $\mathbf{x}_1^q, \dots, \mathbf{x}_{N_q+1}^q$ of G_q to $v_1^q, \dots, v_{N_q+1}^q$ for all regular $q \in J$, and sends the element q of $G_M[q]$ to v^q for all free $q \in J$.

For $q' < q$ in J and $x \in M_{q'}$, we have $\gamma_J(\varphi_{q,q'}(x)) = \phi_{q'}^q(\gamma_J(x))$, where $\phi_{q'}^q: M(E_J)[q'] \rightarrow G(E_J)_q$ is the structural map associated to the \mathcal{J}_J -system coming from the finitely generated conical refinement monoid $M(E_J)$.

In case $J \neq I$, let p be a minimal element of $I \setminus J$ and write $J' = J \cup \{p\}$. We will show that the above statement holds for the lower subset J' in place of J . This clearly establishes the result, because I is a finite poset.

There are two cases to consider:

(1) **p is a minimal element of I :** In this case, we will associate a graph E_p to $G_M[p]$: when p is a regular prime, we define E_p using Lemma 3.1, while in case p is a free prime, we take E_p to be the one-loop graph based at the vertex v^p . Now, let $E_{J'} := E_p \sqcup E_J$. Then

$$M(J') = M(\{p\}) \oplus M(J) \cong M(E_p) \oplus M(E_J) = M(E_{J'})$$

in a canonical way, showing the result.

(2) **p is not a minimal element of I :** In this case, let $L(I, p) = \{q_1, \dots, q_n\}$ be the lower cover of p in I . Here, we will assume that q_1, \dots, q_r are free primes and q_{r+1}, \dots, q_n are regular primes. We will define $E_{J'} = E_J^0 \sqcup E_p^0$, where E_p^0 will be specified later. The edges in the graph $E_{J'}$ will be the edges coming from E_J and a new family of edges that we will describe. Now, we have two different cases to consider.

(i) p is a regular prime: In this case, the set of vertices E_p^0 is defined as in Lemma 3.1, so that

$$E_p^0 = \{v_1^p, \dots, v_{N_p+1}^p, (v^p)_i^j \ (r_p + 1 \leq i \leq N_p, j \geq 1)\}.$$

The relations \mathcal{R} that define the *new* edges of the graph $E_{J'}$, all departing from the vertices in E_p^0 , are as follows:

- (1) $v_{N_p+1}^p = 2v_{N_p+1}^p + \sum_{i=1}^r v_i^p + \sum_{i=r+1}^{N_p} n_{i-r}^p v_i^p$
- (2) For every $r + 1 \leq i \leq N$:
 - (a) $(v^p)_i^1 = n_{i-r}^p (v^p)_i + (v^p)_i^1 + (v^p)_i^3,$
 - (b) for every $j \geq 1$, $(v^p)_i^{2j} = (v^p)_i^{2j-1} + (v^p)_i^{2j},$
 - (c) for every $j \geq 1$, $(v^p)_i^{2j+1} = (v^p)_i^{2j} + (v^p)_i^{2j+1} + (v^p)_i^{2j+3},$
- (3) For every $1 \leq i \leq r$,

$$v_i^p = 2v_i^p + \sum_{j=1, j \neq i}^r v_j^p + \sum_{j=r+1}^{N_p} n_{j-r}^p v_j^p + v_{N_p+1}^p,$$

- (4) For every $r + 1 \leq i \leq N_p$,

$$v_i^p = \sum_{j=1}^r v_j^p + \sum_{j=r+1, 1j \neq i}^{N_p} n_{j-r}^p v_j^p + (n_{i-r}^p + 1)v_i^p + v_{N_p+1}^p + (v^p)_i^1 + A_i + v(i).$$

Here, for each $r + 1 \leq i \leq N_p$:

- A_i is a nonnegative integral linear combination of the vertices $v_1^p, \dots, v_{N_p+1}^p$, which depends on an element $g(i) \in \bigsqcup_{i=1}^n \widehat{G}_{q_i}$, as described below.
- $v(i)$ is a certain vertex in the graph E_J , which will be described below.

There is a map $i \mapsto g(i)$ from $[r_p + 1, r_p + r + \sum_{i=1}^n N_{q_i}] \cap \mathbb{Z}$ to $\bigsqcup_{i=1}^n \widehat{G}_{q_i}$ such that

- sends the set $[r_p + 1, r_p + r] \cap \mathbb{Z}$ bijectively to the set of free primes q_1, \dots, q_r in $L(I, p)$.
- establishes a correspondence between $[r_p + r + 1, r_p + r + \sum_{i=1}^r N_{q_i}] \cap \mathbb{Z}$ and $\bigcup_{j=1}^r \{g_j^{q_i} : j = 1, \dots, N_{q_i}\}$, and
- establishes a correspondence between $[r_p + r + \sum_{i=1}^r N_{q_i} + 1, r_p + r + \sum_{i=1}^n N_{q_i}] \cap \mathbb{Z}$ and $\bigcup_{t=r+1}^n \{\mathbf{x}_j^{q_t} : j = 1, \dots, N_{q_t}\}$.

Of course, we take A_i as the trivial linear combination (with all coefficients being 0) in case i is larger than $r_p + r + \sum_{i=1}^n N_{q_i}$.

Now, we specify the value of the term A_i and the corresponding vertex $v(i)$, which depend on the form of the specific generator $g(i)$. Suppose first that i belongs to the interval $[r_p + r + \sum_{i=1}^r N_{q_i} + 1, r_p + r + \sum_{i=1}^n N_{q_i}]$. In this case $g(i)$ is a canonical *group generator* of a

group G_q corresponding to a regular prime q in the lower cover of p . We write

$$-\varphi_{p,q}(g(i)) = \sum_{j=1}^{N_p+1} a_{ji} \mathbf{x}_j^p$$

for some non-negative integers a_{ji} . Then, we define

$$A_i = \sum_{j=1}^{N_p+1} a_{ji} v_j^p, \text{ and } v(i) = \gamma_J(g(i)).$$

We next consider the case where i belongs to the interval $[r_p + 1, r_p + r]$, so that $g(i) = q$ for a free prime q in the lower cover of p . Then we write

$$-\varphi_{p,q}(q) = \sum_{j=1}^{N_p+1} a_{ji} \mathbf{x}_j^p$$

for some non-negative integers a_{ji} , and we define

$$A_i = \sum_{j=1}^{N_p+1} a_{ji} v_j^p, \text{ and } v(i) = v^q = \gamma_J(q).$$

Finally, we need to consider the case where $g(i)$ is a canonical semigroup generator of the group G_q for a free prime q in the lower cover of p . This means by definition that either $g(i)$ is of the form $\varphi_{q,q'}(h)$, where $q' < q$ is a regular prime and h is a canonical semigroup generator of $G_{q'}$, or that $q' < q$ is a free prime and $g(i) = \varphi_{q,q'}(q')$. In the former case, we set

$$-\varphi_{p,q'}(h) = \sum_{j=1}^{N_p+1} a_{ji} \mathbf{x}_j^p$$

for some non-negative integers a_{ji} , and we define

$$A_i = \sum_{j=1}^{N_p+1} a_{ji} v_j^p, \text{ and } v(i) = \gamma_J(h).$$

In the latter case, we compute the non-negative integers a_{ji} using $-\varphi_{p,q'}(q')$, and we define

$$A_i = \sum_{j=1}^{N_p+1} a_{ji} v_j^p, \text{ and } v(i) = v^{q'} = \gamma_J(q').$$

Note that, in this situation, the same arguments as in Proposition 3.3 give that the subgraph E_p is strongly connected, and that there is a group homomorphism $\gamma_p: G_p \rightarrow M(E_{J'})[f]$ from the group G_p to the archimedean component $M(E_{J'})[f]$ of the graph monoid $M(E_{J'})$ corresponding to the vertices in E_p^0 , where f denotes the neutral element of that component, which sends the canonical semigroup generators $\mathbf{x}_1^p, \dots, \mathbf{x}_{N_p+1}^p$ of G_p to the canonical set $v_1^p, \dots, v_{N_p+1}^p$ of elements of the group $M(E_{J'})[f]$.

Define a map

$$\gamma_{J'}: M(J') \rightarrow M(E_{J'})$$

as follows. The map $\gamma_{J'}$ agrees with the isomorphism γ_J when restricted to the order-ideal $M(J)$ of $M(J')$. The map $\gamma_{J'}$ restricted to $G_{M(J')}[p] = G_M[p] = G_p$ is just the map γ_p above, which sends the canonical semigroup generators of G_p to the elements $v_1^p, \dots, v_{N_p+1}^p$ of $G_{M(E_{J'})}[f]$. In order to prove that $\gamma_{J'}$ gives a well-defined monoid homomorphism, it suffices by [10, Corollary 1.6] to show that if $q < p$ and $x \in G_M[q]$ then $\gamma_{J'}(x) + \gamma_{J'}(p) = \gamma_{J'}(\varphi_{p,q}(x)) + \gamma_{J'}(p)$, that is, $\gamma_{J'}(x) + f = \gamma_{J'}(\varphi_{p,q}(x))$. By the same argument used in the proof of Theorem 3.6, it is enough to consider the case where q belongs to the lower cover of p . We will consider only the case where q is a free prime. (The case where q is a regular prime is easier and is left to the reader). Assume that q is a free prime in the lower cover of p . Consider first the case where $x = q$. There is some i such that $g(i) = q$ and thus $v(i) = v^q$. Observe that, after using the relations \mathcal{R} , relation $\mathcal{R}(4)$ can be expressed as:

$$v_i^p = v_i^q + f + A_i + v(i) = v_i^q + f - \gamma_{J'}(\varphi_{p,q}(q)) + v^q.$$

So, we obtain $\gamma_{J'}(\varphi_{p,q}(q)) = f + v^q$, that is, $f + \gamma_{J'}(x) = \gamma_{J'}(\varphi_{p,q}(x))$, as desired. Now suppose that $x \in M_q$. Write $x = mq + \sum_t \varphi_{q,q'_t}(h_t)$, where $m \in \mathbb{N}$, $q'_t < q$, and h_t is either a canonical semigroup generator of $G_{q'_t}$ (in case q'_t is a regular prime) or q'_t (in case q'_t is a free prime). Assume that we have proven that

$$\gamma_{J'}(\widehat{\varphi}_{p,q}\varphi_{q,q'_t}(h_t)) = f + \gamma_{J'}(h_t) \quad (5.2)$$

for each t . Then, using (5.2), the induction hypothesis and the fact that γ_J and γ_p are semigroup homomorphisms, we get

$$\begin{aligned} \gamma_{J'}(\varphi_{p,q}(x)) &= \gamma_{J'}(\varphi_{p,q}(mq + \sum_t \varphi_{q,q'_t}(h_t))) \\ &= m\gamma_{J'}(\varphi_{p,q}(q)) + \sum_t \gamma_{J'}(\widehat{\varphi}_{p,q}\varphi_{q,q'_t}(h_t)) \\ &= f + m\gamma_J(q) + f + \sum_t \gamma_J(h_t) \\ &= f + \gamma_J(mq + \sum_t h_t) = f + \gamma_J(mq + \sum_t \varphi_{q,q'_t}(h_t)) \\ &= f + \gamma_{J'}(x). \end{aligned}$$

Thus, it remains to prove (5.2). Assume that q'_t is a free prime, so that $h_t = q'_t$. Let i be the index such that $g(i) = \varphi_{q,q'_t}(q'_t)$ and $v(i) = v^{q'_t} = \gamma_{J'}(q'_t)$. Then $\mathcal{R}(4)$ gives again

$$v_i = v_i + f - \gamma_{J'}(\varphi_{p,q'_t}(q'_t)) + \gamma_{J'}(q'_t).$$

Since $\varphi_{p,q'_t}(q'_t) = \widehat{\varphi}_{p,q}\varphi_{q,q'_t}(q'_t)$, we get

$$\gamma_{J'}(\widehat{\varphi}_{p,q}\varphi_{q,q'_t}(q'_t)) = f + \gamma_{J'}(q'_t),$$

as desired. The case where q'_t is a regular prime and h_t is a canonical semigroup generator of $G_{q'_t}$ is treated in the same way.

This shows that the defining relations of $M(J')$ are preserved. So, the map $\gamma_{J'}$ is a well-defined homomorphism from $M(J')$ to $M(E_{J'})$. To build the inverse $\delta_{J'}$ of $\gamma_{J'}$, we follow the idea in the proof of Proposition 3.3. The image by $\delta_{J'}$ of the vertices in E_J is determined by the inverse of γ_J . We define $\delta_{J'}(v_i^p) = \mathbf{x}_i^p$ for $1 \leq i \leq N_p + 1$, and $\delta_{J'}((v^p)_i^j) = p$. It is easily

checked that $\delta_{J'}$ preserves the defining relations of $M(E_{J'})$, and so it gives a well-defined homomorphism from $M(E_{J'})$ to $M(J')$, which is clearly the inverse of $\gamma_{J'}$.

(ii) p is a free prime: In this case, the canonical map $G(\varphi_p): \widetilde{G}_{J_p} \rightarrow G_p$ is an almost isomorphism by hypothesis. Hence, $G(\varphi_p)$ is surjective and its kernel is generated by a strictly positive element x . Note that J_p is a lower subset of J .

By Lemma 5.3, we can write

$$x = \sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i)$$

for some $n_i \in \mathbb{N}$, $i = 1, \dots, r$, and $g_i \in G_{q_i}$, $i = 1, \dots, n$. By Lemma 5.10, for $i \in \{1, \dots, r\}$ we can write

$$g_i = \sum_{q' < q_i, q' \in I_{\text{free}}} a_i^{q'} \varphi_{q_i, q'}(q') + \sum_{q'' < q_i, q'' \in I_{\text{reg}}} \sum_{j=1}^{N_{q''}+1} b_{ji}^{q''} \varphi_{q_i, q''}(\mathbf{x}_j^{q''})$$

for some non-negative integers $a_i^{q'}$ and $b_{ji}^{q''}$, and we can write, for $i \in \{r+1, \dots, n\}$,

$$g_i = \sum_{j=1}^{N_{q_i}+1} a_{ji} \mathbf{x}_j^{q_i},$$

for some non-negative integers a_{ji} . Define the graph $E_{J'}$ with $E_{J'}^0 = E_J^0 \sqcup \{v^p\}$, and with $E_{J'}^1$, the union of E_J^1 and a set of edges starting at v^p , which are determined by the following formula:

$$v^p = v^p + \sum_{i=1}^r n_i v^{q_i} + \sum_{i=1}^r (A_i + B_i) + \sum_{i=r+1}^n C_i, \quad (5.3)$$

where

$$A_i = \sum_{q' < q_i, q' \in I_{\text{free}}} a_i^{q'} v^{q'}, \quad B_i = \sum_{q'' < q_i, q'' \in I_{\text{reg}}} \sum_{j=1}^{N_{q''}+1} b_{ji}^{q''} v_j^{q''}, \quad (i = 1, \dots, r) \quad (5.4)$$

and

$$C_i = \sum_{j=1}^{N_{q_i}+1} a_{ji} v_j^{q_i}, \quad (i = r+1, \dots, n). \quad (5.5)$$

If we define $\widehat{x} := \sum_{i=1}^r n_i v^{q_i} + \sum_{i=1}^r (A_i + B_i) + \sum_{i=r+1}^n C_i$, then we have that $\gamma_J(x) = \widehat{x}$.

The element v_p of $M(E_{J'})$ is a free prime, and so $M(E_{J'})[v_p] = \mathbb{N} \times G'_{v_p}$ for some abelian group G'_{v_p} . It follows easily from the induction hypothesis and the form of the relation (5.3) that the map $\gamma_J|_J$ extends to an order-isomorphism from $J' = J \sqcup \{p\}$ to the set \mathbb{P} of primes of $M(E_{J'})$, by sending p to v^p . Since $M(E_{J'})$ is a finitely generated conical refinement monoid, the map $\phi_p: M(E_{J'})_{\gamma_J(J_p)} \rightarrow G'_{v_p}$ induced by the various semigroup homomorphisms

$$\begin{aligned} \phi_q^p: M(E_{J'})_{\gamma_J(q)} &\rightarrow G'_{v_p} \\ y &\mapsto (v^p + y) - v^p \end{aligned}$$

for $q < p$ is surjective. So, we obtain a surjective group homomorphism

$$G(\phi_p): G(M(E_{J'})_{\gamma_J(J_p)}) \rightarrow G'_{v_p}.$$

In order to somewhat simplify the notation, we will write $M(E_{J'})_{J_p}$ instead of $M(E_{J'})_{\gamma_J(J_p)}$.

Since E_J^0 is a hereditary and saturated subset of $E_{J'}^0$, the order ideal $\mathfrak{J}(E_J^0)$ of $M(E_{J'})$ generated by E_J coincides with the monoid $M(E_J)$ (by Lemma 2.17), and the component $M(E_{J'})_{J_p}$ coincides with the component $M(E_J)_{J_p}$. The monoid isomorphism $\gamma_J: M(J) \rightarrow M(E_J)$ restricts to a semigroup isomorphism $M_{J_p} \rightarrow M(E_J)_{J_p}$, which induces a group isomorphism

$$\tilde{\gamma}_{J_p}: G(M_{J_p}) \rightarrow G(M(E_J)_{J_p})$$

of the Grothendieck groups. Set $K := \ker(G(\phi_p))$, and notice that the relation (5.3) implies that $\tilde{\gamma}_{J_p}(x) = \hat{x} \in K$.

Hence, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle x \rangle & \longrightarrow & G(M_{J_p}) & \xrightarrow{G(\varphi_p)} & G_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \tilde{\gamma}_{J_p} & & \downarrow \gamma_p & & \\ 0 & \longrightarrow & K & \longrightarrow & G(M(E_J)_{J_p}) & \xrightarrow{G(\phi_p)} & G'_{v_p} & \longrightarrow & 0, \end{array} \quad (5.6)$$

where $\gamma_p: G_p \rightarrow G'_{v_p}$ is the map induced from the cokernel of the inclusion $\langle x \rangle \hookrightarrow G(M_J)$ to the cokernel of the inclusion $K \hookrightarrow G(M(E_J)_{J_p})$. Notice that γ_p is an onto map.

Now, we define a map

$$\gamma_{J'}: M(J') \rightarrow M(E_{J'})$$

extending the monoid isomorphism $\gamma_J: M(J) \rightarrow M(E_J)$, and defining $\gamma_{J'}$ on the component $M_p \cong \mathbb{N} \times G_p$ of $M(J')$ by the formula

$$\gamma_{J'}(mp + g) = mv^p + \gamma_p(g),$$

for $m \in \mathbb{N}$ and $g \in G_p$. By [10, Corollary 1.6], to show that $\gamma_{J'}$ is a well-defined monoid homomorphism, it suffices to show that if $q < p$ and $y \in G_M[q] = M_q$ then $\gamma_{J'}(y) + \gamma_{J'}(p) = \gamma_{J'}(\varphi_{p,q}(y) + p)$, that is, $\gamma_J(y) + v^p = \gamma_p(\varphi_{p,q}(y)) + v^p$. As $y \in G_M[q] = M_q$ and we are identifying $G(M(J_p)) \cong G(M_{J_p}) = \tilde{G}_{J_p}$ (Lemma 2.6), there exists a composition map

$$\tau_q: M_q \rightarrow M(J_p) \rightarrow G(M(J_p)) \cong G(M_{J_p})$$

such that $\varphi_{p,q} = G(\varphi_p) \circ \tau_q$. Analogously, we have a map

$$\tau_{\gamma_J(q)}: M(E_{J'})_{\gamma_J(q)} \rightarrow G(M(E_{J'})_{J_p}) = G(M(E_J)_{J_p})$$

such that $\phi_{\gamma_J(q)}^{v^p} = G(\phi_p) \circ \tau_{\gamma_J(q)}$, and clearly $\tilde{\gamma}_{J_p} \circ \tau_q = \tau_{\gamma_J(q)} \circ \gamma_J|_{M_q}$.

Using this fact, and the commutativity of (5.6), we have that

$$\begin{aligned} \gamma_p(\varphi_{p,q}(y)) + v^p &= \gamma_p(G(\varphi_p)(\tau_q(y))) + v^p \\ &= G(\phi_p)(\tilde{\gamma}_{J_p}(\tau_q(y))) + v^p \\ &= G(\phi_p)(\tau_{\gamma_J(q)}(\gamma_J(y))) + v^p \\ &= \phi_{\gamma_J(q)}^{v^p}(\gamma_J(y)) + v^p \\ &= ((v^p + \gamma_J(y)) - v^p) + v^p \\ &= v^p + \gamma_J(y), \end{aligned}$$

as desired.

This shows that there is a well-defined monoid homomorphism $\gamma_{J'}: M(J') \rightarrow M(E_{J'})$ sending the canonical semigroup generators of $M(J')$ to the corresponding canonical sets of vertices seen in $M(E_{J'})$. In particular, $\gamma_{J'}$ is an onto map.

In order to prove the injectivity of $\gamma_{J'}$, we can build an inverse map $\delta_{J'}: M(E_{J'}) \rightarrow M(J')$, as follows. On $M(E_{J'})$ we define $\delta_{J'}$ to be $\gamma_{J'}^{-1}$, while $\delta_{J'}(v_p) := p$. Notice that the only relation on $M(E_{J'})$ not occurring already in $M(E_J)$ is $v^p = v^p + \hat{x}$, where $\gamma_J(x) = \hat{x}$. Thus, $\delta_{J'}(\hat{x}) = x$. But x generates the kernel of the map

$$G(\varphi_p): \tilde{G}_{J_p} \rightarrow G_p \hookrightarrow \hat{G}_p = \mathbb{Z} \times G_p,$$

so that $(p+x) - p$ equals 0 in G_p . Hence, the relation $p = p+x$ holds in $M(J')$. Thus, $\delta_{J'}$ is a well-defined monoid homomorphism, and it is the inverse of $\gamma_{J'}$. This completes the proof of the inductive step, and so the result holds, as desired. \square

As an illustration of Theorem 5.6, we obtain the characterization of antisymmetric finitely generated graph monoids [11].

Corollary 5.13. [11, Theorem 5.1] *Let M be a finitely generated antisymmetric refinement monoid. Then M is a graph monoid if and only if for each free prime p in M the lower cover of p contains at most one free prime.*

Proof. Note that in the antisymmetric case we have that the groups G_q are trivial for all primes q . Consequently the maps $\varphi_{p,q}$, for $q < p$ are all the zero map.

Let p be a free prime of M . If q_1, \dots, q_r are the free primes in the lower cover of p , then the component M_{J_p} is isomorphic to \mathbb{N}^r and thus $G(M_{J_p}) = \mathbb{Z}^r$. The canonical map

$$G(\varphi_p): G(M_{J_p}) \longrightarrow G_p$$

reduces to the trivial map $\mathbb{Z}^r \longrightarrow 0$. If $r > 1$ the kernel of this map cannot be cyclic. Conversely if $r \leq 1$ the kernel of this map is generated by a strictly positive element coming from $\sum_{q \in L(I,p)} q$. \square

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