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The Strong Resolving Graph and the Strong Metric Dimension of Cactus Graphs

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Abstract: A vertex w of a connected graph G strongly resolves two distinct vertices $u, v \in V(G)$, if there is a shortest u, w path containing v , or a shortest v, w path containing u . A set S of vertices of G is a *strong resolving set* for G if every two distinct vertices of G are strongly resolved by a vertex of S . The smallest cardinality of a strong resolving set for G is called the *strong metric dimension* of G . To study the strong metric dimension of graphs, a very important role is played by a structure of graphs called the strong resolving graph. In this work, we obtain the strong metric dimension of some families of cactus graphs, and along the way, we give several structural properties of the strong resolving graphs of the studied families of cactus graphs.

Keywords: strong resolving graph; strong metric dimension; strong resolving set; cactus graphs; unicyclic graphs

MSC: 05C12

1. Introduction

Topics concerning metric dimension and related parameters in graphs are nowadays very common in the research community, probably based on its applicability to diverse practical problems of identification of nodes in networks. One can find in the literature a large number of works dealing with this topic, both from the applied and theoretical points of views. A popular research line in this subject concerns studying different variants of metric dimension in graphs, which have had their beginnings in the seminal standard metric dimension concept. Some of the most recent ones are probably the edge metric dimension [1], the mixed metric dimension [2], the k -metric antidimension [3], the strong partition dimension [4], and the multiset dimension [5,6], just to cite a few recent and remarkable cases. One other interesting version is the strong metric dimension [7], which is now relatively well studied, although a few open questions on this are still open. A fairly complete study on results and open questions concerning the strong metric dimension of graphs can be found in [8].

One significant reason for the interest of several researchers in the strong metric dimension of graphs concerns the closed relationship that exists between such parameter and the very well known vertex cover number of graphs (and thus with the independence number, based on the Gallai's Theorem). To see this relationship, for a given graph G , the construction of a new related graph, called strong resolving graph, was required. This graph transformation clearly raised some other related questions on the transformation itself. That is for instance, given a graph G : can some properties of the strong resolving graph of G be deduced? or; can we realize every graph H as the strong resolving graph of another graph H' ? These ones and several other questions were dealt with in [9], which was the first work paying specific attention to the strong resolving graphs of graphs as a special graph transformation. See also [10], where an open problem from [9] was settled.

Clearly, and as we will further notice, a good knowledge of the strong resolving graph of a graph brings important contributions to studying the strong metric dimension of graphs. In this sense, this work is precisely aimed to study the strong resolving graphs and the strong metric dimension of cactus graphs, with some emphasis on different special structures of such cactus graphs. As one will also note through our exposition, strong resolving graphs are very challenging for those graphs having a large number of induced cycles. Thus, cactus graphs represent a significant example of such a situation. With this work, we also contribute to some open problems presented in [9].

The study of the strong metric dimension of some classes of cactus graphs was started in [11,12] where the authors presented some general results for the strong metric dimension of corona product graph and rooted product graphs, respectively. Clear definitions of these two graph products can be found in [8]. A corona product graph or a rooted product graph can have the structure of a cactus graph, depending on which are the graphs used as factors in the product. For instance, if G is a cycle and H is a graph whose components are only singleton vertices or complete graphs K_2 , then it happens that the corona product graph $G \odot H$ is a cactus graphs. To generate a rooted product graph that is a cactus graph, we may consider for example two graphs G and H which are paths or cycles.

On the other hand, we must mention that the strong metric dimension of unicyclic graphs (which is a cactus graph too) was studied in [13]. There, among other results, several relationships between the strong metric dimension of a unicyclic graph and that of its complement were given. A few other sporadic results can be found in some other articles dealing with related topics that could include examples of cactus graphs. However, we prefer to not include more references that are not essentially connected with this article.

We hence now begin to formalize all the required notations and terminologies that shall be used throughout the document. To this end, for the whole exposition, let G be a connected simple graph with vertex set $V(G)$. For two adjacent vertices $x, y \in V(G)$, we use the notation $x \sim y$. For a vertex x of G , $N_G(x)$ denotes the set of neighbors that x has in G , i.e., $N_G(x) = \{y \in V(G) : y \sim x\}$. The set $N_G(x)$ is called the *open neighborhood of a vertex x* in G and $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neighborhood of a vertex x* in G . The *degree* of the vertex x is $\delta_G(x) = |N_G(x)|$. The diameter of G is defined as $D(G) = \max_{x,y \in V(G)} \{d_G(x,y)\}$, where $d_G(x,y)$ is the length of a shortest path between x and y (a shortest x,y path). Two vertices x,y are called *diametral* if $d_G(x,y) = D(G)$. For a set $S \subset V(G)$, by $\langle S \rangle$ we represent the subgraph induced by S in G .

1.1. Strong Metric Dimension of Graphs

For two distinct vertices $u, v \in V(G)$, a vertex $w \in V(G)$ *strongly resolves* u, v if there is a shortest u, w path containing v , or a shortest v, w path containing u . Note that it could happen $w \in \{u, v\}$. A set S of vertices of G is a *strong resolving set* for G , if every two vertices of G are strongly resolved by some vertex of S . The smallest cardinality among all strong resolving sets for G is called the *strong metric dimension* of G , and is denoted by $dim_s(G)$. We say that a strong resolving set for G of cardinality $dim_s(G)$ is a *strong metric basis* of G . It next appears the value of the strong metric dimension of some basic graphs.

Observation 1. Let G be a connected graph G of order $n \geq 2$.

- (a) $dim_s(G) = n - 1$ if and only if $G \cong K_n$.
- (b) If $G \not\cong K_n$, then $dim_s(G) \leq n - 2$.
- (c) $dim_s(G) = 1$ if and only if $G \cong P_n$.
- (d) If $G \cong C_n$, then $dim_s(G) = \lceil n/2 \rceil$.
- (e) If G is a tree with l leaves, $dim_s(G) = l - 1$.

It is said that a vertex u of G is *maximally distant* from v if for every $w \in N_G(u)$, it happens $d_G(v,w) \leq d_G(u,v)$. If u is maximally distant from v and v is maximally distant from u , then u and v are *mutually maximally distant*, and we write that u, v are MMD in G . The set of MMD vertices of G is

denoted by $\partial(G)$. Note that the set of MMD vertices of a graph G is also known as the *boundary* of G , as defined in [14,15]. An explanation on the equivalence of these two objects can be readily observed, but also found in [16]. From these definitions, the following remarks are straightforward to observe.

Remark 1. *Let G be a connected graph. Then every two vertices with degree 1 are MMD in G .*

For any two mutually maximally distant vertices in G , there is no vertex of G that strongly resolves them, except themselves. This allows to claim the following.

Remark 2. *For every pair of mutually maximally distant vertices x, y of a connected graph G , and for every strong metric basis S of G , it follows that $x \in S$ or $y \in S$.*

1.2. Strong Resolving Graph of a Graph

Given a connected graph G , the *strong resolving graph* of G , denoted by G_{SR} , has vertex set $\partial(G)$ and two vertices u, v are adjacent if and only if u and v are MMD in G . We must remark that the strong resolving graph of a graph G was defined in [7] as the graph with vertex set $V(G)$ and two vertices u, v are adjacent if and only if u and v are MMD in G . Observe that the difference between these two definitions is the existence of isolated vertices in the strong resolving graph from [7]. The main reason of using in this work the slightly different version is to have a simpler notation and more clarity while proving the results. Moreover, this fact does not influence on the computations we made.

For several basic families of graphs, describing their strong resolving graphs is a straightforward problem. We next recall some examples, which will maybe further useful, and to this end, we recall that a vertex v of a graph G is *simplicial*, if its closed neighborhood induces a complete graph, and also that a graph G is *2-antipodal* if every vertex of G is diametral with exactly one other vertex of G .

Observation 2.

- (a) *If $\partial(G)$ equals the set of simplicial vertices of G , then $G_{SR} \cong K_{|\partial(G)|}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T , $T_{SR} \cong K_{I(T)}$.*
- (b) *For any 2-antipodal graph G of order n , $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^k K_2$.*
- (c) *For odd cycles $(C_{2k+1})_{SR} \cong C_{2k+1}$.*
- (d) *For any complete k -partite graph $G = K_{p_1, p_2, \dots, p_k}$ such that $p_i \geq 2, i \in \{1, 2, \dots, k\}$, $G_{SR} \cong \bigcup_{i=1}^k K_{p_i}$.*

In [9], realization and characterization problems of the strong resolving graph of a graph as a graph transformation were firstly dealt with. That is, the following problems were studied.

- **Realization Problem.** Determine which graphs have a given graph as their strong resolving graphs.
- **Characterization Problem.** Characterize those graphs that are strong resolving graphs of some graphs.

For instance, in [9] was proved that complete graphs, paths and cycles of order larger than four are realizable as the strong resolving graph of other graphs. On the other hand, it was also proved in [9] that stars and cycles of order four are not realizable as strong resolving graphs. Based on these two facts, a conjecture concerning the not realization of complete bipartite graphs in general was pointed out. Such conjecture was recently shown in [10].

In connection with these comments, it would be desirable to continue obtaining some realization (and also characterization - although much more complicated) results for the strong resolving graphs of graphs. We are then aimed in this work to present some realization results which are involving cactus graphs.

1.3. Strong Metric Dimension of G versus Vertex Cover Number of G_{SR}

Oellermann and Peters-Fransen [7] showed that the problem of finding the strong metric dimension of graphs can be transformed into the well-known problem regarding the vertex cover of graphs. A set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S . The *vertex cover number* of G , denoted by $\beta(G)$, is the smallest cardinality of a vertex cover of G . We refer to a $\beta(G)$ -set in a graph G as a vertex cover set of cardinality $\beta(G)$.

Theorem 1 ([7]). *For any connected graph G ,*

$$dim_s(G) = \beta(G_{SR}).$$

Recall that the largest cardinality of a set of vertices of G , no two of which are adjacent, is called the *independence number* of G and is denoted by $\alpha(G)$. We refer to an $\alpha(G)$ -set in a graph G as an independent set of cardinality $\alpha(G)$. The following well-known and useful result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

Theorem 2 (Gallai’s theorem). *For any graph G of order n ,*

$$\alpha(G) + \beta(G) = n.$$

Thus, by using Theorems 1 and 2 we immediately obtain the next result.

Corollary 1. *For any graph G ,*

$$dim_s(G) = |\partial(G)| - \alpha(G_{SR}).$$

2. Cactus Graphs: General Issues

A *cactus graph* (also called a cactus tree) is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge of the graph belongs to at most one simple cycle. Next we study the strong metric dimension of cactus graphs, and we first give some necessary terminology. Note that a cycle of two vertices is precisely a path on two vertices. A vertex belonging to at least two simple cycles is a *cut vertex*. A cycle having only one cut vertex is called a *terminal cycle*. In a terminal cycle A , every vertex being diametral, in the subgraph induced by A , with respect to the cut vertex of A is a *terminal vertex*. From now on, $\tau(G)$ denotes the set of terminal vertices of G . Also, $\zeta_2(G)$ denotes the set of vertices v , of degree two, belonging to a cycle of order larger than two, being MMD only with vertices of the same cycle which v belongs. Moreover, $\iota_2(G)$ denotes the set of vertices u , of degree two, belonging to a cycle of order larger than two being MMD with at least one vertex of a different cycle which u belongs. The following remark can be easily observed.

Remark 3. *Let G be a cactus graph. Then, two vertices x, y are MMD in G if and only if $x, y \in \zeta_2(G) \cup \iota_2(G) \cup \tau(G)$.*

Corollary 2. *For any cactus graph G , $\partial(G) = \zeta_2(G) \cup \iota_2(G) \cup \tau(G)$.*

Theorem 3. *Let G be a cactus graph. Then*

$$|\tau(G)| + \left\lfloor \frac{|\zeta_2(G)|}{2} \right\rfloor - 1 \leq dim_s(G) \leq |\tau(G)| + |\iota_2(G)| + \left\lfloor \frac{|\zeta_2(G)|}{2} \right\rfloor.$$

Proof. The lower bound follows from the following facts. Any two terminal vertices of G are MMD on G , and thus, they induce a complete graph of order $|\tau(G)|$. Also, vertices of $\zeta_2(G)$ induce at least

a graph with $\lfloor \frac{|c_2(G)|}{2} \rfloor$ independent edges that need to be covered in G_{SR} . Thus, one needs at least $|\tau(G)| - 1 + \lfloor \frac{|c_2(G)|}{2} \rfloor$ to strongly resolve all the vertices of G .

To see the upper bound, it is only necessary to observe that the set $\tau(G) \cup c_2(G)$ together with half of vertices of the set $c_2(G)$ form a strong resolving set of G , and so, we are done. \square

Despite the fact that the bounds above are easily proved, we might notice that the problem of describing the strong resolving graph, and similarly, of computing the strong metric dimension of cactus graphs seems to be very challenging based on the situation that we can not control things like the orders of the involved cycles, the number of terminal vertices and cut vertices, their adjacencies, etc. In this sense, it is desirable to introduce extra conditions on the cactus graphs to have more possibilities to give some practical results.

3. Strong Resolving Graphs

In this section we aim to describe the structure of the strong resolving graphs of several different families of cactus graphs. We specifically center our attention into unicyclic graphs, bouquet of cycles and chains of even cycles. With some of these results we contribute to the problem of realization of some graphs as strong resolving graphs, that is, to the problems previously presented.

3.1. Unicyclic Graphs

Given a unicyclic graph G different from a cycle, from now on we will denote by $C_r = v_1v_2 \dots v_rv_1$ the subgraph induced by the unique cycle of G . A vertex $v \in V(G)$ of degree one is a *terminal vertex* of G , and $T(G)$ is the set of terminal vertices of G . Note that the terminal vertices defined here represent a particular case of the terminal vertices defined for cactus graphs in general. If the vertex v_i of C_r has degree greater than two, then we say that u_i is a terminal vertex of v_i , if $d_G(u_i, v_i) = \min\{d_G(u_i, v_j) : v_j \neq v_i\}$. The set of terminal vertices of a vertex v_i is denoted by $t(v_i)$. We will denote by $c_2(G)$ the set of vertices of the cycle C_r having degree two. If $v \in c_2(G)$, then we will say that $t(v) = \emptyset$.

Notice that if the unicyclic graph G is isomorphic to the cycle C_n , then for n even $(C_n)_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$ and for n odd $(C_n)_{SR} \cong C_n$ as already presented in Observation 2. Thus, we will study the cases that $G \not\cong C_n$.

We begin with the following straightforward observations that are useful to describe the strong resolving graph of any unicyclic graph.

Remark 4. Let G be a unicyclic graph. For every vertex $x \in c_2(G)$ there exists at least one vertex $y \in c_2(G) \cup T(G)$ such that x, y are MMD in G .

Remark 5. Let G be a unicyclic graph. Then two vertices x, y are MMD in G if and only if $x, y \in c_2(G) \cup T(G)$.

Corollary 3. For any unicyclic graph G , $\partial(G) = c_2(G) \cup T(G)$.

Notice that every two vertices $x, y \in T(G)$ are MMD. Also, every vertex $v \in c_2(G)$ is MMD with every vertex w satisfying one of the following conditions.

- w is a terminal vertex of a vertex u of C_r such that u, v are diametral vertices in C_r .
- w is a diametral vertex with v in C_r and $w \in c_2(G)$.

As a consequence of the above comments, we can deduce the structure of the strong resolving graph of any unicyclic graph G in the following way. First notice that, according to Corollary 3, G_{SR} has vertex set equal to $c_2(G) \cup T(G)$, and to describe the adjacency of vertices in G_{SR} we consider two cases.

G_{SR} for r even.

- The set $T(G)$ forms a clique in G_{SR} and each vertex of $T(G)$ has at most one neighbor in $c_2(G)$.
- If $x, y \in c_2(G)$ are diametral vertices in C_r , then $\langle \{x, y\} \rangle$ is a connected component of G_{SR} isomorphic to K_2 .
- If x, y are diametral vertices in C_r , $x \in c_2(G)$ and $y \notin c_2(G)$, then $\{x\} \cup t(y)$ forms a subgraph of G_{SR} isomorphic to $K_{|t(y)|+1}$ and $N_{G_{SR}}(x) = t(y)$.

As a consequence of the description above, we can observe that $\beta(G_{SR}) \leq \frac{|c_2(G)|-1}{2} + |T(G)|$.

G_{SR} for r odd.

- The set $T(G)$ forms a clique in G_{SR} and each vertex of $T(G)$ has at most two neighbors in $c_2(G)$.
- Let $u \in c_2(G)$ and let x, y being diametral vertices with u in C_r .
 - If $x, y \in c_2(G)$, then $\langle \{u, x, y\} \rangle$ is a subgraph of G_{SR} isomorphic to P_3 , $N_{G_{SR}}(u) = \{x, y\}$ and for every $w \in \{x, y\}$, $\delta_{G_{SR}}(w) \geq 2$.
 - If $x, y \notin c_2(G)$, then $\langle \{u\} \cup t(x) \cup t(y) \rangle$ is a subgraph of G_{SR} isomorphic to $K_{|t(x)|+|t(y)|+1}$, $N_{G_{SR}}(u) = t(x) \cup t(y)$ and for every $w \in t(x) \cup t(y)$, $\delta_{G_{SR}}(w) \geq |t(x)| + |t(y)| + 1$ for $r \geq 5$ (notice that if $r = 3$, then $\delta_{G_{SR}}(w) = |t(x)| + |t(y)|$).
 - If $x \in c_2(G)$ and $y \notin c_2(G)$, then the set $\{u, x\} \cup t(y)$ form a subgraph (not induced) (Notice that the vertices $t(y)$ are adjacent between them in G_{SR} .) of G_{SR} isomorphic to a star graph $S_{1,|t(y)|+1}$ with central vertex u , $N_{G_{SR}}(u) = \{x\} \cup t(y)$, $\delta_{G_{SR}}(x) \geq 2$ and for every $w \in t(y)$, $\delta_{G_{SR}}(w) \geq |t(y)| + 1$.

Similarly to the case when r is even, we can observe here that $\beta(G_{SR}) \leq \frac{|c_2(G)|}{2} + |T(G)|$.

We define the *branch restricted unicyclic graph* $\mathcal{T}(G)$ associated to a unicyclic graph G in the following way. We begin with taking the cycle C_r in G and removing the remaining vertices of G . Then we add $t(v_i)$ pendant edges to every vertex v_i in C_r . Figure 1 shows an example of a unicyclic graph, its branch restricted unicyclic graph and its strong resolving graph.

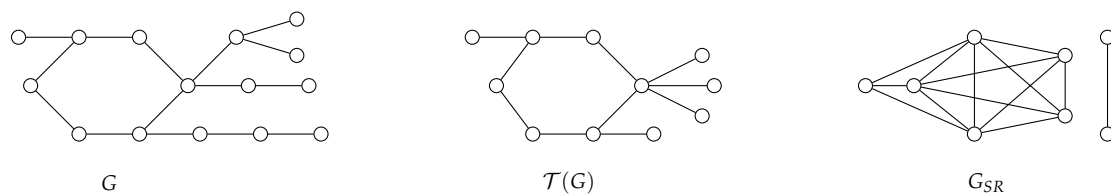


Figure 1. A unicyclic graph G , $\mathcal{T}(G)$ and G_{SR} .

Lemma 1. Let G be a unicyclic graph and $\mathcal{T}(G)$ be its branch restricted unicyclic graph. Then $(\mathcal{T}(G))_{SR}$ is isomorphic to G_{SR}

Proof. From Remarks 4 and 5, and by the definition of the branch restricted unicyclic graph, we deduce that $(\mathcal{T}(G))_{SR}$ is isomorphic to G_{SR} . □

Our next step is dedicated to present a realization result for some corona product graphs, where the solution precisely involves the use of unicyclic graphs. We first recall that the *corona product graph* $G \odot H$ is defined as the graph obtained from a graph G of order n and a graph H , by taking one copy of G and n copies of H , and then joining by an edge each vertex from the i^{th} -copy of H with the i^{th} -vertex of G .

Proposition 1. For any integer $n \geq 3$, there exists a graph G such that $G_{SR} \cong K_n \odot K_1$.

Proof. We consider the unicyclic graph G with a cycle $C_{2n} = v_1v_2 \dots v_{2n}v_1$ such that the vertices v_1, v_2, \dots, v_n form the set $c_2(G)$ and the remaining ones from the cycle have exactly one terminal vertex. Since $2n$ is an even number according to the Description of G_{SR} it clearly follows that G_{SR} is isomorphic to $K_{|T(G)|}$ where each vertex of $T(G)$ has exactly one neighbor in $c_2(G)$. \square

3.2. Bouquet of Cycles

Let $\mathcal{B}_{a,b,c}$ be a family of graphs obtained in the following way. Each graph $B \in \mathcal{B}_{a,b,c}$ is a bouquet of $a + b + c$ cycles where a of them are even cycles (of order at least four), b are odd cycles of order larger than three, c are cycles of order three, $a, b, c \geq 0$, and $a + b + c \geq 2$. All cycles of $B \in \mathcal{B}_{a,b,c}$ have the common vertex w . One example of a bouquet of cycles is given in Figure 2. Let $C_{r_1}, C_{r_2}, \dots, C_{r_a}$ be the even cycles of order at least four in $B \in \mathcal{B}_{a,b,c}$ and $C_{s_1}, C_{s_2}, \dots, C_{s_b}$ be the odd cycles of order larger than three in $B \in \mathcal{B}_{a,b,c}$.

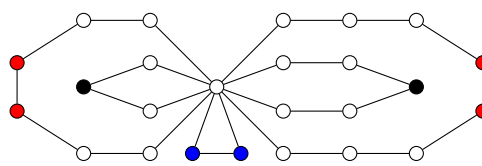


Figure 2. A bouquet of cycles $B \in \mathcal{B}_{2,2,1}$ containing the cycles C_6, C_4, C_9, C_7 and C_3 .

In [17], the authors have described the structure of the strong resolving graph of the graph $B \in \mathcal{B}_{a,b,c}$ as follows. By completeness of our exposition, we copy exactly the description presented there, since it makes no sense to do some changes on it, as it is fairly well written.

- The set of a vertices of the cycles $C_{r_1}, C_{r_2}, \dots, C_{r_a}$ which are diametral with w induces a complete graph in B_{SR} . We denote such set as V_a (in Figures 2 and 3, the black colored vertices).
- The set of $2b$ vertices of the cycles $C_{s_1}, C_{s_2}, \dots, C_{s_b}$ which are diametral with w induces a complete multipartite graph $K_{2, \dots, 2}$ with b bipartition sets each of cardinality two in B_{SR} . We denote such set as V_{2b} (in Figures 2 and 3, the red colored vertices).
- The set of $2c$ vertices of the cycles C_3 different from w induces a complete graph in B_{SR} . We denote such set as V_{2c} (in Figures 2 and 3, the blue colored vertices).
- The set of vertices of each odd cycle $C_{s_i}, i \in \{1, \dots, b\}$, which are different from w induces a path of order $s_i - 1$, in B_{SR} , whose leaves are the two vertices that are diametral with w .
- The set of vertices of each cycle $C_{r_j}, j \in \{1, \dots, a\}$, which are not diametral with w induces a graph isomorphic to the disjoint union of $(r_j - 2)/2$ complete graphs K_2 in B_{SR} .
- Every three vertices x, y, z such that $x \in V_a, y \in V_{2b}$ and $z \in V_{2c}$ are pairwise adjacent.

Figure 3 shows the strong resolving graph of the graph illustrated in Figure 2.

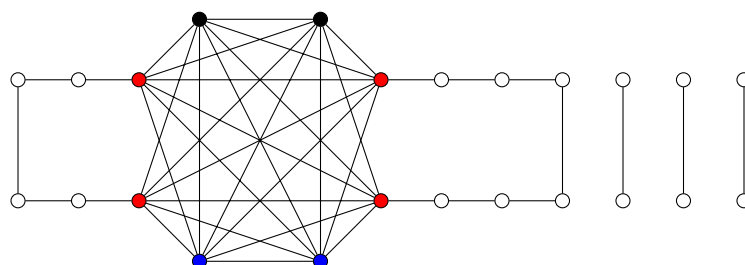


Figure 3. The strong resolving graph B_{SR} of the graph illustrated in Figure 2.

If we study the bouquet of cycles $B \in \mathcal{B}_{a,b,c}$ with $b = 0$ (or equivalently, B has not odd cycles of order larger than three), and $C_{r_1}, C_{r_2}, \dots, C_{r_a}$ are the cycles of even order, then the strong resolving graph B_{SR} is composed by the complete graph K_{a+2c} and $\sum_{i=1}^a \frac{r_i - 2}{2}$ components isomorphic to K_2 .

Now, we again give some realization results for strong resolving graphs. To this end, we need to define a graph structure which we call a *partial multisubdivided complete graph* $K_{2n}(p_1, p_2, \dots, p_n)$. That is, a complete graph K_{2n} where each edge of a perfect matching of this graph is subdivided $p_i \geq 0$ times for $i \in \{1, 2, \dots, n\}$ (the case when some $p_i = 0$ means that the edge corresponding to p_i is not subdivided). Moreover, recall that the *cocktail party graph* R_n , also called the hyperoctahedral graph, is a $n - 2$ regular graph on n vertices.

Proposition 2. *For any integer $n \geq 2$, there exists a graph G such that G_{SR} is isomorphic to $K_{2n}(p_1, p_2, \dots, p_n)$.*

Proof. We consider the bouquet of cycles $B \in \mathcal{B}_{a,b,c}$ with $a, c = 0, b = n$ and $C_{p_1+3}, C_{p_2+3}, \dots, C_{p_n+3}$ are the cycles of odd order larger than three. According to the construction of the strong resolving graph B_{SR} , the subgraph $\langle V_b \rangle$ is isomorphic to R_{2n} and the set of vertices of each odd cycle C_{p_i+3} , $i \in \{1, \dots, n\}$, which are different from w induces a path of order $p_i + 2$, in B_{SR} , whose leaves are the two vertices of this cycle that are not adjacent in R_{2n} . \square

Corollary 4. *For any integer $n \geq 2$, there exists a graph G such that G_{SR} contains the cocktail party graph R_{2n} as an induced subgraph.*

3.3. Chains of Even Cycles

A *chain of cycles* is a cactus graph in which, every cycle has order at least three and there are only two terminal cycles. Notice that in such case every non-terminal cycle has exactly two cut vertices, such that each cut vertex belongs to exactly two cycles. We next center our attention into the case of chains of even cycles. To this end, we need some terminology and notation. A chain of even cycles is a *straight chain*, if the cut vertices of every cycle in the chain are diametral in the cycle. Note that each straight chain contains two diametral vertices, which are the unique terminal vertices of this chain.

For the purposes of simplifying, given an integer $k \geq 0$, we shall define the next family \mathcal{F}_k of graphs. Each graph $F \in \mathcal{F}_k$ is a chain of even cycles constructed as follows.

- We begin with $k + 1$ straight chains of even cycles, say G_0, \dots, G_k , satisfying that the last cycle of the straight chain G_i is isomorphic to the first cycle of the straight chain G_{i+1} for every $i \in \{0, \dots, k - 1\}$.
- Assume that the last cycle of each straight chain G_i is $C_r^i = v_0^i v_1^i \dots v_{r-1}^i v_0^i$, for every $i \in \{0, \dots, k\}$. By the item above, this C_r^i (in G_i) is isomorphic to the first cycle of the straight chain G_{i+1} with $i \in \{0, \dots, k - 1\}$.
- Assume also that the terminal vertices of each straight chain G_i are a_i, b_{k-i} , for every $i \in \{0, \dots, k\}$.
- To construct our chain of even cycles $F \in \mathcal{F}_k$, for every $i \in \{0, \dots, k - 1\}$, we identify the last cycle C_r^i of G_i with the first cycle C_r^{i+1} of G_{i+1} (that are isomorphic) as follows. Every vertex v_j^i of C_r^i is identified with the vertex v_{j+t}^{i+1} for some $t \neq 0$ and every $j \in \{0, r - 1\}$ (operations with the subindex of v are done modulo r).

Notice that for instance, for the chain of even cycles $F \in \mathcal{F}_k$ described above, the two terminal vertices of it are a_0 and b_0 . Figure 4 shows a fairly representative example of a chain of even cycles. Recall that the way of drawing such graph (with respect to directions of the “turns” in the chain) does not influence in our purposes. The chain of even cycles $F \in \mathcal{F}_k$ presented in the Figure 4 has four straight chains of even cycles: G_0 contains C^1 and C^2 , G_1 contains C^2, C^3 and C^4 , G_2 contains C^4, C^5 and C^6 , and G_3 contains C^6, C^7 and C^8 .

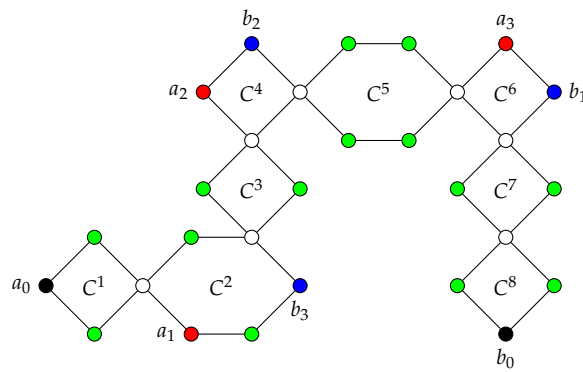


Figure 4. A chain of cycles $F \in \mathcal{F}_3$ containing six cycles C_4 and two cycles C_6 .

We next describe the strong resolving graph of a chain of even cycles $F \in \mathcal{F}_k$. We need first the following observations.

Remark 6. For any chain of even cycles $F \in \mathcal{F}_k$, a vertex x belongs to $\partial(F)$ if and only if x has degree two.

Remark 7. In a straight chain of cycles, the two terminal vertices form a pair of MMD vertices, as well as each pair of diametral vertices in each cycle.

Observation 3. For a chain of even cycles $F \in \mathcal{F}_k$, and for every $i \in \{0, \dots, k\}$ and $j \in \{i, \dots, k\}$ it follows.

- The terminal vertex a_i of the straight chain G_i is MMD with every vertex b_{k-j} of the straight chain G_j .
- The terminal vertex b_i of the straight chain G_{k-i} is MMD with every vertex a_{k-j} of the straight chain G_{k-j} .
- In any cycle of F , any pair of diametral (in the cycle) vertices being not cut nor terminal vertices of F are MMD.

For instance, in Figure 4, the red vertex a_1 is MMD with the blue vertices b_2, b_1, b_0 , while the blue vertex b_1 is MMD with the red vertices a_2, a_1, a_0 . Moreover, again in Figure 4, any pair of green diametral vertices belonging to the same cycle are MMD in F .

With these observations above, we are able to describe the structure of F_{SR} for every chain of even cycles $F \in \mathcal{F}_k$. To do so, we shall need the following construction, which represents a bipartite graph J_r of order $2r + 2$ for some $r \geq 3$. The two bipartition sets of the bipartite graph J_r are the sets $U = \{a_0, \dots, a_r\}$ and $V = \{b_0, \dots, b_r\}$. The edges of J_r are as follows. For every $i \in \{0, \dots, \lfloor r/2 \rfloor\}$ and every $j \in \{0, \dots, r - i\}$, there exist the edges $a_i b_j$ and $b_i a_j$.

- The set of vertices a_i and b_i , with $i \in \{0, \dots, k\}$, forms a component of the graph F_{SR} isomorphic to a bipartite graph J_k .
- In each cycle of F , each pair of diametral vertices in the cycle, not including terminal nor cut vertices, induces a graph isomorphic to K_2 in F_{SR} .

We may remark that, the strong resolving graph of a straight chain of cycles is simply a union of several complete graphs K_2 . The strong resolving graph of the chain of even cycles shown in Figure 4 is drawn in Figure 5.

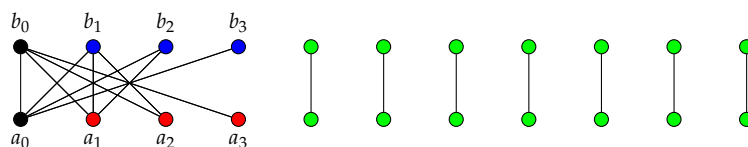


Figure 5. The strong resolving graph F_{SR} of the graph illustrated in Figure 4.

We end this subsection by giving a realization result for strong resolving graphs involving chains of even cycles.

Corollary 5. *For any integer $k \geq 2$, there exists a chain of even cycles $F \in \mathcal{F}_k$ such that F_{SR} contains the bipartite graph J_k as a component.*

4. The Strong Metric Dimension

We are next centered into computing or bounding the strong metric dimension of the cactus graphs which we have studied in the previous section.

4.1. Unicyclic Graphs

Our first results shows the relationship between the strong metric dimension of a unicyclic graph and that of its branch restricted unicyclic graph.

Lemma 2. *Let G be a unicyclic graph and $\mathcal{T}(G)$ be its branch restricted unicyclic graph. Then*

$$dim_s(G) = dim_s(\mathcal{T}(G)).$$

Proof. By Lemma 1 and Theorem 1, we derive that $dim_s(G) = \beta(G_{SR}) = \beta((\mathcal{T}(G))_{SR}) = dim_s(\mathcal{T}(G))$ and the proof is complete. \square

Theorem 4. *Let G be a unicyclic graph with unique cycle C_r . Then*

$$\max \left\{ \left\lceil \frac{r}{2} \right\rceil, |T(G)| - 1 \right\} \leq dim_s(G) \leq |T(G)| + \left\lfloor \frac{|c_2(G)|}{2} \right\rfloor.$$

Proof. From Remark 1 we have that every strong resolving basis must contain at least $|T(G)| - 1$ vertices of degree one. So, $dim_s(G) \geq |T(G)| - 1$. On the other hand, for every vertex $i \in \{1, \dots, r\}$ there exists at least a vertex $w_i \in t(v_i) \cup \{v_i\}$ such that $w_i \in \partial(G)$ (notice that it could happen $w_i = v_i$). Thus we have that $dim_s(G) = \beta(G_{SR}) \geq \frac{|\partial(G)|}{2} \geq \left\lceil \frac{r}{2} \right\rceil$.

On the other side, since $T(G)$ forms a clique in G_{SR} and for every $u \in c_2(G)$ there exists at least one vertex $v \in \partial(G)$ such that they are MMD, according to the description of G_{SR} presented in the previous section, we have $dim_s(G) = \beta(G_{SR}) \leq |T(G)| + \left\lfloor \frac{|c_2(G)|}{2} \right\rfloor$. Therefore the proof is complete. \square

As we can see in the following, the bounds above are tight. In particular, we characterize all the unicyclic graphs having a unique cycle of even order that are attaining the upper bound.

Theorem 5. *Let G be a unicyclic graph with a unique cycle C_r of even order. Then $dim_s(G) = |T(G)| + \left\lfloor \frac{|c_2(G)|}{2} \right\rfloor$ if and only if $|c_2(G)| = r - 1$.*

Proof. (\Leftarrow) We assume $|c_2(G)| = r - 1$. Let v be the only vertex of C_r with degree greater than two, and let u be the diametral vertex with v in C_r . So, every two vertices in $t(v) \cup \{u\}$ are MMD. Also, every two diametral vertices in $c_2(G) - \{u\}$ are MMD. Thus, G_{SR} is formed by $\frac{r-2}{2} = \frac{|c_2(G)|-1}{2}$ connected components isomorphic to K_2 and one component isomorphic to $K_{|t(v)|+1}$. Since $T(G) = t(v)$, we have that

$$dim_s(G) = \beta(G_{SR}) = \beta(K_{|T(G)|+1}) + \beta \left(\bigcup_{i=1}^{\frac{|c_2(G)|-1}{2}} K_2 \right) = |T(G)| + \frac{|c_2(G)| - 1}{2} = |T(G)| + \left\lfloor \frac{|c_2(G)|}{2} \right\rfloor.$$

(\Rightarrow) We assume now that $dim_s(G) = |T(G)| + \lfloor \frac{|c_2(G)|}{2} \rfloor$ is satisfied. If $|c_2(G)| < r - 1$, then there are at least two vertices x, y such that $t(x) \geq 1$ and $t(y) \geq 1$. We consider two cases.

Case 1: x, y are diametral in C_r . Hence, $t(x) \cup t(y)$ forms a clique in G_{SR} of cardinality $|t(x)| + |t(y)|$. Also, the vertices in $t(x) \cup t(y)$ have no neighbor from $c_2(G)$ in G_{SR} . Note that, there could be some other vertices in $T(G)$ having neighbors from $c_2(G)$ in G_{SR} , and if there is one of such vertices, say z , then $|t(z)| \geq 1$ and $t(x) \cup t(y) \cup t(z)$ is also a clique in G_{SR} . However, this will not influence on the fact that, in order to cover the edges of G_{SR} , one can leave one vertex w of $t(x) \cup t(y)$ outside of the vertex cover set, by simply taking $T(G) \setminus \{w\}$ as a part of such vertex cover set. Thus, we have that $\beta(G_{SR}) \leq |T(G)| - 1 + \lfloor \frac{|c_2(G)|}{2} \rfloor$, a contradiction.

Case 2: x, y are not diametral in C_r . Let $x', y' \in c_2(G)$ being diametral vertices with x, y , respectively. Hence, $t(x) \cup t(y)$, $t(x) \cup \{x'\}$ and $t(y) \cup \{y'\}$ form cliques in G_{SR} . Also, x', y' have no neighbor in G_{SR} other than that ones in $t(x), t(y)$, respectively. Thus, in order to cover the edges of G_{SR} , we can leave outside of the vertex cover set both vertices x', y' , by simply taking $T(G)$ in such vertex cover set. On the other hand, to cover the remaining vertices in $c_2(G) \setminus \{x', y'\}$ we will need at most $\lfloor \frac{|c_2(G)|-2}{2} \rfloor$. We then deduce that $\beta(G_{SR}) \leq |T(G)| + \frac{|c_2(G)|-2}{2} = |T(G)| - 1 + |c_2(G)|/2 - 1$, a contradiction again.

Since we have contradiction on both cases above, it must happen that $|c_2(G)| = r - 1$, and the proof is completed. \square

Note that the upper bound of Theorem 4 is also tight when the unique cycle of G is odd, but the characterization of the limit case seems to be a hard working task. For instance, if G has a unique cycle of odd order and $|c_2(G)| = r - 1$, then a “relatively” similar argument to the first part of the proof of Theorem 5 leads to conclude that $dim_s(G) = |T(G)| + \lfloor \frac{|c_2(G)|}{2} \rfloor$. Other cases, when $|c_2(G)| < r - 1$ can be hand computed, and we leave this to the reader.

Proposition 3. *Let G be a unicyclic graph with a unique cycle C_r of even order. Then $dim_s(G) = \frac{r}{2}$ if and only if the following items hold.*

- (i) $|t(x)| \leq 1$ for every x of C_r .
- (ii) There is at most one pair of diametral vertices in C_r each one having one terminal vertex.

Proof. (\Rightarrow) Assume $dim_s(G) = \frac{r}{2}$. If $|t(x)| > 1$ for some x of C_r , then let x' be the vertex of C_r being diametral with x in C_r . Hence, $t(x) \cup t(x')$ (or $t(x) \cup \{x'\}$ if $x' \in c_2(G)$) is a clique in G_{SR} , and so, in order to cover the edges of G_{SR} , we need at most $\frac{r}{2}$ in connection with pairs of diametral vertices in C_r together with at least one extra vertex from $t(x)$, since $|t(x)| > 1$ (there are at least two MMD vertices in $t(x)$). Thus, (i) follows.

Now, let a be the number of pairs of diametral vertices in C_r each one having one terminal vertex. Suppose that $a \geq 2$. Also, let b be the number of pairs of diametral vertices in C_r , in which one of them has one terminal vertex and the other one belongs to $c_2(G)$, and let c be the number of pairs of diametral vertices in C_r , each one belonging to $c_2(G)$. Note that the $a + b + c = \frac{r}{2}$ and that $|T(G)| = 2a + b$. Also, the $2a$ vertices and the b vertices of $T(G)$, corresponding to that pairs mentioned above, form a clique in G_{SR} such that the $2a$ vertices has no neighbors other than that ones in such clique, and such that each of the b vertices has exactly one other neighbor from $c_2(G)$ in G_{SR} . Moreover, the c pairs of vertices also mentioned above, form c components of G_{SR} isomorphic to K_2 . In consequence, we observe that $\beta(G_{SR}) = 2a - 1 + b + c = \frac{r}{2} - 1 + a \geq \frac{r}{2} + 1$. This is a contradiction, and the proof of (ii) is complete.

(\Leftarrow) Assume on the other hand that G satisfies (i) and (ii). We shall use the same notation of a, b and c from the implication above. By (ii), $0 \leq a \leq 1$. If $a = 1$, then $dim_s(G) = \beta(G_{SR}) = 2a - 1 + b + c = \frac{r}{2} - 1 + a = \frac{r}{2}$ (note that the equality $\beta(G_{SR}) = 2a - 1 + b + c$ follows by using (i)). Also, if $a = 0$, then $dim_s(G) = \beta(G_{SR}) = b + c = \frac{r}{2}$ (we again use (i) as explained before). \square

To conclude this section, we next show that the differences between the lower (partially) and upper bounds of Theorem 4, and the real value of the strong metric dimension of some unicyclic graphs can be as large as possible.

We consider the unicyclic graph G^k with a cycle $C_{2n} = v_1v_2 \cdots v_{2n}v_1$ and $1 \leq k \leq n$ such that the vertices $v_{k+1}, v_{k+2}, \dots, v_{2n}$ form the set $c_2(G)$, and each vertex v_i for $i \in \{1, \dots, k\}$ has one terminal vertex denoted by x_i . Since $2n$ is an even number, and according to the description of the strong resolving graph of a unicyclic graph, it clearly follows that $(G^k)_{SR}$ consists of a graph isomorphic to $K_{|T(G)|} \odot K_1$ and $\lfloor \frac{|c_2(G)|-k}{2} \rfloor$ graphs isomorphic to K_2 . Thus, $dim_s(G^k) = |T(G)| + \lfloor \frac{|c_2(G)|-k}{2} \rfloor$. Since $1 \leq k \leq |c_2(G)|$, we can easily observe that $|T(G)| + \lfloor \frac{|c_2(G)|}{2} \rfloor - dim_s(G^k)$ and $dim_s(G^k) - (|T(G)| - 1)$ can be arbitrarily large.

4.2. Bouquet of Cycles

For the results of this subsection, we use the terminology and notations given in Section 3.2.

Theorem 6. For any bouquet of cycles $B \in \mathcal{B}_{a,b,c}$,

$$dim_s(B) = a + \sum_{i=1}^a \frac{r_i - 2}{2} + 2b + \sum_{j=1}^b \frac{s_j - 3}{2} + 2c - 1.$$

Proof. According to the description of B_{SR} presented before, it follows that B_{SR} consist of a graph isomorphic to $K_{a+2b+2c}(s_1 - 3, s_2 - 3, \dots, s_b - 3, 0, 0, \dots, 0)$ and $\sum_{i=1}^a \frac{r_i-2}{2}$ graphs isomorphic to K_2 . First we consider the subgraph H induced by $N_{B_{SR}}[V_a \cup V_{2b} \cup V_{2c}]$. Notice that $\beta(H) = a + 2b + 2c - 1$. In order to compute $\beta(K_{a+2b+2c}(s_1 - 3, s_2 - 3, \dots, s_b - 3, 0, 0, \dots, 0))$ we need to cover the remaining edges in B_{SR} corresponding to edges of the odd cycles in B . Since for each odd cycle C_{s_i} for $i \in \{1, \dots, b\}$, two edges of it are already considered in H , it remains to cover $s_i - 4$ edges which are inducing a path of order $s_i - 3$. Thus, to cover each cycle C_{s_i} we need $\frac{s_i-3}{2}$ vertices.

On the other hand, to cover the $\sum_{i=1}^a \frac{r_i-2}{2}$ graphs isomorphic to K_2 , $\sum_{i=1}^a \frac{r_i-2}{2}$ extra vertices are needed. The sum of these three quantities above gives the vertex cover number of B_{SR} , and also the strong metric dimension of B , by using Corollary 1, which completes the proof. \square

4.3. Chains of Even Cycles

In order to give a formula for the strong metric dimension of chains of even cycles, we need to first compute the value of the vertex cover number of a bipartite graph J_r as described in Section 3.3.

Lemma 3. For any bipartite graph J_r , $\beta(J_r) = r + 1$.

Proof. We first note that if r is an even integer, then the set of edges $E_r = \{a_0b_r, a_1b_{r-1}, \dots, a_{r/2}b_{r/2}\} \cup \{b_0a_r, b_1a_{r-1}, \dots, b_{r/2-1}a_{r/2+1}\}$ is a maximum matching in J_r of cardinality $r/2 + 1 + r/2 = r + 1$.

On the other hand, if r is odd, then the set of edges $E_r = \{a_0b_r, a_1b_{r-1}, \dots, a_{(r-1)/2}b_{(r+1)/2}\} \cup \{b_0a_r, b_1a_{r-1}, \dots, b_{(r-1)/2}a_{(r+1)/2}\}$ is a maximum matching in J_r of cardinality $(r - 1)/2 + 1 + (r - 1)/2 + 1 = r + 1$.

Thus, since J_r is bipartite, by using the famous König’s Theorem, we obtain the required result. \square

Theorem 7. For any chain of even cycles $F \in \mathcal{F}_k$ of order n with c cut vertices,

$$dim_s(F) = \frac{n - c}{2}.$$

Proof. According to the description of F_{SR} presented before, the vertices a_i and b_i , with $i \in \{0, \dots, k\}$, forms a component of the graph F_{SR} isomorphic to a bipartite graph J_k of order $2k + 2$. For completing

the graph F_{SR} , we need to add $\frac{n-c-2k-2}{2}$ graphs isomorphic to K_2 . Hence, by using Theorem 1, Lemma 3 and Observation 1, we have $\dim_s(F) = \beta(J_k) + \frac{n-c-2k-2}{2}\beta(K_2) = \frac{n-c}{2}$. \square

5. Concluding Remarks

We have studied the strong metric dimension of cactus graphs in this work. Along the way, we have given several contributions to the realization and characterization results of strong resolving graphs involving cactus graphs. The results shown allow to observe that working in this topic for the specific case of cactus graphs is very challenging, although some particular structures of such graphs can be easier handled. These are the cases of unicyclic graphs, chains of even cycles and bouquet of cycles, for which we have given the constructions of their strong resolving graphs and bounds or closed formulas for the values of their strong metric dimensions. As a consequence of this study, the following open questions are raised.

- Describe the structure of the strong resolving graphs of some classes of cactus graphs, and compute the strong metric dimension of the graphs in such families.
- Apply the results concerning the descriptions of the strong resolving graphs of the graphs given in the work to other problems, like for instance computing the strong partition dimension (see [4]) of such graphs.
- Continue the lines of this study for other more general families than cactus graphs. This could include for instance, planar graphs or chordal graphs.

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