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# First Integrals of Differential Operators from $SL(2,\mathbb{R})$ Symmetries

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**Abstract:** The construction of first integrals for $SL(2,\mathbb{R})$-invariant $n$th-order ordinary differential equations is a non-trivial problem due to the nonsolvability of the underlying symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$. Firstly, we provide for $n = 2$ an explicit expression for two non-constant first integrals through algebraic operations involving the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$, and without any kind of integration. Moreover, although there are cases when the two first integrals are functionally independent, it is proved that a second functionally independent first integral arises by a single quadrature. This result is extended for $n > 2$, provided that a solvable structure for an integrable distribution generated by the differential operator associated to the equation and one of the prolonged symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ is known. Several examples illustrate the procedures.

**Keywords:** differential operator; first integral; solvable structure; integrable distribution

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## 1. Introduction

The study of $n$th-order ordinary differential equations admitting the unimodular Lie group $SL(2,\mathbb{R})$ is a non-trivial problem due to the nonsolvability of the underlying symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$. This problem has been tackled by different approaches during the last decades, next we review only some of the most relevant for the purposes of this work.

Most results in the literature refer to third-order $SL(2,\mathbb{C})$-invariant ODEs, which can be solved via a pair of quadratures and the solution to a Riccati equation (or, equivalently, to a second-order linear ODE). This result can be proved as a consequence of the study preformed by Clarkson and Olver [1], who connected the three inequivalent actions of $SL(2,\mathbb{C})$ in the complex plane via the standard prolongation process. It was demonstrated in Reference [2] that the fourth action that appears in the real case can be obtained from the same source, and this study was extended to 2D and 3D Lie algebras of symmetries, providing interesting results on the linearization of second-order ODEs via contact transformations [2,3]. A different approach [4] exploits the existence of nonlocal symmetries, latter connected with $\mathcal{C}^\infty$-symmetries in Reference [5], where the fourth realization that appears in the real case was also discussed. By using techniques based on solvable structures [6–12], the general solution in parametric form for each one of the four canonical third-order $SL(2,\mathbb{R})$-invariant ODEs were obtained in References [13,14]. Such solution is given in terms of a fundamental set of solutions to a second-order linear ODE, which is explicitly given for each one of the four different actions of $SL(2,\mathbb{R})$ on the real plane [14]. The concept of generalized solvable structure was introduced in Reference [15] in order to extend the study to $SL(2,\mathbb{R})$-invariant ODEs of arbitrary order $n$. The main result in this regard states that when a generalized solvable structure is known, then a complete set of first integrals can be constructed by quadrature, and given in terms of a fundamental set of solutions to a second-order linear ODE depending on $n-3$ parameters (see Reference [15] for further details).

In this paper we present new relevant results about first integrals of differential operators associated to $SL(2,\mathbb{R})$-invariant ODEs, showing how some of these first integrals can be constructed without any kind of integration at all. The paper is organized as follows. After a preliminary section setting the notation and necessary previous results, we consider the case of second-order equations as a starting point. For this class of equations, instead of using a two-dimensional subgroup for its integration, we provide two first integrals given straightforwardly through the symmetry generators, without integration. Although we are able to prove that both first integrals are not constant, we show with an example that they need not to be functionally independent functions (in contrast to what happens in the case of the rotation symmetry group [16]). Nevertheless, we prove that in this unfavourable situation a functionally independent first integral always arises by a single quadrature.

These results are extended for equations of arbitrary order in Section 4. In the general case, we use a solvable structure for the integrable distribution generated by the differential operator associated to the equation and one of the prolonged symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$. It is worth mentioning that the solvable structure of such distribution needs not to be either a solvable structure or a generalized solvable structure for the equation. We also present illustrative examples in order to show how these new results can be applied in practice.

## 2. Preliminaries

### 2.1. Distributions of Vector Fields and Their Symmetries

Given a set of vector fields $\{A_1,\ldots,A_{n-k}\}$ on a $n$-dimensional manifold $N$, we denote by $\mathcal{A} := \langle A_1,\ldots,A_{n-k}\rangle$ the distribution of vector fields generated by $\{A_1,\ldots,A_k\}$. Similarly, given a set of 1-forms $\{\beta_1,\ldots,\beta_k\}$, we denote by $\boldsymbol{\beta} := \langle \beta_1,\ldots,\beta_k\rangle$ the corresponding Pfaffian system (i.e., the sub-module over $C^\infty(N)$ generated by $\{\beta_1,\ldots,\beta_k\}$) [17,18].

The distribution $\mathcal{A}$ is *integrable* (in Frobenius sense) if and only if the Lie bracket $[A,B]\in\mathcal{A}$, for each $A,B\in\mathcal{A}$ [6,7,11,12]. If $U$ is a open domain of $N$ where the vector fields $\{A_1,\ldots,A_{n-k}\}$ are pointwise linearly independent, we say that $\mathcal{A}$ is a distribution of *maximal rank* $n-k$ (or of *codimension* $k$) on $U$. It is well known that an integrable distribution $\mathcal{A}$ of maximal rank determines a $(n-k)$-dimensional foliation of $U\subseteq N$ [17,18]. If this foliation is described though the vanishing of $k$ functions of the form $I_h - c_h$, where $I_h\in\mathcal{C}^\infty(U)$ and $c_h\in\mathbb{R}$, we can choose $\langle dI_1,\ldots dI_k\rangle$ as generators for the Pfaffian system annihilating the distribution $\mathcal{A}$. A submanifold $S\subset N$ is an *integral manifold* for $\mathcal{A}$ if $\mathcal{A}|_S\subseteq TS$. If, in particular, $\mathcal{A}|_S = TS$ we say that $S$ is a *maximal integral manifold* of $\mathcal{A}$ [18].

Let $\mathcal{A}$ and $\mathcal{B}$ be two distributions on $N$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *transversal* at $p\in N$ if they do not vanish at $p$ and $\mathcal{A}(p)\cap\mathcal{B}(p)=\{0\}$. Analogously, $\mathcal{A}$ and $\mathcal{B}$ are transversal in $U$ if they are transversal at any point of $U$.

Given a distribution $\mathcal{A}$, a vector field $X$ is a *symmetry* of $\mathcal{A}$ if $[X,A]\in\mathcal{A}$, for any $A\in\mathcal{A}$ [6,7,11,12]. An algebra $\mathcal{G}$ of symmetries for a distribution $\mathcal{A}$ is *non-trivial* if $\mathcal{G}$ generates a distribution which is transversal to $\mathcal{A}$. Analogously, given an ideal of differential forms $I$, a vector field $X$ is a symmetry of $I$ if $L_X I\subset I$, where $L$ denotes the Lie derivative [6].

Next we prove two results on symmetries of distributions of vector fields that will be used throughout the paper. In what follows, the Einstein summation convention is used and $\lrcorner$ stands for the interior product (contraction) of vector fields and differential forms [18]. Throughout the paper we assume that we are working on an open simply connected domain $U$ of $N$, functions are usually assumed to be smooth and well defined on $U$, the vector fields and forms are not allowed to vanish at any point of $U$. Where necessary, the reader should assume that the domains have been restricted accordingly.

**Proposition 1.** *Let $\mathcal{A} := \langle A_1, \ldots A_{n-k}\rangle$ be an integrable distribution on an n-dimensional manifold N. If $\Omega$ is a volume form on N, and $\alpha$ is the k-form on N defined by*

$$\alpha := A_1 \lrcorner \ldots \lrcorner A_{n-k} \lrcorner \Omega, \tag{1}$$

*then $A_i$ is a symmetry of $\alpha$, for $i = 1, \ldots, n-k$.*

**Proof.** Since $\Omega$ is a volume form over $N$, $L_{A_i}\Omega = (\mathrm{div}A_i)\Omega$, for $i = 1, \ldots, n-k$. Since $\mathcal{A}$ is an integrable distribution, there exists a family of functions $f_{il}^j \in C^\infty(N)$ such that $[A_i, A_l] = f_{il}^j A_j$, for $1 \leq i, j, l \leq n-k$. By properties of the Lie derivative $L$ and the interior product $\lrcorner$ we can write

$$\begin{aligned} L_{A_i}\alpha &= L_{A_i}(A_1 \lrcorner \ldots \lrcorner A_{n-k} \lrcorner \Omega) \\ &= [A_i, A_1] \lrcorner A_2 \lrcorner \ldots \lrcorner A_{n-k} \lrcorner \Omega + A_1 \lrcorner [A_i, A_2] \lrcorner \ldots \lrcorner A_{n-k} \lrcorner \Omega + \ldots \\ &\quad + A_1 \lrcorner \ldots \lrcorner [A_i, A_{n-k}] \lrcorner \Omega + A_1 \lrcorner \ldots \lrcorner A_{n-k} \lrcorner L_{A_i}\Omega \\ &= (f_{ij}^j + \mathrm{div}A_i)\alpha = H_i\alpha, \end{aligned}$$

where $H_i = (f_{ij}^j + \mathrm{div}A_i)$, for $i = 1, \ldots, n-k$. This proves that $L_{A_i}\alpha \in \langle\alpha\rangle$. $\square$

A similar proof can be used to demonstrate the following result:

**Proposition 2.** *Let $\mathcal{A} = \langle A_1, \ldots, A_{n-k}\rangle$ be a distribution of maximal rank on a n-dimensional manifold N. If X is a symmetry of $\mathcal{A}$ and $\alpha$ is the k-form defined by (1), then X is a symmetry of $\alpha$.*

### *2.2. Solvable Structures for Integrable Distributions*

It is well known that, given an integrable distribution $\mathcal{A}$ of maximal rank $(n-k)$ on an $n$-dimensional manifold $N$, the knowledge of a solvable $k$-dimensional algebra $\mathcal{G}$ of non-trivial symmetries for $\mathcal{A}$ guarantees that $\mathcal{A}$ can be integrated, at least locally, by quadratures alone [8,19]. Solvable structures provide an extension of this classical result, significantly enlarging the class of vector fields which can be used to integrate by quadratures a distribution of vector fields. In this section we recall basic definitions and results on solvable structures. The interested reader is referred to References [6,8,11,12] for further details.

**Definition 1.** *Let $\mathcal{A}$ be an integrable distribution of maximal rank $n-k$ on an n-dimensional manifold N. For a set of vector fields $\{Y_1, Y_2, \ldots, Y_k\}$ we denote $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_h = \mathcal{A} \oplus \langle Y_1, \ldots, Y_h\rangle$, for $h \leq k$. The vector fields $\{Y_1, Y_2, \ldots, Y_k\}$ form a solvable structure for $\mathcal{A}$ in a open neighbourhood $U \subseteq N$ if and only if the following conditions are satisfied:*

1. *the distribution $\langle Y_1, Y_2, \ldots, Y_h\rangle$ has maximal rank h and is transversal to $\mathcal{A}$ in U, for $h \leq k$;*
2. *$\mathcal{A}_h$ is distribution of maximal rank $n-k+h$ on U;*
3. *$L_{Y_h}\mathcal{A}_{h-1} \subset \mathcal{A}_{h-1}$, for $h \in \{1, \ldots, k\}$.*

In the next theorem we collect the main results on the integrability of integrable distributions by means of solvable structures [8,11]. The proof can be consulted, for instance in the following references: [6] (Theorem 3.15), [8] (Proposition 3), [11] (Proposition 5), [12] (Proposition 4.7):

**Theorem 1.** *Let $\mathcal{A} = \langle A_1, \ldots, A_{n-k}\rangle$ be an integrable distribution of maximal rank $n-k$ defined on an orientable n-dimensional manifold N and let $\{Y_1, ..., Y_k\}$ be a solvable structure for $\mathcal{A}$. Let $\Omega$ be a volume form on N and define the k-form $\alpha$ as in (1). The distribution $\mathcal{A}$ is the annihilator of the Pfaffian system generated by*

$$\omega_i = \frac{1}{\Delta}(Y_1 \lrcorner \ldots \lrcorner \widehat{Y_i} \lrcorner \ldots \lrcorner Y_k \lrcorner \alpha), \qquad i = 1, \ldots, k, \tag{2}$$

*where the hat denotes omission of the corresponding vector field and* $\Delta$ *is the function on* $N$ *defined by*

$$\Delta = Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_k \lrcorner \alpha.$$

*Moreover, for* $i \in \{1, ..., k-1\}$, *the 1-forms* $\omega_i$ *satisfy*

$$\begin{aligned} d\omega_k &= 0, \\ d\omega_i &= 0 \quad mod\ \{\omega_{i+1}, \ldots, \omega_k\}. \end{aligned}$$

*In consequence, the integral manifolds of the distribution* $\mathcal{A}$ *can be described in implicit form as level manifolds* $I_1 = c_1,\ I_2 = c_2,\ \ldots,\ I_k = c_k$, *where*

$$\omega_k = dI_k,\ \ \omega_{k-1}|_{\{I_k=c_k\}} = dI_{k-1},\ \ \ldots,\ \ \omega_1|_{\{I_k=c_k, I_{k-1}=c_{k-1},\ldots,I_2=c_2\}} = dI_1.$$

We remark that, if the distribution $\mathcal{A}$ admits an abelian Lie algebra of symmetries generated by the vector fields $Y_1, \ldots, Y_k$, the 1-forms $\omega_i$ are closed, that is, the function $1/\Delta$ provides an integrating factor for all the 1-forms

$$Y_1 \lrcorner \ldots \lrcorner \widehat{Y}_i \lrcorner \ldots \lrcorner Y_k \lrcorner \alpha, \qquad i = 1, \ldots, k.$$

The main difference between a solvable structure and a solvable symmetry algebra for an integrable distribution $\mathcal{A}$ is that the fields belonging to a solvable structure do not need to be symmetries of $\mathcal{A}$. This, of course, gives more freedom in the choice of the vector fields which can be used to find integral manifolds of $\mathcal{A}$ by quadratures.

### 2.3. Jacobi Multipliers for Integrable Distributions

In this section we recall the definition and some properties of Jacobi multipliers [20–22] for an integrable distribution $\mathcal{A}$.

**Definition 2.** *Let* $\mathcal{A} := \langle A_1, \ldots A_{n-k} \rangle$ *be an integrable distribution of codimension* $k$ *on an* $n$*-dimensional manifold* $N$. *If* $\Omega$ *is a volume form on* $N$, *and* $\alpha$ *is the* $k$*-form defined by* (1), *we say that a function* $M \in C^\infty(N)$ *is a Jacobi multiplier for* $\mathcal{A}$ *if*

$$d(M\alpha) = 0,$$

*where* $d$ *denotes the exterior derivative.*

It is well known that Jacobi multipliers are related with first integrals for the distribution $\mathcal{A}$, and in particular we have the following:

**Lemma 1.** *Given two Jacobi multipliers* $M_1$ *and* $M_2$ *for an integrable distribution* $\mathcal{A}$, *the function* $I := M_1/M_2$ *is a first integral for* $\mathcal{A}$.

**Proof.** By definition of Jacobi multipliers we have that $d(M_1\alpha) = d(M_2\alpha) = 0$, which implies

$$d\left(\frac{M_1}{M_2}\right) \wedge \alpha = 0.$$

Therefore, if we consider the interior product of this $(k+1)$-form with any vector field $A \in \mathcal{A}$, we get

$$\left[A \lrcorner\, d\left(\frac{M_1}{M_2}\right)\right] \alpha - d\left(\frac{M_1}{M_2}\right) \wedge (A \lrcorner\, \alpha) = \left[A \lrcorner\, d\left(\frac{M_1}{M_2}\right)\right] \alpha = 0,$$

leading to $A\left(\frac{M_1}{M_2}\right) = 0.$ $\square$

The following theorem provides a link between Jacobi multipliers and symmetries of the distribution $\mathcal{A}$.

**Theorem 2.** *Let $\mathcal{A} := \langle A_1, \dots A_{n-k}\rangle$ be an integrable distribution of codimension $k$ on $N$, admitting a $k$-dimensional algebra $\mathcal{G} := \langle X_1, \dots, X_k\rangle$ of non-trivial symmetries. Then a Jacobi multiplier for $\mathcal{A}$ is given by the function*

$$M := \frac{1}{X_1 \lrcorner \dots \lrcorner X_k \lrcorner \alpha}.$$

**Proof.** The proof is a direct computation. We start by rewriting $d(M\alpha) = 0$ in the equivalent form

$$(X_1 \lrcorner X_2 \lrcorner \dots \lrcorner X_k \lrcorner \alpha) d\alpha - d(X_1 \lrcorner X_2 \lrcorner \dots \lrcorner X_k \lrcorner \alpha) \wedge \alpha = 0. \tag{3}$$

The integrability of $\mathcal{A}$ ensures that $\alpha$ is a decomposable $k$-form such that $d\alpha = \rho \wedge \alpha$ for a suitable 1-form $\rho$. In particular, given $k-1$ vector fields $B_i$, we have

$$(B_1 \lrcorner \dots \lrcorner B_{k-1} \lrcorner \alpha) \wedge \alpha = 0,$$

and this fact, together with a repeated use of the formula

$$L_Y \beta = Y \lrcorner d\beta + d(Y \lrcorner \beta),$$

allow us to rewrite the left hand side of (3) as

$$(X_1 \lrcorner X_2 \lrcorner \dots \lrcorner X_k \lrcorner \alpha) d\alpha - (X_1 \lrcorner X_2 \lrcorner \dots \lrcorner X_k \lrcorner d\alpha) \wedge \alpha. \tag{4}$$

Finally, from $d\alpha = \rho \wedge \alpha$ it follows that (4) vanishes, concluding the proof. □

## 3. Second-Order $SL(2,\mathbb{R})$-Invariant ODEs

In this section we consider the trivially integrable one-dimensional distribution $\mathcal{A} = \langle A\rangle$ defined by a differential operator

$$A = \partial x + u_1 \partial u + F(x, u, u_1)\partial u_1, \tag{5}$$

associated to a second order $SL(2,\mathbb{R})$-invariant ordinary differential equation

$$u_2 = F(x, u, u_1), \tag{6}$$

where $u$ stands for the dependent variable, $x$ is the independent variable and subscript $i$ indicates the derivative of order $i$ of $u$ with respect to $x$.

Throughout this section $N$ denote an open set of the first-order jet bundle $J^1(\mathbb{R},\mathbb{R})$ where $A$ is well defined. Since $J^1(\mathbb{R},\mathbb{R})$ is a three dimensional manifold, Theorem 2 ensures that with any pair of symmetries of (6) we can associate a Jacobi multiplier. In particular, the knowledge of three symmetries for Equation (6) allows us to obtain three Jacobi multipliers for the one-dimensional integrable distribution $\mathcal{A} = \langle A\rangle$ and, by Lemma 1, two corresponding first integrals. Unfortunately, if no special assumptions on the structure of the symmetry algebra are made, we have no guarantees about the non-triviality and functional independence of the first integrals thus obtained. In this section we investigate how to construct non-constant first integrals and complete sets of two functionally independent first integrals.

In the following we denote by $X_1, X_2, X_3$ the generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ satisfying the commutation relations

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = X_1, \quad [X_3, X_2] = X_2. \tag{7}$$

It must be said that since $X_1$ and $X_3$ (or $X_2$ and $X_3$) form a solvable Lie algebra, any of such pair of symmetries can be used to integrate Equation (6) by following any of the strategies described in Reference [16] (Chapter 7). However, next we present a new approach to construct non-trivial first integrals without integration, somehow similar to the strategy followed in Reference [16] (Chapter 8) to integrate a second order ODE admitting a nonsolvable group isomorphic to the group of rotations of a three-dimensional space.

We first prove that the symmetry generators of the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ can be used to construct, without any kind of integration, non-constant first integrals for the differential operator (5) associated to any second-order $SL(2,\mathbb{R})$-invariant equation:

**Theorem 3.** *Let $\mathcal{A} = \langle A \rangle$ be the integrable distribution generated by the vector field (5) associated to any $SL(2,\mathbb{R})$-invariant second-order ODE (6). Assume that for $i,j = 1,2,3$ the distributions $\langle A, X_i, X_j \rangle$ have maximal rank in an open set $U \subset J^1(\mathbb{R},\mathbb{R})$, where $X_1, X_2, X_3$ denote the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$. Then two (possibly not functionally independent) non-constant first integrals $I_1$ and $I_2$ of Equation (6) arise without any kind of integration.*

**Proof.** Let us consider the volume form $\Omega = dx \wedge du \wedge du_1$ on $J^1(\mathbb{R},\mathbb{R})$ and the two-form $\alpha = A \lrcorner \Omega$. The two functions

$$I_1 = \frac{X_1 \lrcorner X_2 \lrcorner \alpha}{X_1 \lrcorner X_3 \lrcorner \alpha}, \quad I_2 = \frac{X_1 \lrcorner X_2 \lrcorner \alpha}{X_2 \lrcorner X_3 \lrcorner \alpha} \tag{8}$$

are well-defined and non-vanishing functions owing to the fact that $\langle A, X_i, X_j \rangle$ have maximal rank for $i,j = 1,2,3$. Moreover, Theorem 2 and Lemma 1 ensure that $I_1$ and $I_2$ are first integrals of the differential operator (5), because they are ratios of Jacobi multipliers. Therefore, we have only to prove that $I_1$ and $I_2$ are not constant.

By contradiction, let us suppose that $I_1$ is constant, so that $X_1(I_1) = L_{X_1}(I_1) = 0$. Thus

$$\left[L_{X_1}(X_1 \lrcorner X_2 \lrcorner \alpha)\right](X_1 \lrcorner X_3 \lrcorner \alpha) - (X_1 \lrcorner X_2 \lrcorner \alpha)\left[L_{X_1}(X_1 \lrcorner X_3 \lrcorner \alpha\right] = 0,$$

and, using Proposition 2 and the commutation relations (7), we can write

$$\begin{aligned}&(2X_1 \lrcorner X_3 \lrcorner \alpha + X_1 \lrcorner X_2 \lrcorner G_1 \alpha)(X_1 \lrcorner X_3 \lrcorner \alpha) - (X_1 \lrcorner X_2 \lrcorner \alpha)(X_1 \lrcorner X_3 \lrcorner G_1 \alpha)\\ &= 2(X_1 \lrcorner X_3 \lrcorner \alpha)^2 = 0.\end{aligned}$$

Then $X_1 \lrcorner X_3 \lrcorner \alpha = 0$ which contradicts the hypothesis on the maximal rank of $\langle A, X_1, X_3 \rangle$.

Analogously we can prove that $I_2$ is not constant, considering $X_2(I_2) = 0$. □

**Example 1.** *Let $A$ denote the corresponding differential operator (5) associated to the second-order equation*

$$u_2 = \frac{u_1^2 + 2k}{2u}, \quad k \in \mathbb{R}, \tag{9}$$

*defined in $N = \{(x,u,u_1) \in J^1(\mathbb{R},\mathbb{R}) : u \neq 0\}$. It is easy to check that the symmetry algebra of Equation (9) is isomorphic to the nonsolvable algebra $\mathfrak{sl}(2,\mathbb{R})$ with symmetry generators*

$$\mathbf{v}_1 = \partial_x, \mathbf{v}_2 = x^2\partial_x + 2xu\partial_u, \mathbf{v}_3 = x\partial_x + u\partial_u, \tag{10}$$

*whose first-order prolongations $X_i = \mathbf{v}_i^{(1)}$, for $i = 1,2,3$, satisfy the commutation relations (7). It can be also checked that the distributions $\langle A, X_1, X_2 \rangle$, $\langle A, X_1, X_3 \rangle$, and $\langle A, X_2, X_3 \rangle$, are of maximal rank in $U = \{(x,u,u_1) \in N : (u_1 x - 2u)^2 + 2kx^2 \neq 0, u_1^2 + 2k \neq 0, xu_1^2 + 2kx - 2uu_1 \neq 0\}$.*

*The corresponding 2-form (1) becomes*

$$\alpha = \frac{u_1^2 + 2k}{2u} dx \wedge du - u_1 dx \wedge du_1 + du \wedge du_1.$$

*By Theorem 3, the functions*

$$I_1 = \frac{X_1 \lrcorner X_2 \lrcorner \alpha}{X_1 \lrcorner X_3 \lrcorner \alpha} = \frac{2xu_1^2 + 4kx - 4uu_1}{u_1^2 + 2k}, \quad I_2 = \frac{X_1 \lrcorner X_2 \lrcorner \alpha}{X_2 \lrcorner X_3 \lrcorner \alpha} = -\frac{2((u_1x - 2u)u_1 + 2kx)}{(u_1x - 2u)^2 + 2kx^2} \tag{11}$$

*are two non-constant first integrals for Equation (9). It can be checked that $I_1$ and $I_2$ are functionally independent in $U$ if and only if $k \neq 0$. Consequently, the general solution of Equation (9) when $k \neq 0$, arises immediately by setting $I_1 = C_1, I_2 = C_2$, for $C_1, C_2 \in \mathbb{R}$. The possible singular solutions whose first-order prolongations are in set where the first integrals are not defined or vanish (i.e., the set where the distributions $\langle A, X_1, X_2 \rangle$, $\langle A, X_1, X_3 \rangle$, and $\langle A, X_2, X_3 \rangle$ are not of maximal rank) should be analyzed separately. This also applies for all the examples presented in the paper.*

*We want to note that Equation (9) can be integrated by the standard Lie method, because it admits a two-dimensional symmetry group, although there is no guarantee that the quadratures can be performed in closed form. However, in the above process, no integration has been made at all.*

*In contrast to the case of second-order ODEs admitting $SO(3)$ as symmetry group ([16] Section 8.3), for second-order equations with symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$ we cannot ensure that the first integrals $I_1$ and $I_2$ given by (8) are always functionally independent. This can be seen, for instance, by considering Equation (9) for $k = 0:$ the first integrals (11) become*

$$I_1 = \frac{2(xu_1 - 2u)}{u_1}, \quad I_2 = -\frac{2u_1}{u_1x - 2u}, \tag{12}$$

*which are clearly functionally dependent because the relation $I_1 I_2 + 4 = 0$ holds.*

Although, as we have shown, the first integrals produced in Theorem 3 may be functionally dependent, next we prove that a second functionally independent first integral arises by quadrature:

**Theorem 4.** *In the same conditions than in Theorem 3, the 1-form*

$$\beta_k = \frac{X_k \lrcorner \alpha}{X_k \lrcorner X_3 \lrcorner \alpha} \tag{13}$$

*for $k = 1$ or for $k = 2$, is locally exact. Moreover, a corresponding primitive $J_k$ is a non-constant first integral of Equation (9) which is functionally independent of the respective first integral $I_k$ given in (8).*

**Proof.** For $k = 1$, the form (13) is closed because the symmetry generators $X_1$ and $X_3$ span a (solvable) two-dimensional symmetry algebra for the differential operator $A$ given in (5). By Poincaré lemma, $\beta_1$ is locally exact and a corresponding primitive $J_1$, which is a first integral of the integrable distribution $\langle A, X_1 \rangle$, arises by quadrature.

In order to prove that $I_1$ and $J_1$ are functionally independent, let us check that $dI_1 \wedge dJ_1 \neq 0$ by proceeding by contradiction. If $dI_1 \wedge dJ_1 = 0$, we can consider the interior product of the vector fields $X_3$ and $X_1$ with the two-form $dI_1 \wedge dJ_1$ which becomes

$$X_3 \lrcorner X_1 \lrcorner (dI_1 \wedge dJ_1) = X_1(I_1) X_3(J_1) - X_3(I_1) X_1(J_1).$$

Using the fact that $dJ_1 = \beta_1$, where $\beta_1$ is given by (13) for $k = 1$, we have $X_3(J_1) = X_3 \lrcorner dJ_1 = -1$ and $X_1(J_1) = X_1 \lrcorner dJ_1 = 0$. Therefore previous equality implies that

$$X_3 \lrcorner X_1 \lrcorner (dI_1 \wedge dJ_1) = -X_1(I_1) = 0.$$

We continue as in the proof of Theorem 3 to get the identity $X_1 \lrcorner X_3 \lrcorner \alpha = 0$, which contradicts the hypothesis on the maximal rank of the distribution $\langle A, X_1, X_3 \rangle$.

A similar proof can be used to show that the 1-form $\beta_2 = \dfrac{X_2 \lrcorner \alpha}{X_2 \lrcorner X_3 \lrcorner \alpha}$ is locally exact and a corresponding primitive $J_2$ is functionally independent of the first integral $I_2$ given in (8). $\square$

**Example 2.** *We recall that the first integrals (12) obtained by application of Theorem 3 to Equation (9) for $k = 0$ were functionally dependent. In this situation, we can use Theorem 4 in order to complete the integration. For instance, the corresponding closed 1-form (13)*

$$\beta_1 = \frac{1}{u}du - \frac{2}{u_1}du_1$$

*provides by quadrature a primitive $J_1 = \ln\left(\dfrac{u}{u_1^2}\right)$. The function $J_1$ (or the equivalent invariant $u/u_1^2$) is a first integral of Equation (9) for $k = 0$, which is functionally independent of the function $I_1$ obtained in (12).*

### *3.1. On the Functional Independence of the First Integrals (8)*

Example 1 shows that the functional independence or the functional dependency of the first integrals produced in Theorem 3 can occur for equations admitting the same realization of the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$.

In this regard, we firstly observe that when $k \neq 0$, Equation (9) does not admit additional Lie point symmetries, whereas the symmetry algebra in the case $k = 0$ is eight-dimensional and hence isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. These are the only possible symmetry algebras for second-order $SL(2,\mathbb{R})$-invariant ODEs [23] (Proposition 2), corresponding the case of $\mathfrak{sl}(3,\mathbb{R})$ to linearizable equations by a point transformation [24] (p. 405), [25] (Theorem 8) (see also References [26,27]). In this section we investigate the functional independence of the first integrals (8) separately for these two types of equations.

#### 3.1.1. Non-Linearizable Equations

According to Reference [28], there are two equivalence classes of second-order ODEs possessing only $\mathfrak{sl}(2,\mathbb{R})$ as symmetry algebra. The respective representations of $\mathfrak{sl}(2,\mathbb{R})$ and the representative equations are

$$\text{Type I:} \quad \mathbf{v}_1 = \partial_u, \mathbf{v}_2 = 2xu\partial_x + (u^2 - x^2)\partial_u, \mathbf{v}_3 = x\partial_x + u\partial_u. \tag{14}$$

$$xu_2 = u_1^3 + u_1 + a(1 + u_1^2)^{3/2}, \quad a \neq 0. \tag{15}$$

$$\text{Type II:} \quad \mathbf{v}_1 = \partial_u, \mathbf{v}_2 = 2xu\partial_x + u^2\partial_u, \mathbf{v}_3 = x\partial_x + u\partial_u, \tag{16}$$

$$xu_2 = au_1^3 - \frac{1}{2}u_1, \quad a \neq 0. \tag{17}$$

Let $U_1$ and $U_2$ denote the respective open sets where the distributions $\langle A, X_i, X_j \rangle$, $1 \leq i,j,k \leq 3$, corresponding to (14) and (16) are of maximal rank. Theorem 3 gives the following first integrals of Equations (15) and (17), respectively:

$$\text{Type I:} \quad I_1 = 2u + \frac{2x}{a\sqrt{u_1^2+1} + u_1}, \quad I_2 = -\frac{2au\sqrt{u_1^2+1} + 2(x + uu_1)}{a(x^2+u^2)\sqrt{u_1^2+1} + u_1(u^2 - x^2) + 2ux}. \tag{18}$$

$$\text{Type II:} \quad I_1 = 2u + \frac{4xu_1}{2au_1^2 - 1}, \quad I_2 = -\frac{4auu_1^2 + 4xu_1 - 2u}{(2au_1^2 - 1)u^2 - 4x^2u_1^2 + 4xuu_1}. \tag{19}$$

It can be easily checked that (19) are functionally independent in $U_2$. This proves the local functional independence of the first integrals (8) for any not linearizable second-order $SL(2,\mathbb{R})$-invariant ODE that can be mapped, through a local change of variables, into the representative

Equation (17). As an example, Equation (1) for $k \neq 0$, corresponds to type II, by means of the change of variables $\{x = u, u = x\}$, and for $a = -k$.

The first integrals (18) are functionally independent in $U_1$ if and only if $a \neq \pm 1$. In consequence, the first integrals (8) are functionally independent for any not linearizable second-order $SL(2, \mathbb{R})$-invariant ODE that is in the equivalence class of Equation (15) for $a \neq \pm 1$. In order to find a second functionally independent first integral for Equation (17) when $a = \pm 1$, we can apply Theorem 4. The corresponding 1-form $\beta_1$ defined in (13) is closed, and hence locally exact, that is, $\beta_1 = dJ_1$. Consequently, two functionally independent first integrals for Equation (15) for $a \neq \pm 1$ are given by

$$I_1 = 2u \pm \frac{2x}{\sqrt{u_1^2+1}+u_1}, J_1 = \ln\left(x\sqrt{u_1^2+1}\right) \mp \operatorname{arcsinh}(u_1), \quad \text{for} \quad a = \pm 1. \tag{20}$$

In (20), $J_1$ is equivalent to the algebraic invariant $\tilde{J}_1 = \exp(J_1)$ :

$$\tilde{J}_1 = x\sqrt{u_1^2+1}\left(u_1 + \sqrt{u_1^2+1}\right)^{\mp 1}, \quad \text{for} \quad a = \pm 1. \tag{21}$$

### 3.1.2. Linearizable Equations

We first observe that the vector field $\mathbf{v}_4 = u\partial_u$, is one of the additional Lie point symmetries admitted by Equation (9) for $k = 0$, and it commutes with all symmetry generators (10). In other words, when $k = 0$ Equation (9) admits the four-dimensional subalgebra $\mathcal{G} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, isomorphic to $\mathfrak{gl}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, which is one the subalgebras of the overall symmetry algebra $\mathfrak{sl}(3, \mathbb{R})$ [29].

In this situation, it is possible to prove that the first integrals (8) become always functionally dependent: since $Y = \mathbf{v}_4^{(1)}$ satisfies $[Y, X_i] = 0$, for $i = 1, 2, 3$, and by Proposition 2, $L_Y(\alpha) \in \langle\alpha\rangle$, we can write

$$\begin{aligned} L_Y(I_1) &= \frac{1}{(X_1 \lrcorner X_3 \lrcorner \alpha)^2}\left[L_Y(X_1 \lrcorner X_2 \lrcorner \alpha)(X_1 \lrcorner X_3 \lrcorner \alpha) - (X_1 \lrcorner X_2 \lrcorner \alpha)L_Y(X_1 \lrcorner X_3 \lrcorner \alpha)\right] \\ &= \frac{1}{(X_1 \lrcorner X_3 \lrcorner \alpha)^2}\left[(X_1 \lrcorner X_2 \lrcorner L_Y(\alpha))(X_1 \lrcorner X_3 \lrcorner \alpha) - (X_1 \lrcorner X_2 \lrcorner \alpha)(X_1 \lrcorner X_3 \lrcorner L_Y(\alpha))\right] = 0. \end{aligned} \tag{22}$$

Similarly, $L_Y(I_2) = 0$. In consequence, $I_1, I_2$ are first integrals of the integrable distribution $\langle A, Y\rangle$ (of maximal rank), which proves that $I_1$ and $I_2$ are functionally dependent.

Previous discussion applies, not only for the particular Equation (9) with $k = 0$, but for any second-order ODE that is linearizable by a point transformation and hence admits the maximum eight-dimensional Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. When the first integrals (8) are calculated by using the symmetry generators of $\mathfrak{sl}(2, \mathbb{R})$ included in the symmetry subalgebra $\mathfrak{gl}(2, \mathbb{R})$, relation (22) proves that such first integrals become always functionally dependent.

This result explains why the first integrals (12) are functionally dependent, but also raises the next question: could there exist other symmetry generators of $\mathfrak{sl}(2, \mathbb{R})$ for linearizable equations that produce functionally independent first integrals?

To answer this question, we consider the unique possible realization of the symmetry algebra $\mathfrak{sl}(3, \mathbb{R})$ in the real plane [30,31], which is given by:

$$\begin{aligned} &\mathbf{w}_1 = \partial_x, \mathbf{w}_2 = \partial_u, \mathbf{w}_3 = x\partial_x, \mathbf{w}_4 = u\partial_x, \mathbf{w}_5 = x\partial_u, \\ &\mathbf{w}_6 = u\partial_u, \mathbf{w}_7 = x^2\partial_x + xu\partial_u, \mathbf{w}_8 = xu\partial_x + u^2\partial_u. \end{aligned} \tag{23}$$

Any second-order ODE admitting a symmetry algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ can be mapped through a point transformation into $u_2 = 0$, which admit the symmetry generators (23) [24,32]. It can be checked that $\mathbf{v}_1 = \mathbf{w}_1, \mathbf{v}_2 = \mathbf{w}_7, \mathbf{v}_3 = \mathbf{w}_3 + \mathbf{w}_6/2$ and $\mathbf{v}_4 = \mathbf{w}_6$ span a symmetry subalgebra isomorphic to $\mathfrak{gl}(2, \mathbb{R})$, where $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \simeq \mathfrak{sl}(2, \mathbb{R})$. It follows from (22) that the first integrals (8) of $u_2 = 0$ calculated by using $X_i = \mathbf{v}_i^{(1)}$, for $i = 1, 2, 3$, are functionally dependent.

Nevertheless, it can be checked that the symmetry generators

$$\begin{array}{rcl} \mathbf{v}_1 & = & \mathbf{w}_1 + \mathbf{w}_5 = \partial_x + x\partial_u, \\ \mathbf{v}_2 & = & 2(\mathbf{w}_7 - \mathbf{w}_4) = 2(x^2 - u)\partial_x + 2xu\partial_u, \\ \mathbf{v}_3 & = & \mathbf{w}_3 + 2\mathbf{w}_6 = x\partial_x + 2u\partial_u, \end{array} \tag{24}$$

also satisfy the commutation relations (7) and hence span a symmetry subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The associated first integrals (8) become

$$I_1 = 2u_1, \quad I_2 = \frac{u_1}{u - xu_1}, \tag{25}$$

and are functionally independent on the open set $U$ where the distributions $\langle A, X_i, X_j \rangle$, $1 \leq i,j,k \leq 3$, are of maximal rank. This proves that also in the linearizable case there exist symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ for which the first integrals (8) are functionally independent.

As an illustration, symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ satisfying (7) that produce functionally independent first integrals for Equation (9) when $k = 0$ are

$$\mathbf{v}_1 = 2\sqrt{u}\partial_x + \sqrt{u}\partial_u, \quad \mathbf{v}_2 = 4x\sqrt{u}\partial_x + 2\sqrt{u}(4u - x)\partial_u, \quad \mathbf{v}_3 = 2x\partial_x + 2u\partial_u. \tag{26}$$

The corresponding first integrals (8) become

$$I_1 = \frac{2\sqrt{u}}{u_1}, \quad I_2 = \frac{\sqrt{u}}{xu_1 - 2u}, \quad u > 0.$$

In summary, the first integrals obtained in Theorem 3 may be functionally dependent for linearizable equations, but it is always possible to select symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ for which the associated first integrals become functionally independent.

## 4. First Integrals for $n$th-Order $SL(2,\mathbb{R})$-Invariant ODEs

In the following we aim at extending Theorems 3 and 4 to higher order equations admitting $\mathfrak{sl}(2,\mathbb{R})$ as symmetry algebra. This extension requires the use of the notion of solvable structure introduced in Section 2.2, allowing the sets to have $s < k$ vector fields satisfying the same conditions as in Definition 1.

Given an $n$-order $SL(2,\mathbb{R})$-invariant ODE of the form

$$u_n = F(x, u, u_1, \ldots, u_{n-1}) \tag{27}$$

and the corresponding vector field

$$A = \partial x + u_1\partial u + \ldots + F(x, u, u_1, \ldots, u_{n-1})\partial u_{n-1} \tag{28}$$

on the jet bundle $J^{n-1}(\mathbb{R},\mathbb{R})$, we have the following result, that shows how to construct (possibly not functionally independent) non-trivial first integrals for (27) without any kind of integration.

**Theorem 5.** *Let $A$ be the differential operator (28) associated to the nth-order ODE (27), which admits the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ with generators $X_1, X_2, X_3$ satisfying the commutation relations (7). If $\{Y_1, \ldots Y_{n-2}\}$ is a solvable structure for the integrable distribution $\langle A, X_k \rangle$ for $k = 1$ or for $k = 2$, and the distributions $\langle A, X_i, X_j, Y_1, \ldots, Y_{n-2} \rangle$ have maximal rank for $i, j = 1, 2, 3$, then the function*

$$I_k = \frac{Y_{n-2} \lrcorner \ldots Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y_{n-2} \lrcorner \ldots Y_1 \lrcorner X_k \lrcorner X_3 \lrcorner \alpha}, \tag{29}$$

*is a non-constant first integral for the operator (28).*

**Proof.** We demonstrate the results only for $k = 1$, as the corresponding proofs for $k = 2$ follow exactly the same line. If we consider the volume form on $J^{n-1}(\mathbb{R}, \mathbb{R})$ given by $\Omega = dx \wedge du \wedge du_x \wedge \ldots \wedge du_{n-1}$ and define the $n$-form

$$\alpha = A \lrcorner \Omega$$

we have to prove that function (29) is a non-trivial first integral of (27).

We remark that, since $Y_1, \ldots Y_{n-2}$ are not (in general) symmetries for (27), but they form a solvable structure for $\langle A, X_1 \rangle$, we cannot use Theorem 2 and Lemma 1 to ensure that $I_1$ is a first integral. Therefore we proceed by explicit computation and we rewrite $A(I_1) = 0$ in the equivalent form

$$\begin{aligned} &L_A(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) + \\ &-(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha) L_A(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) = 0. \end{aligned} \tag{30}$$

Since, by definition of solvable structure

$$L_A(Y_l) = -L_{Y_l} A \in \langle A, X_1, Y_1, \ldots, Y_{l-1} \rangle,$$

for any $l = 1, \ldots n-2$ we have that

$$L_A(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha) = \mathrm{div}(A)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha),$$

where we use the fact that $X_1$ and $X_2$ are symmetries for $A$.

Analogously we have

$$L_A(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) = \mathrm{div}(A)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha),$$

and rewriting (30) as

$$\begin{aligned} &\mathrm{div}(A)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) + \\ &-(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha) \mathrm{div}(A)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) = 0 \end{aligned} \tag{31}$$

we get $A(I_1) = 0$.

The second step is to prove that the first integral $I_1$ is not constant. By contradiction, let us suppose that $I_1$ is constant so that $X_1(I_1) = L_{X_1}(I_1) = 0$. Using the expression (29) we get

$$\begin{aligned} &L_{X_1}(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) + \\ &-(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha) L_{X_1}(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) = 0. \end{aligned} \tag{32}$$

Therefore, the definition of solvable structure and the commutation relations (7) allow us to rewrite (32) as

$$\begin{aligned} &(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner 2X_3 \lrcorner \alpha + Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner G_1 \alpha)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha) + \\ &-(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha)(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner G_1 \alpha) = \\ &2(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha)^2 = 0, \end{aligned}$$

which contradicts the hypothesis that the distribution $\langle A, X_1, X_3, Y_1, \ldots, Y_{n-2} \rangle$ has maximal rank. □

According to Proposition 4 in Reference [8], if the $p$ independent integrals of an involutive system $\mathcal{A}$ of $n-p$ vector fields are known, then one can construct a local coordinate system in which there exist $p$ independent, commuting symmetries of the system $\mathcal{A}$, which constitute, in particular, a (special) case of solvable structure for $\mathcal{A}$. Since for $k = 1, 2$, $\mathcal{A}_k = \{A, X_k\}$ is an integrable distribution, Frobenius' Theorem ensures the (local) existence of $n-1$ functionally independent common first integrals [18,19,33]. Therefore, the result in Reference [8] (Proposition 4) guarantees the (local) existence of $n-1$ commuting symmetries $Z_i$ of the system $\mathcal{A}_k$. From the conditions $L_{Z_i}(\mathcal{A}_k) \subset \mathcal{A}_k$ and

$[Z_i, Z_j] = 0$ it follows that any set with $n-2$ of these commuting symmetries satisfies the conditions of Theorem 5. Of course, the construction of such specific solvable structure of commuting symmetries is not obvious at all, and it can be as difficult as solving the given equation, but theoretically its local existence can be ensured.

**Example 3.** *The third-order ODE*

$$u_3 = \frac{3u_2^2}{2u_1} - 2u_1^3 C(u), \quad u_1 \neq 0, \tag{33}$$

*where $C(u)$ is an arbitrary function depending on $u$, admits the symmetry generators*

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = x^2\partial_x, \quad \mathbf{v}_3 = x\partial_x, \tag{34}$$

*whose respective second-order prolongations $X_i = \mathbf{v}_i^{(2)}$*

$$\begin{array}{rcl} X_1 & = & \partial_x, \\ X_2 & = & x^2\partial_x - 2xu_1\partial_{u_1} - 2(2xu_2 - u_1)\partial_{u_2}, \\ X_3 & = & x\partial_x - u_1\partial_{u_1} - 2u_2\partial_{u_2}, \end{array} \tag{35}$$

*satisfy the commutation relations (7).*

*Apart from (34), Equation (33) for arbitrary $C(u)$ admits the symmetry generators $f_i(u)\partial_u$, where $f_i$, for $i = 1, 2, 3$, are three independent solutions of the linear third-order ODE $f'''(u) + 4C(u)f'(u) + 2C'(u)f(u) = 0$. The structure of the whole symmetry algebra is different depending on the explicit form of $C(u)$. In this example we consider two particular cases for which the equation admits an additional Lie point symmetry $\mathbf{v}$ such that $Y = \mathbf{v}^{(2)}$ satisfies*

$$[A, Y] = [X_i, Y] = 0, \quad i = 1, 2, 3. \tag{36}$$

*This situation is very favourable, because $Y$ is a solvable structure with respect to $\langle A, X_1\rangle$ and also with respect to $\langle A, X_2\rangle$. In this case, according to Theorem 5, a single vector field $Y$ permits to calculate two non-constant first integrals without any kind of integration. We present two different examples, in order to show that both first integrals may be functionally dependent or independent.*

*In what follows, $\alpha = A \lrcorner \Omega$ is given by*

$$\begin{array}{c} \alpha = -\left(\dfrac{3u_2^2}{2u_1} - 2u_1^3 C(u)\right) dx \wedge du \wedge du_1 + \\ u_2 dx \wedge du \wedge du_2 - u_1 dx \wedge du_1 \wedge du_2 + du \wedge du_1 \wedge du_2. \end{array} \tag{37}$$

1. *For $C(u) = 1$, Equation (33) admits, apart from (34), the Lie point symmetry $\mathbf{v} = \partial_u$. It is easy to check that the prolongation $Y = \mathbf{v}^{(2)} = \partial_u$ satisfies the commutation relations (36). Let $U$ denote the open set of $N = \{(x, u, u_1, u_2) \in J^2(\mathbb{R}, \mathbb{R}) : u_1 \neq 0\}$, where the distributions $\langle A, X_i, X_j, Y\rangle$, for $i = 1, 2, 3$, have maximal rank. By Theorem 5, the symmetry generators (35) and $Y$ is all we need to calculate the two non-constant first integrals (29), for $k = 1, 2$ :*

$$\begin{array}{rcl} I_1 & = & \dfrac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha} = \dfrac{2(4xu_1^4 + xu_2^2 + 2u_1u_2)}{u_2^2 + 4u_1^4}, \\ I_2 & = & \dfrac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_2 \lrcorner X_3 \lrcorner \alpha} = -\dfrac{2(4xu_1^4 + xu_2^2 + 2u_1u_2)}{4x^2u_1^4 + x^2u_2^2 + 4xu_1u_2 + 4u_1^2}, \end{array} \tag{38}$$

*defined for* $(x, u, u_1, u_2) \in U$*. The Jacobian determinant*

$$\frac{\partial(I_1, I_2)}{\partial(u_1, u_2)} = \frac{256u_1^6(4xu_1^4 + xu_2^2 + 2u_1u_2)}{(u_2^2 + 4u_1^4)^2(4x^2u_1^4 + x^2u_2^2 + 4xu_1u_2 + 4u_1^2)^2}$$

*does not vanish in* $U$*, and hence the functions* $I_1, I_2$ *given in (38) are functionally independent in* $U$*.*

2. *It can be checked that for* $C(u) = 3u - \frac{9}{4}u^4$ *Equation (9) admits, apart from (34), the Lie point symmetry*

$$\mathbf{v} = \exp(-u^3)\partial_u,$$

*whose prolongation*

$$Y = \exp(-u^3)\left(\partial_u - 3u^2u_1\partial_{u_1} + (3(3u^3 - 2)uu_1^2 - 3u_2u^2)\partial_{u_2}\right), \tag{39}$$

*satisfies the commutation relations (36).*

*Such first integrals are constructed through (29) by using the corresponding 2-form (37) and the symmetry generators (34)–(39). The corresponding first integrals become:*

$$\begin{aligned} I_1 &= \frac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha} = \frac{2u_2x + 6u^2u_1^2 + 4u_1}{u_2 + 3u^2u_1^2}, \\ I_2 &= \frac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_2 \lrcorner X_3 \lrcorner \alpha} = -\frac{2u_2 + 6u_1^2u^2}{u_2x + 3xu^2u_1^2 + 2u_1}. \end{aligned} \tag{40}$$

*Both first integrals are functionally dependent, because the relation* $I_1I_2 + 4 = 0$ *holds.*

This example shows that, as in the case of second order equations, the first integrals $I_1$ and $I_2$ may be functionally dependent. In the next Theorem we prove that under suitable additional hypotheses on the vector fields $Y_1, \ldots, Y_{n-2}$, we are able to construct a second functionally independent first integral with $I_k$, for $k = 1$ or $k = 2$, integrating by quadrature a closed 1-form. This extends Theorem 4 for $SL(2, \mathbb{R})$-invariant ODEs of arbitrary order $n$.

**Theorem 6.** *Let* $A$ *be a vector field on* $J^{n-1}(\mathbb{R}, \mathbb{R})$ *of the form (28), associated with the nth-order ODE (27) which admits the symmetry algebra* $\mathfrak{sl}(2, \mathbb{R})$ *with generators* $X_1, X_2, X_3$ *satisfying the commutation relations (7). If* $\{Y_1, \ldots Y_{n-2}\}$ *is a solvable structure for* $\langle A, X_k\rangle$ *for* $k = 1$ *or for* $k = 2$*, the distributions* $\langle A, X_i, X_j, Y_1, \ldots, Y_{n-2}\rangle$ *have maximal rank for* $i, j = 1, 2, 3$*, and* $\{Y_1, \ldots Y_{n-2}\}$ *is a solvable structure for the three dimensional distribution* $\langle A, X_1, X_3\rangle$*, then the first integral* $I_k$ *given by (29) and the first integral* $J_k$ *obtained by integrating by quadrature the closed 1-form*

$$\beta_k = \frac{Y_{n-3} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_3 \lrcorner X_k \lrcorner \alpha}{Y_{n-2} \lrcorner Y_{n-3} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_3 \lrcorner X_k \lrcorner \alpha} \tag{41}$$

*are functionally independent.*

**Proof.** We write the proof for $k = 1$, the case $k = 2$ is similar. First note that, since $X_1$ and $X_3$ form a solvable algebra of Lie symmetries for $A$, the vector fields $X_1, X_3, Y_1, \ldots Y_{n-2}$ form a solvable structure for $\langle A\rangle$, and the 1-form $\beta_1$ is closed due to Theorem 1. In order to prove that $I_1$ and $J_1$ are functionally independent, we suppose by contradiction that $dI_1 \wedge dJ_1 = 0$ and consider the interior product of the two-form $dI_1 \wedge dJ_1 = 0$ with the vector fields $X_1$ and $Y_{n-2}$. Since $Y_{n-2} \lrcorner dJ_1 = 1$ and $X_1 \lrcorner dJ_1 = 0$ we get

$$Y_{n-2} \lrcorner X_1 \lrcorner (dI_1 \wedge dJ_1) = X_1(I_1)Y_{n-2}(J_1) - X_1(J_1)Y_{n-2}(I_1) = X_1(I_1) = 0. \tag{42}$$

By proceeding as in the proof of Theorem 5, from $X_1(I_1) = 0$ it follows that $(Y_{n-2} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha)^2 = 0$, which contradicts the hypothesis that the distribution $\langle A, X_1, X_3, Y_1, \ldots, Y_{n-2}\rangle$ has maximal rank. □

**Example 4.** *In this example we apply Theorem 6 in order to find, by quadrature, additional first integrals for the the particular cases of Equation (33) considered in Example 3.*

1. *We recall that (38) are two functionally independent first integrals of Equation (33) when $C(u) = 1$. Therefore, we only need to find one additional first integral, than can be done by using any of the corresponding 1-forms (41) for $k = 1$, or for $k = 2$. For instance, the 1-form*

$$\beta_1 = \frac{X_3 \lrcorner X_1 \lrcorner \alpha}{Y \lrcorner X_3 \lrcorner X_1 \lrcorner \alpha} = \frac{1}{u_2^2 + 4u_1^4}\left(du - 4u_2 u_1 du_1 + 2u_1^2 du_2\right) \tag{43}$$

*is closed, and hence locally exact. A corresponding primitive arises by direct integration:*

$$J_1 = u - \arctan\left(\frac{2u_1^2}{u_2}\right). \tag{44}$$

*According to Theorem 5, $\{I_1, I_2, J_1\}$, where $I_1, I_2$ are given in (38), is a complete set of first integrals for Equation (33) when $C(u) = 1$.*

2. *In order to complete the integration of the second particular case of Equation (33) considered in Example 3, we need two additional independent first integrals, because the functions (40) are functionally dependent.*

*With this aim, we use the two 1-forms (41), for $k = 1$ and $k = 2$. By using the 3-form* (37) *and the symmetry generators* (35) *we find that:*

$$\beta_1 = \frac{\exp(u^3)}{(u_2 + 3u_1^2 u^2)^2}\left((u_2^2 + 12u_1^4 u - 9u^3)du - 4u_2 u_1 du_1 + 2u_1^2 du_2\right). \tag{45}$$

*This 1-form is closed, and hence locally exact. A corresponding primitive arises by direct integration:*

$$J_1 = -\frac{2\exp(u^3)u_1^2}{u_2 + 3u^2 u_1^2} + \phi(u), \tag{46}$$

*where $\phi(u)$ satisfies*

$$\phi'(u) = \exp(u^3). \tag{47}$$

*Function $\phi(u)$ can be expressed in terms of the Gamma function as follows: $\phi(u) = \frac{1}{3}\Gamma\left(\frac{1}{3}, -u^3\right)$.*

*According to Theorem 6, the first integral (46) and the first integral $I_1$ (or $I_2$) given in (40) are functionally independent.*

*Similarly, a primitive of the corresponding 1-form (41) for $k = 2$*

$$\begin{aligned}\beta_2 = &\frac{\exp(u^3)}{(xu_2 + 3xu^2u_1^2 + 2u_1)^2}(-4u_1^3 dx + (x^2u_2^2 + 4xu_1u_2 + 4u_1^2 \\ &- (9u^4 - 12u)x^2u_1^4)du - 4xu_1(xu_2 + u_1)du_1 + 2x^2u_1^2 du_2)\end{aligned} \tag{48}$$

*becomes*

$$J_2 = -\frac{2\exp(u^3)xu_1^2}{xu_2 + 2u_1 + 3xu^2u_1^2} + \varphi(u), \tag{49}$$

*where $\varphi(u)$ satisfies condition (47), i.e., $\varphi'(u) = \exp(u^3)$.*

*A rational first integral can be calculated by using $J_1$ and $J_2$ :*

$$J_1 - J_2 = \frac{-4u_1^3 \exp(u^3)}{(xu_2 + 3xu^2u_1^2 + 2u_1)(u_2 + 3u^2u_1^2)}.$$

*Any of the sets $\{I_i, J_1, J_2\}$ or $\{I_i, J_1 - J_2, J_i\}$, for $i = 1, 2$ are complete sets of first integrals for the differential operator associated to Equation (3) for $C(u) = 3u - \frac{9}{4}u^4$, which is now completely integrated.*

Next we present an example of a third-order $SL(2, \mathbb{R})$-invariant ODE which only admits an additional Lie point symmetry.

**Example 5.** *Let $A$ be the differential operator associated to the third-order ordinary differential equation*

$$u_3u^2 + \left(u_1^2 - 2uu_2\right)^{3/2} = 0, \tag{50}$$

*defined in $N = \left\{(x, u, u_1, u_2) \in J^2(\mathbb{R}, \mathbb{R}) : u \neq 0, u_1^2 - 2uu_2 > 0\right\}$. The algebra of Lie point symmetries of Equation (50) is four-dimensional and spanned by*

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{v}_3 = x\partial_x + u\partial_u, \quad \mathbf{v}_4 = u\partial_u. \tag{51}$$

*The symmetry generators $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ correspond to the fourth realization of $\mathfrak{sl}(2, \mathbb{R})$ in Reference [30] (Table 6) and hence span a symmetry subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The corresponding prolongations $X_i = \mathbf{v}_i^{(2)}$, $i = 1, 2, 3$*

$$X_1 = \partial_x, \quad X_2 = x^2\partial_x + 2xu\partial_u + 2u\partial_{u_1} - (2xu_2 - 2u_1)\partial_{u_2}, \quad X_3 = x\partial_x + u\partial_u - u_2\partial_{u_2} \tag{52}$$

*satisfy the commutation relations (7). The Lie point symmetry $\mathbf{v}_4$ commutes with $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, which implies that $Y = \mathbf{v}_4^{(2)}$ forms a solvable structure with respect to $\langle A, X_1\rangle$ and also with respect to $\langle A, X_2\rangle$. Let $U$ be the open set $U \subset N$ where the integrable distributions $\langle A, X_1, X_2, Y\rangle$, $\langle A, X_2, X_3, Y\rangle$ and $\langle A, X_1, X_3, Y\rangle$ are of maximal rank. By Theorem 5, the Lie point symmetry $\mathbf{v}_4$ can be exploited to find two non-constant first integrals of Equation (50) without any kind of integration. Such first integrals become:*

$$\begin{aligned} I_1 &= \frac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_1 \lrcorner X_3 \lrcorner \alpha} = \frac{2\left((xu_1 - u)\sqrt{u_1^2 - 2uu_2} + uu_1 - u_2ux\right)}{u_1\sqrt{u_1^2 - 2uu_2} - uu_2}, \\ I_2 &= \frac{Y \lrcorner X_1 \lrcorner X_2 \lrcorner \alpha}{Y \lrcorner X_2 \lrcorner X_3 \lrcorner \alpha} = \frac{2\left((xu_1 - u)\sqrt{u_1^2 - 2uu_2} + uu_1 - u_2ux\right)}{(2xu - x^2u_1)\sqrt{u_1^2 - 2uu_2} + 2u^2 + x^2u_2u - 2u_1ux}. \end{aligned} \tag{53}$$

*It can be checked that the Jacobian determinant*

$$\frac{\partial(I_1, I_2)}{\partial(u_1, u_2)} = \frac{-16u^4\left((xu_1 - u)\sqrt{u_1^2 - 2uu_2} + uu_1 - u_2ux\right)}{\left(u_1\sqrt{u_1^2 - 2uu_2} - uu_2\right)^2\left((x^2 - 2xu)\sqrt{u_1^2 - 2xu} - 2u^2 - x^2u_2u + 2u_1ux\right)^2}$$

*does not vanish for $(x, u, u_1, u_2) \in U$ and hence the first integrals (53) are functionally independent in $U$.*

*In order to find a third functionally independent first integral by using Theorem 6 we consider the corresponding 1-form $\beta_1$ defined in (41)*

$$\beta_1 = \frac{\left(uu_2^2du + \left((u_1^2 - 2uu_2)^{3/2} - uu_1u_2\right)du_1 + u^2u_2du_2\right)}{(2uu_2 - u_1^2)\left(u_2u - u_1\sqrt{u_1^2 - 2uu_2}\right)},$$

*which is closed and then locally exact. A corresponding function $J_1$ such that $dJ_1 = \beta_1$ is locally given by:*

$$J_1 = \sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}\left(u_1 + \sqrt{u_1^2 - 2uu_2}\right)}{2u_1}\right) + \ln\left(\sqrt{u_1^2 - 2uu_2}\right). \tag{54}$$

*According to Theorem 6, $\{I_1, I_2, J_1\}$, where $I_1, I_2, J_2$ are given in (53) and (54), is a complete set of functionally independent first integrals that completes the integration of Equation (50).*

## 5. Conclusions and Further Extensions

New methods to construct first integrals of differential operators associated to $SL(2,\mathbb{R})$-invariant ODEs have been introduced. Some of these first integrals can be computed through the symmetry generators without any kind of integration. It is possible to prove that they are never constant, although in some cases may be functionally dependent. In this last situation, it is possible to find an independent first integral by quadrature alone. The theoretical results show that the vector fields that can be used to integrate the equation by quadrature are not limited to symmetries or solvable structures.

To the best of our knowledge, it is the first time that these techniques are applied to provide first integrals for $SL(2,\mathbb{R})$-invariant ODEs without integration. In the case of second-order ODEs, only the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ are used in the procedures. Our results apply in particular for the class of second-order equations that are linearizable by a point transformation, which admit the maximal symmetry algebra $\mathfrak{sl}(3,\mathbb{R})$. Second-order ODEs admitting two commuting and noncommuting unnconnected point symmetries were investigated with a view to linerization in References [26,27], and several equivalent characterizations of linearization for second-order ODEs are well known in the literature [25]. For all these equations we have shown how to construct directly two independent first integrals, given just in terms of appropriate symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$, without any integration at all.

The $SL(2,\mathbb{R})$-invariant second-order ODEs that do not pass Lie's test of linearization cannot admit additional symmetries, apart from the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$. They are in the class of second-order ODEs with only three symmetries that have been considered in Reference [28] with a view to linearization by not point transformation, relationships with the complete symmetry group [32,34], and the Painlevé property [35–37]. Our investigation reveals that for this class of equations the first integrals obtained without integration in Theorem 3 generally are functionally independent. In fact there are only two possible equations which fail (considered as the representatives of the corresponding equivalence classes). We do not know if such equations have special features which explain this peculiar situation.

The potential application of our results to investigate the linearization under non-point transformations needs to be investigated further. The relationships with linear equations of higher and lower order found in Reference [28] suggest to investigate the role of different types of symmetries in the construction of the solvable structures used in Theorem 5. This includes nonlocal symmetries, hidden symmetries of types I and II [38], contact symmetries and generalized symmetries. By other hand, the presence of fundamental sets of solutions of second-order linear equations that have been recently found in the first integrals and in the general parametric solutions of $SL(2,\mathbb{R})$-invariant ODEs [13–15] may help to establish other connections with linear equations by means of new types of transformations.

Finally, we would like to note that several further questions remain open. Between them, it would be of great interest to study the functional independence of the first integrals obtained in Theorem 5, extending the study performed for $n = 2$ in Section 3.1 to equations of arbitrary order $n > 2$. In this regard, the results obtained in Example 3 seem to indicate that the representation of the symmetry algebra is not as relevant as the structure of possible additional symmetries admitted by the equation. A detailed analysis of each of the four canonical third-order equations associated to the nonequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ in the real plane [13,14] could be a good starting point for this research.

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