Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 442 (2022) 92–108

FUZZY sets and systems

www.elsevier.com/locate/fss

Bipolar equations on complete distributive symmetric residuated lattices: The case of a join-irreducible right-hand side

M. Eugenia Cornejo [a], David Lobo [a,*], Jesús Medina [a], Bernard De Baets [b]

[a] *Department of Mathematics, University of Cádiz, Spain*
[b] *KERMIT, Department of Data Analysis and Mathematical Modelling, Ghent University, Belgium*

Received 8 March 2021; received in revised form 19 October 2021; accepted 1 February 2022
Available online 7 February 2022

Abstract

Bipolar max-$*$ equations, with $*$ a triangular norm, have recently become a popular research topic embedded in the broad field of fuzzy relational equations. In this paper, we lift the work from the restrictive setting of the real unit interval — obfuscating the underlying lattice-theoretical essence — to the general setting of complete distributive symmetric residuated lattices, allowing to build upon the vast body of knowledge on unipolar sup-$*$ equations on complete distributive residuated lattices. We determine the full solution set, with particular emphasis on the extremal solutions, of a bipolar sup-$*$ equation in case the right-hand side is a join-irreducible element. The results are illustrated by means of ample examples.

Keywords: Bipolar equation; Distributive symmetric residuated lattice; Negation operator; Irreducible element

1. Introduction

Bipolarity in human reasoning may emerge in many different ways, from pros and cons in decision making to the processing of contradictory information. Hence, any intelligent system that aims to mimic human reasoning should be able to deal with bipolar information. As extensively discussed in [19], different types of bipolarity have been recognized and studied in the literature.

Within this context of bipolarity, bipolar fuzzy relational equations were introduced in [20] as a generalization of fuzzy relational equations [30,31] (one of the core research topics in fuzzy set theory [15,18]), where the unknown variables and their negations appear simultaneously. Most, if not all, publications on bipolar fuzzy relational equations are restricted to the real unit interval [20,26,27,33], as was the case for fuzzy relational equations in the early days. Moreover, the equations considered in the literature are all of the type max-$*$, with $*$ a triangular norm (t-norm, for short). In particular, the different approaches use the three basic t-norms: the minimum operator [20,24,25], the

* Corresponding author.
E-mail addresses: mariaeugenia.cornejo@uca.es (M.E. Cornejo), david.lobo@uca.es (D. Lobo), jesus.medina@uca.es (J. Medina), bernard.debaets@ugent.be (B. De Baets).

https://doi.org/10.1016/j.fss.2022.02.003

algebraic product [7,11] and the Łukasiewicz t-norm [26,32,33]. Moreover, in all of these works, negation is modelled by the standard negation. Only in [8,9] a different negation is considered, namely the one derived from the residual implication of the algebraic product.

When looking back in retrospect at the development of the field of fuzzy relational equations, it is clear that a major breakthrough was their framing in a non-linear lattice-theoretical context [10,15,18]. The main objective of this paper is to lift the study of bipolar equations to this more general framework, while at the same time considering the most general class of t-norms possible. This does not only allow to expand their scope, but also allows to exploit the vast body of knowledge in the field of fuzzy relational equations. Indeed, when restricting to the unit interval, one is often distracted by the specific analytical expression of the operations considered, thus missing the bigger picture.

As in the study of unipolar fuzzy relational equations, here we focus on bipolar sup-$*$ equations, as they are the building stones of systems of bipolar equations, and ultimately, bipolar fuzzy relational equations [15]. In view of the knowledge on unipolar sup-$*$ equations, and the fact a bipolar sup-$*$ equation involves negation, we will first identify the proper lattice-theoretical setting in which bipolar sup-$*$ equations should be considered. As it will turn out, this will be the setting of complete distributive symmetric residuated lattices, which, interestingly, will be shown to be nothing else but complete distributive residuated lattices endowed with an involutive negation $\neg$. This clearly extends the algebraic framework of bipolar equations defined on the unit interval and based on some particular t-norms (the minimum operator [20,24,25], the algebraic product [7,11] and the Łukasiewicz t-norm [26,32,33]).

It will also not come as a surprise that the morphism behaviour of the operation $*$ will play a crucial role. However, the generality of this class of structures will be contrasted by the restriction imposed on the right-hand side of the equations considered. Indeed, as in [14], in this first contribution, we will restrict our attention to bipolar sup-$*$ equations with a join-irreducible right-hand side, deferring the case of a join-decomposable right-hand side (as well as systems of such equations) to future work.

Our approach to solving bipolar sup-$*$ equations follows a traditional recipe. First, we determine the solution set, from which a necessary condition for solvability of such equation is deduced. Furthermore, a simple characterization of the solvability of a bipolar sup-$*$ equation is provided. Next, the algebraic structure of the solution set is investigated, providing conditions under which there exists a greatest (resp. smallest) solution or maximal (resp. minimal) solutions. All these results largely generalize those available in the literature [7,11,20,24–26,32,33] related to bipolar sup-$*$ equations on the unit interval. Finally, ample examples are provided to illustrate the wide applicability of the approach presented.

2. Preliminary notions

This section recalls some basic lattice-theoretical notions that will be used throughout this manuscript. In what follows, the symbols $\vee$ and $\wedge$ represent supremum (i.e., the smallest upper bound) and infimum (i.e., the greatest lower bound), respectively. For more details on lattice theory, the reader is referred to monographs on the topic [4,12,21].

Definition 1. Let $(L, \preceq)$ be a partially ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$, then $(L, \preceq)$ is called a *lattice*.
(ii) If the supremum and the infimum of S exist for all $S \subseteq L$, then $(L, \preceq)$ is called a *complete lattice*.
(iii) If $(L, \preceq)$ is a lattice and the following equality holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$, then $(L, \preceq)$ is called a *distributive lattice*.
(iv) Let $(L, \preceq)$ be a lattice and K be a nonempty subset of L. If $x \vee y \in K$ and $x \wedge y \in K$ for all $x, y \in K$, then $(K, \preceq)$ is said to be a *sublattice* of $(L, \preceq)$.

The well-known M_3–N_5 Theorem states that a lattice is distributive if and only if it does not contain a diamond M_3 or a pentagon N_5 as a sublattice (see Fig. 1 for a Hasse diagram of these lattices). Obviously, any chain is a distributive lattice.

Theorem 2. *A lattice $(L, \preceq)$ is distributive if and only if neither M_3 nor N_5 is a sublattice of $(L, \preceq)$.*

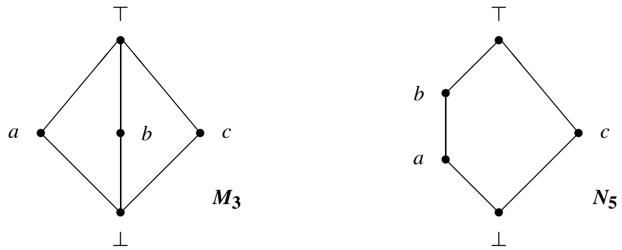

Fig. 1. A Hasse diagram of the lattices M_3 and N_5.

Join-irreducible elements of a lattice are remarkable elements that will play a key role in this paper. As the name suggests, a join-irreducible element cannot be written as the supremum of two (or more) other elements of the lattice [4,21]. Note that some sources exclude the bottom element (if it exists) [12], but this is irrelevant for the discussion in this paper. Obviously, all elements of a chain, in particular of the real unit interval equipped with the natural order, are join-irreducible.

Definition 3 *([4])*. Let $(L, \preceq)$ be a lattice. An element $x \in L$ is said to be *join-irreducible* if $x = a \vee b$ implies $x = a$ or $x = b$, for all $a, b \in L$.

Next, we recall the notions of supremum-morphism, infimum-morphism and (lattice) homomorphism.

Definition 4. Let $(L, \preceq)$ be a complete lattice. A mapping $f : L \to L$ is called:

(i) a *supremum-morphism* if $f(\sup A) = \sup(f(A))$, for all $A \in \mathcal{P}(L) \setminus \{\varnothing\}$.
(ii) an *infimum-morphism* if $f(\inf A) = \inf(f(A))$, for all $A \in \mathcal{P}(L) \setminus \{\varnothing\}$.
(iii) a *(lattice) homomorphism* if it is both a supremum-morphism and an infimum-morphism.

Consider an increasing (i.e., order-preserving) mapping $f : L \to L$ and a subset $A \in \mathcal{P}(L) \setminus \{\varnothing\}$. If $\sup A \in A$, then it trivially holds that $f(\sup A) = \sup(f(A))$; similarly, if $\inf A \in A$, then $f(\inf A) = \inf(f(A))$.

3. Complete distributive symmetric residuated lattices

The term *symmetric residuated lattice* was introduced in [6] to refer to a bounded residuated lattice equipped with a De Morgan involution, i.e., an involutive operator satisfying the De Morgan laws. These concepts are recalled below.

Definition 5. A *residuated lattice* is an algebraic structure $(L, \preceq, *, \to)$ such that

(i) $(L, \preceq)$ is a lattice with top element $\top$;
(ii) $(L, *, \top)$ is a commutative monoid;
(iii) $(*, \to)$ is a residuated pair, i.e., $x * y \preceq z$ if and only if $x \preceq y \to z$, for all $x, y, z \in L$.

A residuated lattice with bottom element $\bot$ is called a *bounded residuated lattice*.

For a complete residuated lattice $(L, \preceq, *, \to)$, due to the commutativity of $*$, if the first partial mappings of $*$, i.e. the mappings $*_y(x) = x * y$ with $y \in L$, are homomorphisms, then the same holds for the second partial mappings of $*$, i.e. for the mappings $*_x(y) = x * y$ with $x \in L$, and vice versa. Further on, we will simply refer to the condition that the partial mappings of $*$ are homomorphisms. Continuous triangular norms on the real unit interval, such as the three basic t-norms (the Gödel, product and Łukasiewicz t-norms), are key examples of such operations $*$ with partial mappings that are homomorphisms. In fuzzy set theory, the binary operation $*$ of a bounded residuated lattice is generally referred to as a t-norm.

Definition 6. Consider a bounded residuated lattice $(L, \preceq, *, \rightarrow)$. If a unary operator $\neg : L \rightarrow L$ satisfies the properties[1]

(i) $\neg\neg x = x$, for all $x \in L$ (involutivity),
(ii) $\neg(x \vee y) = \neg x \wedge \neg y$, for all $x, y \in L$ (first De Morgan law),
(iii) $\neg(x \wedge y) = \neg x \vee \neg y$, for all $x, y \in L$ (second De Morgan law),

then $(L, \preceq, *, \rightarrow, \neg)$ is called a *symmetric residuated lattice*. Moreover, if $(L, \preceq)$ is a complete distributive lattice, then $(L, \preceq, *, \rightarrow, \neg)$ is called a *complete distributive symmetric residuated lattice (CDSRL)*.

Note that for a CDSRL $(L, \preceq, *, \rightarrow, \neg)$ with top element $\top$ and bottom element $\bot$, it holds that $(L, \vee, \wedge, \bot, \top, \neg)$ is a De Morgan algebra [1,5,29]. Such algebras were first presented by Moisil [28], and then studied in depth by Kalman [23] under the name of distributive i-lattices, and by Bialynicki–Birula and Rasiowa [2,3] under the name of quasi-Boolean algebras. According to the mentioned literature, under the assumptions of distributivity and involutivity, the two De Morgan laws are equivalent. Furthermore, they are equivalent to $\neg$ being order-reversing (i.e., decreasing).

Actually, as we will show next, distributivity is not necessary for the equivalence between the De Morgan laws and the decreasingness of $\neg$, and only involutivity is effectively required.

Proposition 7. *Let $(L, \preceq, \bot, \top)$ be a bounded lattice and $\neg\colon L \rightarrow L$ an involutive mapping. The following statements are equivalent:*

(1) $\neg(x \vee y) = \neg x \wedge \neg y$, *for all* $x, y \in L$*;*
(2) $\neg(x \wedge y) = \neg x \vee \neg y$, *for all* $x, y \in L$*;*
(3) $\neg\top = \bot$, $\neg\bot = \top$ *and* $\neg$ *is order-reversing.*

Proof.

(1)⇒(2) Suppose that $\neg(u \vee v) = \neg u \wedge \neg v$ for all $u, v \in L$. Let $x, y \in L$ and consider $u = \neg x$ and $v = \neg y$, then it holds that

$$\neg(\neg x \vee \neg y) = \neg\neg x \wedge \neg\neg y\,.$$

Since $\neg$ is involutive, this leads to

$$\neg(\neg x \vee \neg y) = x \wedge y\,.$$

Hence, applying $\neg$ to both sides of this equality, we get

$$\neg(x \wedge y) = \neg\neg(\neg x \vee \neg y) = \neg x \vee \neg y\,.$$

(2)⇒(3) Suppose that $\neg(u \wedge v) = \neg u \vee \neg v$ for all $u, v \in L$. The following chain of equalities is easily verified:

$$\neg\top = \neg\top \vee \bot = \neg\top \vee \neg\neg\bot = \neg(\top \wedge \neg\bot) = \neg\neg\bot = \bot\,.$$

By involutivity of $\neg$, we then also get that $\neg\bot = \neg\neg\top = \top$. It remains to show that $\neg$ is order-reversing. Let $x, y \in L$ such that $x \preceq y$, then $x = x \wedge y$ and

$$\neg x = \neg(x \wedge y) = \neg x \vee \neg y\,.$$

By definition of supremum, it holds that $\neg y \preceq \neg x$.

(3)⇒(1) Let $x, y \in L$. By definition of supremum, $x \preceq x \vee y$ and $y \preceq x \vee y$. Since $\neg$ is order-reversing, it follows that $\neg(x \vee y) \preceq \neg x$ and $\neg(x \vee y) \preceq \neg y$, and thus

$$\neg(x \vee y) \preceq \neg x \wedge \neg y\,. \tag{1}$$

[1] In order to simplify the notation, we will write $\neg x$ instead of $\neg(x)$ whenever no confusion is possible.

Similarly, by definition of infimum, it holds that $\neg x \wedge \neg y \preceq \neg x$ and $\neg x \wedge \neg y \preceq \neg y$. Since $\neg$ is order-reversing and involutive, it follows that:

$$x = \neg\neg x \preceq \neg(\neg x \wedge \neg y)$$

$$y = \neg\neg y \preceq \neg(\neg x \wedge \neg y)\,.$$

This leads to $x \vee y \preceq \neg(\neg x \wedge \neg y)$. Once more invoking involutivity of $\neg$, we conclude that

$$\neg x \wedge \neg y = \neg\neg(\neg x \wedge \neg y) \preceq \neg(x \vee y)\,. \tag{2}$$

From inequalities (1) and (2), we conclude that $\neg(x \vee y) = \neg x \wedge \neg y$, which concludes the proof. □

Consequently, a symmetric residuated lattice can be characterized as a bounded residuated lattice with an involutive negation.

Definition 8. Consider a bounded lattice $(L, \preceq, \perp, \top)$. A unary operator $\neg : L \to L$ is called an involutive negation if it satisfies the properties:

(i) $\neg\top = \perp$ and $\neg\perp = \top$ (Boolean boundary conditions);
(ii) $\neg$ is order-reversing;
(iii) $\neg$ is involutive.

Corollary 9. *Consider a bounded residuated lattice $(L, \preceq, *, \to)$, then $(L, \preceq, *, \to, \neg)$ is a symmetric residuated lattice if and only if $\neg : L \to L$ is an involutive negation.*

Obviously, a CDSRL is then nothing else but a complete distributive residuated lattice endowed with an involutive negation.

Corollary 10. *A CDSRL is a complete distributive residuated lattice endowed with an involutive negation.*

4. Bipolar equations

4.1. Problem formulation

Various studies have been devoted to the resolution of bipolar max-$*$ equations on the real unit interval, in particular when $*$ is one of the three basic t-norms: the minimum operation [24,25], the algebraic product [9,11,22] and the Łukasiewicz t-norm [26,32,33]. In all cases, the involutive negation considered is the standard negation $\neg_S$ defined by $\neg_S x = 1 - x$.

In this paper, we focus on the resolution of bipolar sup-$*$ equations (also briefly called *bipolar equations*) on a CDSRL, i.e., a complete distributive residuated lattice endowed with an involutive negation, thus not only covering, but also going largely beyond the state-of-the-art.

Definition 11. Let $(L, \preceq, *, \to, \neg)$ be a CDSRL and $a_j^+, a_j^-, b \in L$, for each $j \in \{1, \dots, m\}$. A bipolar sup-$*$ equation is an equation of the type:

$$\bigvee_{j\in\{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) = b \tag{3}$$

in the unknowns $x_1, \dots, x_m$ in L.

Since a CDSRL is a complete lattice, the suprema involved in a bipolar sup-$*$ equation always exist and Eq. (3) is well defined. Recall that the residuated implication $\to$ is given by

$$a \to b = \sup\{x \in L \mid a * x \preceq b\}\,.$$

Following the approach in [13], we associate a second operation $\rightsquigarrow$ with $*$, defined by

$$a \rightsquigarrow b = \inf\{x \in L \mid b \preceq a * x\}\,.$$

These operations will play a key role in expressing the extremal solutions of a bipolar sup-$*$ equation in a compact manner. Note that both operations $\rightarrow$ and $\rightsquigarrow$ are well defined on a complete lattice. Both operations are decreasing in the first argument (antecedent) and increasing in the second argument (consequent); however, they do not share the same boundary conditions. For instance, we can easily check that:

$$\begin{aligned}
\bot \rightarrow \bot &= \sup\{x \in L \mid \bot * x \preceq \bot\} = \sup\{x \in L \mid \bot \preceq \bot\} = \sup L = \top \\
\bot \rightsquigarrow \bot &= \inf\{x \in L \mid \bot \preceq \bot * x\} = \inf\{x \in L \mid \bot \preceq \bot\} = \inf L = \bot\,.
\end{aligned}$$

This implies, in particular, that $\rightsquigarrow$ cannot be seen as a fuzzy implication [16], in the sense of a hybrid monotone mapping $\mathcal{I}$ (decreasing first and increasing second partial mappings) such that $\mathcal{I}(\top, \bot) = \bot$, $\mathcal{I}(\top, \top) = \top$ and $\mathcal{I}(\bot, \bot) = \top$.

4.2. Solution set of a bipolar equation

Our approach to solving bipolar sup-$*$ equations will rely on the following key result, which provides the solution set of a (unipolar) sup-$*$ inequality or (unipolar) sup-$*$ equation. More specifically, Proposition 16 and Theorem 2 in [13] can be summarized as follows.

Theorem 12 *([13]). Let $(L, \preceq, *, \rightarrow, \neg)$ be a complete distributive residuated lattice, $a_1, \dots, a_m \in L$ and b a join-irreducible element of L. If the partial mappings of $*$ are homomorphisms, then the following statements hold:*

(a) The solution set of the sup-$*$ *inequality*

$$\bigvee_{j \in \{1,\dots,m\}} a_j * x_j \preceq b \tag{4}$$

is equal to $[0, g]$, where 0 represents the tuple $(\bot, \dots, \bot)$ and $g = (g_1, \dots, g_m)$ with $g_j = a_j \rightarrow b$, for each $j \in \{1, \dots, m\}$.

(b) If the solution set of the sup-$*$ *equation*

$$\bigvee_{j \in \{1,\dots,m\}} a_j * x_j = b \tag{5}$$

is not empty, then it is equal to $\bigcup_{s \in S} [s, g]$, where $S = \{s^j \mid b \preceq a_j$ and $a_j \rightsquigarrow b \preceq a_j \rightarrow b\}$ and $s^j = (s^j_1, \dots, s^j_m)$ with

$$s^j_k = \begin{cases} a_j \rightsquigarrow b & \text{if} \quad k = j \\ \bot & \text{otherwise} \end{cases}\,.$$

Since a CDSRL is a particular complete distributive residuated lattice, Theorem 12 remains valid in that context as well. Let us provide some more background on this theorem, in particular on the condition that the partial mappings of $*$ should be homomorphisms. Note that $a_j * \bot = \bot \preceq b$, and thus the set $\{x \in L \mid a_j * x \preceq b\}$ is not empty for each $j \in \{1, \dots, m\}$. If the partial mappings of $*$ are supremum-morphisms, then for each $j \in \{1, \dots, m\}$, we obtain

$$\begin{aligned}
a_j * (a_j \rightarrow b) &= a_j * \sup\{x \in L \mid a_j * x \preceq b\} \\
&= \sup\{a_j * x \mid x \in L \text{ and } a_j * x \preceq b\} \\
&\preceq b\,.
\end{aligned}$$

Hence, the tuple $g = (a_1 \rightarrow b, \dots, a_m \rightarrow b)$ is a solution of inequality (4). Clearly, it is the greatest solution, leading to Statement (a) in Theorem 12.

Obviously, if the solution set of Eq. (5) is not empty, then g is also the greatest solution of that equation. The fact that b is join-irreducible then implies that there exists $k \in \{1, \dots, m\}$ such that $b = a_k * g_k = a_k * (a_k \to b) \preceq a_k$. This implies that the set $\{x \in L \mid b \preceq a_k * x\}$ is not empty and that $a_k \rightsquigarrow b \preceq a_k \to b$; hence, the set S is not empty.

Now consider $j \in \{1, \dots, m\}$ such that $b \preceq a_j$ and $a_j \rightsquigarrow b \preceq a_j \to b$. Note that $b \preceq a_j = a_j * \top$, and thus the set $\{x \in L \mid b \preceq a_j * x\}$ is not empty. If the partial mappings of $*$ are infimum-morphisms, then we obtain:

$$\begin{aligned}
b &\preceq \inf\{a_j * x \mid x \in L \text{ and } b \preceq a_j * x\}\\
&= a_j * \inf\{x \in L \mid b \preceq a_j * x\}\\
&= a_j * (a_j \rightsquigarrow b)\\
&\preceq a_j * (a_j \to b)\\
&\preceq b\,.
\end{aligned}$$

The chain of inequalities thus becomes a chain of equalities. It is then immediate that s_j is a minimal solution of Eq. (5). The distributivity of $(L, \preceq)$ plays a crucial role in order to demonstrate that all minimal solutions are obtained in this way and that each solution is bounded from below by a minimal solution. For more details, we refer to [13].

In order to discuss the resolution of bipolar equations, we introduce the following notations related to Eq. (3):

(i) $S^+ = \{s^{j+} \mid b \preceq a_j^+ \text{ and } a_j^+ \rightsquigarrow b \preceq a_j^+ \to b\}$, where $s^{j+} = (s_1^{j+}, \dots, s_m^{j+})$ with

$$s_k^{j+} = \begin{cases} a_j^+ \rightsquigarrow b & \text{if } \ k = j\\ \perp & \text{otherwise}\end{cases}\,;$$

(ii) $S^- = \{s^{j-} \mid b \preceq a_j^- \text{ and } a_j^- \rightsquigarrow b \preceq a_j^- \to b\}$, where $s^{j-} = (s_1^{j-}, \dots, s_m^{j-})$ with

$$s_k^{j-} = \begin{cases} a_j^- \rightsquigarrow b & \text{if } \ k = j\\ \perp & \text{otherwise}\end{cases}\,;$$

(iii) $g^+ = (g_1^+, \dots, g_m^+)$ with $g_j^+ = a_j^+ \to b$, for each $j \in \{1, \dots, m\}$;
(iv) $g^- = (g_1^-, \dots, g_m^-)$ with $g_j^- = a_j^- \to b$, for each $j \in \{1, \dots, m\}$;
(v) For any $x = (x_1, \dots, x_m) \in L^m$, $\neg x = (\neg x_1, \dots, \neg x_m)$;
(vi) For any $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m) \in L^m$,

$$x \wedge y = (x_1 \wedge y_1, \dots, x_m \wedge y_m)$$

$$x \vee y = (x_1 \vee y_1, \dots, x_m \vee y_m)\,.$$

Using the above notations, the following theorem describes the solution set of a bipolar equation.

Theorem 13. *Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. If Eq. (3) is solvable, then its solution set is equal to*

$$D = \Big(\bigcup_{s \in S^+} [s \vee \neg g^-, g^+]\Big) \cup \Big(\bigcup_{s \in S^-} [\neg g^-, g^+ \wedge \neg s]\Big)\,.$$

Proof. Since b is a join-irreducible element of L, we can assert that Eq. (3) is solvable if and only if one of the following systems is solvable:

$$\begin{cases} \displaystyle\bigvee_{j \in \{1,\dots,m\}} a_j^+ * x_j = b & \text{(a)}\\ \bigvee_{j \in \{1,\dots,m\}} a_j^- * \neg x_j \preceq b & \text{(b)}\end{cases} \tag{6}$$

or

$$\begin{cases} \bigvee_{j\in\{1,\dots,m\}} a_j^+ * x_j \preceq b & \text{(a)} \\ \bigvee_{j\in\{1,\dots,m\}} a_j^- * \neg x_j = b\,. & \text{(b)} \end{cases} \tag{7}$$

Therefore, the solution set of Eq. (3) is equal to the union of the solution set of the first system, denoted as D_1, and the solution set of the second system, denoted as D_2. Next, we construct both sets.

As far as the first system is concerned, according to Statement (b) of Theorem 12, a tuple x satisfies Eq. (6a) if and only if

$$x \in \bigcup_{s\in S^+} [s, g^+]\,.$$

Furthermore, Statement (a) of Theorem 12 implies that a tuple x satisfies inequality (6b) if and only if $\neg x \in [0, g^-]$, or, equivalently, since $\neg$ is an involutive negation, if and only if $\neg g^- \preceq x \preceq \neg 0 = 1$, where 1 represents the tuple $(\top, \dots, \top)$. Hence, we conclude that

$$D_1 = \Big(\bigcup_{s\in S^+} [s, g^+]\Big) \cap [\neg g^-, 1]\,.$$

Based on elementary set operations and the fact that $g^+ \preceq 1$, we deduce that

$$D_1 = \bigcup_{s\in S^+} \Big([s, g^+] \cap [\neg g^-, 1]\Big) = \bigcup_{s\in S^+} [s \vee \neg g^-, g^+]\,.$$

As far as the second system is concerned, according to Statement (a) of Theorem 12, a tuple x satisfies inequality (7a) if and only if $x \in [0, g^+]$. Furthermore, Statement (b) of Theorem 12 implies that a tuple x satisfies Eq. (7b) if and only if

$$\neg x \in \bigcup_{s\in S^-} [s, g^-]\,.$$

Since $\neg$ is an involutive negation, the chain of inequalities $s \preceq \neg x \preceq g^-$ holds if and only if $\neg g^- \preceq x \preceq \neg s$. Therefore, we obtain

$$D_2 = [0, g^+] \cap \Big(\bigcup_{s\in S^-} [\neg g^-, \neg s]\Big)\,.$$

Using the fact that $0 \preceq \neg g^-$ and applying elementary set operations, the set D_2 can be rewritten as

$$D_2 = \bigcup_{s\in S^-} \Big([0, g^+] \cap [\neg g^-, \neg s]\Big) = \bigcup_{s\in S^-} [\neg g^-, g^+ \wedge \neg s]\,.$$

As a result, we conclude that the solution set of Eq. (3) is equal to

$$D = \Big(\bigcup_{s\in S^+} [s \vee \neg g^-, g^+]\Big) \cup \Big(\bigcup_{s\in S^-} [\neg g^-, g^+ \wedge \neg s]\Big)\,. \quad \square$$

Corollary 14. *Let $(L, \preceq, *, \rightarrow, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. If Eq. (3) is solvable, then the inequality $\neg g^- \preceq g^+$ holds.*

Proof. Follows directly from the expression of the solution set in Theorem 13. $\square$

As a consequence of Theorem 13, the solution set of Eq. (3) can be split into two parts: $\bigcup_{s\in S^+}[s \vee \neg g^-, g^+]$ and $\bigcup_{s\in S^-}[\neg g^-, g^+ \wedge \neg s]$. Hence, based on the expression of these two sets, we can obtain a characterization of the solvability of Eq. (3) in terms of the tuples g^+ and $\neg g^-$.

Theorem 15. *Let $(L, \preceq, *, \rightarrow, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. Equation (3) is solvable if and only if either g^+ or $\neg g^-$ is a solution.*

Proof. According to Theorem 13, the solution set of Eq. (3) is equal to $D_1 \cup D_2$, where

$$D_1 = \bigcup_{s \in S^+} [s \vee \neg g^-, g^+]$$

and

$$D_2 = \bigcup_{s \in S^-} [\neg g^-, g^+ \wedge \neg s].$$

Clearly, $D_1 \neq \varnothing$ if and only if $g^+ \in D_1$, and $D_2 \neq \varnothing$ if and only if $\neg g^- \in D_2$. Hence, the solution set of Eq. (3) is not empty if and only if $g^+ \in D_1$ or $\neg g^- \in D_2$. In other words, Eq. (3) is solvable if and only if either g^+ or $\neg g^-$ is a solution. $\square$

4.3. Extremal elements of the solution set

The algebraic structure of the solution set of a sup-$*$ equation has been widely studied in the literature [14,17]. In what follows, we provide a characterization of the structure of the solution set of a bipolar equation.

Theorem 16. *Let $(L, \preceq, *, \rightarrow, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. Suppose that Eq. (3) is solvable, then the following statements hold:*

(1) If g^+ is a solution of Eq. (3), then g^+ is the greatest solution of Eq. (3). Otherwise, the set of maximal solutions of Eq. (3) is equal to the set $\{g^+ \wedge \neg s \mid s \in S^-, \neg g^- \preceq g^+ \wedge \neg s\}$.
(2) If $\neg g^-$ is a solution of Eq. (3), then $\neg g^-$ is the smallest solution of Eq. (3). Otherwise, the set of minimal solutions of Eq. (3) is equal to the set $\{s \vee \neg g^- \mid s \in S^+, s \vee \neg g^- \preceq g^+\}$.

Proof. According to Theorem 13, the solution set of Eq. (3) is equal to $D_1 \cup D_2$, where

$$D_1 = \bigcup_{s \in S^+} [s \vee \neg g^-, g^+]$$

and

$$D_2 = \bigcup_{s \in S^-} [\neg g^-, g^+ \wedge \neg s].$$

Clearly, if g^+ is a solution of Eq. (3), then $x \preceq g^+$ for each $x \in D_1$. Besides, as $g^+ \wedge \neg s \preceq g^+$ for each $s \in S^-$, we can assert that $x \preceq g^+$ for each $x \in D_2$. Therefore, g^+ is the greatest solution of Eq. (3).

Next, suppose that g^+ is not a solution of Eq. (3). Obviously, then $D_1 = \varnothing$ and thus $D = D_2$. As a result, the maximal solutions of Eq. (3) coincide with the maximal elements of the set D_2. Given $s \in S^-$, if $\neg g^- \npreceq g^+ \wedge \neg s$, then the interval $[\neg g^-, g^+ \wedge \neg s]$ is empty. Thus, the set of maximal elements of D_2 is contained in the set $M = \{g^+ \wedge \neg s \mid s \in S^-, \neg g^- \preceq g^+ \wedge \neg s\} \subseteq D_2$. In what follows, we will see that every $g^+ \wedge \neg s \in M$ is actually a maximal element of D_2. We will give a proof by contradiction.

Suppose that there exists $g^+ \wedge \neg s^{j_1^-} \in M$ such that it is not a maximal element of D_2. Obviously, there exists $x \in D_2$ such that $g^+ \wedge \neg s^{j_1^-} \prec x$. According to the expression of D_2, there exists $s^{j_2^-} \in S^-$ with $j_1^- \neq j_2^-$ such that $x \preceq g^+ \wedge \neg s^{j_2^-}$. Thus, $g^+ \wedge \neg s^{j_1^-} \prec g^+ \wedge \neg s^{j_2^-}$. Next, we will deduce that $g^+ \wedge \neg s^{j_2^-} = g^+$, i.e., $g_j^+ \wedge \neg s_j^{j_2^-} = g_j^+$ for each $j \in \{1, \dots, m\}$. This fact will lead to the desired contradiction, since $g^+ \wedge \neg s^{j_2^-} \in D$ while g^+ is not a solution of Eq. (3) by hypothesis.

On the one hand, for each $j \in \{1, \ldots, m\}$ with $j \neq j_2$, by definition of $s^{j_2^-}$, it holds that $s_j^{j_2^-} = \bot$. Thus, the following chain of equalities holds

$$g_j^+ \wedge \neg s_j^{j_2^-} = g_j^+ \wedge \neg\bot = g_j^+ \wedge \top = g_j^+ .$$

On the other hand, as $g^+ \wedge \neg s^{j_1^-} \prec g^+ \wedge \neg s^{j_2^-}$, the inequality $g_j^+ \wedge \neg s_j^{j_1^-} \preceq g_j^+ \wedge \neg s_j^{j_2^-}$ holds, for each $j \in \{1, \ldots, m\}$. In particular, $g_{j_2}^+ \wedge \neg s_{j_2}^{j_1^-} \preceq g_{j_2}^+ \wedge \neg s_{j_2}^{j_2^-}$. Now, according to the definition of the tuple g^+ and the fact that $j_1 \neq j_2$, we obtain:

$$g_{j_2}^+ = a_{j_2}^+ \to b$$
$$s_{j_2}^{j_1^-} = \bot .$$

Therefore, the following chain of equalities holds

$$a_{j_2}^+ \to b = (a_{j_2}^+ \to b) \wedge \top = (a_{j_2}^+ \to b) \wedge \neg\bot = g_{j_2}^+ \wedge \neg s_{j_2}^{j_1^-} \preceq g_{j_2}^+ \wedge \neg s_{j_2}^{j_2^-} . \tag{8}$$

Furthermore, as $g_{j_2}^+ = a_{j_2}^+ \to b$, by definition of $\wedge$, it holds that

$$g_{j_2}^+ \wedge \neg s_{j_2}^{j_2^-} = (a_{j_2}^+ \to b) \wedge \neg s_{j_2}^{j_2^-} \preceq a_{j_2}^+ \to b . \tag{9}$$

Hence, from Eqs. (8) and (9), we deduce that $g_{j_2}^+ \wedge \neg s_{j_2}^{j_2^-} = a_{j_2}^+ \to b$. In other words, $g_{j_2}^+ \wedge \neg s_{j_2}^{j_2^-} = g_{j_2}^+$.

Consequently, we can assert that $g_j^+ \wedge \neg s_j^{j_2^-} = g_j^+$, for each $j \in \{1, \ldots, m\}$, i.e., $g^+ \wedge \neg s^{j_2^-} = g^+$, the desired contradiction. Hence, we conclude that every $g^+ \wedge \neg s \in M$ is a maximal element of D_2. This completes the proof of Statement (1).

As far as Statement (2) is concerned, if $\neg g^-$ is a solution of Eq. (3), then $\neg g^- \preceq x$ for each $x \in D_2$. Besides, as $\neg g^- \preceq s \vee \neg g^-$ for each $s \in S^+$, we can assert that $\neg g^- \preceq x$ for each $x \in D_1$. Therefore, $\neg g^-$ is the smallest solution of Eq. (3).

Next, suppose that $\neg g^-$ is not a solution of Eq. (3). Following a similar reasoning as for Statement (1), we obtain that the set of minimal solutions of Eq. (3) coincides with the set $\{s \vee \neg g^- \mid s \in S^+, s \vee \neg g^- \preceq g^+\}$. □

The involutivity of the negation plays a key role in the algebraic structure of the solution set of Eq. (3). For instance, the following result shows that every solution of Eq. (3) is smaller than or equal to a maximal solution and greater than or equal to a minimal solution of the equation. This might seem a straightforward result, but it does not hold for a bipolar equation with a non-involutive negation, to which Theorem 16 obviously does not apply. See, for instance, Example 18 in [9].

Proposition 17. *Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \ldots, m\}$, and b a join-irreducible element of L. Given the sets $\hat{M}$ and $\check{M}$ of maximal and minimal solutions of Eq. (3), respectively, then for any solution x of Eq. (3) there exist $\hat{m} \in \hat{M}$ and $\check{m} \in \check{M}$ such that $\check{m} \preceq x \preceq \hat{m}$.*

Proof. According to Theorem 13, the solution set of Eq. (3) is equal to $D_1 \cup D_2$, where

$$D_1 = \bigcup_{s \in S^+} [s \vee \neg g^-, g^+]$$

and

$$D_2 = \bigcup_{s \in S^-} [\neg g^-, g^+ \wedge \neg s] .$$

Suppose that $x \in D_1$, then there exists $s \in S^+$ such that $x \in [s \vee \neg g^-, g^+] \subseteq D_1$. Obviously, g^+ then is a solution of Eq. (3). On the one hand, if $\neg g^-$ is also a solution of Eq. (3), then Theorem 16 ensures that g^+ and $\neg g^-$ are the

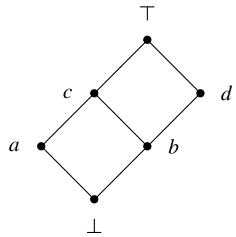

Fig. 2. Hasse diagram of the lattice $(L, \preceq)$ in Subsection 4.4.

Table 1
The t-norm $*$ in Subsection 4.4.

$*$	$\bot$	a	b	c	d	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	$\bot$	a
b	$\bot$	$\bot$	$\bot$	$\bot$	b	b
c	$\bot$	a	$\bot$	a	b	c
d	$\bot$	$\bot$	b	b	d	d
$\top$	$\bot$	a	b	c	d	$\top$

Table 2
The involutive negation $\neg$ in Subsection 4.4.

$\neg$	$\bot$	a	b	c	d	$\top$
	$\top$	d	c	b	a	$\bot$

greatest and the smallest solution of Eq. (3), respectively. Since $\neg g^- \preceq s \vee \neg g^- \preceq x \preceq g^+$, taking $\check{m} = \neg g^-$ and $\hat{m} = g^+$, we obtain that $\check{m} \preceq x \preceq \hat{m}$. On the other hand, if $\neg g^-$ is not a solution of Eq. (3), then Theorem 16 implies that $s \vee \neg g^-$ is a minimal solution of Eq. (3). Hence, taking $\check{m} = s \vee \neg g^-$ and $\hat{m} = g^+$, we obtain that $\check{m} \preceq x \preceq \hat{m}$, with $\hat{m} \in \hat{M}$ and $\check{m} \in \check{M}$.

Assume next that $x \notin D_1$, and thus $x \in D_2$. Following a similar reasoning as above, there exists $s \in S^-$ such that $x \in [\neg g^-, g^+ \wedge \neg s] \subseteq D_2$ and $\neg g^-$ is a solution of Eq. (3). On the one hand, if g^+ is a solution of Eq. (3), taking $\check{m} = \neg g^-$ and $\hat{m} = g^+$, we obtain that $\check{m} \preceq x \preceq \hat{m}$. On the other hand, if g^+ is not a solution of Eq. (3), then Theorem 16 implies that $g^+ \wedge \neg s$ is a maximal solution of Eq. (3). As a result, we obtain that $\check{m} \preceq x \preceq \hat{m}$, with $\check{m} = \neg g^-$ and $\hat{m} = g^+ \wedge \neg s$. $\square$

4.4. Examples of bipolar equations on a finite CDSRL

In this section, several examples of bipolar equations on a finite CDSRL are examined in detail in order to illustrate the wide range of solution sets that can be obtained. In particular, we discuss the case of a non-chain, a setting not covered by the existing literature.

Let $(L, \preceq)$ be the finite lattice with Hasse diagram shown in Fig. 2, and consider the operations $*\colon L \times L \to L$ (t-norm) $\neg\colon L \to L$ (involutive negation) given by Table 1 and Table 2, respectively.

We will first verify that the main hypotheses of Theorems 13, 15 and 16 are satisfied. First of all, since $(L, \preceq)$ is a finite lattice and M_3 and N_5 are not sublattices of $(L, \preceq)$, Theorem 2 implies that $(L, \preceq)$ is a distributive complete lattice. Note that its join-irreducible elements are given by a, b and d. Clearly, $\neg$ is a decreasing mapping with $\neg\bot = \top$ and $\neg\top = \bot$, and $\neg\neg x = x$ for each $x \in L$, i.e., $\neg$ is an involutive negation. Furthermore, the operation $*$ can be shown to be a t-norm and to have a residuated implication. Hence, we can conclude that the tuple $(L, \preceq, *, \to, \neg)$ is a CDSRL. The residuated implication $\to$ and the operation $\rightsquigarrow$ are shown in Table 3.

It remains to verify that the partial mappings of $*$ are homomorphisms. To that end, it is sufficient to check that $\sup_{x\in A}\{x * z\} = \sup A * z$ and $\inf_{x\in A}\{x * z\} = \inf A * z$, for each $z \in L$ and each $A \in \{\{c, d\}, \{a, b\}, \{a, d\}\}$. Note that any subset of L without a greatest or smallest elements contains one of the pairs $\{c, d\}$, $\{a, b\}$ and $\{a, d\}$. It

Table 3
The operations $\rightarrow$ and $\rightsquigarrow$ in Subsection 4.4.

$\rightarrow$	$\bot$	a	b	c	d	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	d	$\top$	d	$\top$	d	$\top$
b	c	c	$\top$	$\top$	$\top$	$\top$
c	b	c	d	$\top$	d	$\top$
d	a	a	c	c	$\top$	$\top$
$\top$	$\bot$	a	b	c	d	$\top$

$\rightsquigarrow$	$\bot$	a	b	c	d	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	a	$\top$	$\top$	$\top$	$\top$
b	$\bot$	$\top$	d	$\top$	$\top$	$\top$
c	$\bot$	a	d	$\top$	$\top$	$\top$
d	$\bot$	$\top$	b	$\top$	d	$\top$
$\top$	$\bot$	a	b	c	d	$\top$

is then a matter of direct verification that the desired equalities hold. Hence, we conclude that the hypotheses of Theorems 13, 15 and 16 are satisfied.

Now consider the bipolar equation in the two unknown variables x and y given by:

$$(a * x) \vee (d * \neg x) \vee (c * y) \vee (b * \neg y) = d\,. \tag{10}$$

Note that the right-hand side is a join-irreducible element.

In order to apply the mentioned theorems, we first determine the tuples g^+ and $\neg g^-$. This results in:

$$g^+ = (a \rightarrow d, c \rightarrow d) = (d, d)$$

$$g^- = (d \rightarrow d, b \rightarrow d) = (\top, \top)\,.$$

Hence, $\neg g^- = (\bot, \bot)$.

According to Theorem 13, it holds that Eq. (10) is solvable if and only if g^+ or $\neg g^-$ is a solution. Indeed, although g^+ is not a solution of the equation, the tuple $\neg g^- = (\bot, \bot)$ effectively is a solution of Eq. (10), as we demonstrate next:

$$\begin{aligned} &(a * \bot) \vee (d * \neg\bot) \vee (c * \bot) \vee (b * \neg\bot) \\ =\,& (a * \bot) \vee (d * \top) \vee (c * \bot) \vee (b * \top) \\ =\,& \bot \vee d \vee \bot \vee b \\ =\,& d\,. \end{aligned}$$

As a result, Eq. (10) is solvable and its solution set is given in terms of the sets S^+ and S^-:

$$S^+ = \varnothing$$

$$S^- = \{(d, \bot)\}\,.$$

According to Theorem 13, the solution set of Eq. (10) is then given by:

$$\begin{aligned} D &= \Big(\bigcup_{s \in S^+} [s \vee \neg g^-, g^+]\Big) \cup \Big(\bigcup_{s \in S^-} [\neg g^-, g^+ \wedge \neg s]\Big) \\ &= \varnothing\ \cup [\neg g^-, g^+ \wedge \neg(d, \bot)] \\ &= [(\bot, \bot), (d, d) \wedge (a, \top)] \\ &= [(\bot, \bot), (\bot, d)] \\ &= \{(\bot, \bot), (\bot, b), (\bot, d)\}\,. \end{aligned}$$

Obviously, $\neg g^- = (\bot, \bot)$ is the smallest solution and $(\bot, d)$ is the greatest solution of Eq. (10). The solution set forms a chain in L^2 and its Hasse diagram is depicted in Fig. 3.

Clearly, the solvability of Eq. (10) is strongly influenced by the coefficients and the right-hand side of the equation. In fact, if the coefficients are slightly modified, then the bipolar equation becomes unsolvable. For instance, consider the following bipolar equation:

$$(a * x) \vee (d * \neg x) \vee (c * y) \vee (a * \neg y) = d\,. \tag{11}$$

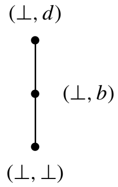

Fig. 3. Hasse diagram of the solution set of Eq. (10).

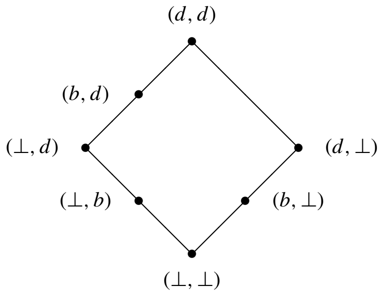

Fig. 4. Hasse diagram of the solution set of Eq. (12).

Note that compared to Eq. (10), only the coefficient of $\neg y$ has been modified from b to a. The corresponding tuples g^+, g^- and $\neg g^-$ are given by:

$$\begin{aligned} g^+ &= (a \to d, c \to d) = (d, d) \\ g^- &= (d \to d, a \to d) = (\top, d) \\ \neg g^- &= (\bot, a)\,. \end{aligned}$$

Since $\neg g^- \not\preceq g^+$, it follows from Corollary 14 that Eq. (11) is not solvable.

As shown above, the solution set of Eq. (10) is a chain. Of course, this is not necessarily always the case. For instance, consider the following bipolar equation:

$$(a * x) \vee (d * \neg x) \vee (\top * y) \vee (d * \neg y) = d\,. \tag{12}$$

Both $g^+ = (d, d)$ and $\neg g^- = (\bot, \bot)$ are solutions of Eq. (12). Moreover, from Theorem 16, it follows that $g^+ = (d, d)$ and $\neg g^- = (\bot, \bot)$ are the greatest and the smallest solution of Eq. (12), respectively. Finally, Theorem 13 implies that the full solution set of Eq. (12) is the lattice with Hasse diagram shown in Fig. 4. Note that it is not a sublattice of L^2.

Finally, note that Figs. 3 and 4 both represent sets with a smallest element. However, in view of Theorem 16, this is not necessarily always the case. Next, we will present a solvable bipolar equation whose solution set has a greatest solution, but it does not have a smallest one. Consider the bipolar equation:

$$(d * x) \vee (b * \neg x) \vee (\top * y) \vee (b * \neg y) = d\,. \tag{13}$$

One easily verifies that the tuple $g^+ = (\top, d)$ is a solution of Eq. (13), while the tuple $\neg g^- = (\bot, \bot)$ is not. To obtain the full solution set, we determine the sets S^+ and S^-:

$$\begin{aligned} S^+ &= \{(d, \bot), (\bot, d)\} \\ S^- &= \varnothing\,. \end{aligned}$$

Therefore, according to Theorem 16, the tuple $g^+ = (\top, d)$ is the greatest solution of Eq. (13), whereas the tuples

$$(d, \bot) \vee \neg g^- = (d, \bot) \vee (\bot, \bot) = (d, \bot)$$

$$(\bot, d) \vee \neg g^- = (\bot, d) \vee (\bot, \bot) = (\bot, d)$$

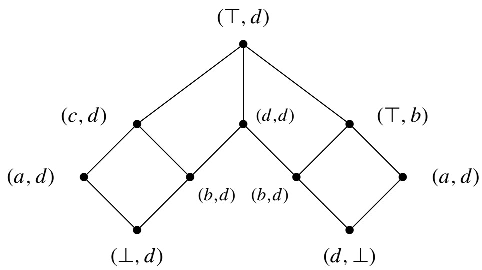

Fig. 5. Hasse diagram of the solution set of Eq. (13).

are minimal solutions of Eq. (13). According to Theorem 13, the full solution set of Eq. (13) is given by:

$$
\begin{aligned}
D &= \Big(\bigcup_{s\in S^+}[s\vee\neg g^-, g^+]\Big)\cup\Big(\bigcup_{s\in S^-}[\neg g^-, g^+\wedge\neg s]\Big)\\
&= \Big([(d,\bot),(\top,d)]\cup[(\bot,d),(\top,d)]\Big)\cup\ \varnothing\\
&= \{(d,\bot),(d,b),(d,d),(\top,\bot),(\top,b),(\top,d)\}\\
&\quad\cup\{(\bot,d),(a,d),(b,d),(c,d),(d,d),(\top,d)\}\\
&= \{(d,\bot),(d,b),(d,d),(\top,\bot),(\top,b),(\top,d),(\bot,d),(a,d),(b,d),(c,d)\}\,.
\end{aligned}
$$

A Hasse diagram of this poset D is shown in Fig. 5.

4.5. Examples of bipolar equations on the real unit interval

As mentioned in the introduction, the real unit interval is the most common lattice on which bipolar equations have been considered so far in the literature. In this section, we illustrate how the results presented in this paper can be applied to a bipolar equation on the real unit interval $[0, 1]$ equipped with the natural order $\leq$. Note that $([0, 1], \leq)$ is a distributive complete lattice containing join-irreducible elements only. Moreover, the structure $([0, 1], \leq, \wedge, \rightarrow_\wedge, \neg_S)$ is a CDSRL, with the residuated implication $\rightarrow_\wedge$ given by

$$
\begin{aligned}
a\rightarrow_\wedge b &= \sup\{x\in[0,1]\mid a\wedge x\leq b\}\\
&= \begin{cases} 1 & \text{if } a\leq b\\ b & \text{if } b<a\end{cases}
\end{aligned}
$$

and the (almost identical) operation $\rightsquigarrow_\wedge$ by

$$
\begin{aligned}
a\rightsquigarrow_\wedge b &= \inf\{x\in[0,1]\mid b\leq a\wedge x\}\\
&= \begin{cases} 1 & \text{if } a<b\\ b & \text{if } b\leq a\end{cases}\,.
\end{aligned}
$$

Moreover, as the partial mappings of $\wedge$ are homomorphisms (since the minimum operation is a continuous t-norm), we conclude that the hypotheses of Theorems 13, 15 and 16 are satisfied.

Now consider the following bipolar max-min equation with the standard negation $\neg_S$ in the three unknown variables x, y and z:

$$(0.4\wedge x)\vee(0.5\wedge\neg_S x)\vee(0.6\wedge y)\vee(0.4\wedge\neg_S y)\vee(0.7\wedge z)\vee(0.8\wedge\neg_S z)=0.6\,.$$

Introducing the shorthand notation

$$(u,v,w)\odot(x,y,z)=(u\wedge x)\vee(v\wedge y)\vee(w\wedge z)\,,$$

this equation can be written in a more compact form as

$$(0.4, 0.6, 0.7) \odot (x, y, z) \vee (0.5, 0.4, 0.8) \odot \neg_S(x, y, z) = 0.6 \,. \tag{14}$$

First, we compute the tuples g^+, g^- and $\neg_S\, g^-$:

$$\begin{aligned} g^+ &= (0.4 \to_\wedge 0.6, 0.6 \to_\wedge 0.6, 0.7 \to_\wedge 0.6) = (1, 1, 0.6) \\ g^- &= (0.5 \to_\wedge 0.6, 0.4 \to_\wedge 0.6, 0.8 \to_\wedge 0.6) = (1, 1, 0.6) \\ \neg_S g^- &= (\neg_S 1, \neg_S 1, \neg_S 0.6) = (0, 0, 0.4) \,. \end{aligned}$$

As shown below, both g^+ and $\neg_S g^-$ are solutions of Eq. (14):

$$\begin{aligned} &(0.4, 0.6, 0.7) \odot (1, 1, 0.6) \vee (0.5, 0.4, 0.8) \odot \neg_S(1, 1, 0.6) \\ &= (0.4, 0.6, 0.7) \odot (1, 1, 0.6) \vee (0.5, 0.4, 0.8) \odot (0, 0, 0.4) \\ &= (0.4 \vee 0.6 \vee 0.6) \vee (0 \vee 0 \vee 0.4) = 0.6 \vee 0.4 = 0.6 \end{aligned}$$

and

$$\begin{aligned} &(0.4, 0.6, 0.7) \odot (0, 0, 0.4) \vee (0.5, 0.4, 0.8) \odot \neg_S(0, 0, 0.4) \\ &= (0.4, 0.6, 0.7) \odot (0, 0, 0.4) \vee (0.5, 0.4, 0.8) \odot (1, 1, 0.6) \\ &= (0 \vee 0 \vee 0.4) \vee (0.5 \vee 0.4 \vee 0.6) = 0.4 \vee 0.6 = 0.6 \,. \end{aligned}$$

Obviously then, the equation is solvable, and according to Theorem 16, the tuple $g^+ = (1, 1, 0.6)$ is the greatest solution and $\neg_S g^- = (0, 0, 0.4)$ is the smallest solution.

Finally, we determine the full solution set of Eq. (14). To that end, we first compute the sets S^+ and S^-:

$$\begin{aligned} S^+ &= \{(0, 0.6 \rightsquigarrow_\wedge 0.6, 0), (0, 0, 0.7 \rightsquigarrow_\wedge 0.6)\} = \{(0, 0.6, 0), (0, 0, 0.6)\} \\ S^- &= \{(0, 0, 0.8 \rightsquigarrow_\wedge 0.6)\} = \{(0, 0, 0.6)\} \,. \end{aligned}$$

According to Theorem 13, the solution set of Eq. (14) is then given by:

$$\begin{aligned} D &= \Big(\bigcup_{s \in S^+} [s \vee \neg_S\, g^-, g^+] \Big) \cup \Big(\bigcup_{s \in S^-} [\neg_S\, g^-, g^+ \wedge \neg_S\, s] \Big) \\ &= \Big([(0, 0.6, 0) \vee (0, 0, 0.4), (1, 1, 0.6)] \cup [(0, 0, 0.6) \vee (0, 0, 0.4), (1, 1, 0.6)] \Big) \\ &\quad \cup [(0, 0, 0.4), (1, 1, 0.6) \wedge \neg_S(0, 0, 0.6)] \\ &= [(0, 0.6, 0.4), (1, 1, 0.6)] \cup [(0, 0, 0.6), (1, 1, 0.6)] \cup \; [(0, 0, 0.4), (1, 1, 0.4)] \,. \end{aligned}$$

As mentioned before, the coefficients play an important role in the solvability of a bipolar equation. Equally important is the involutive negation. To illustrate this, we consider Eq. (14) in which the standard negation $\neg_S$ is replaced by another involutive negation. More specifically, we consider the Yager negation with parameter $\lambda = 2$, i.e., the involutive negation $\neg_Y \colon [0, 1] \to [0, 1]$ defined by $\neg_Y\, t = (1 - t^2)^{1/2}$; thus $([0, 1], \leq, \wedge, \to_\wedge, \neg_Y)$ is a CDSRL.

We now study the bipolar equation:

$$(0.4, 0.6, 0.7) \odot (x, y, z) \vee (0.5, 0.4, 0.8) \odot \neg_Y(x, y, z) = 0.6 \,. \tag{15}$$

Evidently, the tuples g^+ and g^- remain unchanged. However, $\neg_Y\, g^- \neq \neg_S\, g^-$, since

$$\neg_Y\, g^- = (\neg_Y\, 1, \neg_Y\, 1, \neg_Y\, 0.6) = (0, 0, 0.8) \,.$$

Since $\neg_Y\, g^- = (0, 0, 0.8) \not\leq (1, 1, 0.6) = g^+$, we conclude from Corollary 14 that Eq. (15) is not solvable.

5. Conclusions and future work

In this paper, we have carried out an in-depth study of the resolution of bipolar sup-$*$ equations on a CDSRL with a join-irreducible right-hand side. As in the case of unipolar sup-$*$ equations, dual results can be obtained for bipolar inf-$*$ equations on a CDSRL with a meet-irreducible right-hand side. Note that the work presented here heavily relies on the fact that the right-hand side is irreducible. More specifically, being able to split Eq. (3) into two systems has been instrumental. As a consequence, our approach is not valid when the right-hand side is decomposable. The solvability of bipolar equations with a decomposable right-hand side will therefore be the subject of future work, followed by the study of systems of bipolar equations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

Partially supported by the 2014–2020 ERDF Operational Programme in collaboration with the State Research Agency (AEI) in projects TIN2016-76653-P and PID2019-108991GB-I00, and with the Department of Economy, Knowledge, Business and University of the Regional Government of Andalusia in project FEDER-UCA18-108612, and by the European Cooperation in Science & Technology (COST) Action CA17124. BDB received funding from the Flemish Government under the “Onderzoeksprogramma Artificiële Intelligentie (AI) Vlaanderen” programme.

References

[1] R. Balbes, P. Dwinger, Distributive Lattices, Univ. of Missouri Press, 1975.
[2] A. Bialynicki-Birula, Remarks on quasi-Boolean algebras, Bull. Acad. Polon. Sci., Cl. III 5 (1957) 615–619.
[3] A. Bialynicki-Birula, H. Rasiowa, On the representation of quasi-Boolean algebras, Bull. Acad. Polon. Sci., Cl. III 5 (1957) 259–261.
[4] G. Birkhoff, Lattice Theory, third edition, American Mathematical Society, 1967.
[5] R. Cignoli, Injective De Morgan and Kleene algebras, Proc. Am. Math. Soc. 47 (2) (1975) 269.
[6] R. Cignoli, F. Esteva, Commutative integral bounded residuated lattices with an added involution, Ann. Pure Appl. Log. 161 (2) (2009) 150–160.
[7] M.E. Cornejo, D. Lobo, J. Medina, Bipolar fuzzy relation equations based on the product t-norm, in: Proceedings of the 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2017, pp. 1–6.
[8] M.E. Cornejo, D. Lobo, J. Medina, Bipolar fuzzy relation equations systems based on the product t-norm, Math. Methods Appl. Sci. 42 (17) (2019) 5779–5793.
[9] M.E. Cornejo, D. Lobo, J. Medina, On the solvability of bipolar max-product fuzzy relation equations with the product negation, J. Comput. Appl. Math. 354 (2019) 520–532.
[10] M.E. Cornejo, D. Lobo, J. Medina, Solving generalized equations with bounded variables and multiple residuated operators, Mathematics 8 (11) (1992) 1–22, 2020.
[11] M.E. Cornejo, D. Lobo, J. Medina, On the solvability of bipolar max-product fuzzy relation equations with the standard negation, Fuzzy Sets Syst. 410 (2021) 1–18.
[12] B. Davey, H. Priestley, Introduction to Lattices and Order, second edition, Cambridge University Press, 2002.
[13] B. De Baets, An order-theorethic approach to solving sup-T equations, in: Fuzzy Set Theory and Advanced Mathematical Applications, in: International Series in Intelligent Technologies, vol. 4, Springer US, 1995, pp. 67–87.
[14] B. De Baets, Sup-t equations: state of the art, in: Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications, in: NATO ASI Series, vol. 162, Springer, Berlin, Heidelberg, 1998, pp. 80–93.
[15] B. De Baets, Analytical solution methods for fuzzy relational equations, in: Fundamentals of Fuzzy Sets, in: D. Dubois, H. Prade (Eds.), The Handbooks of Fuzzy Sets Series, vol. 1, Kluwer Academic Publishers, 2000, pp. 291–340, Chapter 6.
[16] B. De Baets, J. Fodor, Residual operators of uninorms, Soft Comput. 3 (2) (1999) 89–100.
[17] A. Di Nola, S. Sessa, W. Pedrycz, Fuzzy relation equation under a class of triangular norms: a survey and new results, Stochastica 8 (2) (1984) 99–145.
[18] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy Relation Equations and Their Applications to Knowledge Engineering, Theory and Decision Library. Series D. System Theory, Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, 1989.
[19] D. Dubois, H. Prade, An introduction to bipolar representations of information and preference, Int. J. Intell. Syst. 23 (8) (2008) 866–877.
[20] S. Freson, B. De Baets, H. De Meyer, Linear optimization with bipolar max–min constraints, Inf. Sci. 234 (2013) 3–15.
[21] G. Grätzer, General Lattice Theory, Birkhäuser Verlag, 1998.

[22] C.-K. Hu, F.-B. Liu, C.-F. Hu, Solving bipolar max-t_p equation constrained multi-objective optimization problems, Int. J. Soft Comput. 7 (4) (2016) 11–23.
[23] J.A. Kalman, Lattices with involution, Trans. Am. Math. Soc. 87 (2) (1958) 485.
[24] P. Li, Q. Jin, On the resolution of bipolar max-min equations, Kybernetika 52 (4) (2016) 514–530.
[25] C.-C. Liu, Y.-Y. Lur, Y.-K. Wu, Some properties of bipolar max-min fuzzy relational equations, in: Proceedings of the International MultiConference of Engineers and Computer Scientists 2015, in: Lecture Notes in Engineering and Computer Science, vol. 2222, 2015, pp. 955–969.
[26] C.-C. Liu, Y.-Y. Lur, Y.-K. Wu, Linear optimization of bipolar fuzzy relational equations with max-łukasiewicz composition, Inf. Sci. 360 (2016) 149–162.
[27] D. Lobo, M.E. Cornejo, J. Medina, Abductive reasoning in normal residuated logic programming via bipolar max-product fuzzy relation equations, in: 2019 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), in: Atlantis Studies in Uncertainty Modelling, vol. 1, Atlantis Press, 2019, pp. 588–594.
[28] G.C. Moisil, Recherches sur l'algèbre de la logique, Ann. Sci. Univ. Jassy 22 (1935) 1–117.
[29] A. Monteiro, Matrices de morgan caractéristiques pour le calcul propositionnel classique, An. Acad. Bras. Ciênc. 32 (1960) 1–7.
[30] E. Sanchez, Resolution of composite fuzzy relation equations, Inf. Control 30 (1) (1976) 38–48.
[31] E. Sanchez, Inverses of fuzzy relations. Application to possibility distributions and medical diagnosis, Fuzzy Sets Syst. 2 (1) (1979) 75–86.
[32] X.-P. Yang, Resolution of bipolar fuzzy relation equations with max-Łukasiewicz composition, Fuzzy Sets Syst. 397 (2020) 41–60.
[33] J. Zhou, Y. Yu, Y. Liu, Y. Zhang, Solving nonlinear optimization problems with bipolar fuzzy relational equation constraints, J. Inequal. Appl. 2016 (2016) 126.