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Variational λ-symmetries and exact solutions to Euler-Lagrange equations lacking standard symmetries

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Variational λ-symmetries are used to find exact solutions to second- and fourth-order Euler-Lagrange equations associated to variational problems for which standard procedures fail. A one-parameter family of exact solutions in terms of Bessel functions is obtained for a first-order variational problem whose Euler-Lagrange equation does not admit Lie symmetries. A family of second-order equations, involving arbitrary functions and parameters, is first written in variational form. The variational λ-symmetry method successes in finding one-parameter families of exact solutions, despite the lack of Lie point and variational symmetries. A three-parameter family of exact solutions for a fourth-order equation with absence of Lie point symmetries is also deduced.

KEYWORDS

Euler-Lagrange equation, variational λ-symmetry, variational problem, variational symmetries

MSC CLASSIFICATION

34A05, 34A26, 70H03

1 | INTRODUCTION

The calculus of variations dates back to the work of Euler and the Bernoullis in the eighteenth century, although it was deeper formalized in the nineteenth century by Weierstrass and Hilbert.[1] Roughly speaking, the calculus of variations refers to the problem of determining the extreme values of a functional, which is a rule that associates a real number to each function belonging to a determined set[2] (called the set of admissible functions). This problem is usually known in the literature as variational problem.

It is well known that if a function is an extreme value of the considered functional, then it has to satisfy a system of differential equations, which are known as the Euler-Lagrange equations associated to the variational problem.[2,3] Therefore, the development of methods for finding exact solutions to differential equations plays an important role in the calculus of variations. In this regard, Sophus Lie developed a prominent theory based on symmetry groups of transformations providing a wide range of tools for solving differential equations, which have been intensively studied in recent decades.[1,4–8] Lie also tackled the application of symmetry groups to variational problems, giving rise to the notion of variational symmetry group.[1,4] Later on, in 1918, Emmy Noether published the famous theorem that establishes a one-to-one correspondence between one-parameter variational symmetry groups and conservation laws for Euler-Lagrange equations.

In this work, we deal with nondegenerate nth-order scalar variational problems involving a single variable integral for which the associated Euler-Lagrange equation turns out to be a $2n$th-order ordinary differential equation. In this framework, the knowledge of a one-parameter symmetry group for the $2n$th-order Euler-Lagrange equation permits to reduce the order of the equation by one, and the general solution can be retrieved from the general solution of the reduced equation by a single quadrature.[1,5,7,8] Moreover, if the symmetry group is also variational, then the order of the corresponding Euler-Lagrange equation can be reduced by two.[1,4,9] In this sense, variational symmetries double the power of standard symmetries. Nevertheless, there exist many cases in which the standard Lie symmetry method is no longer applicable for finding exact solutions to Euler-Lagrange equations. The possible obstacles might be one of the followings: (i) the Euler-Lagrange equation does not admit Lie symmetries, which implies that there not exist variational symmetries, (ii) the Euler-Lagrange equation admits only one Lie point symmetry that is not variational, and the corresponding $(2n-1)$th-order reduced equation is difficult to solve. This motivated the need of considering vector fields that are more general than Lie point symmetries. In this context, the notion of λ-symmetry (or $\mathcal{C}^\infty$-symmetry) was introduced in Muriel and Romero[10] by considering a new way of prolonging vector fields. Its extension to variational problems has been also addressed in the recent literature leading to the so-called variational λ-symmetry (or variational $\mathcal{C}^\infty$-symmetry).[11,12]

In this work, we apply the variational λ-symmetry method to find exact solutions of Euler-Lagrange equations for which the standard (Lie and variational) methods fail. This paper is organized as follows. First, in Section 2, we review the theoretical background regarding variational problems, symmetry groups of differential equations, variational symmetries, and divergence symmetries. We also recall the concept of variational λ-symmetry, as well as its application to reduce the order of Euler-Lagrange equations.[11,12] In Section 3, three applications of the variational λ-symmetry method are presented. The first example corresponds to a first-order variational problem lacking both variational and divergence symmetries. Despite the absence of symmetries, we are able to compute a one-parameter family of exact solutions to the associated Euler-Lagrange equation in terms of Bessel functions. In the second example, a family of second-order equations involving arbitrary functions and parameters is considered. Remarkably, the family of equations does not admit Lie point symmetries in the general case. After writing the given class of equations in variational form, a one-parameter family of exact solutions is determined by using a variational λ-symmetry. Particular equations of special interest that belong to the family are also analyzed. Finally, a fourth-order equation which is the Euler-Lagrange equation associated to a second-order Lagrangian is studied. Although the fourth-order equation does not admit Lie symmetries, a three-parameter family of exact solutions can be obtained by means of the variational λ-symmetry method.

2 | THEORETICAL BACKGROUND

In this section, with the aim of being self-contained, we recall the basics regarding jet spaces, symmetry groups of differential equations, and variational problems. The reader is referred to Olver,[1,4] Logan,[2] and Gelfand and Fomin[3] for a detailed study of the previous concepts.

2.1 | Variational symmetries

For simplicity, we will work in the Euclidean space $X = \mathbb{R}$, with coordinate x representing the independent variable and $u \in U = \mathbb{R}$ the dependent variable. The corresponding nth-order jet space[13] will be denoted by $J^n(\mathbb{R},\mathbb{R})$, which is equipped with coordinates $(x, u, u_1, \dots, u_n)$, where for $i = 1, \dots, n$, u_i denotes the derivative of u with respect to x of order i. Let $\Omega \subset X$ be an open and connected subset. We consider an nth-order variational problem

$$\mathcal{L}[u] = \int_\Omega L(x, u, u_1, \dots, u_n)dx, \tag{1}$$

where the integrand L, called the Lagrangian of the variational problem, is a smooth function defined on some open set $N \subset J^n(\mathbb{R},\mathbb{R})$. Let $M \subset \Omega \times U \subset X \times U$ be such that $M^{(n)} \subset N$, where $M^{(i)}$ denotes the corresponding jet space of order i, for $i = 1, \dots, n$.

The Euler operator or variational derivative is given by

$$E_u = \sum_{i=0}^{n} (-\mathbf{D}_x)^i \partial_{u_i},$$

where $\mathbf{D}_x = \partial_x + u_1\partial_u + u_2\partial_{u_1} + \dots$ stands for the total derivative operator with respect to x. The possible functions $u = f(x)$ that maximize or minimize the functional (1) must be solutions of the Euler-Lagrange equation

$$E_u[L] = \sum_{i=0}^{n} (-\mathbf{D}_x)^i \left(\frac{\partial L}{\partial u_i} \right) = 0, \tag{2}$$

which turns out to be, in general, a $2n$th-order ordinary differential equation.

In general terms, a variational symmetry group is a local group of transformations G that leaves $\mathcal{L}[u]$ invariant (modulo boundary terms) when evaluated on functions $u = f(x)$ whose graph is transformed by the action of the group on M. A connected group of transformations G is a variational symmetry group of the variational problem (1) if, for each infinitesimal generator of G of the form

$$\mathbf{v} = \xi(x,u)\partial_x + \eta(x,u)\partial_u, \tag{3}$$

the following infinitesimal condition holds:

$$\mathbf{v}^{(n)}(L) + L\mathbf{D}_x\xi = 0, \tag{4}$$

where $\mathbf{v}^{(n)}$ refers to the standard nth-order prolongation[1] of the vector field $\mathbf{v}$. It is well known that the presence of a one-parameter variational symmetry group permits to reduce the order of the Euler-Lagrange Equation (2) by two. Moreover, the general solution of the equation can be reconstructed from the solution of such $(2n-2)$th-order reduced equation by quadrature (see Olver[1] and references therein).

The connection between variational symmetries and first integrals of the corresponding Euler-Lagrange equation was established by Emmy Noether by means of her celebrated theorem: If a vector field $\mathbf{v}$ of the form (3) is a variational symmetry, then there exists a differential function $P = P(x, u, \dots, u_{2n-1})$ such that

$$QE_u[L] = \mathbf{D}_x(P), \tag{5}$$

where $Q = \eta(x,u) - u_1\xi(x,u)$ is the characteristic of the corresponding variational symmetry [14] (see also Olver[1]). Indeed, the condition of variational symmetry can be relaxed in order to deduce the existence of a conservation law, giving rise to the notion of divergence symmetry: A vector field $\mathbf{v}$ of the form (3) is a divergence symmetry of the variational problem (1) if there exists a differential function $B = B(x, u, \dots, u_{n-1})$ such that

$$\mathbf{v}^{(n)}(L) + L\mathbf{D}_x\xi = \mathbf{D}_x(B). \tag{6}$$

Both variational symmetries and divergence symmetries are always Lie point symmetries of the associated Euler-Lagrange equation. Then, in practice, variational symmetries (resp. divergence symmetries) can be found by computing the general Lie symmetry group of the Euler-Lagrange Equation (2) and then checking if the corresponding infinitesimal generators satisfy condition (4) (resp. (6)).

2.2 | Variational λ-symmetries

The concept of variational symmetry was generalized in Muriel et al[11] by means of the notion of variational λ-symmetry (also known as variational $\mathcal{C}^\infty$-symmetries). This new concept was introduced by considering a new way of prolonging vector fields: given a smooth vector field of the form (3) defined on M and a smooth function $\lambda \in \mathcal{C}^\infty(M^{(1)})$, the nth-order λ-prolongation of $\mathbf{v}$ is defined as the vector field on $M^{(n)}$

$$\mathbf{v}^{[\lambda,(n)]} = \xi(x,u)\partial_x + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x, u^{(i)})\partial_{u_i}, \tag{7}$$

where $u_0 = u$, $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and, for $1 \le i \le n$,

$$\eta^{[\lambda,(i)]}(x, u^{(i)}) = \mathbf{D}_x \left(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)}) \right) - \mathbf{D}_x(\xi(x,u))u_i + \lambda \left(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)}) - \xi(x,u)u_i \right). \tag{8}$$

Some important considerations regarding the λ-prolongation of a vector field are the following:

- When $\lambda = 0$, the λ-prolonged vector field (7) agrees with the standard nth-order prolongation of $\mathbf{v}$.
- $\mathbf{v}^{[\lambda,(n)]}$ is the unique vector field that satisfies the commutation relation[10, Theorem 2.1]:

$$[\mathbf{v}^{[\lambda,(n)]}, \mathbf{D}_x] = \lambda \mathbf{v}^{[\lambda,(n)]} + \rho \mathbf{D}_x, \tag{9}$$

where $\rho = -(\mathbf{D}_x + \lambda)\xi$.

A pair $(\mathbf{v}, \lambda)$, where $\mathbf{v}$ is a smooth vector of the form (3) and $\lambda \in C^\infty(M^{(1)})$, is a variational λ-symmetry (or variational C^∞ symmetry) of the variational problem (1) provided that the following infinitesimal criteria is satisfied[11,12]:

$$\mathbf{v}^{[\lambda,(n)]}(L) + L(\mathbf{D}_x + \lambda)\xi = (\mathbf{D}_x + \lambda)(B), \tag{10}$$

for some function $B = B(x, u, \dots, u_{n-1})$. If $B = 0$, then $(\mathbf{v}, \lambda)$ is called a strict variational λ-symmetry. The term generalized variational λ-symmetry is used when the infinitesimals of $\mathbf{v}$ or the function λ depend on derivatives of higher order (see Muriel et al.[11, Section 3]).

The effectiveness of variational λ-symmetries to find exact solutions to Euler-Lagrange equations associated to variational problems has been proved in Muriel et al[11] and Ruiz et al.[12] It has been demonstrated that if a variational problem of the form (1) admits a variational λ-symmetry, then the order of the associated Euler-Lagrange Equation (2) can be reduced by two. Also, a $(2n-1)$-parameter family of exact solutions can be reconstructed from the solution of the corresponding reduced equation by solving an associated first-order equation.

The loss of a one-parameter family of solutions can also be interpreted in terms of the following formula, which was proved in Muriel et al[11] as a version of Noether's theorem in the variational λ-symmetry context:

$$QE_u[L] = (\mathbf{D}_x + \lambda)(P), \tag{11}$$

for some differential function $P = P(x, u, \dots, u_{2n-1})$ and where Q is the characteristic of $\mathbf{v}$. As we can see in (11), the general solution of the Euler-Lagrange equation cannot be reconstructed from the general solution of $P(x, u, \dots, u_{n-1}) = C$, with C an arbitrary constant, unless $C = 0$.

Next, we sketch the reduction method associated to variational λ-symmetries for nth-order variational problems. The reader is referred to Muriel et al[11] for the details and proofs:

- In a neighborhood where $\mathbf{v}$ does not vanish, it is possible to introduce suitable coordinates $(\tilde{x}, \tilde{u})$ such that $\mathbf{v} = \partial_{\tilde{u}}$. With the aim of simplifying the notation, we continue denoting such rectifying coordinates as (x, u).
- Consider the Lagrangian

$$\hat{L} = L + \mathbf{D}_x(A), \tag{12}$$

where L is the Lagrangian function appearing in the variational problem (1) and the function A is such that $B = -\frac{\partial A}{\partial u}$. The Lagrangian $\hat{L}$ is equivalent to L in the sense that $E_u[L] = E_u[\hat{L}]$.
- Determine a function $w = w(x, u, u_1)$ such that

$$\mathbf{v}^{[\lambda,(1)]}(w) = 0.$$

- Compute by successive differentiation

$$w_i = \mathbf{D}_x(w_{i-1}), \ \ i = 1, \dots, n-1.$$

The set $\{x, w, \dots, w_{n-1}\}$ forms a complete system of differential invariants of $\mathbf{v}^{[\lambda,(n)]}$.
- The Lagrangian (12) can be expressed in terms of $\{x, w, \dots, w_{n-1}\}$, leading to a reduced Lagrangian of the form

$$\tilde{L} = \tilde{L}(x, w, \dots, w_{n-1}), \tag{13}$$

whose associated Euler-Lagrange equation

$$E_w[\tilde{L}] = 0 \tag{14}$$

is of order $2n-2$. The relation between the Euler-Lagrange equation $E_u[\hat{L}]$ and the corresponding reduced Euler-Lagrange Equation (14) (written in terms of the original variables $\{x, u, u_1 \dots, u_{2n}\}$) is the following (Mauriel et al. [11, eq. 29]):

$$E_u[\hat{L}] = (\mathbf{D}_x + \lambda)\left(-\frac{\partial w}{\partial u_1} E_w[\tilde{L}]\right). \tag{15}$$

- Let $w = G(x; C_1, \dots, C_{2n-2})$ $(2n-2)$th-order equation $E_w[\tilde{L}] = 0$. When w is written in terms of $\{x, u, u_1\}$, we obtain a first-order ordinary differential equation, called the auxiliary equation:

$$w(x, u, u_1) = G(x; C_1, \dots, C_{2n-2}), \tag{16}$$

and denote by $u = g(x; C_1, \dots, C_{2n-1})$ its general solution. When the second-member of (15) evaluated on $u = g(x; C_1, \dots, C_{2n-1})$ is well defined, it must vanish because $w = G(x; C_1, \dots, C_{2n-2})$ is the solution of the $(2n-2)$th-order equation $E_w[\tilde{L}] = 0$. This proves that $u = g(x; C_1, \dots, C_{2n-1})$ provides a $(2n-1)$-parameter family of solutions to the original Euler-Lagrange equation $E_u[\hat{L}] = 0$ (and hence of $E_u[L] = 0$).

Let us observe that for the particular case of first-order Lagrangian functions, the corresponding reduced Euler-Lagrange Equation (14) is of order zero, i.e., an algebraic equation.

3 | APPLICATIONS OF VARIATIONAL λ-SYMMETRIES

In this section, we aim to show different situations in which variational λ-symmetries can be successfully applied for obtaining exact solutions of Euler-Lagrange equations.

3.1 | A first-order variational problem lacking symmetries

Let us consider the first-order variational problem of the form (1) with Lagrangian function

$$L(x, u, u_1) = \left(u_1 - \frac{u}{x} + u^2 + 1\right)^2 + 2uu_1, \tag{17}$$

defined on $N \subset J^1(\mathbb{R}, \mathbb{R})$ such that $x \neq 0$. The corresponding Euler-Lagrange equation becomes

$$E_u[L] = \frac{2}{x}\left(xu_2 - 2xu - 2xu^3 + 3u^2 + 1\right) = 0,$$

i.e.,

$$xu_2 - 2xu^3 + 3u^2 - 2xu + 1 = 0. \tag{18}$$

It can be checked that the second-order Equation (18) does not admit Lie point symmetries, which implies that the associated variational problem does not present variational symmetries. Therefore, the methods of point, variational, and divergence symmetries do not apply to find exact solutions to Equation (18).

It can be checked that the pair $(\mathbf{v}, \lambda)$, where

$$\mathbf{v} = \partial_u, \qquad \lambda = \frac{1 - 2xu}{x},$$

satisfies the condition

$$\mathbf{v}^{[\lambda,(1)]}(L) = (\mathbf{D}_x + \lambda)(2u)$$

and hence (10) holds for $B = 2u$. This means that $(\mathbf{v}, \lambda)$ is a variational λ-symmetry of the variational problem associated to the Lagrangian (17).

Next, we show how this variational λ-symmetry can be used to obtain a one-parameter family of exact solutions to Equation (18). A function A such that $B = -\frac{\partial A}{\partial u}$ can be chosen as $A = -u^2$. The equivalent Lagrangian (12) becomes

$$\hat{L} = L + \mathbf{D}_x(A) = \left(u_1 - \frac{u}{x} + u^2 + 1\right)^2. \tag{19}$$

A complete set of first-order invariants for $\mathbf{v}^{[\lambda,(1)]}$ is given by

$$y = x, \qquad w = u_1 - \frac{u}{x} + u^2. \tag{20}$$

Thus, the Lagrangian (19) can be expressed as the reduced Lagrangian

$$\tilde{L} = (w+1)^2,$$

whose Euler-Lagrange equation becomes the algebraic equation

$$2(w+1) = 0,$$

with solution $w = -1$. Taking into account the expression of the invariant w given in (20), we deduce that a one-parameter family of solutions to (18) can be found by solving the auxiliary equation

$$u_1 - \frac{u}{x} + u^2 = -1,$$

which turns out to be a Riccati equation with general solution

$$u(x) = \frac{C\mathrm{J}_0(x) + \mathrm{Y}_0(x)}{C\mathrm{J}_1(x) + \mathrm{Y}_1(x)}, \qquad C \in \mathbb{R}, \tag{21}$$

where J_ν and Y_ν denote the Bessel functions of first and second kinds, respectively, of order ν for $\nu = 0, 1$. It can be checked that (21) satisfies Equation (18); therefore, a one-parameter family of exact solutions has been explicitly computed by means of the variational λ-symmetry approach. The particular solutions corresponding to $C = 10$ and $C = -10$ are graphically represented in the figure 1.

3.2 | A family of second-order equations with insufficient Lie symmetries

Consider the second-order ordinary differential equations that belong to the family

$$u_2 = b(x)^2 \alpha u^{2\alpha-1} + \big((\alpha+1)b(x)a(x) - b'(x)\big)\, u^\alpha + (a(x)^2 - a'(x))u, \tag{22}$$

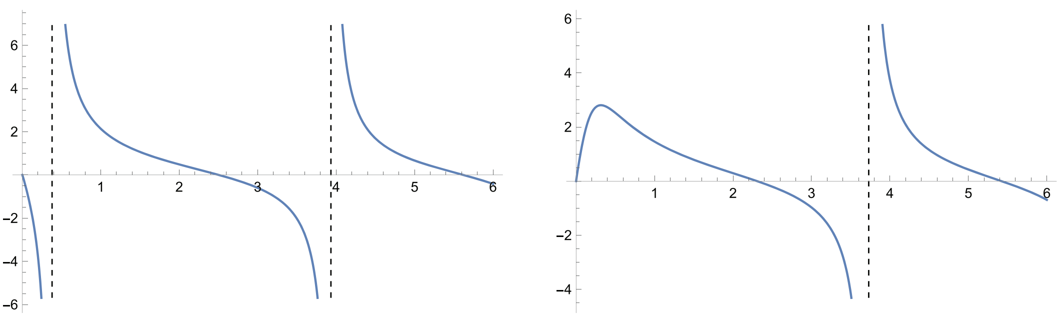


FIGURE 1 Exact solutions (21) of Equation (18) for $C = 10$ and $C = -10$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

where a, b are arbitrary smooth functions of the independent variable x and $\alpha \in \mathbb{R}$ with $\alpha \neq 1$. It can be checked that Equation (22) does not admit Lie point symmetries for arbitrary functions a, b and for arbitrary values of α. Therefore, the determination of exact solutions appears to be a challenging task.

It is well known that a second-order ordinary differential equation of the form

$$u_2 = \varphi(x, u, u_1) \tag{23}$$

is locally equivalent to an Euler-Lagrange equation (see Olver[1, Exercise 5.48]); i.e., there always exists a Lagrangian function L and a non-vanishing function μ such that

$$E_u[L] = \mu(u_2 - \varphi).$$

In this case, it can be checked that (22) is, locally, the Euler-Lagrange equation associated to the Lagrangian:

$$L(x, u, u_1) = (u_1 + a(x)u + b(x)u^{\alpha})^2 + \frac{1}{u} - \frac{xu_1}{u^2}. \tag{24}$$

The next stage of our approach is to try to compute a variational λ-symmetry for the Lagrangian (24). The pair forms by

$$\mathbf{v} = \partial_u, \qquad \text{and} \qquad \lambda = \frac{u_1(a(x) + b(x)\alpha u^{\alpha-1}) + (\alpha+1)a(x)b(x)u^{\alpha} + a(x)^2 u + \alpha b(x)^2 u^{2\alpha-1}}{u_1 + a(x)u + b(x)u^{\alpha}} \tag{25}$$

satisfies condition (10) for $B = -x/u^2$ and, therefore, is a variational λ-symmetry of the corresponding variational problem. A corresponding function A such that $B = -\frac{\partial A}{\partial u}$ can be taken as $A = -x/u$, and hence, the Lagrangian (12) becomes

$$\hat{L} = L + \mathbf{D}_x(A) = (u_1 + a(x)u + b(x)u^{\alpha})^2.$$

It can be checked that a complete system of first-order differential invariants for the corresponding first-order λ-prolongation is given by

$$x, \;\; w(x, u, u_1) = u_1 + a(x)u + b(x)u^{\alpha}.$$

As a consequence, the Lagrangian $\hat{L}$ can be expressed as the reduced Lagrangian

$$\tilde{L} = w^2,$$

whose Euler-Lagrange equation turns out to be the algebraic equation $w = 0$. Therefore, a one-parameter family of solutions for the original Euler-Lagrange equation, i.e., for the family of Equations (22), can be reconstructed by solving the first-order auxiliary equation

$$u_1 + a(x)u + b(x)u^{\alpha} = 0. \tag{26}$$

Equation (26) is of Bernoulli-type and its general solution is given by

$$u(x) = \frac{\exp\{-H(x)\}}{((\alpha-1)R(x) + C)^{\frac{1}{\alpha-1}}}, \tag{27}$$

where

$$R'(x) = b(x)\exp\{H(x)(1-\alpha)\}, \;\; H'(x) = a(x), \;\; C \in \mathbb{R}.$$

It can be checked that (27) satisfies the second-order Equation (22) and, therefore, it provides a one-parameter family of exact solutions for the family of second-order equations considered. In the next subsection, we present some particular cases of (22) which are of special interest because either the equation does not admit Lie symmetries or the admitted symmetry algebra is one-dimensional but cannot be used to determine exact solutions of the equation.

3.2.1 | Particular case for $\alpha = -2$, $a(x) = x$ and $b(x) = x$

In this particular situation, we have that the second-order Equation (22) is

$$u_2 = (x^2 - 1)u - (x^2 + 1)u^{-2} - 2x^2u^{-5}, \tag{28}$$

defined on $N \subset J^1(\mathbb{R}, \mathbb{R})$ such that $u \neq 0$. In this case, it can be checked that Equation (28) does not admit Lie symmetries. The corresponding Lagrangian function (24) is

$$L = \left(u_1 + xu + \frac{x}{u^2}\right)^2 + \frac{1}{u} - \frac{xu_1}{u^2},$$

and the corresponding variational λ-symmetry (25) becomes

$$\mathbf{v} = \partial_u, \qquad \lambda = \frac{x(2 - u^3)}{u^3}.$$

In spite of the absence of symmetries, we have been able to determine a one-parameter family of solutions given, according to (27), by

$$u(x) = \left(Ce^{-3x^2/2} - 1\right)^{1/3}, \qquad C \in \mathbb{R}. \tag{29}$$

The solutions for two particular values of the arbitrary constant C are displayed in Figure 2.

3.2.2 | Particular case for $\alpha = 2$, $a(x) = -1/x$, and $b(x) = 1$

In this particular case, the second-order Equation (22) becomes

$$u_2 = -\frac{3}{x}u^2 + 2u^3, \tag{30}$$

defined on an open set $N \subset J^1(\mathbb{R}, \mathbb{R})$ such that $x \neq 0$.

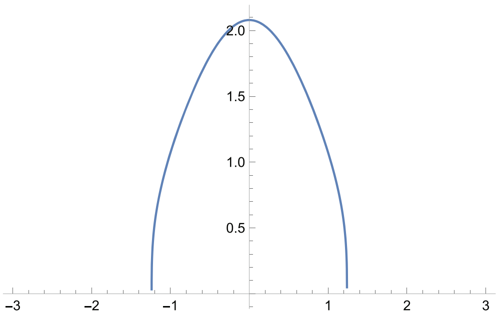
Equation (30) admits $\mathbf{v} = x\partial_x - u\partial_u$ as Lie point symmetry, which can be used to reduce the order of the equation. It can be checked that by means of the transformation

$$z = xu, \qquad h = \frac{-1}{x(u_1x + u)}, \tag{31}$$

Equation (30) reduces to

$$h_1 = (-2z^3 + 3z^2 + 2z)h^3 + 3h^2. \tag{32}$$

Equation (32) is an Abel equation which seems difficult to solve. In fact, it does not belong to the solvable cases listed in Polyanin and Zaitsev.[15]



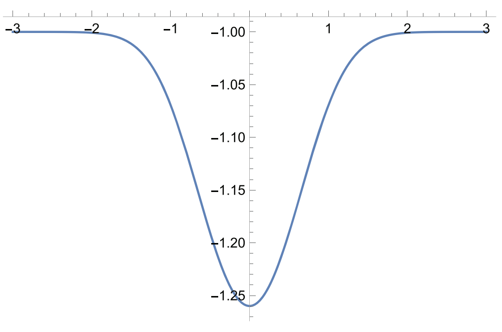


FIGURE 2 Exact solutions (29) of Equation (28) for $C = 10$ and $C = -1$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

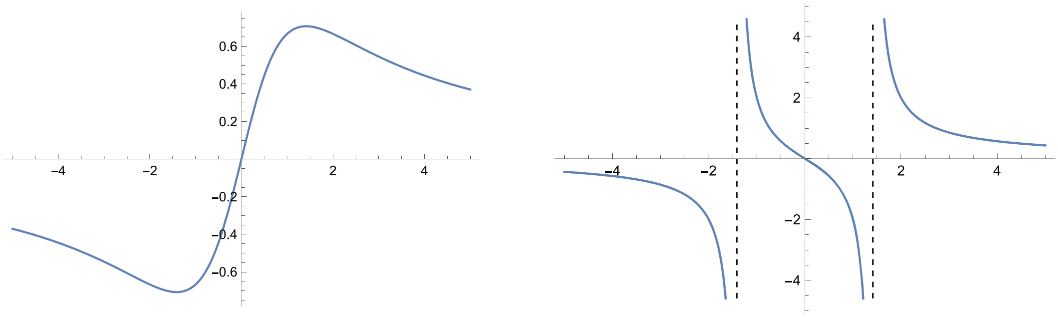


FIGURE 3 Exact solutions (33) of Equation (30) for $C = 2$ and $C = -2$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

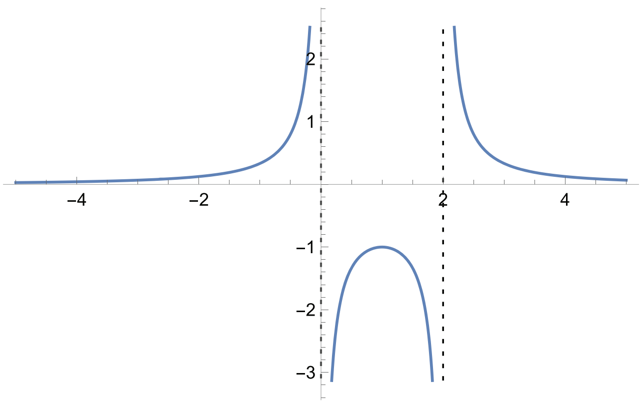


FIGURE 4 Particular solution (35) of the Abel Equation (32) [Colour figure can be viewed at wileyonlinelibrary.com]

As a consequence of the previous discussion, the Lie standard method seems to fail in providing exact solutions of Equation (30). However, the variational λ-symmetry approach yields the one-parameter family of solutions (27) which becomes

$$u(x) = \frac{2x}{x^2 + C}, \qquad C \in \mathbb{R}. \tag{33}$$

Two particular solutions corresponding to the family (33) are graphically represented in Figure 3.

It is important to observe that the presented approach provides an indirect method to calculate an exact solution of the Abel Equation (32): Taking into account the transformation (31) and the expression (33), we deduce that an exact solution of the Abel Equation (32) is given in parametric form by

$$z = \frac{2x^2}{x^2 + C}, \qquad h = \frac{-(x^2 + C)^2}{4Cx^2}, \; C \in \mathbb{R}. \tag{34}$$

The parametric solution (34) yields the following particular solution to the Abel Equation (32) in explicit form:

$$h(z) = \frac{1}{z^2 - 2z}. \tag{35}$$

Such particular solution is graphically represented in Figure 4.

3.2.3 | Particular case for $\alpha = -1$, $a(x) = 1/x$, and $b(x) = 1/x$

In this case, the second-order Equation (22) is

$$u_2 = \frac{1}{x^2}\left(2u + \frac{1}{u} - \frac{1}{u^3}\right), \tag{36}$$

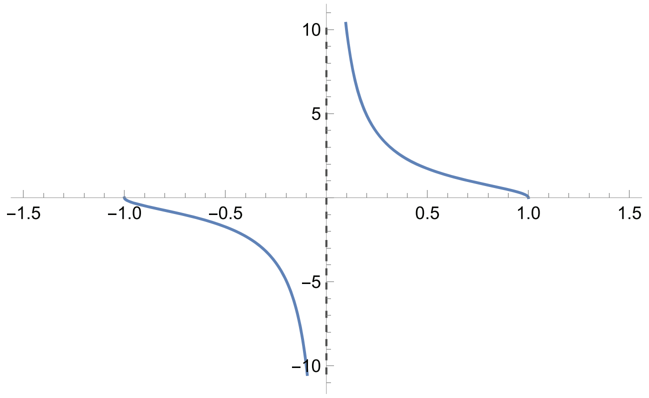


FIGURE 5 Exact solution (39) of Equation 36 for $C = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

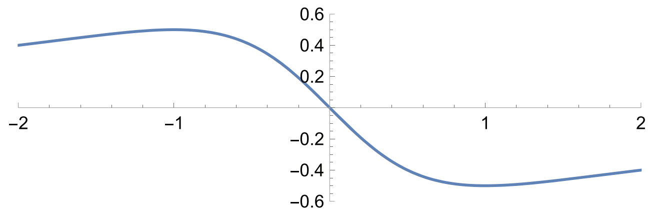


FIGURE 6 Particular solution (41) of the Abel Equation (37) [Colour figure can be viewed at wileyonlinelibrary.com]

defined on a suitable open set $N \subset J^1(\mathbb{R}, \mathbb{R})$ such that $xu \neq 0$. In this case, it can be checked that Equation (36) admits $\mathbf{v} = x\partial_x$ as Lie point symmetry, which yields the reduction to the following Abel equation

$$h_1 = \frac{-2z^4 - z^2 + 1}{z^3} h^3 - h^2, \tag{37}$$

by means of the transformation

$$z = u, \;\; h = \frac{1}{xu_1}. \tag{38}$$

As far as we are aware, finding a closed-form solution for the Abel Equation (37) is a difficult task. Therefore, for Equation (37), the Lie symmetry method is of not help. However, the variational λ-symmetry approach provides exact solutions to Equation (36). For this example, the one-parameter family of solutions (27) becomes

$$u(x) = \frac{\sqrt{C - x^2}}{x}, \qquad C \in \mathbb{R}. \tag{39}$$

A particular solution corresponding to the family (39) is plotted in Figure 5.

As in the previous example, as a byproduct of our approach, the following exact solution to the Abel Equation (37) is obtained in parametric form:

$$z = \frac{\sqrt{C - x^2}}{x}, \qquad h = \frac{-x\sqrt{C - x^2}}{C}, \qquad C \in \mathbb{R}, \tag{40}$$

which yields the following particular solution in explicit form:

$$h(z) = \frac{-z}{z^2 + 1}. \tag{41}$$

Such particular solution is plotted in Figure 6.

3.3 | Example of a second-order Lagrangian

In this subsection, we aim to show the application of variational λ-symmetries to higher order variational problems. Let us consider the following second-order Lagrangian:

$$L(x, u, u_1, u_2) = \frac{\left(u_2 x^2 u - x^2 u_1^2 - x u^2 u_1 + u^3\right)^2}{x^4 u^4} + \frac{x u_1 - u^2}{u}, \tag{42}$$

whose associated Euler-Lagrange equation becomes the fourth-order equation

$$u_4 = \frac{4 u_1 u_3}{u} + \frac{3 u_2^2}{u} + a(x, u, u_1) u_2 + b(x, u, u_1), \tag{43}$$

where

$$\begin{aligned} a(x, u, u_1) &= -\frac{12 x^2 u_1^2 - 3 x u^2 u_1 - u^4 + 3u^3}{x^2 u^2}, \\ b(x, u, u_1) &= \frac{4u^6 + (x^4 - 4 x u_1 - 12) u^5 + (x^4 + 12 x u_1) u^4 - 4 x^3 u^2 u_1^3 + 12 x^4 u_1^4}{2 x^4 u^3}. \end{aligned}$$

It can be checked that Equation (43) does not admit Lie symmetries, which implies that the corresponding variational problem does not present neither variational nor divergence symmetries.

In order to compute a variational λ-symmetry, we consider a pair of the form $(\partial_u, \lambda(x, u, u_1))$ and its corresponding second-order λ-prolongation:

$$(\partial_u)^{[\lambda,(2)]} = \partial_u + \lambda \partial_{u_1} + (\mathbf{D}_x(\lambda) + \lambda^2)\partial_{u_2}. \tag{44}$$

Thus, according to (10) and (44), the pair $(\partial_u, \lambda(x, u, u_1))$ is a variational λ-symmetry provided that the following partial differential equation for the function λ is satisfied:

$$(\lambda_x + u_1 \lambda_u + u_2 \lambda_{u_1} + \lambda^2) L_{u_2} = \lambda(B - L_{u_1}) + \mathbf{D}_x(B) - L_u, \tag{45}$$

for some function $B = B(x, u, u_1)$ and where L is given in (42). By using the *ansatz* $\lambda(x, u, u_1) = \alpha(u) u_1 + \beta(x, u)$ and by setting $B = 0$, the following particular solution to the partial differential Equation (45) can be obtained:

$$\alpha(u) = \frac{1}{u} \text{ and } \beta(x, u) = \frac{u}{x}.$$

Therefore, we conclude that the pair

$$\left(\partial_u, \frac{u_1}{u} + \frac{u}{x}\right)$$

is a variational λ-symmetry for the variational problem under study.

A function $w = w(x, u, u_1)$ such that $\mathbf{v}^{[\lambda,(1)]}(w) = 0$ becomes

$$w(x, u, u_1) = \frac{x u_1 - u^2}{x u}, \tag{46}$$

and a second-order differential invariant for the vector field (44) can be computed by differentiation as follows:

$$w_1 = \mathbf{D}_x(w) = \frac{u_2 u x^2 - x^2 u_1^2 - x u^2 u_1 + u^3}{x^2 u^2}.$$

It can be checked that the original Lagrangian (42) can be expressed in terms of the coordinates $\{x, w, w_1\}$ as the following first-order reduced Lagrangian:

$$\tilde{L} = w_1^2 + x w,$$

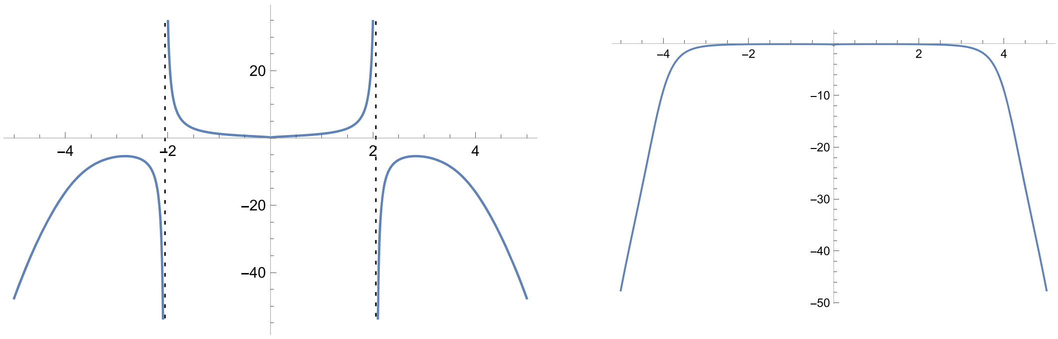


FIGURE 7 Exact solutions (49) of Equation (43) for $C_3 = 0$ and $C_3 = -10$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

whose associated Euler-Lagrange equation turns out to be

$$w_2 = \frac{x}{2}$$

with general solution

$$w(x) = \frac{x^3}{12} + C_1 x + C_2, \qquad C_1, C_2 \in \mathbb{R}.$$

According to (46), a three-parameter family of solutions to the original Equation (43) can be reconstructed by solving the auxiliary equation:

$$\frac{xu_1 - u^2}{xu} = \frac{x^3}{12} + C_1 x + C_2. \tag{47}$$

Equation (47) is of Bernoulli type, and its general solution is given by

$$u(x) = \frac{\exp\left(\frac{x^4}{48} + \frac{C_1 x^2}{2} + C_2 x\right)}{T(x) + C_3}, \tag{48}$$

where

$$T'(x) = -\frac{1}{x}\exp\left(\frac{x^4}{48} + \frac{C_1 x^2}{2} + C_2 x\right),$$

and $C_1, C_2, C_3 \in \mathbb{R}$. For the case $C_1 = C_2 = 0$, a one-parameter family of solutions for Equation (43) can be obtained from (48) expressed in terms of the exponential integral function[16] Ei as follows:

$$u(x) = \frac{4\exp\left(\frac{x^4}{48}\right)}{-\mathrm{Ei}\left(\frac{x^4}{48}\right) + 4C_3}, \quad C_3 \in \mathbb{R}. \tag{49}$$

In Figure 7, some particular solutions to the fourth-order Equation (43) corresponding to (49) are represented.

4 | CONCLUDING REMARKS

The variational λ-symmetry method has been applied to find exact solutions for some second- and fourth-order Euler-Lagrange equations associated to first- and second-order variational problems, respectively. Such solutions are not obtainable by Lie point or variational symmetries.

A one-parameter family of solutions in terms of Bessel functions has been found for Equation (18), even though the equation does not admit Lie point symmetries and, hence, variational symmetries. Most of the equations in the family (22) also lack Lie point symmetries. After writing them in variational form, a variational λ-symmetry has been found, which yields a systematic procedure to calculate explicitly a one-parameter set of exact solutions. They arise by direct integration of a Bernoulli-type equation. Finally, a three-parameter family of exact solutions for the fourth-order Equation (43) has been obtained. Remarkably, Equation (43) does not admit Lie symmetries and therefore neither variational nor divergence symmetries.

Some particular equations in the family (22) admit just one Lie point symmetry, but the reduced equations are of Abel type and do not seem easy to solve. For this reason, the Lie reduction procedure does not provide any solution for the initial second-order ODE. As byproduct of our procedure, we have shown how to use the one-parameter of exact solutions derived from the variational λ-symmetry in order to construct explicitly particular solutions of the Abel-type equations. Therefore, it would be interesting to investigate if the presented method could be applied for finding new cases of Abel equations admitting particular solutions of a specific type. This study is currently in progress.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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