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Rough set decision algorithms for modeling with uncertainty ☆

Fernando Chacón-Gómez, M. Eugenia Cornejo, Jesús Medina *, Eloísa Ramírez-Poussa

Department of Mathematics, University of Cádiz, Spain

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ABSTRACT

The use of decision rules allows to extract information and to infer conclusions from relational databases in a reliable way, thanks to some indicators like support and certainty. Moreover, decision algorithms collect a group of decision rules that satisfies desirable properties to describe the relational system. However, when a decision table is considered within a fuzzy environment, it is necessary to extend all notions related to decision algorithms to this framework. This paper presents a generalization of these notions, highlighting the new definitions of indicators of relevance to describe decision rules and decision algorithms.

1. Introduction

Pawlak [1,2] introduced a formal tool to deal with imprecise or incomplete information contained in databases, which was called Rough Set Theory (RST). This theory considers information systems, which are relational databases composed of a set of objects and a set of attributes which characterizes the objects, and it has attracted the attention of other (related) theories. For example, attribute reduction in the extensions of RST from the philosophy of Formal Concept Analysis (FCA) were studied in [3–6]. The paper [7] combined the K-nearest neighbors algorithm with RST, the algebraic notion of congruence was adapted in order to introduce an optimal attribute reduction based on RST in the FCA framework in [8], fuzzy soft sets were used to propose a combined forecasting approach for complementing a methodology based on rough sets [9], and fuzzy rough sets and probability statistics were studied and combined in [10]. A particular type of information system widely studied within RST is the one provided by decision tables. Decision rules [11–16] can be used to describe decision tables, allowing the extraction of information and the inference of significant conclusions. Decision rules are usually accompanied by some relevance indicators which describe them. In addition, decision algorithm is another important notion related to decision rules. This notion was introduced with the purpose of collecting some desirable properties that a set of decision rules should satisfy in order to describe the decision table in a suitable way.

On the other hand, the management of real databases can be a really difficult task. Fuzzy Rough Set Theory (FRST) [17–19] arises with the goal of increasing the flexibility of RST, addressing a wider range of real applications. FRST is a generalization of RST in the fuzzy environment, essential to deal with the information contained in databases, especially when they are composed by continuous quantities.

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* Corresponding author.
E-mail addresses: fernando.chacon@uca.es (F. Chacón-Gómez), mariaeugenia.cornejo@uca.es (M.E. Cornejo), jesus.medina@uca.es (J. Medina), eloisa.ramirez@uca.es (E. Ramírez-Poussa).

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The aim of this paper is to provide a generalization of the notion of decision algorithm from the classical environment to the fuzzy framework given by the multi-adjoint paradigm. With this goal, all the necessary notions related to decision rules and their corresponding relevance indicators are also extended to the fuzzy environment. We will prove that the new notions introduced in this work are indeed generalizations of the classical ones. This fact shows the possibility of applying this approach to the mechanisms focused on obtaining decision rules, which are mainly based on a crisp point of view. Notice that, for example, the last contributions in the area, such as [20], which considers an equivalent algorithm to the one provided by Pawlak [14] in the crisp case. The authors present a mechanism on a real decision table in which they need to consider a discretization, however, it is more natural to study this table from a fuzzy point of view, as we will do at the end of the paper. Hence, the contribution of this paper will also allow to extend the theory given in [20] to the fuzzy case. Granular computing is also applied to rule acquisition [21] in which the fuzzy point of view will also allow to be more precise and flexible, and give more information about the decision table. Moreover, in [22], the authors consider a variant of the notion of algorithm [14] in crisp incomplete decision systems, in which the approach introduced in this paper also extends the results given in the aforementioned paper to the fuzzy case, being also possible its adaptation to incomplete decision systems. Thus, it is possible to apply the framework introduced in this paper to classification problems with a greater flexibility than the current mechanisms. We will also include some examples to illustrate the content of the paper.

The paper is organized as follows: Section 2 recalls some preliminary classical notions of RST. In Section 3, a study about decision rules in FRST is introduced, including the generalizations of the relevance indicators to the fuzzy framework. Section 4 introduces a new notion of decision algorithm and shows the existing connection between the classical and the fuzzy version. Section 6 finishes with some conclusions and prospects for future work.

2. Basic notions in rough set theory

This section presents some important definitions of RST [13,14]. First of all, it is convenient to recall that databases can be represented as decision tables in this framework.

Definition 1. Let U and $\mathcal{A}$ be non-empty sets of objects and attributes, respectively. A *decision table* is a tuple $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ such that $\mathcal{A}_d = \mathcal{A} \cup \{d\}$ with $d \notin \mathcal{A}$, $\mathcal{V}_{\mathcal{A}_d} = \{V_a \mid a \in \mathcal{A}_d\}$, where V_a is the set of values associated with the attribute a over U, and $\overline{\mathcal{A}_d} = \{\bar{a} \mid a \in \mathcal{A}_d, \bar{a}\colon U \to V_a\}$. In this case, the attributes of $\mathcal{A}$ are called *condition attributes* and d is called *decision attribute*.

This paper will focus on decision tables with one decision attribute, as it is shown in this definition. However, a similar study could be carried out in decision tables with more than one decision attribute.

Now, an equivalence relation is defined on the set of objects of a decision table, in order to compare them by using a given subset of attributes.

Definition 2. Let $(U, \mathcal{A}_d, V_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table. The *indiscernibility mapping* $I\colon \mathcal{P}(\mathcal{A}_d) \to \mathcal{P}(U \times U)$ is defined, for each $B \subseteq \mathcal{A}_d$, as the equivalence relation

$$I(B) = \{(x, y) \in U \times U \mid \bar{a}(x) = \bar{a}(y),\ \text{for all } a \in B\}$$

which is called *B-indiscernibility relation*. Each class of $I(B)$ can be written as $[x]_{I(B)} = \{y \in U \mid (x, y) \in I(B)\}$. The partition determined by $I(B)$ on the set of objects U is denoted as $U/I(B) = \{[x]_{I(B)} \mid x \in U\}$.

The positive region is introduced now. This notion is deeply studied in order to analyze and extract information from decision tables.

Definition 3. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $B \subseteq \mathcal{A}$. The *positive region* of the partition $U/I(\{d\})$ with respect to B is defined as

$$\mathrm{POS}_B(\{d\}) = \bigcup_{X \in U/I(\{d\})} B_*(X)$$

where $B_*(X) = \{x \in U \mid [x]_{I(B)} \subseteq X\}$.

2.1. Decision rules

The previous notions are useful for the management of the information contained in decision tables. Nevertheless, decision rules [14] arise in order to analyze these tables by using logic equivalences. This fact provides easier interpretation of decision tables, describing decisions in terms of conditions that must be satisfied. In order to present the notion of decision rule, we must introduce a formal language to describe approximations in logical terms.

Definition 4. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $B \subseteq \mathcal{A}_d$. The set of formulas associated with B, denoted as $For(B)$, is built from attribute–value pairs(a, v), where $a \in B$ and $v \in V_a$, by means of the conjunction and disjunction logical connectives, $\wedge$ and $\vee$, respectively.

For each $\Phi \in For(B)$, with $\Phi = (a, v)$, the set of objects $X \subseteq U$ that satisfies Φ in S is defined as:

$$\|\Phi\|_S = \|(a, v)\|_S = \{x \in U \mid \bar{a}(x) = v\}$$

Inductively, given $\Phi, \Psi \in For(B)$, the set of objects that satisfies $\Phi \wedge \Psi$ in S is defined as $\|\Phi \wedge \Psi\|_S = \|\Phi\|_S \cap \|\Psi\|_S$ and the set of objects that satisfies $\Phi \vee \Psi$ in S is defined as $\|\Phi \vee \Psi\|_S = \|\Phi\|_S \cup \|\Psi\|_S$.

Now, we introduce the notion of decision rule. As we mentioned above, it is an important pillar of the synthesis of information from relational databases.

Definition 5. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $B \subseteq \mathcal{A}$. A *decision rule* in S is an expression $\Phi \to \Psi$, with $\Phi \in For(B), \Psi \in For(\{d\})$ where Φ and Ψ are the antecedent and the consequent of the decision rule, respectively. In addition, we will say that *an object $x \in U$ satisfies the decision rule $\Phi \to \Psi$* if $x \in \|\Phi \wedge \Psi\|_S$.

As we commented above, decision rules provide a logic representation of decision tables. However, it is necessary to interpret these rules for the draw of conclusions from the decision table. For this purpose, we recall the indicators which describe decision rules from different points of view. Specifically, the *support* considers the number of objects satisfying the decision rule, the *strength* represents the proportion of objects satisfying the decision rule, that is, the representativeness of that rule in the table, the *certainty* provides us with the proportion of objects which satisfy the consequent satisfying the antecedent of the given decision rule and the *coverage* determines the proportion of objects which satisfy the antecedent satisfying the consequent of the given decision rule.

Definition 6. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $B \subseteq \mathcal{A}$ and $\Phi \to \Psi$ be a decision rule in S, with $\Phi \in For(B)$ and $\Psi \in For(\{d\})$. We call

- *support* of the decision rule $\Phi \to \Psi$ to the value:
$$supp_S(\Phi, \Psi) = |\|\Phi \wedge \Psi\|_S|$$
- *strength* of the decision rule $\Phi \to \Psi$ to the value:
$$\sigma_S(\Phi, \Psi) = \frac{supp_S(\Phi, \Psi)}{|U|}$$
- *certainty* of the decision rule $\Phi \to \Psi$ to the value:
$$cer_S(\Phi, \Psi) = \frac{supp_S(\Phi, \Psi)}{|\|\Phi\|_S|}$$
when $|\|\Phi\|_S| \neq 0$.
- *coverage* of the decision rule $\Phi \to \Psi$ to the value:
$$cov_S(\Phi, \Psi) = \frac{supp_S(\Phi, \Psi)}{|\|\Psi\|_S|}$$
when $|\|\Psi\|_S| \neq 0$.

From the notion of certainty, we will say that $\Phi \to \Psi$ is a *true decision rule* if $cer_S(\Phi, \Psi) = 1$. If $cer_S(\Phi, \Psi) = 0$, we say that the decision rule is *false*. Otherwise, it will be called a *not entirely true decision rule*.

These notions are of great importance in the study of decision algorithms and their efficiency, as it is shown in the next section.

2.2. Decision algorithm

Decision algorithms are a collection of decision rules of a decision table verifying certain conditions, focused on providing a global representation of the decision table. The definition of a decision algorithm is given below [14].

Definition 7. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $Dec(S) = \{\Phi_i \to \Psi_i\}_{i \in I}$ be a set of decision rules of S, where the index set is $I = \{1, \dots, m\}$ and $m \geq 2$. We say that:

1. $Dec(S)$ is a set of *pairwise mutually exclusive (independent) decision rules*, if each pair of decision rules $\Phi \to \Psi, \Phi' \to \Psi' \in Dec(S)$ satisfies that $\Phi = \Phi'$ or $\|\Phi \wedge \Phi'\|_S = \varnothing$, and $\Psi = \Psi'$ or $\|\Psi \wedge \Psi'\|_S = \varnothing$.
2. $Dec(S)$ *covers* U, if $\|\bigvee_{i=1}^{m} \Phi_i\|_S = \|\bigvee_{i=1}^{m} \Psi_i\|_S = U$.
3. A decision rule $\Phi \to \Psi \in Dec(S)$ is *admissible in S* if $supp_S(\Phi, \Psi) \neq 0$.

4. *Dec*(S) *preserves the consistency of* S if
$$\mathrm{POS}_{\mathcal{A}}(\{d\}) = \bigvee_{\Phi\to\Psi\in Dec^+(S)} \|\Phi\|_S$$
where $Dec^+(S)$ is the set of true decision rules of $Dec(S)$.

A set of decision rules $Dec(S)$ satisfying properties (1)–(4) is called *decision algorithm in* S and it is denoted as $DA(S)$.

Notice that, the notion of consistency is not defined from the positive region in other works, such as [22]. However, the approach proposed in [22] considers the notions of admissible and redundancy, being this last one closely related to Property (1) in Definition 7. Hence, our contribution also extends the results given in [22], being possible its adaptation to incomplete decision systems.

On the other hand, a pair of conclusions can be extracted from the definition of decision algorithm. By Properties (1) and (2), if $DA(S)$ is a decision algorithm, the antecedents and consequents of the decision rules of $DA(S)$ define a partition of U. Furthermore, by Property (4), the set of true decision rules define a partition of the positive region with respect to $\mathcal{A}$. On the other hand, it is convenient to emphasize that there may exist objects which do not satisfy any decision rule of the decision algorithm $DA(S)$, as it is shown in the following example.

Example 8. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table where the set of objects is $U = \{x_1, x_2, x_3\}$ and the set of attributes is $\mathcal{A} = \{a\}$. This decision table is represented in Table 1.

Table 1
Table associated with the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 8.

	a	d
x_1	0	0
x_2	0	1
x_3	1	1

Consider the set of decision rules $Dec(S) = \{\Phi_1 \to \Psi_1, \Phi_2 \to \Psi_2\}$, denoted as r_1 and r_2 respectively, given as:

$r_1 : (a, 0) \to (d, 0)$
$r_2 : (a, 1) \to (d, 1)$

Hence, it is immediate to check that $\Phi_1 \neq \Phi_2$, $\Psi_1 \neq \Psi_2$ and
$$\|\Phi_1 \wedge \Phi_2\|_S = \|\Phi_1\|_S \cap \|\Phi_2\|_S = \{x_1, x_2\} \cap \{x_3\} = \varnothing$$
$$\|\Psi_1 \wedge \Psi_2\|_S = \|\Psi_1\|_S \cap \|\Psi_2\|_S = \{x_1\} \cap \{x_2, x_3\} = \varnothing$$

In addition, $\|\bigvee_{i=1}^{2}\Phi_i\|_S = \|\bigvee_{i=1}^{2}\Psi_i\|_S = U$ and $supp_S(\Phi_1, \Psi_1) = supp_S(\Phi_2, \Psi_2) = 1 \neq 0$. Finally, $cer_S(\Phi_1, \Psi_1) = 0.5$ and $cer_S(\Phi_2, \Psi_2) = 1$. Therefore,
$$\mathrm{POS}_{\mathcal{A}}(\{d\}) = \{x_3\} = \|\Phi_2\|_S = \bigvee_{\Phi\to\Psi\in Dec^+(S)} \|\Phi\|_S$$

As a consequence, $Dec(S)$ is a decision algorithm and the object x_2 does not satisfy any decision rule of $Dec(S)$. □

Decision algorithms where all the objects of the decision table satisfy a decision rule of that algorithm were also defined by Pawlak in [13] and they are recalled now.

Definition 9. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $DA(S)$ be a decision algorithm. If for each $x \in U$ there exists $\Phi \to \Psi \in DA(S)$ such that $x \in \|\Phi \wedge \Psi\|_S$ then $DA(S)$ it is called a *complete decision algorithm.*

An important consequence of Definition 9, taking into account the first item of Definition 7, is that the addition of the supports of all decision rules is the cardinal of the set of objects, since for each object there exists only one decision rule which is satisfied by that object. Therefore, the addition of the strengths of all decision rules is 1.

3. Decision rules in fuzzy rough set theory

This section will focus on the study of decision tables in the fuzzy environment by using decision rules. With this purpose, it is needed to recall some important notions of FRST [17]. Most of them are the generalization of the previous definitions to the fuzzy framework. The first definition is essential to compare a pair of objects of a decision table according to an attribute. It presents a map that, evaluated on two objects, indicates the relationship between these objects, taking into account the considered attribute.

Definition 10. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $a \in \mathcal{A}_d$. An *a-indiscernibility relation* is a [0, 1]-fuzzy tolerance relation $R_a : U \times U \to [0, 1]$, that is, a reflexive and symmetrical fuzzy relationship.

Depending on the value of $R_a(x, y)$ we can conclude how similar the objects x, y are, according to the attribute a. Next definition is required to compare pairs of objects by using a subset of attributes $B \subseteq \mathcal{A}$ instead of a single attribute. Aggregation operators [17] are indispensable for that.

Definition 11. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $R_a : U \times U \to [0, 1]$ be an a-indiscernibility relation for all $a \in \mathcal{A}$. The *B-indiscernibility relation* $R_B : U \times U \to [0, 1]$ is defined for each pair of objects $x, y \in U$ as

$$R_B(x, y) = @(\mathcal{R}_B^{x,y}(a_1), \dots, \mathcal{R}_B^{x,y}(a_m))$$

where $\mathcal{R}_B^{x,y} : \mathcal{A} \to [0, 1]$ is defined for each $a \in \mathcal{A}$ as

$$\mathcal{R}_B^{x,y}(a) = \begin{cases} R_a(x, y) & \text{if } a \in B \\ 1 & \text{otherwise} \end{cases}$$

and $@ : [0, 1]^m \to [0, 1]$ is an aggregation operator, that is, an increasing operator on each argument satisfying $@(1, \dots, 1) = 1$ and $@(0, \dots, 0) = 0$.

The interpretation of the value given by this mapping is the same as Definition 10. In addition, notice that R_B is a [0, 1]-fuzzy tolerance relation for all $B \subseteq \mathcal{A}$.

In the following, some previous notions given in [17] are recalled, which are necessary to define the positive region in the fuzzy environment. First of all, we introduce the notion of adjoint triple.

Definition 12. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$ be partially ordered sets (posets) and $\& : P_1 \times P_2 \to P_3$, $\swarrow : P_3 \times P_2 \to P_1$ and $\nwarrow : P_3 \times P_1 \to P_2$ be mappings. Then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2 and P_3 if these mappings satisfy the adjoint property, that is, $x \leq_1 z \swarrow y$ iff $x \& y \leq_3 z$ iff $y \leq_2 z \nwarrow x$ for all $x \in P_1$, $y \in P_2$, $z \in P_3$.

Notice that, the mappings $\swarrow, \nwarrow$ can be univocally obtained from the operator &. Therefore, we can refer to the adjoint triple $(\&, \swarrow, \nwarrow)$ simply as &. On the other hand, in the particular case that the mapping & is commutative and $P_1 = P_2$, we obtain that $\swarrow = \nwarrow$. More properties can be seen in [23].

Left-continuous t-norms together with their corresponding residuated implications clearly are particular cases of adjoint triples [24], such as the well-known Gödel, product and Łukasiewicz t-norms. In order to illustrate the impact of the use of adjoint triples in the management of information, a pair of adjoint triples are presented in the following example.

Example 13. We include an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the unit interval in which the mapping & is not commutative. This fact will lead to obtain two different mappings $\swarrow$ and $\nwarrow$.

$$x \& y = x^2 y$$
$$z \swarrow y = \begin{cases} 1 & \text{if } y \leq z \\ \sqrt{z/y} & \text{otherwise} \end{cases}$$
$$z \nwarrow x = \begin{cases} 1 & \text{if } x^2 \leq z \\ z/x^2 & \text{otherwise} \end{cases}$$

From this adjoint triple we can deduce that the values of the variable x are penalized by the square (the square of values less or equal to 1 provides a less value than the original). Hence, we can say that with the use of this conjunctor the variable y has a major impact (for obtaining a large aggregated value) than the variable x. This fact allows us to give different importance to each variable of the corresponding study.

Now, we present the family of Łukasiewicz adjoint triples $(\&_Ł^\beta, \swarrow_Ł^\beta, \nwarrow_Ł^\beta)$, which depends on a parameter $\beta \in \mathbb{N}$:

$$\begin{aligned} x \&_Ł^\beta y &= \max\{0, \sqrt[\beta]{x^\beta + y^\beta - 1}\} \\ z \nwarrow_Ł^\beta x &= \sqrt[\beta]{\min\{1, 1 + z^\beta - x^\beta\}} \end{aligned} \tag{1}$$

Since the mapping $\&_Ł^\beta$ is commutative we obtain that $\swarrow_Ł^\beta = \nwarrow_Ł^\beta$. In this family of adjoint triples, the parameter β affects to the monotonicity. Specifically, it can be checked that $\&_Ł^\beta$ is decreasing in β and $\nwarrow_Ł^\beta$ is increasing in β, as it is shown in Fig. 1. Hence, the user can consider different values of β depending on the objects under consideration, choosing higher or lower values to give them different importance [17]. □

Now, we recall the notion of multi-adjoint property-oriented frame.

Definition 14. Given a poset $(P_1, \leq_1)$, two complete lattices $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$ and adjoint triples $(\&_i, \swarrow^i, \nwarrow_i)$, with respect to P_1, L_2, L_3, for all $i \in \{1, \dots, n\}$, a *multi-adjoint property-oriented frame* is the tuple

$$(P_1, L_2, L_3, \&_1, \dots, \&_n)$$

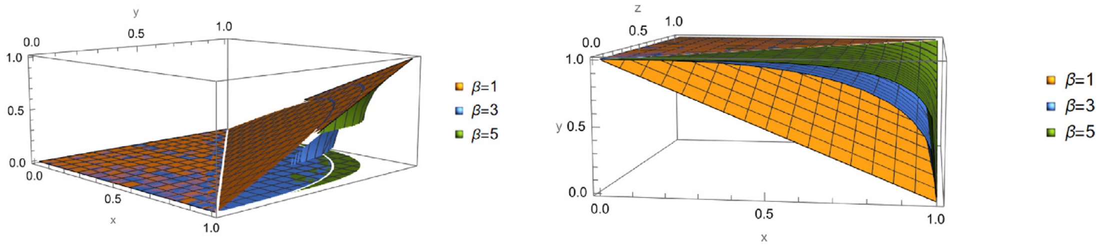

Fig. 1. Operators $\&_{Ł}^{\beta}$ (left) and $\nwarrow_{Ł}^{\beta}$ (right), with $\beta \in \{1, 3, 5\}$.

Next, the notion of context is introduced.

Definition 15. Let A and B be non-empty sets and $(P_1, L_2, L_3, \&_1, \dots, \&_n)$ be a multi-adjoint property-oriented frame. A *context* is a tuple (A, B, R, τ), where R is a P_1-fuzzy relation $R\colon A \times B \to P_1$ and $\tau\colon A \times B \to \{1, \dots, n\}$ is a mapping which associates any pair of elements in $A \times B$ with some particular adjoint triple in the frame.

In the following definition, we recall the generalization of the notion of upper and lower approximation of fuzzy RST given in [17].

Definition 16. Let $(P_1, L_2, L_3, \&_1, \dots, \&_n)$ be a multi-adjoint property oriented frame and (A, B, R, τ) be a context. Given $g \in L_2^B$ and $f \in L_3^A$ we define the *possibility and necessity operators,* ${}^{\uparrow_\pi}\colon L_2^B \to L_3^A$ *and* ${}^{\downarrow^N}\colon L_3^A \to L_2^B$, respectively, as:

$$g^{\uparrow_\pi}(a) = \sup\{R(a, b) \&_{\tau(a,b)} g(b) \mid b \in B\}$$
$$f^{\downarrow^N}(b) = \inf\{f(a) \nwarrow_{\tau(a,b)} R(a, b) \mid a \in A\}$$

where $g^{\uparrow_\pi}$ *is interpreted as the upper approximation of* g *and* $f^{\downarrow^N}$ *as the lower approximation of* f.

Notice that, the use of the mapping τ allows the consideration of different degrees of preference on the sets of objects and attributes [25]. Furthermore, the possibility of taking into account non-commutative operators allows to consider different levels of relevance in the computation of the aggregated value. For example, if the first adjoint triple in Example 13 is considered in the possibility operator, then we are assuming that it is more relevant the values $g(b)$ (the object b belong to the fuzzy subset g) than the values given by the relation.

In this paper, we will focus on a multi-adjoint property-oriented frame $([0, 1], [0, 1], [0, 1], \&_1, \dots, \&_n)$ and a context (U, U, R_B, τ) with $B \subseteq \mathcal{A}$. Finally, we recall the notion of positive region (generalization of Definition 3) in the fuzzy setting when R_d only takes boolean values, which was given in [17].

Definition 17. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $B \subseteq \mathcal{A}$. Given a multi-adjoint property oriented frame $([0, 1], [0, 1], [0, 1], \&_1, \dots, \&_n)$, a context (U, U, R_B, τ) and a boolean relation R_d, the *multi-adjoint fuzzy B-positive region* is defined, for each $y \in U$, as:

$$\mathrm{POS}_B^{\mathrm{f}}(y) = (R_d\, y)^{\downarrow^N}(y) = \inf\{(R_d\, y)(x) \nwarrow_{\tau(x,y)} R_B(x, y) \mid x \in U\}$$

where $R_d\, y\colon U \to \{0, 1\}$ is defined as $(R_d\, y)(x) = R_d(y, x)$ and $\nwarrow_{\tau(x,y)}$ is the left residuated fuzzy implication of $\&_{\tau(x,y)}$ associated with the pair of objects x, y.

The authors in [17] asserted that Definition 17 is a generalization of Definition 3. The following result shows the details of this statement.

Proposition 18. *Let* $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ *be a decision table,* $x \in U$ *and* $R_{\mathcal{A}}$ *and* R_a *be boolean relations for all* $a \in \mathcal{A}_d$. *Then* $x \in POS_{\mathcal{A}}(\{d\})$ *if and only if* $POS_{\mathcal{A}}^{f}(x) = 1$.

Proof. First of all, notice that if $R_{\mathcal{A}}$ is a boolean relation then, given two objects $x, y \in U$, $R_{\mathcal{A}}(x, y) = 1$ if and only if $R_a(x, y) = 1$ for all $a \in \mathcal{A}$, that is, $\bar{a}(x) = \bar{a}(y)$ for all $a \in \mathcal{A}$, and $R_{\mathcal{A}}(x, y) = 0$ otherwise. Let $x \in \mathrm{POS}_{\mathcal{A}}(\{d\})$ and $y \in U$. We distinguish two cases.

- We suppose that $y \in [x]_{I(\mathcal{A})}$. Since $x \in \mathrm{POS}_{\mathcal{A}}(\{d\})$ we obtain that $[x]_{I(\mathcal{A})} \subseteq [x]_{I(d)}$. Therefore, $y \in [x]_{I(d)}$. Hence, $R_a(x, y) = 1$ for all $a \in \mathcal{A}_d$. As a consequence,

$$R_d(y, x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x, y) = 1 \nwarrow_{\tau(x,y)} 1 = 1$$

- Now, we suppose that $y \notin [x]_{I(\mathcal{A})}$. Then there exists $a \in \mathcal{A}$ such that $\bar{a}(x) \neq \bar{a}(y)$. Hence, $R_{\mathcal{A}}(x, y) = 0$. As a consequence,

$$R_d(y, x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x, y) = R_d(y, x) \nwarrow_{\tau(x,y)} 0 = 1$$

In conclusion, $R_d(y,x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x,y) = 1$ for all $y \in U$. Therefore,

$$\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = (R_d\, y)^{\downarrow^N}(y) = \inf\{R_d(y,x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x,y) \mid x \in U\} = 1$$

Now, we suppose that $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = 1$. As a consequence, we obtain that $R_d(y,x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x,y) = 1$ for all $y \in U$. If $x \notin \mathrm{POS}_{\mathcal{A}}(\{d\})$ then there exists $y \in [x]_{I(\mathcal{A})} \setminus [x]_{I(d)}$. Hence, $R_{\mathcal{A}}(x,y) = 1$ and $R_d(x,y) = 0$. Therefore,

$$R_d(y,x) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x,y) = 0 \nwarrow_{\tau(x,y)} 1 = 0,$$

obtaining a contradiction. In conclusion, $x \in \mathrm{POS}_{\mathcal{A}}(\{d\})$. □

Since in the computation of $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(y)$ all the objects of the decision table are taken into account, by using indiscernibility relations and a fuzzy implication, this notion provides a degree of consistency of the object y in the decision table. As a consequence, the higher the value of $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(y)$, the more dependence there will be between the set of attributes $\mathcal{A}$ and the decision attribute d for the object y.

Real datasets usually present numerous attributes, making difficult the management of the information contained in them. The notion of fuzzy decision reduct arises to reduce the number of attributes without losing information, which enables us to deal with them in an easier way. Before introducing this notion, we must recall the following definition.

Definition 19. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $B \subseteq \mathcal{A}$ and $([0,1],[0,1],[0,1],\&_1,\dots,\&_n)$ be a multi-adjoint property oriented frame. A monotonic mapping $m\colon \mathcal{P}(\mathcal{A}) \to [0,1]$ is a $[0,1]$-*valued measure associated with* S if the condition $R_d(x,y) \nwarrow_{\tau(x,y)} R_B(x,y) = R_d(x,y) \nwarrow_{\tau(x,y)} R_{\mathcal{A}}(x,y)$, for all $x, y \in U$ implies $m(B) = 1$.

Now, we can present the notion of fuzzy decision reduct, which is supported by [0, 1]-valued measures.

Definition 20 ([17]). Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $B \subseteq \mathcal{A}$, m be a $[0,1]$-valued measure associated with S and $\alpha \in (0,1]$. The set B is called a *fuzzy m-decision superreduct to degree* α if $\alpha \preccurlyeq m(B)$. Moreover, if $\alpha \not\preccurlyeq m(B')$ for each $B' \subset B$ then B is called *fuzzy m-decision reduct to degree* α.

Now, we recall the notion of cardinal of a fuzzy set, which will be used later. This definition is introduced focusing on the multi-adjoint fuzzy positive region.

Definition 21. Given a universe U and $f\colon U \to [0,1]$, the *cardinal* of f is given as

$$card_F(f) = \sum_{x \in U} f(x)$$

Furthermore, it is necessary to generalize the notions defined in Sections 2.1 and 2.2 to this framework.

First of all, we introduce a particular kind of tolerance relation that will be used in the rest of the paper.

Definition 22 ([26]). Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $a \in \mathcal{A}$ and $T_a\colon V_a \times V_a \to [0,1]$ be a tolerance relation. If $T_a(v,w) = 1$ implies $v = w$ then T_a is called a *separable tolerance relation*.

Notice that, the counterpart is verified for all tolerance relations since they are reflexive. As a consequence, a separable tolerance relation T_a satisfies that $T_a(v,w) = 1$ if and only if $v = w$.

Regarding to the notion of formula in the fuzzy setting, we must emphasize that this notion is the same as Definition 4. The reason is that the value v can be any element of an arbitrary set associated with the attribute a, providing a great level of flexibility, which is in consonance with the fuzzy environment. However, $\|\Phi\|_S$ has a different meaning.

Definition 23. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $B \subseteq \mathcal{A}_d$ and $T = \{T_a\colon V_a \times V_a \to [0,1] \mid a \in \mathcal{A}_d\}$ be a family of separable $[0,1]$-fuzzy tolerance relations. The mapping $\|\cdot\|^T_S\colon For(B) \to [0,1]^U$ is inductively defined as:

$$\|\Phi\|^T_S(x) = T_a(\bar{a}(x), v)$$

for all $x \in U$ and $\Phi = (a,v)$, where $a \in B$ and $v \in V_a$. For every $\Phi, \Psi \in For(B)$, the conjunction and disjunction of formulas are defined, for all $x \in U$, as follows:

$$\begin{aligned}\|\Phi \wedge \Psi\|^T_S(x) &= \inf\{\|\Phi\|^T_S(x), \|\Psi\|^T_S(x)\}\\ \|\Phi \vee \Psi\|^T_S(x) &= \sup\{\|\Phi\|^T_S(x), \|\Psi\|^T_S(x)\}\end{aligned}$$

Therefore, $\|\Phi\|^T_S(x)$ represents *how much the object x satisfies the formula* Φ, through the relationships between the value of the attribute a in the object x and the value of the attribute a in the formula Φ.

Notice that, the infimum and the supremum are the extensions of the intersection and the union of sets to the fuzzy environment, respectively. In this way, $\|\Phi \wedge \Psi\|^T_S(x)$ and $\|\Phi \vee \Psi\|^T_S(x)$ generalize the conjunction and disjunction of formulas in the classical environment given in Definition 4. In addition, Definition 23 generalizes Definition 4 when boolean separable tolerance relations are considered, as we show below.

Table 2
Table associated with the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 25.

	a_1	a_2	a_3	a_4	d
x_1	0.34	0.31	0.75	0.2	0
x_2	0.21	0.71	0.5	0.2	1
x_3	0.52	0.92	1	0.7	0
x_4	0.85	0.65	1	0.7	1
x_5	0.43	0.89	0.5	0.2	0
x_6	0.21	0.47	0.25	0.5	1
x_7	0.09	0.93	0.25	0.5	0

Proposition 24. *Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $B \subseteq \mathcal{A}_d$, $\Phi \in For(B)$ and $x \in U$. If $T = \{T_a : V_a \times V_a \to \{0, 1\} \mid a \in \mathcal{A}_d\}$ is a family of boolean separable tolerance relations then the following property holds:*

$$x \in \|\Phi\|_S \text{ if and only if } \|\Phi\|_S^T(x) = 1$$

Proof. We will consider an attribute–value pair $\Phi = (a, v)$ and T_a a boolean separable tolerance relation. The proof can be extended inductively to the conjunction or disjunction of attribute–value pairs. Suppose that $x \in \|\Phi\|_S = \|(a, v)\|_S$. Then, x satisfies the formula Φ, that is, $\bar{a}(x) = v$. As a consequence, since T_a is reflexive, we obtain that $T_a(\bar{a}(x), v) = 1$. Hence, $\|\Phi\|_S^T(x) = 1$.

The proof of the counterpart is completely analogous taking into account that T_a is a separable relation. □

On the other hand, the definition of decision rules in FRST is the same as Definition 5. An example is presented in order to show the notions of formula and decision rule in FRST.

Example 25. Consider the decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ where the set of objects is $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, the set of attributes is $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$ and $\mathcal{V}_a = [0, 1]$ for all $a \in \mathcal{A}_d$. This decision table is represented in Table 2. In this example, we will show the decision rules that can be obtained from these data considering a reduct instead of all the attributes $\mathcal{A}$. In addition, we will compute how much each object satisfies each antecedent and consequent of the decision rules, respectively.

First of all, we are interested in finding a fuzzy m-decision reduct to degree 1 in order to deal with the decision table S without considering all the attributes while all the information contained in S is preserved. With this purpose, we will use the $[0, 1]$-valued measure $m : \mathcal{P}(\mathcal{A}) \to [0, 1]$ defined for each $B \subseteq \mathcal{A}$ as

$$\begin{aligned} m(B) &= card_F(\mathrm{POS}_B^{\mathrm{f}}) \swarrow^{Ł} card_F(\mathrm{POS}_{\mathcal{A}}^{\mathrm{f}}) &(2)\\ &= \min\left\{1, 1 - \frac{card_F(\mathrm{POS}_{\mathcal{A}}^{\mathrm{f}})}{|U|} + \frac{card_F(\mathrm{POS}_B^{\mathrm{f}})}{|U|}\right\} &(3)\end{aligned}$$

Now, we fix the set $B = \{a_1, a_2, a_3\}$ in order to check if it is possible to discard the attribute a_4 without losing information. Next step is to compute $\mathrm{POS}_{\mathcal{A}}^{\mathrm{f}}(x)$ and $\mathrm{POS}_B^{\mathrm{f}}(x)$, for all $x \in U$. For this purpose, we consider the a-indiscernibility relation $R_a : U \times U \to [0, 1]$ given as:

$$R_a(x, y) = 1 - |\bar{a}(x) - \bar{a}(y)| \tag{4}$$

for all $a \in \mathcal{A}_d$ and $x, y \in U$. Furthermore, we will consider the following indiscernibility relations:

$$R_{\mathcal{A}}(x, y) = \frac{R_{a_1}(x, y) + 2(R_{a_2}(x, y) + R_{a_3}(x, y)) + R_{a_4}(x, y)}{6}$$

$$R_B(x, y) = \frac{R_{a_1}(x, y) + 2(R_{a_2}(x, y) + R_{a_3}(x, y)) + 1}{6}$$

for all $x, y \in U$. Moreover, we define a fuzzy implication in order to calculate the fuzzy positive region in the context $(U, U, R_{\mathcal{A}}, \tau)$. It will depend on the objects under consideration and on a value β given from the mapping $\tau : U \times U \to \{1, 3, 5\}$ defined for each $x_i, x_j \in U$ as:

$$\tau(x_i, x_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are even} \\ 3 & \text{if } i \text{ and } j \text{ are odd} \\ 5 & \text{otherwise} \end{cases}$$

We will consider the fuzzy implication in Eq. (1), that is, $\nwarrow_{Ł}^{\beta} : [0, 1] \times [0, 1] \to$ defined, for each $x_i, x_j \in U$, as:

$$x_i \nwarrow_{Ł}^{\beta} x_j = \sqrt[\beta]{\min\{1, 1 + x_i^{\beta} - x_j^{\beta}\}} \tag{5}$$

Table 3
Relation R_{a_1} of Example 25.

R_{a_1}	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	0.87	0.82	0.49	0.91	0.87	0.75
x_2	0.87	1	0.69	0.36	0.78	1	0.88
x_3	0.82	0.69	1	0.67	0.91	0.69	0.57
x_4	0.49	0.36	0.67	1	0.58	0.36	0.24
x_5	0.91	0.78	0.91	0.58	1	0.78	0.66
x_6	0.87	1	0.69	0.36	0.78	1	0.88
x_7	0.75	0.88	0.57	0.24	0.66	0.88	1

Table 4
Relation R_{a_2} of Example 25.

R_{a_2}	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	0.6	0.39	0.66	0.42	0.84	0.38
x_2	0.6	1	0.79	0.94	0.82	0.76	0.78
x_3	0.39	0.79	1	0.73	0.97	0.55	0.99
x_4	0.66	0.94	0.73	1	0.76	0.82	0.72
x_5	0.42	0.82	0.97	0.76	1	0.58	0.96
x_6	0.84	0.76	0.55	0.82	0.58	1	0.54
x_7	0.38	0.78	0.99	0.72	0.96	0.54	1

Table 5
Relation R_{a_3} of Example 25.

R_{a_3}	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	0.75	0.75	0.75	0.75	0.5	0.5
x_2	0.75	1	0.5	0.5	1	0.75	0.75
x_3	0.75	0.5	1	1	0.5	0.25	0.25
x_4	0.75	0.5	1	1	0.5	0.25	0.25
x_5	0.75	1	0.5	0.5	1	0.75	0.75
x_6	0.5	0.75	0.25	0.25	0.75	1	1
x_7	0.5	0.75	0.25	0.25	0.75	1	1

Table 6
Relation R_{a_4} of Example 25.

R_{a_4}	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	1	0.5	0.5	1	0.7	0.7
x_2	1	1	0.5	0.5	1	0.7	0.7
x_3	0.5	0.5	1	1	0.5	0.8	0.8
x_4	0.5	0.5	1	1	0.5	0.8	0.8
x_5	1	1	0.5	0.5	1	0.7	0.7
x_6	0.7	0.7	0.8	0.8	0.7	1	1
x_7	0.7	0.7	0.8	0.8	0.7	1	1

with $\tau(x_i, x_j) = \beta$. Notice that, the case $\beta = 1$ corresponds to the Łukasiewicz implication. The relations $R_{a_1}, R_{a_2}, R_{a_3}, R_{a_4}$, R_B and $R_{\mathcal{A}}$ given in Tables 3–8 will be needed to compute $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x)$ and $\mathrm{POS}^{\mathrm{f}}_{B}(x)$, for all $x \in U$. In order to illustrate the computations, we include the following examples:

$$\begin{aligned}
R_{a_1}(x_1, x_2) &= 1 - |\bar{a}_1(x_1) - \bar{a}_1(x_2)| = 1 - |0.34 - 0.21| = 0.87 \\
R_{\mathcal{A}}(x_1, x_2) &= \frac{R_{a_1}(x_1, x_2) + 2(R_{a_2}(x_1, x_2) + R_{a_3}(x_1, x_2)) + R_{a_4}(x, y)}{6} \\
&= \frac{0.87 + 2(0.6 + 0.75) + 1}{6} \\
&= \frac{4.57}{6} \approx 0.76
\end{aligned}$$

On the other hand, since d is a boolean attribute, the tolerance relation $R_d(x, y) = 1 - |\bar{d}(x) - \bar{d}(y)|$ is given as follows:

$$R_d(x, y) = \begin{cases} 1 & \text{if } \bar{d}(x) = \bar{d}(y) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Table 7
R_B of Example 25.

R_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	0.76	0.68	0.72	0.71	0.76	0.59
x_2	0.76	1	0.71	0.71	0.9	0.84	0.82
x_3	0.68	0.71	1	0.86	0.81	0.55	0.68
x_4	0.72	0.71	0.86	1	0.68	0.58	0.53
x_5	0.71	0.9	0.81	0.68	1	0.74	0.85
x_6	0.76	0.84	0.55	0.58	0.74	1	0.83
x_7	0.59	0.82	0.68	0.53	0.85	0.83	1

Table 8
$R_{\mathcal{A}}$ of Example 25.

$R_{\mathcal{A}}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	0.76	0.6	0.64	0.71	0.71	0.54
x_2	0.76	1	0.63	0.62	0.9	0.79	0.77
x_3	0.6	0.63	1	0.86	0.73	0.52	0.64
x_4	0.64	0.62	0.86	1	0.6	0.55	0.5
x_5	0.71	0.9	0.73	0.6	1	0.69	0.8
x_6	0.71	0.79	0.52	0.55	0.69	1	0.83
x_7	0.54	0.77	0.64	0.5	0.8	0.83	1

Table 9
$\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}$ of Example 25.

	$\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}$
x_1	0.94
x_2	0.83
x_3	0.89
x_4	0.89
x_5	0.83
x_6	0.91
x_7	0.91

Finally, Tables 7 and 8 show the relations R_B and $R_{\mathcal{A}}$. From Tables 7 and 8 the multi-adjoint fuzzy B-positive region and $\mathcal{A}$-positive region are computed, obtaining that $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = \mathrm{POS}^{\mathrm{f}}_{B}(x)$, for all $x \in U$. These values are shown in Table 9. As a consequence, it is immediately obtained that $m(B) = 1$.

In a similar way it can be computed that $m(\{a_1, a_2\}) = 0.97$, $m(\{a_1, a_3\}) = 0.86$ and $m(\{a_2, a_3\}) = 0.98$. Therefore, by Definition 20, B is a fuzzy m-decision reduct to degree 1. In this way, we can discard the attribute a_4 without losing information. Thus, we will only consider the attributes a_1, a_2 and a_3, and by using the conjunction of formulas, to compute the set of decision rules which will describe the decision table of Table 2. In this way, one decision rule is obtained for each object. For instance, for the object x_1 we obtain the formulas $(a_1, 0.34)$, $(a_2, 0.31)$ and $(a_3, 0.75)$ considering each attribute of $\mathcal{A}$. By using the conjunction of these pairs we obtain the formula $\Phi_1 = (a_1, 0.34) \wedge (a_2, 0.31) \wedge (a_3, 0.75)$. On the other hand, considering the attribute d we obtain the formula $\Psi_1 = (d, 0)$. As a consequence, the decision rule $r_1 = \Phi_1 \to \Psi_1$ associated with the object x_1 is $r_1 : (a_1, 0.34) \wedge (a_2, 0.31) \wedge (a_3, 0.75) \to (d, 0)$. Following an analogous reasoning next decision rules are obtained.

$$
\begin{aligned}
r_1 &: (a_1, 0.34) \wedge (a_2, 0.31) \wedge (a_3, 0.75) \to (d, 0)\\
r_2 &: (a_1, 0.21) \wedge (a_2, 0.71) \wedge (a_3, 0.5) \to (d, 1)\\
r_3 &: (a_1, 0.52) \wedge (a_2, 0.92) \wedge (a_3, 1) \to (d, 0)\\
r_4 &: (a_1, 0.85) \wedge (a_2, 0.65) \wedge (a_3, 1) \to (d, 1)\\
r_5 &: (a_1, 0.43) \wedge (a_2, 0.89) \wedge (a_3, 0.5) \to (d, 0)\\
r_6 &: (a_1, 0.21) \wedge (a_2, 0.47) \wedge (a_3, 0.25) \to (d, 1)\\
r_7 &: (a_1, 0.09) \wedge (a_2, 0.93) \wedge (a_3, 0.25) \to (d, 0)
\end{aligned}
$$

Now, we compute the degree of satisfaction to each antecedent for each object, that is, $\|\Phi_i\|^T_S(x_j)$ with $i, j \in \{1, 2, \ldots, 7\}$. For this purpose, we use the family of separable $[0, 1]$-fuzzy tolerance relation $T = \{T_a : V_a \times V_a \to [0, 1] \mid a \in \mathcal{A}_d\}$ defined as $T_a(\bar{a}(x), v) = 1 - |\bar{a}(x) - v|$, for all $a \in \mathcal{A}_d$, $x \in U$ and $v \in [0, 1]$. Notice that, $\|\Phi_i\|^T_S(x_j) = \|\Psi_i\|^T_S(x_j) = 1$ if $i = j$, since each object generates the decision rule with the same subscript. The computation of $\|\Phi_2\|^T_S(x_1)$ is shown for a non-trivial example.

$$
\begin{aligned}
\|\Phi_2\|^T_S(x_1) &= \|(a_1, 0.21) \wedge (a_2, 0.71) \wedge (a_3, 0.5)\|^T_S(x_1)\\
&= \|(a_1, 0.21)\|^T_S(x_1) \wedge \|(a_2, 0.71)\|^T_S(x_1) \wedge \|(a_3, 0.5)\|^T_S(x_1)
\end{aligned}
$$

Table 10
Degree of satisfaction to each antecedent of decision rules for each object.

	$\|\Phi_1\|_S^T$	$\|\Phi_2\|_S^T$	$\|\Phi_3\|_S^T$	$\|\Phi_4\|_S^T$	$\|\Phi_5\|_S^T$	$\|\Phi_6\|_S^T$	$\|\Phi_7\|_S^T$
x_1	1	0.6	0.39	0.49	0.42	0.5	0.38
x_2	0.6	1	0.5	0.36	0.78	0.75	0.75
x_3	0.39	0.5	1	0.67	0.5	0.25	0.25
x_4	0.49	0.36	0.67	1	0.5	0.25	0.24
x_5	0.42	0.78	0.5	0.5	1	0.58	0.66
x_6	0.5	0.75	0.25	0.25	0.58	1	0.54
x_7	0.38	0.75	0.25	0.24	0.66	0.54	1

$$
\begin{aligned}
&= (1 - |\overline{a_1}(x_1) - 0.21|) \wedge (1 - |\overline{a_2}(x_1) - 0.71|) \wedge (1 - |\overline{a_3}(x_1) - 0.5|)\\
&= (1 - |0.34 - 0.21|) \wedge (1 - |0.31 - 0.71|) \wedge (1 - |0.75 - 0.5|)\\
&= 0.87 \wedge 0.6 \wedge 0.75 = 0.6
\end{aligned}
$$

Following an analogous procedure, the rest of calculations are obtained. They are shown in Table 10.

On the other hand, we compute $\|\Psi_i\|_S^T(x_j) = \|(d, v_i)\|_S^T(x_j)$ with $i, j \in \{1, 2, \ldots, 7\}$. In this case, since d is a boolean attribute the calculations are easier, obtaining

$$
\|\Psi_i\|_S^T(x_j) = \begin{cases} 1 & \text{if } \bar{d}(x_j) = v_i \\ 0 & \text{otherwise} \end{cases} \quad \square \tag{7}
$$

Now, we present the notions of support, strength, certainty and coverage in the fuzzy environment, in order to describe decision rules in this framework. These notions generalize the classical ones, so they have an analogous interpretation.

Definition 26. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\Phi \to \Psi$ be a decision rule in S and $T = \{T_a : V_a \times V_a \to [0, 1] \mid a \in \mathcal{A}_d\}$ be a family of separable [0, 1]-fuzzy tolerance relations. We call:

- T-*support* of the decision rule $\Phi \to \Psi$ to the value:

$$supp_S^T(\Phi, \Psi) = card_F(\|\Phi \wedge \Psi\|_S^T)$$

- T-*strength* of the decision rule $\Phi \to \Psi$ to the value:

$$\sigma_S^T(\Phi, \Psi) = \frac{supp_S^T(\Phi, \Psi)}{|U|}$$

- T-*certainty* of the decision rule $\Phi \to \Psi$ to the value:

$$cer_S^T(\Phi, \Psi) = \frac{supp_S^T(\Phi, \Psi)}{card_F(\|\Phi\|_S^T)}$$

  when $card_F(\|\Phi\|_S^T) \neq 0$.
- T-*coverage* of the decision rule $\Phi \to \Psi$ to the value:

$$cov_S^T(\Phi, \Psi) = \frac{supp_S^T(\Phi, \Psi)}{card_F(\|\Psi\|_S^T)}$$

  when $card_F(\|\Psi\|_S^T) \neq 0$.

Notice that, $cer_S^T(\Phi, \Psi) \leq 1$ for each $\Phi \to \Psi$ in S since $\|\Phi \wedge \Psi\|_S^T(x) = \|\Phi\|_S^T(x) \wedge \|\Psi\|_S^T(x) \leq \|\Phi\|_S^T(x)$ for all $x \in U$. In the same way, it can be concluded that $\sigma_S^T(\Phi, \Psi), cov_S^T(\Phi, \Psi) \leq 1$. Analogously to the classical environment, we characterize decision rules depending on the value of its T-certainty.

Definition 27. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\Phi \to \Psi$ be a decision rule in S and $T = \{T_a : V_a \times V_a \to [0, 1] \mid a \in \mathcal{A}_d\}$ be a family of separable [0, 1]-fuzzy tolerance relations. If $cer_S^T(\Phi, \Psi) = 1$ we will say that $\Phi \to \Psi$ is a T-*true decision rule*. If $cer_S^T(\Phi, \Psi) = 0$, we say that the decision rule is T-*false*. Otherwise, it will be called a *not entirely* T-*true decision rule*.

Now, we prove that these notions generalize to the classical ones.

Proposition 28. *Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $\Phi \to \Psi$ be a decision rule in S. If $T = \{T_a : V_a \times V_a \to \{0, 1\} \mid a \in \mathcal{A}_d\}$ is a family of boolean separable tolerance relations then*

$$supp_S^T(\Phi, \Psi) = supp_S(\Phi, \Psi)$$

$$\begin{aligned}
\sigma_S^T(\Phi,\Psi) &= \sigma_S(\Phi,\Psi)\\
cer_S^T(\Phi,\Psi) &= cer_S(\Phi,\Psi)\\
cov_S^T(\Phi,\Psi) &= cov_S(\Phi,\Psi)
\end{aligned}$$

Proof. Let $B \subseteq \mathcal{A}_d$. Since it is well-known that $card_F$ generalizes the classical definition of cardinality, we have that $|\|\Phi\|_S| = card_F(\|\Phi\|_S^T)$ with $\Phi \in For(B)$. Thus, because of the notions of T-support, T-strength, T-certainty and T-coverage are based on the cardinal of fuzzy sets we obtain that these notions generalize to the classical ones. □

These notions are illustrated in the following example.

Example 29. Coming back to the environment of Example 25, we compute the T-support, T-strength, T-certainty and T-coverage of each decision rule by using Table 10, Eq. (7) and Definition 26. We will only show some calculus since the rest are analogous.

First of all, we will calculate $supp_S^T(\Phi_1,\Psi_1)$.

$$\begin{aligned}
supp_S^T(\Phi_1,\Psi_1) &= card_F(\|\Phi_1\wedge\Psi_1\|_S^T)\\
&= \sum_{x\in U}\|\Phi_1\wedge\Psi_1\|_S^T(x)\\
&= \sum_{x\in U}\left(\|\Phi_1\|_S^T(x)\wedge\|\Psi_1\|_S^T(x)\right)\\
&= \|\Phi_1\|_S^T(x_1)\wedge\|\Psi_1\|_S^T(x_1)+\|\Phi_1\|_S^T(x_2)\wedge\|\Psi_1\|_S^T(x_2)\\
&\quad+\|\Phi_1\|_S^T(x_3)\wedge\|\Psi_1\|_S^T(x_3)+\|\Phi_1\|_S^T(x_4)\wedge\|\Psi_1\|_S^T(x_4)\\
&\quad+\|\Phi_1\|_S^T(x_5)\wedge\|\Psi_1\|_S^T(x_5)+\|\Phi_1\|_S^T(x_6)\wedge\|\Psi_1\|_S^T(x_6)\\
&\quad+\|\Phi_1\|_S^T(x_7)\wedge\|\Psi_1\|_S^T(x_7)\\
&= 1\wedge 1+0.6\wedge 0+0.39\wedge 1+0.49\wedge 0+0.42\wedge 1+0.5\wedge 0+0.38\wedge 1\\
&= 1+0+0.39+0+0.42+0+0.38\\
&= 2.19
\end{aligned}$$

Hence, the T-strength of the decision rule $\Phi_1 \to \Psi_1$ is

$$\sigma_S^T(\Phi_1,\Psi_1) = \frac{supp_S^T(\Phi_1,\Psi_1)}{|U|} = \frac{2.19}{7} \approx 0.31$$

In order to compute the T-certainty of this decision rule we calculate $card_F(\|\Phi_1\|_S^T)$.

$$\begin{aligned}
card_F(\|\Phi_1\|_S^T) &= \sum_{x\in U}\|\Phi_1\|_S^T(x) = \|\Phi_1\|_S^T(x_1)+\|\Phi_1\|_S^T(x_2)+\|\Phi_1\|_S^T(x_3)\\
&\quad+\|\Phi_1\|_S^T(x_4)+\|\Phi_1\|_S^T(x_5)+\|\Phi_1\|_S^T(x_6)+\|\Phi_1\|_S^T(x_7)\\
&= 1+0.6+0.39+0.49+0.42+0.5+0.38\\
&= 3.78
\end{aligned}$$

As a consequence,

$$cer_S^T(\Phi_1,\Psi_1) = \frac{supp_S^T(\Phi_1,\Psi_1)}{card_F(\|\Phi_1\|_S^T)} = \frac{2.19}{3.78} \approx 0.58$$

Now, we calculate the T-coverage of the decision rule $\Phi_1 \to \Psi_1$. With this purpose, we compute $card_F(\|\Psi_1\|_S^T)$.

$$\begin{aligned}
card_F(\|\Psi_1\|_S^T) &= \sum_{x\in U}\|\Psi_1\|_S^T(x) = \|\Psi_1\|_S^T(x_1)+\|\Psi_1\|_S^T(x_2)+\|\Psi_1\|_S^T(x_3)\\
&\quad+\|\Psi_1\|_S^T(x_4)+\|\Psi_1\|_S^T(x_5)+\|\Psi_1\|_S^T(x_6)+\|\Psi_1\|_S^T(x_7)\\
&= 1+0+1+0+1+0+1\\
&= 4
\end{aligned}$$

Hence,

$$cov_S^T(\Phi_1,\Psi_1) = \frac{supp_S^T(\Phi_1,\Psi_1)}{card_F(\|\Psi_1\|_S^T)} = \frac{2.19}{4} \approx 0.55$$

The other indicators are calculated analogously and they are presented in Table 11. □

Table 11
T-support, T-strength, T-certainty and T-coverage of the decision rules of Example 25.

Rule	$supp_S^T$	σ_S^T	cer_S^T	cov_S^T
r_1	2.19	0.31	0.58	0.55
r_2	2.11	0.3	0.45	0.7
r_3	2.14	0.31	0.6	0.54
r_4	1.61	0.23	0.58	0.54
r_5	2.58	0.37	0.58	0.65
r_6	2	0.29	0.52	0.67
r_7	2.29	0.33	0.6	0.57

Table 12
Relation between each pair of antecedents of decision rules of Example 25.

R_{Fd}	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7
Φ_1	1	0.6	0.39	0.49	0.42	0.5	0.38
Φ_2	0.6	1	0.5	0.36	0.78	0.75	0.75
Φ_3	0.39	0.5	1	0.67	0.5	0.25	0.25
Φ_4	0.49	0.36	0.67	1	0.5	0.25	0.24
Φ_5	0.42	0.78	0.5	0.5	1	0.58	0.66
Φ_6	0.5	0.75	0.25	0.25	0.58	1	0.54
Φ_7	0.38	0.75	0.25	0.24	0.66	0.54	1

4. Decision algorithms in fuzzy rough set theory

This section is devoted to the study of decision algorithms in the fuzzy environment. First of all, it is necessary to compare formulas. These will be conjunction of attribute–value pairs because it is the usual form of the antecedent of decision rules. With this purpose, a fuzzy tolerance relation is introduced below.

Definition 30. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $T = \{T_a : V_a \times V_a \to [0,1] \mid a \in \mathcal{A}_d\}$ be a family of separable $[0,1]$-fuzzy tolerance relations and $\Phi, \Phi' \in For(\mathcal{A}_d)$, with $\Phi = (a_1, v_1) \wedge \cdots \wedge (a_n, v_n)$ and $\Phi' = (a'_1, w_1) \wedge \cdots \wedge (a'_m, w_m)$. The *F-indiscernibility relation* is a $[0,1]$-fuzzy tolerance relation $R_{Fd} : For(\mathcal{A}_d) \times For(\mathcal{A}_d) \to [0,1]$ given as

$$R_{Fd}(\Phi, \Phi') = \begin{cases} \bigwedge\limits_{i\in\{1,\ldots,n\}} T_{a_i}(v_i, w_i) & \text{if } n = m \text{ and } a_i = a'_i \text{ for all } i \in \{1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$$

Given $\alpha \in [0,1]$, the $R_{Fd} - \alpha$*-block* is defined for each $\Phi \in For(\mathcal{A}_d)$ as:

$$[\Phi]_\alpha = \{\Phi' \in For(\mathcal{A}_d) \mid \alpha \le R_{Fd}(\Phi, \Phi')\}$$

If $\Phi' \in [\Phi]_\alpha$ then we will say that Φ and Φ' are $R_{Fd} - \alpha$*-related*.

It is important to emphasize that R_{Fd} is not a transitive fuzzy tolerance relation, in general, this is why we have tolerance blocks instead of equivalence classes. In addition, notice that given a pair of formulas Φ, Φ' and a fuzzy tolerance relation R_{Fd}, the value α determines if each formula belongs to the $R_{Fd} - \alpha$-block of the other formula and vice versa.

In the following example, we will compare each pair of antecedents of decision rules of Example 25 by using a F-indiscernibility relation R_{Fd}.

Example 31. We will compute $R_{Fd}(\Phi, \Phi')$ for each Φ, Φ' antecedents of decision rules of Example 25. With this purpose, we fix the $[0,1]$-fuzzy tolerance relation $T_a : V_a \times V_a \to [0,1]$ defined as $T_a(v, w) = 1 - |v - w|$ for all $a \in \mathcal{A}$. In particular, we illustrate the computation of $R_{Fd}(\Phi_1, \Phi_2)$. This is made by using Table 2 and Definition 30.

$$\begin{aligned} R_{Fd}(\Phi_1, \Phi_2) &= \bigwedge_{i\in\{1,2,3\}} T_{a_i}(v_i, w_i) \\ &= T_{a_1}(0.34, 0.21) \wedge T_{a_2}(0.31, 0.71) \wedge T_{a_3}(0.75, 0.5) \\ &= 0.87 \wedge 0.6 \wedge 0.75 = 0.6 \end{aligned}$$

The rest of results are shown in Table 12. □

Before defining the notion of decision algorithm in the fuzzy framework it is necessary to introduce the following definition.

Definition 32. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\alpha \in (0,1]$, $R_{Fd} : For(\mathcal{A}_d) \times For(\mathcal{A}_d) \to [0,1]$ be a F-indiscernibility relation and $Dec(S) = \{\Phi_i \to \Psi_i\}_{i\in I}$ be a set of decision rules of S, where the index set is $I = \{1,\ldots,m\}$

and $m \geq 2$. The set of *α-consistent decision rules* is defined as follows:

$$Dec^+_\alpha(S) = \{\Phi \to \Psi \in Dec(S) \mid \text{if for each } \Phi' \to \Psi' \in Dec(S) \text{ such that } \Phi' \in [\Phi]_\alpha \text{ then } \Psi' \in [\Psi]_\alpha\}$$

Notice that, this set does not contain contradictory decision rules, that is, this set is composed of decision rules that, if their antecedents are $R_{Fd} - \alpha$-related then their consequents are also $R_{Fd} - \alpha$-related. Therefore, it is not possible to find decision rules in $Dec^+_\alpha(S)$ whose antecedents are $R_{Fd} - \alpha$-related but their consequents not.

Finally, it is important to emphasize that we have not defined $Dec^+_\alpha(S)$ as the set of T-true decision rules because, by Definition 26, given $\Phi \to \Psi \in Dec(S)$, we have that $cer^T_S(\Phi, \Psi) = 1$ if and only if $\|\Phi\|_S(x) \leq \|\Psi\|_S(x)$ for all $x \in U$. Hence, in order to avoid a restrictive condition we have adapted the notion of $Dec^+(S)$ to obtain interesting results.

Now, we present the notion of decision algorithm in the fuzzy setting.

Definition 33. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\alpha \in (0, 1]$, $\alpha_1, \alpha_2, \alpha_4 \in [0, 1)$,[1] $\alpha_3 > 0$, $T = \{T_a : V_a \times V_a \to [0, 1] \mid a \in \mathcal{A}_d\}$ be a family of separable tolerance relations and $Dec(S) = \{\Phi_i \to \Psi_i\}_{i \in I}$ be a set of decision rules of S, where the index set is $I = \{1, \dots, m\}$ and $m \geq 2$. We say that:

1. $Dec(S)$ is a set of *$\alpha_1\alpha_2$-pairwise mutually exclusive (independent)* decision rules, if each pair of decision rules $\Phi \to \Psi, \Phi' \to \Psi' \in Dec(S)$ satisfies that $\Phi = \Phi'$ or $\|\Phi \wedge \Phi'\|^T_S(x) \leq \alpha_1$ and $\Psi = \Psi'$ or $\|\Psi \wedge \Psi'\|^T_S(x) \leq \alpha_2$, for all $x \in U$.
2. $Dec(S)$ *covers* U, if $card_F(\|\bigvee_{i=1}^{m} \Phi_i\|^T_S) = card_F(\|\bigvee_{i=1}^{m} \Psi_i\|^T_S) = |U|$.
3. The decision rule $\Phi \to \Psi \in Dec(S)$ is *α_3-admissible in S* if $\alpha_3 < supp^T_S(\Phi, \Psi)$.
4. $Dec(S)$ *preserves the α-consistency of S with a degree α_4* if the next inequality holds for all $x \in U$:

$$\Big|\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) - \bigvee_{\Phi \to \Psi \in Dec^+_\alpha(S)} \|\Phi\|^T_S(x)\Big| \leq \alpha_4$$

The set of decision rules $Dec(S)$ satisfying the previous properties for the values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is called $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-*decision algorithm in S* and it is denoted as $DA_T(S)$.

In the previous definition only α affects α_4, however, since the value α is a 'threshold' that the user must fix from the beginning, it should be known before defining the decision algorithm, and so it appears as subindex of the tuple of thresholds $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

It is important to emphasize that the set $Dec(S)$ in which each decision rule is extracted from each object of the decision table S is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm, taking appropriate values of each α_i, with $i \in \{1, 2, 3, 4\}$ as we will show in Example 38. Each value has a different role. The value α_1 sets a bound to discern each pair of antecedents of the decision rules of $Dec(S)$, except the cases in which they are equals. The interpretation of α_2 is the same considering the consequents instead of the antecedents. Moreover, α_3 ensures a minimum value of satisfaction for all decision rules in $Dec(S)$ taking into account all the objects, which provides a meaningful description of the set of decision rules. Finally, α_4 is established to compare two important notions involved in the study of dependence between attributes, the fuzzy positive region and the decision rules of Dec^+_α.

In order to obtain a relationship between the sets $Dec^+_\alpha(S)$ and $Dec^+(S)$, we introduce the following notion.

Definition 34. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $DA_T(S)$ be a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm. If for each $x \in U$ there exists $\Phi \to \Psi \in DA_T(S)$ such that $\|\Phi \wedge \Psi\|^T_S(x) = 1$ then $DA_T(S)$ is called a *complete* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-*decision algorithm.*

Considering complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithms, it is possible to relate $Dec^+_\alpha(S)$ of Definition 33 and the set $Dec^+(S)$ of Definition 7, as the following result reveals.

Proposition 35. *Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $T = \{T_a : V_a \times V_a \to \{0, 1\} \mid a \in \mathcal{A}_d\}$ be a family of boolean separable tolerance relations. If $\alpha > 0$ and $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm then*

$$Dec^+_\alpha(S) = Dec^+(S)$$

Proof. To begin with, notice that if each T_a tolerance relation of the family T is boolean, then R_{Fd} is also a boolean tolerance relation. In addition, since $\alpha > 0$ we obtain that

$$\begin{aligned}[\Phi]_\alpha &= \{\Phi' \mid \alpha \leq R_{Fd}(\Phi, \Phi'), \Phi' \to \Psi' \in DA_T(S)\} \\ &= \{\Phi' \mid R_{Fd}(\Phi, \Phi') = 1, \Phi' \to \Psi' \in DA_T(S)\} \\ &= \{\Phi' \mid \Phi' = \Phi, \Phi' \to \Psi' \in DA_T(S)\}\end{aligned}$$

[1] The value 1 is not taken into account since these cases are always satisfied and they are valueless.

Hence, $\Phi' \in [\Phi]_\alpha$ if and only if $\Phi = \Phi'$. Therefore, under these hypotheses we obtain that

$$Dec^+_\alpha(S) = \{\Phi \to \Psi \in DA_T(S) \mid \text{if for each } \Phi' \to \Psi' \in DA_T(S) \text{ such that } \Phi' = \Phi \text{ then } \Psi' = \Psi\}$$

First of all, we prove that $Dec^+_\alpha(S) \subseteq Dec^+(S)$. Let $\Phi \to \Psi \in Dec^+_\alpha(S)$. By reductio ad absurdum, we suppose that $\Phi \to \Psi \notin Dec^+(S)$. Then $cer^T_S(\Phi, \Psi) < 1$. Since T is a family of boolean separable tolerance relations, by Proposition 28, we obtain $cer^T_S(\Phi, \Psi) = cer_S(\Phi, \Psi)$. Therefore,

$$\frac{\sum_{x\in U} \|\Phi \wedge \Psi\|^T_S(x)}{\sum_{x\in U} \|\Phi\|^T_S(x)} = \frac{supp^T_S(\Phi, \Psi)}{card_F(\|\Phi\|^T_S)} = cer^T_S(\Phi, \Psi) = cer_S(\Phi, \Psi) < 1$$

As a consequence, there exists $x \in U$ such that $\|\Phi \wedge \Psi\|^T_S(x) < \|\Phi\|^T_S(x)$. Taking into account that T is a family of boolean separable tolerance relations, $\|\Phi\|^T_S(x) = 1$ and $\|\Psi\|^T_S(x) = 0$.

On the other hand, since $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm, there exists $\Phi' \to \Psi' \in DA_T(S)$ such that $\|\Phi' \wedge \Psi'\|^T_S(x) = 1$. Hence, $\|\Phi\|^T_S(x) = \|\Phi'\|^T_S(x) = \|\Phi \wedge \Phi'\|^T_S(x) = 1$. As a result, taking into account that $\alpha_1 < 1$ it is deduced that $\alpha_1 < \|\Phi \wedge \Phi'\|^T_S(x)$, so $\Phi = \Phi'$. In addition, $\|\Psi'\|^T_S(x) = 1$ and $\|\Psi \wedge \Psi'\|^T_S(x) = 0$. Therefore, it must be $\Psi \neq \Psi'$. This fact leads to a contradiction, since $\Phi \to \Psi \in Dec^+_\alpha(S)$, so $\Phi \to \Psi \in Dec^+(S)$. Therefore, we can conclude that $Dec^+_\alpha(S) \subseteq Dec^+(S)$.

Now, we prove that $Dec^+(S) \subseteq Dec^+_\alpha(S)$. Let $\Phi \to \Psi \in Dec^+(S)$. Then $cer_S(\Phi, \Psi) = 1$ and by Proposition 28, $cer^T_S(\Phi, \Psi) = 1$. By reductio ad absurdum, we suppose that $\Phi \to \Psi \notin Dec^+_\alpha(S)$. Then, there exists $\Phi \to \Psi' \in DA_T(S)$ such that $\Psi \neq \Psi'$. Since $DA_T(S)$ is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm, it is obtained that $supp^T_S(\Phi, \Psi), supp^T_S(\Phi, \Psi') > \alpha_3 \geq 0$. In particular, since each mapping of T is boolean and separable, $supp^T_S(\Phi, \Psi), supp^T_S(\Phi, \Psi') \geq 1$. Therefore, there exists $x \in U$ such that $\|\Phi \wedge \Psi'\|^T_S(x) = 1$. Taking into account that $\Psi \neq \Psi'$, it is concluded that $\|\Phi \wedge \Psi\|^T_S(x) = 0$. Hence,

$$cer^T_S(\Phi, \Psi) = \frac{supp^T_S(\Phi, \Psi)}{card_F(\|\Phi\|^T_S)} = \frac{\sum_{x\in U} \|\Phi \wedge \Psi\|^T_S(x)}{\sum_{x\in U} \|\Phi\|^T_S(x)} < 1$$

leading to a contradiction. As a consequence, $\Phi \to \Psi \in Dec^+_\alpha(S)$, and $Dec^+_\alpha(S) = Dec^+(S)$. □

From this result, we can relate the set $Dec^+_\alpha(S)$ to the set of T-true decision rules as it is shown below.

Corollary 36. *Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $T = \{T_a : V_a \times V_a \to \{0, 1\} \mid a \in \mathcal{A}_d\}$ be a family of boolean separable tolerance relations. If $\alpha > 0$ and $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm then*

$$Dec^+_\alpha(S) = \{\Phi \to \Psi \in DA_T(S) \mid cer^T_S(\Phi, \Psi) = 1\}$$

Proof. It is immediately obtained, by using Propositions 28 and 35

$$\begin{aligned} Dec^+_\alpha(S) &= Dec^+(S) \\ &= \{\Phi \to \Psi \in DA_T(S) \mid cer_S(\Phi, \Psi) = 1\} \\ &= \{\Phi \to \Psi \in DA_T(S) \mid cer^T_S(\Phi, \Psi) = 1\} \quad \square \end{aligned}$$

Hence, when boolean and separable relations are considered, the set $Dec^+_\alpha(S)$ coincides with the set of T-true decision rules.

Finally, we show that complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithms generalize complete decision algorithms.

Proposition 37. *Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $T = \{T_a : V_a \times V_a \to \{0, 1\} \mid a \in \mathcal{A}_d\}$ be a family of boolean separable tolerance relations and R_A and R_a be boolean separable tolerance relations for all $a \in \mathcal{A}_d$. Given $DA_T(S) = \{\Phi_i \to \Psi_i\}_{i\in I}$, where the index set is $I = \{1, \dots, m\}$ and $m \geq 2$, $\alpha \in (0, 1]$, $\alpha_1, \alpha_2, \alpha_4 \in [0, 1)$ and $0 \leq \alpha_3 < \min\{supp^T_S(\Phi, \Psi) \mid \Phi \to \Psi \in DA_T(S)\}$, then $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm if and only if it is a complete decision algorithm.*

Proof. Suppose that $DA_T(S)$ is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm. We will prove the equivalence between each item of Definitions 7 and 33.

1. Let $\Phi \to \Psi, \Phi' \to \Psi' \in DA_T(S)$ with $\Phi \neq \Phi'$. If $\|\Phi \wedge \Phi'\|^T_S(x) \leq \alpha_1 < 1$ for all $x \in U$, taking into account that T is a family of boolean separable tolerance relations, it will be $\|\Phi \wedge \Phi'\|^T_S(x) = 0$ for all $x \in U$. Then, by Proposition 24, $x \notin \|\Phi \wedge \Phi'\|_S$ for all $x \in U$. As a consequence, $\|\Phi \wedge \Phi'\|_S = \varnothing$. Analogously, if $\|\Psi \wedge \Psi'\|^T_S(x) \leq \alpha_2 < 1$ for all $x \in U$ then $\|\Psi \wedge \Psi'\|_S = \varnothing$.

2. We suppose that $card_F(\|\bigvee_{i=1}^{m}\Phi_i\|_S^T) = |U|$. By the definition of cardinal of a fuzzy set and the disjunction of formulas we obtain

$$card_F(\|\bigvee_{i=1}^{m}\Phi_i\|_S^T) = \sum_{x\in U}\|\bigvee_{i=1}^{m}\Phi_i\|_S^T(x) = \sum_{x\in U}\bigvee_{i=1}^{m}\|\Phi_i\|_S^T(x) = |U|$$

Since $0 \leq \bigvee_{i=1}^{m}\|\Phi_i\|_S^T(x) \leq 1$ and that addition has $|U|$ addends, it is obtained that $\bigvee_{i=1}^{m}\|\Phi_i\|_S^T(x) = 1$, for all $x \in U$. Therefore, for each $x \in U$, there exists $k \in \{1, 2, \dots, m\}$ such that $\|\Phi_k\|_S^T(x) = 1$. Hence, by Proposition 24, $x \in \|\Phi_k\|_S \subseteq \|\bigvee_{i=1}^{m}\Phi_i\|_S$. Consequently, $U \subseteq \|\bigvee_{i=1}^{m}\Phi_i\|_S$. The other inclusion is immediately obtained because of Definition 4. It can be proved analogously that $card_F(\|\bigvee_{i=1}^{m}\Psi_i\|_S^T) = |U|$ implies that $\|\bigvee_{i=1}^{m}\Psi_i\|_S = U$.

3. The third item is obtained immediately since, by Proposition 28, $supp_S^T(\Phi, \Psi) = supp_S(\Phi, \Psi)$ for each decision rule $\Phi \to \Psi$ when T is a family of boolean separable tolerance relations.

4. For the fourth item notice that $Dec_\alpha^+(S) = Dec^+(S)$ by Proposition 35. Since $\alpha_4 < 1$ and $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x)$, $\bigvee_{\Phi\to\Psi\in Dec_\alpha^+(S)}\|\Phi\|_S^T(x) \in \{0, 1\}$ for all $x \in U$, we obtain that

$$\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = \bigvee_{\Phi\to\Psi\in Dec_\alpha^+(S)}\|\Phi\|_S^T(x)$$

for all $x \in U$. Now, we will prove that $\mathrm{POS}_{\mathcal{A}}(\{d\}) = \bigvee_{\Phi\to\Psi\in Dec^+(S)}\|\Phi\|_S$. Let $x \in \mathrm{POS}_{\mathcal{A}}(\{d\})$. By Proposition 18, $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = 1$. Hence, $\bigvee_{\Phi\to\Psi\in Dec_\alpha^+(S)}\|\Phi\|_S^T(x) = 1$. As a consequence, $x \in \bigvee_{\Phi\to\Psi\in Dec^+(S)}\|\Phi\|_S$ and $\mathrm{POS}_{\mathcal{A}}(\{d\}) \subseteq \bigvee_{\Phi\to\Psi\in Dec^+(S)}\|\Phi\|_S$.

Suppose now that $x \in \bigvee_{\Phi\to\Psi\in Dec^+(S)}\|\Phi\|_S$. Then, $\bigvee_{\Phi\to\Psi\in Dec_\alpha^+(S)}\|\Phi\|_S^T(x) = 1$. As a consequence, $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) = 1$. Therefore, by Proposition 18 $x \in \mathrm{POS}_{\mathcal{A}}(\{d\})$ and $\bigvee_{\Phi\to\Psi\in Dec^+(S)}\|\Phi\|_S \subseteq \mathrm{POS}_{\mathcal{A}}(\{d\})$.

Finally, we suppose that $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm. Then, for each $x \in U$ there exists $\Phi \to \Psi \in DA_T(S)$ such that $\|\Phi \wedge \Psi\|_S^T(x) = 1$. Taking into account that T is a family of boolean separable tolerance relations and applying Proposition 24, we obtain that $x \in \|\Phi \wedge \Psi\|_S$. As a consequence, $DA_T(S)$ is a complete decision algorithm.

The counterpart can be analogously proved, taking into account the requirement of the value α_3 given in the hypothesis. □

This connection between complete decision algorithms in the classical and fuzzy frameworks gives more importance to Definition 33, since it provides more flexibility than Definition 7, thanks to the family of tolerance relations T and the values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

The following example is introduced to illustrate the notion of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm.

Example 38. Now, we will calculate maximum/minimum values of α_k with $k \in \{1, 2, 3, 4\}$ to obtain a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm composed of the decision rules $Dec(S) = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ of Example 25. Firstly, we compute α_1 by using Table 10 and Definition 23. From the first item of Definition 33, we compute the value

$$\begin{aligned}\max\{\|\Phi \wedge \Phi'\|_S^T(x) \mid \Phi \to \Psi, \Phi' \to \Psi' \in Dec(S), \Phi \neq \Phi', x \in U\} &= \|\Phi_2 \wedge \Phi_5\|_S(x_2)\\ &= 1 \wedge 0.78\\ &= 0.78\end{aligned}$$

Consequently, any value $\alpha_1 \geq 0.78$ verifies that $\|\Phi \wedge \Phi'\|_S^T(x) \leq \alpha_1$ for each Φ, Φ' antecedents of decision rules of $Dec(S)$. The condition $\alpha_2 \geq 0$ clearly arises because d is a boolean attribute. Hence, $Dec(S)$ is a set of $\alpha_1\alpha_2$-pairwise mutually exclusive decision rules.

On the other hand, notice that each decision rule of $Dec(S)$ satisfies that $\|\Phi_i \wedge \Psi_i\|_S^T(x_i) = \|\Phi_i\|_S^T(x_i) \wedge \|\Psi_i\|_S^T(x_i) = 1$, that is, $\|\Phi_i\|_S^T(x_i) = \|\Psi_i\|_S^T(x_i) = 1$ for all $i \in \{1, 2, \dots, 7\}$. As a consequence,

$$card_F(\|\bigvee_{i=1}^{7}\Phi_i\|_S^T) = \sum_{x\in U}\|\bigvee_{i=1}^{7}\Phi_i\|_S^T(x) = \sum_{x\in U}\bigvee_{i=1}^{7}\|\Phi_i\|_S^T(x) = \sum_{j=1}^{|U|}\|\Phi_j\|_S^T(x_j) = \sum_{j=1}^{|U|}1 = |U|$$

Analogously, $card_F(\|\bigvee_{i=1}^{m}\Psi_i\|_S^T) = |U|$. Therefore, according to Definition 33, $Dec(S)$ covers U.

Now, for the computation of α_3 we take into account the third item of Definition 33, obtaining:

$$\min\{supp_S^T(\Phi, \Psi) \mid \Phi \to \Psi \in Dec(S)\} = supp_S^T(\Phi_4, \Psi_4) = 1.61$$

Therefore, any value $\alpha_3 < 1.61$ satisfies that $\alpha_3 < supp_S^T(\Phi, \Psi)$ for all $\Phi \to \Psi \in Dec(S)$. Consequently, each decision rule of $Dec(S)$ is α_3-admissible.

Finally, the value α_4 is computed. In Table 9, we showed the values of $\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x)$, for all $x \in U$. Now, we must obtain the set $Dec_\alpha^+(S)$ to compute $\bigvee_{\Phi\to\Psi\in Dec_\alpha^+(S)} \|\Phi\|_S^T(x)$ for all $x \in U$. In order to consider a good level of flexibility and a value close to 1 at the same time, we choose the value $\alpha = 0.75$. Hence, according to Definition 33, it is necessary to calculate the $R_{Fd} - 0.75$-block of each antecedent. By using Table 12, these blocks are

$$\begin{array}{lllllllll} [\Phi_1]_{0.75} & = & \{\Phi_1\} & & & & & & \\ [\Phi_2]_{0.75} & = & \{\Phi_2, \Phi_5, \Phi_6, \Phi_7\} & [\Phi_4]_{0.75} & = & \{\Phi_4\} & [\Phi_6]_{0.75} & = & \{\Phi_2, \Phi_6\} \\ [\Phi_3]_{0.75} & = & \{\Phi_3\} & [\Phi_5]_{0.75} & = & \{\Phi_2, \Phi_5\} & [\Phi_7]_{0.75} & = & \{\Phi_2, \Phi_7\} \end{array}$$

Analogously, the $R_{Fd} - 0.75$-block of each consequent is given as

$$[\Psi_i]_{0.75} = \begin{cases} \{\Psi_1, \Psi_3, \Psi_5, \Psi_7\} & \text{if } i \text{ is odd} \\ \{\Psi_2, \Psi_4, \Psi_6\} & \text{if } i \text{ is even} \end{cases}$$

Consequently, from these blocks we obtain that $Dec^+_{0.75}(S) = \{r_1, r_3, r_4, r_6\}$ because $\Phi_5, \Phi_7 \in [\Phi_2]_{0.75}$ but $\Psi_5, \Psi_7 \notin [\Psi_2]_{0.75}$. For the fourth item of Definition 33 consider the value

$$\begin{aligned} &\max\{\left|\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}(x) - \bigvee_{\Phi\to\Psi\in Dec^+_{0.75}(S)} \|\Phi\|_S^T(x)\right| \mid x \in U\} \\ &= \max\{|0.96 - 1|, |0.86 - 0.75|, |0.91 - 1|, |0.91 - 1|, |0.86 - 0.58|, \\ &\qquad |0.93 - 1|, |0.93 - 0.54|\} \\ &= 0.39 \end{aligned}$$

Hence, any value $\alpha_4 \geq 0.39$ makes $Dec(S)$ preserves the 0.75-consistency of S to degree α_4.

Therefore, $Dec(S) = DA_T(S)$ is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm. In fact, it is easy to see that $DA_T(S)$ is a complete $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_\alpha$-decision algorithm since $\|\Phi_i \wedge \Psi_i\|_S^T(x_i) = 1$ for all $i \in \{1, 2, \ldots, 7\}$. □

In conclusion, in this section we have presented the fuzzy notion of decision rule and all the rest notions related to it such as the relevance indicators which describe them and decision algorithms. It is important to emphasize that all these notions generalize the classical ones and they present interesting properties thanks to the flexibility the fuzzy framework offers. Next, we present an example with real information, to illustrate all of these definitions, showing in detail some calculus as a sample.

5. A toy example

This section applies the developed theory to analyze a real dataset. We include an example based on the knowledge system displayed in Table 13, whose dataset has been extracted from Statistical Yearbook of Zhejiang Province (2016) [20], in order to optimize of water conservancy project investment decision-making. It is important to mention that the knowledge system, obtained from the previously mentioned data, was already studied by using classical RST techniques in [20]. Now, we will carry out its study by using the notions and results obtained in the introduced fuzzy framework.

Specifically, the knowledge system collects eight project evaluations, based on the attributes construction expense (CE), financial income (FI), strategy benefit (SB) and external influence (EI), together with the corresponding decisions made by an investor, that is, to make an investment (I), a delayed investment (DI) or not to invest (NI). Every unit in Table 13 correspond to 10.000 yuan. With the goal of applying the proposed fuzzy RST technique, we transform the data given in Table 13 into the decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ represented in Table 14, where the set of objects is $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$, the set of attributes is $\mathcal{A} = \{CE, FI, SB, EI\}$, $\mathcal{V}_a = [0, 1]$, for all $a \in \mathcal{A}$ and $\mathcal{V}_d = \{I, NI, DI\}$. Notice that, we have applied a normalization factor by columns so that the domain is the interval [0, 1]. As the values of the first three columns are expected significantly higher than the values of the last one, we have divided by 1000 in the first case and by 500 in the second one. We could have normalized from the maximum and minimum values in each column of the decision table. However, we have considered a bigger range in order to have the possibility of taking into consideration projects with amounts greater and lower than these maximums and minimums.

To begin with, we compute a fuzzy m-decision reduct to degree 1 in order to consider less attributes in the decision rules, without losing information. The corresponding [0, 1]-valued measure $m : \mathcal{P}(\mathcal{A}) \to [0, 1]$ is the one given in Eq. (2)

Table 13
Knowledge system on water conservancy project investment decision-making.

	CE	FI	SB	EI	Decision (d)
Project 1 (p_1)	500	200	690	150	*DI*
Project 2 (p_2)	700	470	650	440	*NI*
Project 3 (p_3)	300	410	500	280	*NI*
Project 4 (p_4)	200	455	550	290	*I*
Project 5 (p_5)	250	260	490	105	*DI*
Project 6 (p_6)	510	380	430	130	*DI*
Project 7 (p_7)	350	550	255	145	*I*
Project 8 (p_8)	650	600	570	120	*NI*

Table 14
Decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ extracted from Table 13.

	CE	*FI*	*SB*	*EI*	*d*
p_1	0.5	0.2	0.69	0.3	*DI*
p_2	0.7	0.47	0.65	0.88	*NI*
p_3	0.3	0.41	0.5	0.56	*NI*
p_4	0.2	0.455	0.55	0.58	*I*
p_5	0.25	0.26	0.49	0.21	*DI*
p_6	0.51	0.38	0.43	0.26	*DI*
p_7	0.35	0.55	0.255	0.29	*I*
p_8	0.65	0.6	0.57	0.24	*NI*

and the a-indiscernibility relation $R_a : U \times U \to [0, 1]$ is the one given in Eq. (4), for all $a \in \mathcal{A}$. For the d-indiscernibility relation R_d, we define the following mapping, for all $x, y \in U$:

$$R_d(x, y) = \begin{cases} 1 & \text{if } \bar{d}(x) = \bar{d}(y) \\ 0.5 & \text{if } \bar{d}(x) = DI \text{ and } \bar{d}(y) \neq DI \\ 0 & \text{otherwise} \end{cases}$$

It is important to recall that R_d is a symmetrical relation. On the other hand, since we are interested in all attributes have the same impact in the evaluation of construction projects, we will use the B-indiscernibility relation $R_B : U \times U \to [0, 1]$ defined, for each $B \subseteq \mathcal{A}$, $x, y \in U$, as:

$$R_B(x, y) = \frac{\mathcal{R}_B^{x,y}(CE) + \mathcal{R}_B^{x,y}(FI) + \mathcal{R}_B^{x,y}(SB) + \mathcal{R}_B^{x,y}(EI)}{4}$$

where $\mathcal{R}_B^{x,y}(a)$ was given in Definition 11, for all $a \in \mathcal{A}$. Finally, the multi-adjoint fuzzy $\mathcal{A}$-positive region is computed from the fuzzy implication given in Eq. (5), considering the following symmetrical mapping $\tau : U \times U \to \{1, 3, 5\}$ defined, for each $x, y \in U$, as:

$$\tau(x, y) = \begin{cases} 1 & \text{if } \bar{d}(x) = \bar{d}(y) \\ 3 & \text{if } \bar{d}(x) = DI \text{ and } \bar{d}(y) \neq DI \\ 5 & \text{if } \bar{d}(x) = I \text{ and } \bar{d}(y) = NI \end{cases}$$

We would like to cushion the relatively big differences between the decision attribute values. For example, if we consider p_2 and p_7, we obtain $R_d(p_2, p_7) = 0$, which has a great impact in the computation of the implication $R_d(p_2, p_7) \nwarrow_{\tau(p_2,p_7)} R_{\mathcal{A}}(p_2, p_7)$, and so in the positive region. Hence, the selection of the implication $\tau(p_2, p_7) = 5$ reduces the fact they have some similarity on $\mathcal{A}$ and gives a greater value to the implication.

Fig. 2 shows the computed multi-adjoint fuzzy $\mathcal{A}$-positive region. The values of the multi-adjoint fuzzy $\mathcal{A}$-positive region allow us to compute $m(B)$, for all $B \subseteq \mathcal{A}$. The most remarkable results are also presented in Fig. 2.

We have shown the [0, 1]-valued measure m for all the subsets of three attributes and the three subsets of two attributes with the greatest [0, 1]-valued measure. It is important to emphasize the relevance of the attribute *CE* in the decision table since, as we can see in Fig. 2, it is the only attribute which belongs to the two subsets with two attributes of greater [0, 1]-valued measure. Therefore, when a project is evaluated to invest or not, it is fundamental to pay attention to the construction expense, which is also very natural. However, from these values we can conclude that $\mathcal{A}$ is the only fuzzy m-decision reduct to degree 1, so all its attributes are indispensable to process all the information contained in

	$\mathrm{POS}^{\mathrm{f}}_{\mathcal{A}}$
p_1	0.83
p_2	0.9
p_3	0.75
p_4	0.75
p_5	0.79
p_6	0.78
p_7	0.76
p_8	0.78

$$\begin{aligned} m(\{CE,FI,EI\}) &= 0.95\\ m(\{FI,SB,EI\}) &= 0.94\\ m(\{CE,SB,EI\}) &= 0.94\\ m(\{CE,FI,SB\}) &= 0.93\\ m(\{CE,FI\}) &= 0.88\\ m(\{CE,EI\}) &= 0.87\\ m(\{FI,FI\}) &= 0.85 \end{aligned}$$

Fig. 2. Multi-adjoint fuzzy $\mathcal{A}$-positive region.

Table 15
Degree of satisfaction for each antecedent in $Dec(S)$ for each object.

	$\|\Phi_1\|^T_S$	$\|\Phi_2\|^T_S$	$\|\Phi_3\|^T_S$	$\|\Phi_4\|^T_S$	$\|\Phi_5\|^T_S$	$\|\Phi_6\|^T_S$	$\|\Phi_7\|^T_S$	$\|\Phi_8\|^T_S$
p_1	1	0.42	0.74	0.7	0.75	0.74	0.565	0.6
p_2	0.42	1	0.6	0.5	0.33	0.38	0.41	0.36
p_3	0.74	0.6	1	0.9	0.65	0.7	0.73	0.65
p_4	0.7	0.5	0.9	1	0.63	0.68	0.705	0.55
p_5	0.75	0.33	0.65	0.63	1	0.74	0.71	0.6
p_6	0.74	0.38	0.7	0.68	0.74	1	0.825	0.78
p_7	0.565	0.41	0.73	0.705	0.71	0.825	1	0.685
p_8	0.6	0.36	0.65	0.55	0.6	0.78	0.685	1

Table 16
Degree of satisfaction for each consequent in $Dec(S)$ for each object.

	$\|\Psi_1\|^T_S$	$\|\Psi_2\|^T_S$	$\|\Psi_3\|^T_S$	$\|\Psi_4\|^T_S$	$\|\Psi_5\|^T_S$	$\|\Psi_6\|^T_S$	$\|\Psi_7\|^T_S$	$\|\Psi_8\|^T_S$
p_1	1	0.5	0.5	0.5	1	1	0.5	0.5
p_2	0.5	1	1	0	0.5	0.5	0	1
p_3	0.5	1	1	0	0.5	0.5	0	1
p_4	0.5	0	0	1	0.5	0.5	1	0
p_5	1	0.5	0.5	0.5	1	1	0.5	0.5
p_6	1	0.5	0.5	0.5	1	1	0.5	0.5
p_7	0.5	0	0	1	0.5	0.5	1	0
p_8	0.5	1	1	0	0.5	0.5	0	1

Table 14. As a consequence, to evaluate projects in this framework it is necessary to take into account the whole set of attributes to make a decision. We obtain the following decision rules:

$$\begin{aligned}
r_1 &: (CE, 0.5) \wedge (FI, 0.2) \wedge (SB, 0.69) \wedge (EI, 0.3) \to (d, DI)\\
r_2 &: (CE, 0.7) \wedge (FI, 0.47) \wedge (SB, 0.65) \wedge (EI, 0.88) \to (d, NI)\\
r_3 &: (CE, 0.3) \wedge (FI, 0.41) \wedge (SB, 0.5) \wedge (EI, 0.56) \to (d, NI)\\
r_4 &: (CE, 0.2) \wedge (FI, 0.455) \wedge (SB, 0.55) \wedge (EI, 0.58) \to (d, I)\\
r_5 &: (CE, 0.25) \wedge (FI, 0.26) \wedge (SB, 0.49) \wedge (EI, 0.21) \to (d, DI)\\
r_6 &: (CE, 0.51) \wedge (FI, 0.38) \wedge (SB, 0.43) \wedge (EI, 0.26) \to (d, DI)\\
r_7 &: (CE, 0.35) \wedge (FI, 0.55) \wedge (SB, 0.255) \wedge (EI, 0.29) \to (d, I)\\
r_8 &: (CE, 0.65) \wedge (FI, 0.6) \wedge (SB, 0.57) \wedge (EI, 0.24) \to (d, NI)
\end{aligned}$$

We denote this set of decision rules as $Dec(S)$. The degree of satisfaction for each antecedent and consequent, for each object, is computed by using the family of separable [0, 1]-fuzzy tolerance relations $T = \{T_a : V_a \times V_a \to [0,1] \mid a \in \mathcal{A}_d\}$ defined as $T_a(\bar{a}(x), v) = 1 - |\bar{a}(x) - v|$, for all $a \in \mathcal{A}$, $T_d = R_d$, $x \in U$ and $v \in [0, 1]$. All the results are shown in Tables 15 and 16.

Now, we compute the T-support, T-strength, T-certainty and T-coverage of each decision rule, which are values obtained from new measures presented in this paper, that could not be taken into account in the framework of [20], and that offer more information about the set of decision rules. The results are presented in Table 17.

From Table 17 we can extract some conclusions. First of all, we can detect that the rules with largest T-support are r_1, r_5 and r_6, whose decision attribute takes the value DI. This fact arises because of the three rules have DI as decision attribute value and DI has a level of similarity with the other decision attribute values. Moreover, these rules have also the largest T-certainty, showing that the decision DI is the best one to recommend in case of doubt about the investment.

Table 17
T-support, T-strength, T-certainty and T-coverage of each rule in $Dec(S)$.

Rule	$supp_S^T$	σ_S^T	cer_S^T	cov_S^T
r_1	4.91	0.61	0.89	0.89
r_2	3.09	0.39	0.77	0.69
r_3	3.75	0.47	0.63	0.83
r_4	3.21	0.4	0.57	0.92
r_5	4.82	0.6	0.89	0.88
r_6	4.86	0.61	0.83	0.88
r_7	3.21	0.4	0.57	0.92
r_8	3.51	0.44	0.67	0.78

Table 18
Relation between each pair of antecedents of $Dec(S)$.

R_{Fd}	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
Φ_1	1	0.42	0.74	0.7	0.75	0.74	0.565	0.6
Φ_2	0.42	1	0.6	0.5	0.33	0.38	0.41	0.36
Φ_3	0.74	0.6	1	0.9	0.65	0.7	0.73	0.65
Φ_4	0.7	0.5	0.9	1	0.63	0.68	0.705	0.55
Φ_5	0.75	0.33	0.65	0.63	1	0.74	0.71	0.6
Φ_6	0.74	0.38	0.7	0.68	0.74	1	0.825	0.78
Φ_7	0.565	0.41	0.73	0.705	0.71	0.825	1	0.685
Φ_8	0.6	0.36	0.65	0.55	0.6	0.78	0.685	1

Table 19
Relation between each pair of consequents of $Dec(S)$.

R_{Fd}	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ_8
Ψ_1	1	0.5	0.5	0.5	1	1	0.5	0.5
Ψ_2	0.5	1	1	0	0.5	0.5	0	1
Ψ_3	0.5	1	1	0	0.5	0.5	0	1
Ψ_4	0.5	0	0	1	0.5	0.5	1	0
Ψ_5	1	0.5	0.5	0.5	1	1	0.5	0.5
Ψ_6	1	0.5	0.5	0.5	1	1	0.5	0.5
Ψ_7	0.5	0	0	1	0.5	0.5	1	0
Ψ_8	0.5	1	1	0	0.5	0.5	0	1

On the other hand, the rules r_4 and r_7, which correspond to the decision I, have the lowest values in the T-certainty. This is due to the fact that no project presents properties that make it a safe investment, since the objects associated with these rules are similar to others with a different decision. However, these rules have the greatest T-coverage because, in case that having to invest in some projects, they are the best candidates according to their condition attributes. Finally, the decision rules with lowest T-coverage are r_2, r_3 and r_8, in which their decision attribute is NI. This fact shows that there are reasons not to invest in the projects related to the rules r_2, r_3 and r_8.

Now, we are going to prove whether the considered set of decision rules forms a decision algorithm (Definition 33) and, in affirmative case, the associated thresholds $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ will be analyzed, if the tolerance value is fixed at 0.75. The relationship between each pair of antecedents and consequents is computed based on Definition 30. The obtained results are shown in Tables 18 and 19.

As in Example 38, we have set the tolerance value $\alpha = 0.75$ to compute the $R_{Fd} - \alpha$-blocks of the antecedents and consequents. By using Tables 15 and 16, it can be checked that these blocks are the following:

$$\begin{array}{lllllllll} [\Phi_1]_{0.75} & = & \{\Phi_1, \Phi_5\} & [\Phi_4]_{0.75} & = & \{\Phi_3, \Phi_4\} & [\Phi_7]_{0.75} & = & \{\Phi_6, \Phi_7\} \\ [\Phi_2]_{0.75} & = & \{\Phi_2\} & [\Phi_5]_{0.75} & = & \{\Phi_1, \Phi_5\} & [\Phi_8]_{0.75} & = & \{\Phi_6, \Phi_8\} \\ [\Phi_3]_{0.75} & = & \{\Phi_3, \Phi_4\} & [\Phi_6]_{0.75} & = & \{\Phi_6, \Phi_7, \Phi_8\} & & & \end{array}$$

$$[\Psi_1]_{0.75} = [\Psi_5]_{0.75} = [\Psi_6]_{0.75} = \{\Psi_1, \Psi_5, \Psi_6\}$$
$$[\Psi_2]_{0.75} = [\Psi_3]_{0.75} = [\Psi_8]_{0.75} = \{\Psi_2, \Psi_3, \Psi_8\}$$
$$[\Psi_4]_{0.75} = [\Psi_7]_{0.75} = \{\Psi_4, \Psi_7\}$$

Consequently, by Definition 32, we deduce that $Dec^{+}_{0.75}(S) = \{r_1, r_2, r_5\}$. Following an analogous procedure to the one given in Example 38, we obtain that $Dec(S)$ is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_{0.75}$-decision algorithm, with $\alpha_1 \geq 0.9$, $\alpha_2 \geq 0.5$, $\alpha_3 < 3.09$ and $\alpha_4 \geq 0.21$. From the bound α_1 we deduce that it is difficult to discern objects of the table, which shows the complexity in the decision-making process for these evaluation projects. In fact, by Table 18 we know this value has been obtained from $\|\Phi_3 \wedge \Phi_4\|_S^T(p_3) = 0.9$, where $\Psi_3 = (d, NI)$ and $\Psi_4 = (d, I)$, which are totally contrary decisions. Therefore, this dataset presents inconsistencies, which were not detected in [20].

Table 20
Degree of satisfaction for each antecedent in $Dec(S')$ for each object.

	$\|\Phi'_1\|_S^T$	$\|\Phi'_2\|_S^T$	$\|\Phi'_3\|_S^T$	$\|\Phi'_4\|_S^T$	$\|\Phi'_5\|_S^T$	$\|\Phi'_6\|_S^T$	$\|\Phi'_7\|_S^T$	$\|\Phi'_8\|_S^T$
p_1	1	0.42	0.74	0.7	0.75	0.82	0.65	0.6
p_2	0.42	1	0.6	0.5	0.33	0.38	0.41	0.36
p_3	0.74	0.6	1	0.9	0.65	0.7	0.73	0.65
p_4	0.7	0.5	0.9	1	0.63	0.68	0.71	0.55
p_5	0.75	0.33	0.65	0.63	1	0.74	0.71	0.6
p_6	0.82	0.38	0.7	0.68	0.74	1	0.83	0.78
p_7	0.65	0.41	0.73	0.71	0.71	0.83	1	0.7
p_8	0.6	0.36	0.65	0.55	0.6	0.78	0.7	1

Table 21
T-support, T-strength, T-certainty and T-coverage of each rule in $Dec(S')$.

Rule	$supp_S^T$	σ_S^T	cer_S^T	cov_S^T
r'_1	4.99	0.62	0.88	0.91
r'_2	3.09	0.39	0.77	0.69
r'_3	3.75	0.47	0.63	0.83
r'_4	3.21	0.4	0.57	0.92
r'_5	4.82	0.6	0.89	0.88
r'_6	4.94	0.62	0.83	0.9
r'_7	3.21	0.4	0.56	0.92
r'_8	3.51	0.44	0.67	0.78

On the other hand, due to the decision attribute is not fuzzy, it is easy to discern objects with different decisions, as the bound α_2 shows. With respect to the bound α_3, it illustrates that all the rules have a high level of T-support, taking into account the reduced number of objects. This aspect is in line with the similarity in the values of the attributes. Notice that the T-support will be reduced if the normalization is done through the maximum and the minimum values in the decision table. However, as we previously commented, we have considered the range from 0 to 1.000 in order to have the possibility of taking into consideration projects with amounts lower than 2 millions of yuan and greater than 7 millions of yuan. Finally, α_4 exposes that the consistency of the dataset, provided by the fuzzy positive region, and the consistency of the considered set of decision rules in the algorithm is similar, taking into account that the tolerance value has been set in $\alpha = 0.75$. As a consequence, the chosen decision algorithm is suitable to study the corresponding dataset.

Notice that, if we avoid the inclusion of the decision rule associated with the object p_4 (a similar study can be done with respect to object p_3), then the inconsistency decreases. Specifically, the new obtained thresholds are: $\alpha_1 \geq 0.83$, $\alpha_2 \geq 0.5$, $\alpha_3 < 3.09$ and $\alpha_4 \geq 0.29$. However, the new set of rules does not form an algorithm, because of no formula Φ exists such that $\|\Phi\|_S^T(p_4) = 1$, and so the new set of rules does not cover U.

Although we remarked previously that only the set $\mathcal{A}$ is a fuzzy m-decision reduct to degree 1, we also showed the importance of the set of attributes $B = \{CE, FI, EI\}$. In fact, it can be checked that it is a fuzzy m-decision reduct to degree α with $0.94 < \alpha \leq 0.98$. Therefore, we will also study the set of decision rules in which the antecedents are the conjunction of the attributes of B, comparing the obtained results with those of the previous rules. The new set of rules is as follows:

$$r'_1 : (CE, 0.5) \wedge (FI, 0.2) \wedge (EI, 0.3) \rightarrow (d, DI)$$
$$r'_2 : (CE, 0.7) \wedge (FI, 0.47) \wedge (EI, 0.88) \rightarrow (d, NI)$$
$$r'_3 : (CE, 0.3) \wedge (FI, 0.41) \wedge (EI, 0.56) \rightarrow (d, NI)$$
$$r'_4 : (CE, 0.2) \wedge (FI, 0.455) \wedge (EI, 0.58) \rightarrow (d, I)$$
$$r'_5 : (CE, 0.25) \wedge (FI, 0.26) \wedge (EI, 0.21) \rightarrow (d, DI)$$
$$r'_6 : (CE, 0.51) \wedge (FI, 0.38) \wedge (EI, 0.26) \rightarrow (d, DI)$$
$$r'_7 : (CE, 0.35) \wedge (FI, 0.55) \wedge (EI, 0.29) \rightarrow (d, I)$$
$$r'_8 : (CE, 0.65) \wedge (FI, 0.6) \wedge (EI, 0.24) \rightarrow (d, NI)$$

This set of decision rules will be denoted as $Dec(S')$, where S' is the decision table obtained from S removing the attribute SB. Considering the same family of separable $[0, 1]$-fuzzy tolerance relations T that in the previous study, all the degrees of satisfaction for each antecedent in $Dec(S')$ and object are shown in Table 20.

Notice that, the degrees of satisfaction for the consequents do not change due to the same decision attribute is considered. Table 21 shows the T-support, T-strength, T-certainty and T-coverage of the decision rules of $Dec(S')$.

Comparing Tables 17 and 21 we notice very similar values, emphasizing the increment of the T-support of some rules, given that we have considered three attributes for the antecedents of the rules instead of the four, increasing the degrees of satisfaction for the antecedents.

Analogously, we compute the relationship between each pair of antecedents of the rules of $Dec(S')$. The results are shown in Table 22.

Table 22
Relation between each pair of antecedents of decision rules of $Dec(S')$.

R_{Fd}	Φ'_1	Φ'_2	Φ'_3	Φ'_4	Φ'_5	Φ'_6	Φ'_7	Φ'_8
Φ'_1	1	0.42	0.74	0.7	0.75	0.82	0.65	0.6
Φ'_2	0.42	1	0.6	0.5	0.33	0.38	0.41	0.36
Φ'_3	0.74	0.6	1	0.9	0.65	0.7	0.73	0.65
Φ'_4	0.7	0.5	0.9	1	0.63	0.68	0.71	0.55
Φ'_5	0.75	0.33	0.65	0.63	1	0.74	0.71	0.6
Φ'_6	0.82	0.38	0.7	0.68	0.74	1	0.83	0.78
Φ'_7	0.65	0.41	0.73	0.71	0.71	0.83	1	0.7
Φ'_8	0.6	0.36	0.65	0.55	0.6	0.78	0.7	1

Once again, we consider the value $\alpha = 0.75$ to compute the $R_{Fd}-\alpha$-blocks of the antecedents, which are the following:

$$\begin{array}{lll} [\Phi'_1]_{0.75} = \{\Phi'_1, \Phi'_5, \Phi'_6\} & [\Phi'_4]_{0.75} = \{\Phi'_3, \Phi'_4\} & [\Phi'_7]_{0.75} = \{\Phi'_6, \Phi'_7\} \\ [\Phi'_2]_{0.75} = \{\Phi'_2\} & [\Phi'_5]_{0.75} = \{\Phi'_1, \Phi'_5\} & [\Phi'_8]_{0.75} = \{\Phi'_6, \Phi'_8\} \\ [\Phi'_3]_{0.75} = \{\Phi'_3, \Phi'_4\} & [\Phi'_6]_{0.75} = \{\Phi'_1, \Phi'_6, \Phi'_7, \Phi'_8\} & \end{array}$$

It can be checked that $Dec^+_{0.75}(S') = \{r'_1, r'_2, r'_5\}$ and that $Dec(S')$ is a $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_{0.75}$-decision algorithm with $\alpha_1 \geq 0.9$, $\alpha_2 \geq 0.5$, $\alpha_3 < 3.09$ and $\alpha_4 \geq 0.21$. These bounds are the same that in the algorithm $Dec(S)$. This fact shows that we do not lose relatively much information considering in the antecedents of the rules the attributes $\{CE, FI, EI\}$ instead of $\{CE, FI, SB, EI\}$. As a consequence, it is possible not to take into account the attribute SB to decide whether to invest or not in these kinds of projects, taking into account the tolerance value $\alpha = 0.75$, which can be increased to reduce this tolerance. Moreover, recall that $\{CE, FI, EI\}$ is not a fuzzy m-decision reduct to degree 1, so that from this point of view, some information can also be lost. Although this loss of information has not impact in the obtained decision algorithms. In the future, the relationships between the degree of the m-reducts, the tolerance value and the decision algorithms will be studied in depth.

6. Conclusions and future work

Decision rules and decision algorithms have been studied in the fuzzy environment in this paper. In particular, a generalization to the fuzzy framework of the classic notions related to decision rules, including the support, strength, certainty and coverage have been introduced. Furthermore, the fuzzy notion of decision algorithms have also been presented. This notion will allow us to study the efficiency of a decision algorithm in this framework, which will be developed in future works. In addition, some examples have been introduced in order to illustrate all these new definitions.

In the future, we are interested in studying a generalized notion of efficiency which can be used to compare different decision algorithms. In addition, the results shown in this work can be applied to the decision rules obtained from mixed contexts [27]. However, the obtained algorithms would not take into account the attribute dependency given in this kind of contexts. Therefore, the consideration of mixed contexts will also be studied in the future. Moreover, we will apply the obtained theoretical developments in real examples whose decision attribute is not boolean. In order to evaluate the usefulness of the new notion of decision algorithm we will explore diverse examples that contain a large number of decision rules. In addition, it will be compared and complemented with other notions in the literature frameworks, such as logic program in fuzzy logic programming [28,29] and attribute implicational system in formal concept analysis [30,31].

Data availability

No data was used for the research described in the article

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