Fuzzy Optim Decis Making (2018) 17:177–193
https://doi.org/10.1007/s10700-017-9269-9

Different optimum notions for fuzzy functions and optimality conditions associated

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Published online: 15 March 2017

Abstract Fuzzy numbers have been applied on decision and optimization problems in uncertain or imprecise environments. In these problems, the necessity to define optimal notions for decision-maker's preferences as well as to prove necessary and sufficient optimality conditions for these optima are essential steps in the resolution process of the problem. The theoretical developments are illustrated and motivated with several numerical examples.

Keywords Fuzzy numbers · Crisp order relation · Interval order relation · Differentiable fuzzy mappings · Stationary fuzzy point · Fuzzy optimization

The research in this paper has been supported by MTM2015-66185 (MINECO/FEDER, UE) and Fondecyt-Chile, Project 1151154.

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1 Introduction

In conventional mathematical programming, the problem coefficients are assumed to be deterministic and fixed in value. But there are many situations where this assumption is not valid because of uncertain or imprecise environments. Fuzzy sets theory, and particularly the concept of fuzzy number, provides an appropriate theoretical framework to model quantities that are imprecise because of their own nature or measurement errors.

Fuzzy numbers have been applied on decision and optimization problems. In these, the procedures to rank fuzzy numbers are necessary. Ranking fuzzy numbers is a complex issue. All the proposed methods can be classified as corresponding to two different approaches:

1. Ranking fuzzy numbers using crisp relations (see Yager 1981; Campos-Ibáñez and González-Muñoz 1989). These procedures are based on a ranking function and they provide a crisp total order relation between fuzzy numbers.
2. Using ordering relations between compact intervals in $\mathbb{R}$, using fuzzy numbers characterization by their level sets (see Ishibuchi and Tanaka 1990; Wu 2007a, b).

Since comparing results in real problems affect implicated individuals, their subjectivity should be reflected in the method for ranking. When one is faced with deciding whether a fuzzy number is greater than, equal to or less than another, one introduces a subjective element set that needs to be considered in the mathematical model of decision process.

We present the optimum definitions for fuzzy functions using the different ordering relations defined; and we present and relate the optimality conditions for the different optimum definitions given. This allows us to interpret the ranking process and make a comparative study.

2 Preliminaries

A fuzzy set on $\mathbb{R}^n$ is a mapping defined as $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u, we denote its α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$. We denote the support of u by $supp(u)$ where $supp(u) = \{x \in \mathbb{R}^n | \, u(x) > 0\}$. The closure of $supp(u)$ defines the 0-level set of u, i.e. $[u]^0 = \overline{\{x \in \mathbb{R}^n | \, u(x) > 0\}}$.

Definition 1 A fuzzy interval or fuzzy number is a fuzzy set, u, defined on $\mathbb{R}$ satisfying the following conditions:

1. u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2. $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}, \ \forall x, y \in \mathbb{R}, \ \forall \lambda \in [0, 1]$;
3. u is upper semicontinuous, i.e., $\{x : u(x) \geq \alpha\}$ is a closed set for all $\alpha \in [0, 1]$;
4. $[u]^0$ is compact.

Let $\mathcal{F}_C$ denotes the family of all fuzzy numbers or fuzzy intervals. By definition the α-level sets of a fuzzy number are closed real intervals. Let $\mathcal{K}_C$ denotes the family of all bounded closed intervals in $\mathbb{R}$.

$$u \in \mathcal{F}_C \Rightarrow [u]^\alpha \in \mathcal{K}_C$$

Theorem 1 (Goestschel and Voxman (1986) and Stefanini and Bede (2009)) *A fuzzy interval is completely determined by any pair $u = (\underline{u}, \overline{u})$ of functions $\underline{u}, \overline{u} : [0, 1] \to \mathbb{R}$, defining the endpoints of the α-level sets,*

$$[u]^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)] = [\underline{u}_\alpha, \overline{u}_\alpha]$$

satisfying the following three conditions:

- $\underline{u}(\alpha) = \underline{u}_\alpha \in \mathbb{R}$ *is a bounded nondecreasing left-continuous function in* $(0, 1]$ *and it is right-continuous at* 0*;*
- $\overline{u}(\alpha) = \overline{u}_\alpha \in \mathbb{R}$ *is a bounded nonincreasing left-continuous function in* $(0, 1]$ *and it is right-continuous at* 0*;*
- $\underline{u}(\alpha) \leq \overline{u}(\alpha)$*, for all* $\alpha \in [0, 1]$.

We denote by $\mathcal{F}_C^C$ the all level-continuous fuzzy intervals family. Thus $u \in \mathcal{F}_C^C$ if the application $\alpha \mapsto [u]^\alpha$ is continuous (Román-Flores and Rojas-Medar 2002).

Proposition 1 (Chalco-Cano et al. (2013)) *Let $u = (\underline{u}, \overline{u}) \in \mathcal{F}_C^C$ be a fuzzy interval. Then, $u \in \mathcal{F}_C^C$ if and only if $\underline{u}$ and $\overline{u}$ are continuous functions with respect to α.*

This paper is organized as follows. In Sect. 3 we present ordering relations based on average functions and in Sect. 4 we propose order relations between alternatives which represent the decision-maker's preference when the cost of each alternative is known only to lie in an interval. In Sect. 5, associated with the average index ordering relation and intervals ordering relation, we give the minimum definitions for fuzzy mappings. In Sect. 6, we give necessary and sufficient optimality conditions based on appropriate stationary point definitions for minimum concepts defined in the previous section. In Sect. 7 we include numerical examples to illustrate the results obtained and main conclusions are presented in Sect. 8.

3 Ordering relations based on average functions

In Campos-Ibáñez and González-Muñoz (1989) a ranking function is defined to compare fuzzy numbers. This function was called "average index" because it can be interpreted as a weighted average in the following way: first, the decision-maker chooses a subset Y of the unit interval, so that the associated level sets contain the information which is considered outstanding about the imprecise quantity. Next, he assigns a weight, represented by a probability distribution P, to the different elements or measurable subsets of Y. Also, the decision-maker determines a position function, $f_u(\alpha)$ giving to each associated level set a real number. Finally, the index is defined as an average of level set positions in Y using P (Campos-Ibáñez and González-Muñoz 1989). Some well-known indexes are included in this schema (Campos and Gonzalez 1994; Fachinetti et al. 2002).

Definition 2 Let u be a fuzzy number, Y a subset of $[0, 1]$, P a probability distribution on Y and $f_u : [0, 1] \to \mathbb{R}$. The real number

$$V_P(u) = \int_Y f_u(\alpha) dP(\alpha)$$

is called average index of u.

The average index for every fuzzy number is defined by means of a integration process of a function representing the position of every α-cut in $\mathbb{R}$, and through a subjective assignation of weights related to the relative importance of levels α sets. The average index represents a mean value of the different α-cut positions through a measure P in Y. In fact, average index represents a mean value of the fuzzy number. In González (1990) the definition of the mean value of a fuzzy number (Dubois and Prade 1987) is interpreted as a particular case of the average index. The average index maps fuzzy numbers set into $\mathbb{R}$ in such a manner that resulting numbers give us a meaningful manner for ordering the original fuzzy numbers.

By means of $V_P(\cdot)$ a comparison relation on $\mathcal{F}_C$ is built:

Definition 3 For all $u, v \in \mathcal{F}_C$

- $u \preceq_V v \Leftrightarrow V_P(u) \leq V_P(v)$,
- $u \prec_V v$ if $u \preceq_V v$ and $V_P(u) \neq V_P(v)$.

This ranking function has been studied by many authors as a method for ordering fuzzy numbers. In Campos and Gonzalez (1994), authors study and interpret different parameters used to define the average index and the authors show how these parameters can be adapted to the decision-maker's preferences. For example, let us suppose the case in what the decision maker's preference is that all membership degrees have relevant information, then $Y = [0, 1]$. Each level set is represented by its midpoint, so $f_u(\alpha) = \frac{(\underline{u}(\alpha)+\overline{u}(\alpha))}{2}$ and we may use the following preferences:

– $P(\alpha) = \alpha^2$ gives more weight to the high α,

$$V_P(u) = \int_0^1 \alpha(\underline{u}(\alpha) + \overline{u}(\alpha))d\alpha.$$

– $P(\alpha) = \alpha^{\frac{1}{2}}$ gives more weight to the low α,

$$V_P(u) = \frac{1}{4}\int_0^1 \frac{1}{\sqrt{\alpha}}(\underline{u}(\alpha) + \overline{u}(\alpha))d\alpha.$$

– $P(\alpha) = \alpha$ gives equal weight to all α values,

$$V_P(u) = \int_0^1 (\underline{u}(\alpha) + \overline{u}(\alpha))d\alpha.$$

In Campos-Ibáñez and González-Muñoz (1989) the authors propose to choose one point included in each level set of u as the value for f_u:

$$f_u^\lambda : Y \to \mathbb{R}$$
$$f_u^\lambda(\alpha) = \lambda\overline{u}(\alpha) + (1-\lambda)\underline{u}(\alpha)$$

where $\lambda \in [0, 1]$ is an optimism-pessimism degree, which must be selected by the decision maker: when the most advantageous decision is to choose the greatest quantity, an optimistic person would think about the upper extreme of the interval $\overline{u}$ ($\lambda = 1$), which reflects the greatest profit. To the contrary, a pessimistic person would prefer the lower extreme of the interval $\underline{u}$ ($\lambda = 0$), which represents the least he can win.

When the most advantageous decision is to choose the least quantity, the interpretation is the opposite, with $\lambda = 0$ for the optimism and $\lambda = 1$ for pessimism. Thus, if the optimism-pessimism degree of the decision-maker is $\mu \in [0, 1]$, the parameter λ for the function f_u^λ is

$$\lambda = \begin{cases} \mu & \text{if the “best” is the “greatest”,} \\ 1 - \mu & \text{if the “best” is the “least”.} \end{cases}$$

Between the two extreme values $\lambda = 0$ and $\lambda = 1$ there is an attitude scale for the uncertainty for each decision-maker.

In general, the definition $f_u(\alpha)$ could be made arbitrarily by the decision-maker. However the use of $f_u^\lambda(\alpha)$ presents the following advantages: First, its easy construction since $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ are known and then λ allows us to use the decision-maker's subjectivity. In Campos-Ibáñez and González-Muñoz (1989) the authors prove that $f_u^\lambda(\alpha)$ has good properties and it generalizes some well-known procedures.

When $f_u(\alpha) = f_u^\lambda(\alpha)$, we denote the average index by $V_P^\lambda(u)$. If we consider $u \in \mathcal{F}_C^C$ then the integral $V_P^\lambda(u)$ is well defined.

4 Ordering relations based on intervals

In Ishibuchi and Tanaka (1990) the authors propose order relations between alternatives which represent the decision-maker's preference when the cost of each alternative is known only to lie in an interval.

Definition 4 Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ be two closed intervals in $\mathbb{R}$. The center and the width of an interval may be calculated as $A_C = (\overline{a} + \underline{a})/2$, $A_W = \overline{a} - \underline{a}$. Let us define the following order relations $\preceqq_{LW}$, $\preceqq_{LR}$ and $\preceqq_{CW}$:

1.
   - $A \preceqq_{LW} B \Rightarrow \underline{a} \leq \underline{b}$ and $A_W \leq B_W$,
   - $A \preceq_{LW} B \Rightarrow A \preceqq_{LW} B$ and $A \neq B$, i.e., $\underline{a} \leq \underline{b}$ and $A_W \leq B_W$, with some strict inequality.
2.
   - $A \preceqq_{LR} B \Rightarrow \underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$,
   - $A \preceq_{LR} B \Rightarrow A \preceqq_{LR} B$ and $A \neq B$, i.e., $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$, with some strict inequality.
3.
   - $A \preceqq_{CW} B \Rightarrow A_C \leq B_C$ and $A_W \leq B_W$,
   - $A \preceq_{CW} B \Rightarrow A \preceqq_{CW} B$ and $A \neq B$, i.e., $A_C \leq B_C$ and $A_W \leq B_W$, with some strict inequality.

Based on Theorem 1, we can order fuzzy numbers using the previous definition. We denote $\preceqq_*$ any of the orders defined.

Definition 5 For $u, v \in \mathcal{F}_C$

- $u \preceqq_* v$ if $[u]^\alpha \preceqq_* [v]^\alpha, \quad \forall \alpha \in [0, 1]$,
- $u \preceq_* v$ if $[u]^\alpha \preceqq_* [v]^\alpha, \forall \alpha \in [0, 1]$ and $\exists \alpha_0 / [u]^{\alpha_0} \preceq_* [v]^{\alpha_0}$,
- $u \prec_* v$ if $[u]^\alpha \preceq_* [v]^\alpha \quad \forall \alpha \in [0, 1]$.

The order $\preceq_{LR}$ represents the decision-maker's preference for the alternative with lower minimum cost and maximum cost. The order $\preceq_{LW}$ represents the decision-maker's preference for the alternative with lower minimum cost and less uncertainty since the width of an interval can be regarded as an uncertainty risk or a type of variance. And $\preceq_{CW}$ represents the preference for the alternative with lower expected value and less uncertainty.

Now we establish relationships between the different orders.

Proposition 2 *Let $u, v \in \mathcal{F}_C$, if*

$$u \preceqq_{LW} (\preceq_{LW}, \prec_{LW}) \; v$$

then

$$u \preceqq_{LR} (\preceq_{LR}, \prec_{LR}) \; v.$$

Proof If $u \preceqq_{LW} v$ then $\underline{u}(\alpha) \leq \underline{v}(\alpha)$ and $\overline{u}(\alpha) - \underline{u}(\alpha) \leq \overline{v}(\alpha) - \underline{v}(\alpha), \forall \alpha \in [0, 1]$. Summing, $\overline{u}(\alpha) \leq \overline{v}(\alpha)$, then $[u]^\alpha \preceqq_{LR} [v]^\alpha, \quad \forall \alpha \in [0, 1]$ and so $u \preceqq_{LR} v$.

Similarly it would be proved for $\preceq_{LW}$ and $\prec_{LW}$ orders. □

The reciprocal of the previous result is not true, in general.

Example 1 Let u and v be fuzzy intervals defined by their level-sets, such that $u \preceqq_{LR} v$.

$$[u]^\alpha = [-1, 1 - \alpha], \quad \alpha \in [0, 1],$$
$$[v]^\alpha = \left[\frac{\alpha}{2}, 1\right], \quad \alpha \in [0, 1].$$

It holds that $\underline{u}(\alpha) = -1 \leq \frac{\alpha}{2} = \underline{v}(\alpha)$ and $\overline{u}(\alpha) - \underline{u}(\alpha) \in [1, 2] \geq \overline{v}(\alpha) - \underline{v}(\alpha) \in [\frac{1}{2}, 1]$. Therefore $u \preceqq_{LW} v$ does not hold.

Proposition 3 *Let $u, v \in \mathcal{F}_C$, if*

$$u \preceqq_{LW} (\preceq_{LW}, \prec_{LW}) \; v$$

then

$$u \preceqq_{CW} (\preceq_{CW}, \prec_{CW}) \; v.$$

Proof If $u \preceqq_{LW} v$ then $\underline{u}(\alpha) \leq \underline{v}(\alpha)$ and $\overline{u}(\alpha) - \underline{u}(\alpha) \leq \overline{v}(\alpha) - \underline{v}(\alpha), \forall \alpha \in [0, 1]$. Summing, $\overline{u}(\alpha) \leq \overline{v}(\alpha)$ and so

$$\frac{\underline{u}(\alpha) + \overline{u}(\alpha)}{2} \leq \frac{\underline{v}(\alpha) + \overline{v}(\alpha)}{2}.$$

Therefore $[u]^\alpha \underline{\preceq}_{CW} [v]^\alpha$, $\forall \alpha \in [0, 1]$ and so $u \underline{\preceq}_{CW} v$. Similarly it would be proved for $\preceq_{CW}$ and $\prec_{CW}$ orders. □

The reciprocal of the previous proposition is not true, in general.

Example 2 Let u and v be fuzzy intervals defined by their level-sets,

$$[u]^\alpha = [-1, 1-\alpha], \quad \alpha \in [0, 1],$$
$$[v]^\alpha = [-2, 4-\alpha], \quad \alpha \in [0, 1].$$

Then

$$\frac{\underline{u}(\alpha) + \overline{u}(\alpha)}{2} = -\frac{\alpha}{2} \in \left[-\frac{1}{2}, 0\right],$$
$$\frac{\underline{v}(\alpha) + \overline{v}(\alpha)}{2} = \frac{2-\alpha}{2} \in \left[\frac{1}{2}, 1\right].$$

and

$$\overline{u}(\alpha) - \underline{u}(\alpha) = 2 - \alpha \in [1, 2],$$
$$\overline{v}(\alpha) - \underline{v}(\alpha) = 6 - \alpha \in [5, 6].$$

We have that $u \underline{\preceq}_{CW} v$, but $\underline{u}(\alpha) \geq \underline{v}(\alpha)$, $\forall \alpha$, then $u \underline{\npreceq}_{LW} v$ and $u \underline{\npreceq}_{LR} v$.

And the following example proves that in general, $u \underline{\preceq}_{LR} v$ does not imply $u \underline{\preceq}_{CW} v$ either.

Example 3 Let u and v be fuzzy intervals defined by their level-sets,

$$[u]^\alpha = [0, 1-\alpha], \quad \alpha \in [0, 1],$$
$$[v]^\alpha = \left[2, \frac{5}{2}\right], \quad \alpha \in [0, 1].$$

Then $u \underline{\preceq}_{LR} v$ but $\overline{u}(0) - \underline{u}(0) = 1 > \overline{v}(0) - \underline{v}(0) = \frac{1}{2}$, so $u \underline{\npreceq}_{CW} v$.

Proposition 4 *Let $u, v \in \mathcal{F}_C^C$, if $u \underline{\preceq}_{LR} v$ then $u \preceq_V v$ $\forall \lambda \in [0, 1]$, $\forall P$.*

Proof If $u \underline{\preceq}_{LR} v$ then $\underline{u}(\alpha) \leq \underline{v}(\alpha)$ and $\overline{u}(\alpha) \leq \overline{v}(\alpha)$, $\forall \alpha \in [0, 1]$. Then

$$(1-\lambda)\underline{u}(\alpha) + \lambda \overline{u}(\alpha) \leq (1-\lambda)\underline{v}(\alpha) + \lambda \overline{v}(\alpha), \quad \forall \alpha \in Y, \quad \forall \lambda \in [0, 1].$$
$$f_u^\lambda(\alpha) \leq f_v^\lambda(\alpha), \quad \forall \alpha \in Y, \quad \forall \lambda \in [0, 1].$$

Hence

$$V_P^\lambda(u) \leq V_P^\lambda(v), \quad \forall \lambda \in [0, 1], \quad \forall P \Rightarrow u \preceq_V v, \quad \forall \lambda \in [0, 1], \quad \forall P.$$

□

Proposition 5 *Let $u, v \in \mathcal{F}_C^C$, if $u \prec_{LR} v$ then $u \prec_V v$, $\forall \lambda \in (0,1)$, $\forall P$.*

Proof If $u \prec_{LR} v$ then $\underline{u}(\alpha) \leq \underline{v}(\alpha)$ and $\overline{u}(\alpha) \leq \overline{v}(\alpha)$, with any strict inequality $\forall \alpha \in [0,1]$. Then

$$\begin{aligned} &(1-\lambda)\underline{u}(\alpha) + \lambda\overline{u}(\alpha) < (1-\lambda)\underline{v}(\alpha) + \lambda\overline{v}(\alpha), \quad \forall \alpha \in Y, \quad \forall \lambda \in (0,1).\\ &f_u^\lambda(\alpha) < f_v^\lambda(\alpha), \quad \forall \alpha \in Y, \quad \forall \lambda \in (0,1). \end{aligned}$$

Therefore

$$V_P^\lambda(u) < V_P^\lambda(v), \quad \forall \lambda \in (0,1), \quad \forall P \Rightarrow u \prec_V v, \quad \forall \lambda \in (0,1), \quad \forall P.$$

□

Corollary 1 *Let $u, v \in \mathcal{F}_C^C$, then $u \preceqq_{LW} v$ ($\prec_{LW}$) and then $u \preceq_V v$ ($\prec_V$), $\forall \lambda \in [0,1]$ ($\forall \lambda \in (0,1)$), $\forall P$.*

Proof This proof is immediate using Propositions 2 and 4 (Propositions 2 and 5). □

Example 4 If $\lambda = \frac{1}{2}$, $P(\alpha) = \alpha^2$ and $Y = [0,1]$, we obtain the following ranking value function:

$$V_P(u) = \int_0^1 \alpha(\underline{u}(\alpha) + \overline{u}(\alpha))d\alpha.$$

For the Example 2, $V_P(u) = -\frac{1}{3}$ and $V_P(v) = \frac{2}{3}$, so $u \preceq_V v$, but it is not true that $u \preceqq_{LW} v$.

Proposition 6 *Let $u, v \in \mathcal{F}_C^C$, if $u \preceqq_{CW} v$ then $u \preceq_V v$, $\forall \lambda \in [1/2, 1]$, $\forall P$.*

Proof It holds that

$$f_u^\lambda(\alpha) = \frac{\underline{u}(\alpha) + \overline{u}(\alpha)}{2} + \left(\lambda - \frac{1}{2}\right)(\overline{u}(\alpha) - \underline{u}(\alpha)).$$

So, if $u \preceqq_{CW} v$ and $\lambda \in [1/2, 1]$ then $f_u^\lambda(\alpha) \leq f_v^\lambda(\alpha), \forall P$. And the proof is completed. □

Corollary 2 *Let $u, v \in \mathcal{F}_C^C$, if $u \prec_{CW} v$ then $u \prec_V v$, $\forall \lambda \in (1/2, 1]$.*

5 Fuzzy optimization: minimum definitions

A mapping $G : K \subset \mathbb{R}^n \to \mathcal{F}_C$ is said to be a fuzzy mapping, and so $G(x)$ is a fuzzy interval, that is uniquely determined by two functions such that

$$[G(x)]^\alpha = [\underline{g}_\alpha(x), \bar{g}_\alpha(x)] = [\underline{g}(\alpha, x), \bar{g}(\alpha, x)], \quad \forall \alpha \in [0,1], \quad \forall x \in K.$$

Then for G, we define the interval-valued functions family $G_\alpha : K \to \mathcal{K}_C$ given by $G_\alpha(x) = [G(x)]^\alpha$, for any $\alpha \in [0,1]$, where $\mathcal{K}_C$ is the family of all bounded closed real intervals. Here, for each $\alpha \in [0, 1]$, the endpoint functions $\underline{g}_\alpha, \overline{g}_\alpha : K \to \mathbb{R}$ are called lower and upper functions of G, respectively. Let us consider the fuzzy functions $G : K \subseteq \mathbb{R} \to \mathcal{F}_C^C$ and so $V_P(G(x))$ is well defined.

Associated with the average index ordering relation and intervals ordering relation we give the following minimum definitions for fuzzy mappings considering that $N_\delta(\bar{x})$ denotes a δ-neighborhood of $\bar{x}$.

Definition 6 Let $G : K \subseteq \mathbb{R}^n \to \mathcal{F}_C^C$ be a fuzzy mapping:

- $\bar{x}$ belonging to K is a (local) minimum^V for G if (there exists $N_\delta(\bar{x})$ such that) $G(\bar{x}) \preceq_V G(x)$ $(\forall x \in K \cap N_\delta(\bar{x}))$ $\forall x \in K$,
- $\bar{x}$ belonging to K is a (local) strict minimum^V for G if (there exists $N_\delta(\bar{x})$ such that) $G(\bar{x}) \prec_V G(x)$ $(\forall x \in K \cap N_\delta(\bar{x}))$ $\forall x \in K$.

Remark 1 It is clear that if $\bar{x}$ belonging to K is a (strict) minimum^V for G then it is a local (strict) minimum^V.

Moreover if $\bar{x}$ belonging to K is a (local) strict minimum^V it is a (local) minimum^V for G.

Definition 7 Let $G : K \subseteq \mathbb{R}^n \to \mathcal{F}_C$ be a fuzzy mapping:

- $\bar{x}$ belonging to K is a (local) minimum* for G if there does not exist $(x \in K \cap N_\delta(\bar{x})))$ $x \in K$ such that $G(x) \preceq_* G(\bar{x})$,
- $\bar{x}$ belonging to K is a (local) strict minimum* for G if there does not exist $(x \in K \cap N_\delta(\bar{x})))$ $x \in K$ such that $G(x) \underline{\preceq}_* G(\bar{x})$,
- $\bar{x}$ belonging to K is a (local) weak minimum* for G if there does not exist $(x \in K \cap N_\delta(\bar{x})))$ $x \in K$ such that $G(x) \prec_* G(\bar{x})$.

Remark 2 It is clear that if $\bar{x}$ belonging to K is a (weak, strict) minimun* for G then it is a local (weak, strict) minimum* for G.

Moreover if $\bar{x}$ belonging to K is a (local) strict minimum* for G then it is a (local) minimum* $\Rightarrow$ (local) weak minimum* for G.

Theorem 2 *If $\bar{x}$ belonging to K is a strict minimum^V for G for any $\lambda \in [0, 1]$ and any P, then $\bar{x}$ is a strict minimum^{LR} for G and then it is a strict minimum^{LW} for G.*

Proof Let us suppose that $\bar{x}$ is a strict minimum^V and now let us suppose that $\bar{x}$ is not a strict minimum^{LR}, then there exists another $x \in K$ such that $G(x) \underline{\preceq}_{LR} G(\bar{x})$. From Proposition 4, $G(x) \preceq_V G(\bar{x})$, and this is a contradiction.

If we suppose that there exists another $x \in K$ such that $G(x) \underline{\preceq}_{LW} G(\bar{x})$, then from Proposition 2, $G(x) \underline{\preceq}_{LR} G(\bar{x})$ and this is a contradiction. So, the proof is completed. □

Similarly to previous proof and from Propositions 2, 3, 4, 5, 6, Corollaries 1, 2 and Remarks 1, 2, we establish the relations among the different minimum types (Figs. 1). Above results are also true if we consider the local nature of the minima.

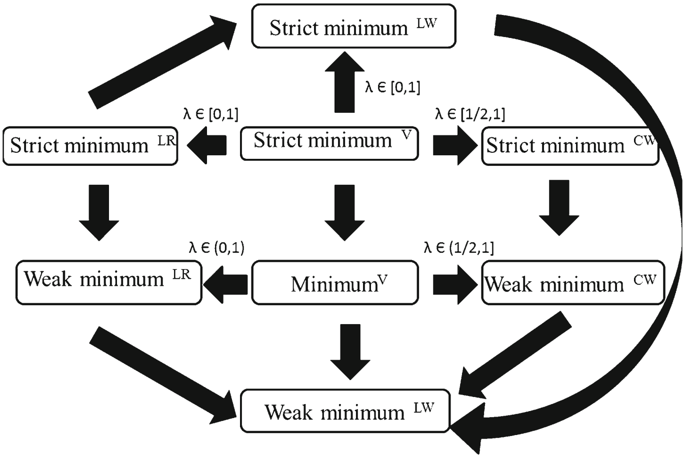

Fig. 1 Relationships between minimum concepts

6 Necessary and sufficient fuzzy optimality conditions

In classical optimization methods, it is well known that the stationary point concept (the one that cancels the derivative) plays a crucial role as a necessary optimality condition for problems defined by differentiable functions, since it allows to identify the potential candidates to be optimal solutions. Different fuzzy stationary point definitions (Panigrahi et al. 2008; Wu 2007a, b, 2009a, b) can be found in literature.

In previous works, either necessary optimality conditions are not proved, or they are proved under restrictive conditions (comparable functions), or they are complicated conditions to check (Panigrahi et al. 2008; Wu 2007a, b, 2009a, b). So, it is important to prove more adequate necessary optimality conditions for fuzzy functions. The necessary optimality conditions are based on a stationary point notion. So far, in literature there are different definitions for fuzzy functions that generalize the stationary point definition given in classical mathematical programming. The stationary point notion that we define is clearly different from that studied until now in the literature or it is more general (Chalco-Cano et al. 2013). Also we establish relationships between them. They present conceptual and specially computational checking advantages, since we demonstrate that they are equivalent to simply check that zero belongs to an interval or they are valid for modelling more general decision-maker's preferences than those found in previous works.

In this section, firstly we give necessary optimality conditions based on appropriate stationary point definitions for minimum concepts defined in the previous section. We establish more efficient conditions from computational point of view than those found in previous works on the subject. In Chalco-Cano et al. (2013) optimality conditions are studied for a particular case of ranking function; and (Osuna-Gómez et al. 2016) stationary point notion is defined adequately for a fuzzy function on $\mathbb{R}$. So far, optimality conditions had not been studied jointly for both orders.

Let us consider hereafter fuzzy functions $G : S \subseteq \mathbb{R}^n \to \mathcal{F}_C^C$ where S is an open set and we suppose that G is a level-wise differentiable fuzzy mapping, (Wu 2007a, b) and we will assume that $\frac{\partial f^{\lambda}_{G(x)}}{\partial x_i}(\alpha)$ are continuous on Y for all $x \in S$ and $i = 1, \ldots, n$ when it is necessary.

Definition 8 A fuzzy function $G : S \subseteq \mathbb{R}^n \to \mathcal{F}_C^C$ defined on an open set S, is said to be a level-wise differentiable fuzzy function if and only the endpoint functions associated are differentiable and

$$\left[\frac{\partial G}{\partial x_i}(x_0)\right]^{\alpha} = \left[\min\left\{\frac{\partial \underline{g}_{\alpha}}{\partial x_i}(x_0), \frac{\partial \overline{g}_{\alpha}}{\partial x_i}(x_0)\right\}, \max\left\{\frac{\partial \underline{g}_{\alpha}}{\partial x_i}(x_0), \frac{\partial \overline{g}_{\alpha}}{\partial x_i}(x_0)\right\}\right].$$

And so,

$$\nabla G(x_0) = \left(\frac{\partial G}{\partial x_1}(x_0), \ldots, \frac{\partial G}{\partial x_n}(x_0)\right) \in (\mathcal{F}_C)^n.$$

Remark 3 In order to guarantee that $\nabla V_P^{\lambda}(G(\cdot))$ exists we suppose that $\nabla f^{\lambda}_{G(\cdot)}(\alpha)$ is continuous with respect to α, and together with P monotonicity we can ensure the existence of the integral.

On the other hand, if $\lambda = \frac{1}{2}$ we can relax the hypotheses considering gH-differentiable fuzzy functions (Chalco-Cano et al. 2016), to ensure the differentiability of $f^{\frac{1}{2}}_{G(\cdot)}(\alpha)$.

Associated with average functions we define:

Definition 9 It is said that $\bar{x} \in S$ is a **stationary point**^{V} for G if $\nabla V_P^{\lambda}(G(\bar{x})) = 0$ for some λ and P.

Theorem 3 *If $\bar{x}$ is a local minimum^V for G then it is a stationary point^V for G.*

Proof As a consequence of Definition 3, $G(\bar{x}) \preceq_V G(x) \Leftrightarrow V_P^{\lambda}(G(x)) \leq V_P^{\lambda}(G(\bar{x}))$, then $\bar{x} \in S$ is a local minimum^V if and only if $\bar{x}$ is a local minimun for $V_P^{\lambda}(G(\cdot))$ and from necessary optimality condition for real functions, $\bar{x}$ is a stationary point for $V_P^{\lambda}(G(\cdot))$, i.e. $\nabla V_P^{\lambda}(G(\bar{x})) = 0$ for some λ and P. □

Associated with the intervals ordering relation we give the following definition for fuzzy functions:

Definition 10 It is said that $\bar{x} \in S$ is a **stationary point**^{LR} for G if $(0)^n \in [\nabla G(\bar{x})]^0$.

Theorem 4 *If $\bar{x}$ is a local weak minimum^{LR} for G then it is a stationary point^{LR} for G.*

Proof Arguing by contradiction, let us suppose that $(0)^n \notin [\nabla G(\bar{x})]^0$, then there exists i such that $0 \notin \left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^0$ and then $0 \notin \left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^{\alpha}$ for any α. Hence two cases are possible:

(i) $\left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^\alpha \subset \mathbb{R}^+$ for all $\alpha \in [0, 1]$, or
(ii) $\left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^\alpha \subset \mathbb{R}^-$ for all $\alpha \in [0, 1]$.

In any case, there exists $y_i \in \mathbb{R}$ such that

$$y_i \left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^\alpha \prec [0, 0], \quad \forall \alpha \in [0, 1]. \tag{1}$$

From Definition 8

$$\left[\frac{\partial G}{\partial x_i}(x_0)\right]^\alpha = \left[\min\left\{\frac{\partial \underline{g}_\alpha}{\partial x_i}(x_0), \frac{\partial \overline{g}_\alpha}{\partial x_i}(x_0)\right\}, \max\left\{\frac{\partial \underline{g}_\alpha}{\partial x_i}(x_0), \frac{\partial \overline{g}_\alpha}{\partial x_i}(x_0)\right\}\right]. \tag{2}$$

Above (1) and (2) implies that $\forall \alpha \in [0, 1]$

$$\begin{aligned} y_i \frac{\partial \underline{g}_\alpha}{\partial x_i}(\bar{x}) &< 0 \quad \text{and} \\ y_i \frac{\partial \overline{g}_\alpha}{\partial x_i}(\bar{x}) &< 0. \end{aligned}$$

Then

$$\begin{aligned} y_i \frac{\partial \underline{g}_\alpha}{\partial x_i}(\bar{x}) &= \lim_{h \to 0} \frac{\underline{g}_\alpha(\bar{x_1}, \ldots, \bar{x_i} + y_i h, \ldots, \bar{x_n}) - \underline{g}_\alpha(\bar{x})}{h} < 0, \quad \forall \alpha \in [0, 1], \\ y_i \frac{\partial \overline{g}_\alpha}{\partial x_i}(\bar{x}) &= \lim_{h \to 0} \frac{\overline{g}_\alpha(\bar{x_1}, \ldots, \bar{x_i} + y_i h, \ldots, \bar{x_n}) - \overline{g}_\alpha(\bar{x})}{h} < 0, \quad \forall \alpha \in [0, 1]. \end{aligned}$$

implies that $\exists \underline{\epsilon_i} > 0$ such that $\forall |h| < \underline{\epsilon_i}$ holds

$$\underline{g}_\alpha(\bar{x_1}, \ldots, \bar{x_i} + y_i h, \ldots, \bar{x_n}) - \underline{g}_\alpha(\bar{x}) < 0, \quad \forall \alpha \in [0, 1]. \tag{3}$$

And $\exists \overline{\epsilon_i} > 0$ such that $\forall |h| < \overline{\epsilon_i}$ holds

$$\overline{g}_\alpha(\bar{x_1}, \ldots, \bar{x_i} + y_i h, \ldots, \bar{x_n}) - \overline{g}_\alpha(\bar{x}) < 0, \quad \forall \in [0, 1]. \tag{4}$$

Then, if $\epsilon = \min\{\underline{\epsilon_i}, \overline{\epsilon_i}, \delta, \quad i = 1, \ldots, n\}$, $|h| < \epsilon$ and called $x = (\bar{x_1}, \ldots, \bar{x_i} + y_i h, \ldots, \bar{x_n})$, from (3), (4), we have that

$$\begin{aligned} \underline{g}_\alpha(x) - \underline{g}_\alpha(\bar{x}) &< 0, \\ \overline{g}_\alpha(x) - \overline{g}_\alpha(\bar{x}) &< 0, \quad \forall \alpha \in [0, 1]. \end{aligned}$$

So, $x \in N_\delta(\bar{x}) \cap S$ and $G(x) \prec_{LR} G(\bar{x})$ contradict that $\bar{x}$ is a local weak minimum^{LR} and the proof is completed. □

Now we give the relations between the stationary point definitions:

Theorem 5 *$\bar{x}$ is a stationary point^V for any $\lambda \in [0,1]$ and P, then $\bar{x}$ is a stationary point^{LR}.*

Proof If $\nabla V_P^\lambda(G(\bar{x}) = 0$ then for each $i = 1, \ldots, n$

$$(1-\lambda)\frac{\partial \underline{g}}{\partial x_i}(\bar{x},\alpha) + \lambda\frac{\partial \overline{g}}{\partial x_i}(\bar{x},\alpha) = 0 \quad \text{for some } \alpha.$$

If not,

$$(1-\lambda)\frac{\partial \underline{g}}{\partial x_i}(\bar{x},\alpha) + \lambda\frac{\partial \overline{g}}{\partial x_i}(\bar{x},\alpha) > 0, \quad \forall\alpha \Rightarrow \frac{\partial V_P^\lambda(G)}{\partial x_i}(\bar{x}) > 0.$$

Or

$$(1-\lambda)\frac{\partial \underline{g}}{\partial x_i}(\bar{x},\alpha) + \lambda\frac{\partial \overline{g}}{\partial x_i}(\bar{x},\alpha) < 0, \quad \forall\alpha \Rightarrow \frac{\partial V_P^\lambda(G)}{\partial x_i}(\bar{x}) < 0.$$

Then, if there exists α^i, for each i, such that $(1-\lambda)\frac{\partial \underline{g}}{\partial x_i}(\bar{x},\alpha^i) + \lambda\frac{\partial \overline{g}}{\partial x_i}(\bar{x},\alpha^i) = 0$, we have that $0 \in \left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^{\alpha^i} \subseteq \left[\frac{\partial G}{\partial x_i}(\bar{x})\right]^0$. Then $(0)^n \in [\nabla G(\bar{x})]^0$. □

Example 5 The reciprocal of the above theorem is not true in general. In fact, let $G(x)$ be a given function by $[G(x)]^\alpha = [\alpha, 2-\alpha] \cdot x^2$ and $S = (0, +\infty)$. Then, G is level-wise differentiable on S and $[\nabla G(x)]^\alpha = 2 \cdot [\alpha, 2-\alpha] \cdot x$. Now $\forall x \in S$, $[\nabla G(x)]^0 = [0, 2x]$ and $0 \in [\nabla G(x)]^0$, therefore, any value of S is a stationary point^{LR}.

On the other hand, $f_{G(x)}^\lambda(\alpha) = (1-\lambda)\alpha x^2 + \lambda(2-\alpha)x^2$ and $(f_{G(x)}^\lambda)'(\alpha) = 2(1-\lambda)\alpha x + 2\lambda(2-\alpha)x$.

It is verified that in $x = 1$, $(f_{(G(1))}^\lambda)'(\alpha) = 2\alpha + 4\lambda - 4\lambda\alpha$ and for $P(\alpha) = \alpha$ and $Y = [0,1]$ [index proposed by Tsumura et al. (1981)]

$$V_P^\lambda(G(1)) = 1 + 3\lambda \neq 0 \quad \forall\lambda \in [0,1].$$

Another main part in optimization theory is establishing sufficient optimality conditions. It is also known that not all stationary points are optimal, so it is necessary to use some problem intrinsic properties for eliminating the non-optimal candidates. Convexity or generalized convexity hypotheses made on the functions that define the problem are some of these properties.

Hanson (1981) defined a new class of functions.

Definition 11 Let $\phi(x)$ be a real differentiable function defined on a set $C \subseteq \mathbb{R}^n$, then $\phi(x)$ is called invex if

$$\phi(x_1) - \phi(x_2) \geq \eta^T(x_1, x_2)\nabla\phi(x_2) \quad \text{for all} \quad x_1, x_2, \in C, \tag{5}$$

for some arbitrary given vector function $\eta(x_1, x_2)$ defined on $C \times C$.

If the inequality (5) is strict, then we say that ϕ is a strictly invex function on C.

Theorem 6 *If $\underline{g}_\alpha$ and $\overline{g}_\alpha$ are invex functions with respect to the same η and $\bar{x}$ is a stationary point^V, then $\bar{x}$ is a minimum^V for G,.*

Proof If $\underline{g}_\alpha$ and $\overline{g}_\alpha$ are invex with respect to η then $f^\lambda_{G(\cdot)}$ is a invex function with respect to η, and so $V^\lambda_P(G(\cdot))$ is invex with respect to η also, due to the linearity and monotonicity of integral R-S because P is non-decreasing.

If $\bar{x}$ is a stationary point^V, then $\bar{x}$ is a stationary point for $V^\lambda_P(G(\cdot))$ also, and as $V^\lambda_P(G(\cdot))$ is an invex function, hence $\bar{x}$ is a minimum for $V^\lambda_P(G(\cdot))$, then it is a minimum^V for G. □

Theorem 7 *If $\underline{g}_\alpha$ and $\overline{g}_\alpha$ are strictly invex functions for all α with respect to the same η, then all stationary point^{LR} is a weak minimum^{LR} for G.*

Proof Let us suppose that there exists $x \in S$ such that $G(x) \prec_{LR} G(\bar{x})$ then, $\forall\alpha \in [0, 1]$

$$\left.\begin{array}{l} 0 \geq \underline{g}_\alpha(x) - \underline{g}_\alpha(\bar{x}) > \eta(x,\bar{x})^T \nabla \underline{g}_\alpha(\bar{x}) \\ 0 \geq \overline{g}_\alpha(x) - \overline{g}_\alpha(\bar{x}) > \eta(x,\bar{x})^T \nabla \overline{g}_\alpha(\bar{x}) \end{array}\right\}.$$

Therefore, there exists $\eta(x,\bar{x}) \in \mathbb{R}^n$ such that $\eta(x,\bar{x})^T[\nabla G(\bar{x}]^\alpha \prec_{LR} [0, 0]$, $\forall\alpha \in [0, 1]$. Particularly for $\alpha = 0$, and thus $0 \notin [\nabla G(\bar{x}]^0$ and so $\bar{x}$ is not a stationary point^{LR}, that it is a contradiction. □

Example 6 If we consider Example 5, it is immediate to check that all points are stationary points^{LR}, however there is no weak minimum^{LR}, because $\underline{g}$ and $\overline{g}$ are not strictly invex for all α ($\underline{g}(0, x) = 0$, and this is not a strictly invex function).

On the other hand, there is no stationary point^V because there is no minimum^V.

Figure 2 shows the relationships between stationary points and minima proved in the previous theorems.

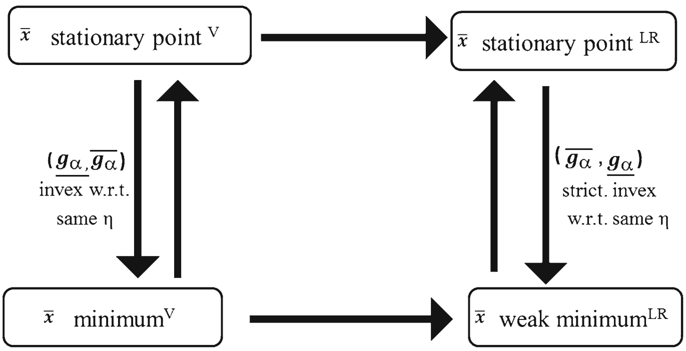

Fig. 2 Relationships between stationary points and minima

7 Numerical examples

In this section we present some examples to illustrate the importance of our results. In addition, through these examples we show that the results are very useful.

Example 7 Let $G : (0, \infty) \to \mathcal{F}_C^C$ represents the reproduction rate of some germ:

$$G(x)(t) = \begin{cases} \frac{1}{x^2}, & t \in [0, x^2]; \\ 1 - \frac{t-x^2}{x^2}, & t \in (x^2, 2x^2]; \\ 0, & t \notin [0, 2x^2]. \end{cases}$$

where x represents the predicted quantity and t represents the actual reproduction quantity.

Then $[G(x)]^\alpha = [\alpha x^2, (2-\alpha)x^2]$, $\alpha \in [0, 1]$ and for $Y = [0, 1]$ and $P(\alpha) = \alpha^2$ (index based on measure Stieltjes Campos and Gonzalez (1994)),

$$V_P^\lambda(G(x)) = \frac{2-\lambda}{3}x^2.$$

Thus, there does not exist local weak minima^{LR}, weak minima^{LW}, weak minima^{CW} and local minima^V either.

Furthermore, $[\nabla G(x)]^0 = [0, 4x]$, so for all $x \in (0, +\infty)$ is a stationary point^{LR}. On the other hand, $\underline{g}_\alpha = \alpha x^2$ is not a strictly invex function for $\alpha = 0$, and as $\nabla V_P^\lambda(G(x)) = 4(1-\alpha)x$, there does not exist any stationary point^V.

Example 8 Let the fuzzy mapping $G : (0, +\infty) \to \mathcal{F}_C^C$ represent a variable degree of temperature in a wellbore during injection, which is used in the drilling of petroleum

$$G(x)(t) = \begin{cases} -1 + \frac{t}{x}, & t \in [x, 2x]; \\ 2 - \frac{t^x}{2x}, & t \in [2x, 4x]; \\ 0, & t \notin [x, 4x], \end{cases}$$

where x represents the earth temperature and t represents the temperature at injection. For each $\alpha \in [0, 1]$ we have

$$[G(x)]^\alpha = [(1+\alpha)x, 2(2-\alpha)x].$$

Then

$$[\nabla G(x)]^\alpha = [1+\alpha, 2(2-\alpha)].$$

Thus, by Theorem 4 we can affirm that local weak minima^{LR} do not exist. And by Theorems 5 and 3, local minima^V do not exist also for any λ and any P.

8 Conclusions

In decision and optimization problems, the necessity of the procedures to rank fuzzy numbers is obvious but it is a complex issue. In this work, we present the optimum definitions for fuzzy functions using the different ordering relations defined which represent decision-maker's preferences and we provide relations between them. Also, we present and relate the necessary and sufficient optimality conditions for the different optimum definitions given. The examples given show the advantages of the results proved in this paper which relate optima according to the decision-maker's preferences.

Constrained optimization problems involving fuzzy functions are an interesting field for futher researchs. For example, finance represents a good field to implement models for sensitive analysis through fuzzy mathematics. The fuzzy representation of the parameters has to answer a fundamental demand: the possibility of a flexible shape of fuzzy number in order to capture all the stylized facts of financial markets. For example, Guerra et al. (2011) introduced the so-called LU-representation of fuzzy numbers, based on the use of parametrized lower and upper monotonic functions and generalizing the LR-fuzzy setting in the direction of the shape preservation and they also allow easy error-controlled approximations. Several authors are working hard to shape sources of uncertainty: prices, interest rates, volatilities, etc. (see Guerra et al. 2011; Buckley 1987). The idea is to advance in this field. Therefore, extending ideas developed in this paper to constrained fuzzy optimization problems is a very interesting issue. However, the authors find serious difficulties mainly with the formulation of the feasible set based on fuzzy functions and the use of the alternative theorems in the presence of fuzzy constraints.

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