

# On the Strong Metric Dimension of Cartesian and Direct Products of Graphs

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## Abstract

Let  $G$  be a connected graph. A vertex  $w$  *strongly resolves* a pair  $u, v$  of vertices of  $G$  if there exists some shortest  $u - w$  path containing  $v$  or some shortest  $v - w$  path containing  $u$ . A set  $W$  of vertices is a *strong resolving set* for  $G$  if every pair of vertices of  $G$  is strongly resolved by some vertex of  $W$ . The smallest cardinality of a strong resolving set for  $G$  is called the *strong metric dimension* of  $G$ . It is known that the problem of computing the strong metric dimension of a graph is NP-hard. In this paper we obtain closed formulae for the strong metric dimension of several families of the Cartesian product of graphs and the direct product of graphs.

*Keywords:* Strong resolving set; strong metric dimension; Cartesian product of graphs; direct product of graphs; strong resolving graph.

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## 1 Introduction

A *generator* of a metric space is a set  $S$  of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of  $S$ . Given a

simple and connected graph  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ , we consider the metric  $d_G : V(G) \times V(G) \rightarrow \mathbb{R}^+$ , where  $d_G(x, y)$  is the length of a shortest path between  $x$  and  $y$ . The pair  $(V(G), d_G)$  is readily seen to be a metric space. A vertex  $v \in V(G)$  is said to distinguish two vertices  $x$  and  $y$  if  $d_G(v, x) \neq d_G(v, y)$ . In the sequel we will assume that all graphs are connected. We use the notation  $u \sim v$  if  $u$  and  $v$  are adjacent vertices. For a vertex  $v$  of a graph  $G$ ,  $N_G(v)$  will denote the set of neighbors of  $v$  in  $G$ , i.e.,  $N_G(v) = \{u \in V(G) : u \sim v\}$ .

A set  $S \subset V(G)$  is said to be a *metric generator* for  $G$  if any pair of vertices of  $G$  is distinguished by some element of  $S$ . If  $S = \{w_1, w_2, \dots, w_k\}$  is an (ordered) set of vertices, then the *metric vector* of a vertex  $v \in V(G)$  relative to  $S$  is the vector  $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ . Thus,  $S$  is a metric generator if distinct vertices have distinct metric vectors relative to  $S$ . A minimum metric generator is called a *metric basis* and its cardinality, the *metric dimension* of  $G$ , is denoted by  $\dim(G)$ . Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of the metric dimension of a graph was introduced by Slater in [34, 35], where the metric generators were called *locating sets*. The metric dimension of a graph was introduced independently by Harary and Melter in [16], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [23] and applications to chemistry in [21, 22]. This invariant has been further studied in a number of other papers including for example, [3, 4, 5, 7, 8, 18, 28, 37, 41]. Several variations of metric generators and the metric dimension have been studied in [2, 10, 11, 13, 17, 26, 31, 32, 40].

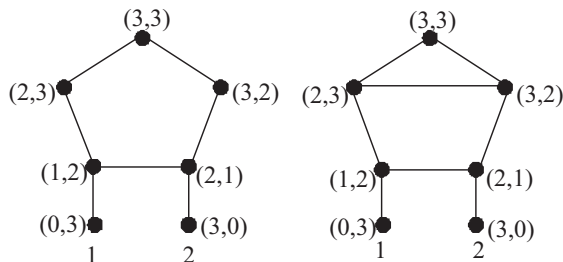


Figure 1: Non-isomorphic graphs with the same metric vectors

In [33] the authors asked, for a given metric generator  $T$  of a graph  $H$ , the following question: whenever  $H$  is a subgraph of a graph  $G$  and the metric vectors of the vertices of  $H$  relative to  $T$  agree in both  $H$  and  $G$ , is  $H$  an isometric subgraph of  $G$ ? Even though the metric vectors relative to a metric generator of a graph distinguish all pairs of vertices in the graph, they do not uniquely determine all distances in a graph as was first shown in [33]. Fig. 1 shows two graphs of order 7 having the same metric vectors relative to the metric generator  $\{1, 2\}$ , but for which the distances between pairs of vertices having the same metric vector are not the same. It was observed in [33] that, if “metric generator” is replaced with a stronger notion, namely that of “strong resolving set”, then the above question can be answered in the affirmative. For two vertices  $u$  and  $v$  in a connected graph  $G$ , the interval  $I_G[u, v]$  between  $u$  and  $v$  is defined as the collection of all vertices that belong to some shortest  $u - v$  path. A vertex  $w$  *strongly resolves* two vertices  $u$  and  $v$  if  $w \in I_G[u, w]$  or  $u \in I_G[v, w]$ . A set  $S$  of vertices in a connected graph  $G$  is a *strong resolving set*

for  $G$  if every two vertices of  $G$  are strongly resolved by some vertex of  $S$ . The smallest cardinality of a strong resolving set of  $G$  is called its *strong metric dimension* and is denoted by  $\dim_s(G)$ . We say that a strong resolving set for  $G$  of cardinality  $\dim_s(G)$  is a *strong metric basis* of  $G$ .

Since every strong resolving set is also a resolving set,  $\dim(G) \leq \dim_s(G)$ . It was shown in [8] that  $\dim(G) = 1$  if and only if  $G$  is a path. It now readily follows that  $\dim_s(G) = 1$  if and only if  $G$  is a path. At the other extreme we see that  $\dim_s(G) = n - 1$  if and only if  $G$  is the complete graph of order  $n$ . For the cycle  $C_n$  of order  $n$ , the strong metric dimension is  $\dim_s(C_n) = \lceil n/2 \rceil$ , and if  $T$  is a tree with  $l(T)$  leaves<sup>1</sup>, then its strong metric dimension equals  $l(T) - 1$  (see [33]).

We now describe the approach developed in [30] of transforming the problem of finding the strong metric dimension of a graph to the vertex cover problem. A vertex  $u$  of  $G$  is *maximally distant* from  $v$  if for every vertex  $w$  in the open neighborhood of  $u$ ,  $d_G(v, w) \leq d_G(u, v)$ . The collection of all vertices of  $G$  that are maximally distant from some vertex of the graph is called the *boundary* of the graph, see [1, 6], and is denoted by  $\partial(G)$ <sup>2</sup>. If  $u$  is maximally distant from  $v$  and  $v$  is maximally distant from  $u$ , then we say that  $u$  and  $v$  are *mutually maximally distant*. If  $u$  is maximally distant from  $v$ , and  $v$  is not maximally distant from  $u$ , then  $v$  has a neighbour  $v_1$ , such that  $d(v_1, u) > d(v, u)$ , i.e.,  $d(v_1, u) = d(v, u) + 1$ . It is easily seen that  $u$  is maximally distant from  $v_1$ . If  $v_1$  is not maximally distant from  $u$ , then  $v_1$  has a neighbour  $v_2$ , such that  $d(v_2, u) > d(v_1, u)$ . Continuing in this manner we construct a sequence of vertices  $v_1, v_2, \dots$  such that  $d(v_{i+1}, u) > d(v_i, u)$  for every  $i$ . Since  $G$  is finite this sequence terminates with some  $v_k$ . Thus for all neighbours  $x$  of  $v_k$  we have  $d(v_k, u) \geq d(x, u)$ . Thus  $v_k$  is maximally distant from  $u$  and  $u$  is maximally distant from  $v_k$ . Hence every boundary vertex belongs to the set  $S = \{u \in V(G) : \text{there exists } v \in V(G) \text{ such that } u, v \text{ are mutually maximally distant}\}$ . Certainly every vertex of  $S$  is a boundary vertex. For some basic graph classes, such as complete graphs  $K_n$ , complete bipartite graphs  $K_{r,s}$ , cycles  $C_n$  and hypercube graphs  $Q_k$ , the boundary is simply the whole vertex set. It is not difficult to see that this property also holds for all 2-antipodal<sup>3</sup> graphs and for all distance-regular graphs. Notice that the boundary of a tree consists of its leaves. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph  $G$ , we denote by  $\sigma(G)$  the set of simplicial vertices of  $G$ . It is readily seen that  $\sigma(G) \subseteq \partial(G)$ .

We use the notion of “strong resolving graph” based on a concept introduced in [30]. The *strong resolving graph* of  $G$ , denoted by  $G_{SR}$ , has vertex set  $V(G_{SR}) = \partial(G)$  where two vertices  $u, v$  are adjacent in  $G_{SR}$  if and only if  $u$  and  $v$  are mutually maximally distant in  $G$ <sup>4</sup>.

A set  $S$  of vertices of  $G$  is a *vertex cover* of  $G$  if every edge of  $G$  is incident with at least one vertex of  $S$ . The *vertex cover number* of  $G$ , denoted by  $\beta(G)$ , is the smallest cardinality

<sup>1</sup>The notation  $l(T)$  for the number of leaves of a tree  $T$  is used throughout the article.

<sup>2</sup>In fact, the boundary  $\partial(G)$  of a graph was defined first in [9] as the subgraph of  $G$  induced by the set mentioned in our article with the same notation. We follow the approach of [1, 6] where the boundary of the graph is just the subset of the boundary vertices defined in this article.

<sup>3</sup>The diameter of  $G = (V(G), E(G))$  is defined as  $D(G) = \max_{u,v \in V(G)} \{d(u, v)\}$ . We recall that  $G = (V(G), E(G))$  is 2-antipodal if for each vertex  $x \in V(G)$  there exists exactly one vertex  $y \in V(G)$  such that  $d_G(x, y) = D(G)$ .

<sup>4</sup>In fact, according to [30] the strong resolving graph  $G'_{SR}$  of a graph  $G$  has vertex set  $V(G'_{SR}) = V(G)$  and two vertices  $u, v$  are adjacent in  $G'_{SR}$  if and only if  $u$  and  $v$  are mutually maximally distant in  $G$ . So, the strong resolving graph defined here is a subgraph of the strong resolving graph defined in [30] and can be obtained from the latter graph by deleting its isolated vertices.

of a vertex cover of  $G$ . We refer to a  $\beta(G)$ -set in a graph  $G$  as a vertex cover of cardinality  $\beta(G)$ . Oellermann and Peters-Fransen [30] showed that the problem of finding the strong metric dimension of a connected graph  $G$  can be transformed to the problem of finding the vertex cover number of  $G_{SR}$ .

**Theorem 1.** [30] *For any connected graph  $G$ ,  $\dim_s(G) = \beta(G_{SR})$ .*

It was shown in [30] that the problem of computing  $\dim_s(G)$  is NP-hard. This suggests finding the strong metric dimension for special classes of graphs or obtaining good bounds on this invariant. The strong metric dimension of the Cartesian product of two cycles was obtained in [30] and for distance hereditary graphs an efficient procedure for finding the strong metric dimension was described in [27]. The metric dimension of Cartesian product of graphs has interesting applications. Several such applications are surveyed in [3]. In this paper we study the problem of finding exact values or sharp bounds for the strong metric dimension of Cartesian product and direct product of graphs and express these in terms of invariants of the factor graphs.

The *Cartesian product of two graphs*  $G$  and  $H$  is the graph  $G \square H$ , such that  $V(G \square H) = V(G) \times V(H)$  and two vertices  $(a, b), (c, d)$  are adjacent in  $G \square H$  if and only if, either  $(a = c$  and  $bd \in E(H))$  or  $(b = d$  and  $ac \in E(G))$ . Also, the *direct product of two graphs*  $G$  and  $H$  is the graph  $G \times H$ , such that  $V(G \times H) = V(G) \times V(H)$  and two vertices  $(a, b), (c, d)$  are adjacent in  $G \times H$  if and only if  $ac \in E(G)$  and  $bd \in E(H)$ .

There are some families of graphs for which the strong resolving graphs can be obtained relatively easily. We state some of these here since we need to refer to these in other sections of paper.

**Observation 2.** (a) *If  $\partial(G) = \sigma(G)$ , then  $G_{SR} \cong K_{\partial(G)}$ . In particular,  $(K_n)_{SR} \cong K_n$  and for any tree  $T$ ,  $(T)_{SR} \cong K_{l(T)}$ .*

(b) *For any 2-antipodal graph  $G$  of order  $n$ ,  $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$ . Even cycles are 2-antipodal. Thus,  $(C_{2k})_{SR} \cong \bigcup_{i=1}^k K_2$ .*

(c) *For odd cycles  $(C_{2k+1})_{SR} \cong C_{2k+1}$ .*

## 2 The strong metric dimension of Cartesian product of graphs

We begin this section by establishing an interesting connection between the strong resolving graph of the Cartesian product of two graphs and the direct product of the strong resolving graphs of its factors.

**Theorem 3.** *Let  $G$  and  $H$  be two connected graphs. Then  $(G \square H)_{SR} \cong G_{SR} \times H_{SR}$ .*

*Proof.* Let  $(g, h), (g', h')$  be any two vertices of  $G \square H$ . Then, we have  $d_{G \square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')$ . Thus, if  $g'' \sim g'$  and  $d_G(g, g'') = d_G(g, g') + 1$ , then  $(g', h') \sim (g'', h')$  and  $d_{G \square H}((g, h), (g'', h')) = d_G(g, g'') + d_H(h, h') + 1 = d_{G \square H}((g, h), (g', h')) + 1$ .

Using these observations, it is readily seen that  $(g, h)$  and  $(g', h')$  are mutually maximally distant if and only if  $g$  and  $g'$  are mutually maximally distant in  $G$  and  $h$  and  $h'$  are mutually maximally distant in  $H$ . Moreover,  $(g, h)(g', h') \in E((G \square H)_{SR})$  if and only if  $gg' \in E(G_{SR})$  and  $hh' \in E(H_{SR})$ . Thus

$$V((G \square H)_{SR}) = \partial(G \square H) = \partial(G) \times \partial(H) = V(G_{SR} \times H_{SR}),$$

and

$$(G \square H)_{SR} \cong G_{SR} \times H_{SR}.$$

□

The following result, which is obtained by using Theorem 1 and Theorem 3, will be the main tool of this section.

**Corollary 4.** *Let  $G$  and  $H$  be two connected graphs. Then  $\dim_s(G \square H) = \beta(G_{SR} \times H_{SR})$ .*

Now we consider some cases in which we can compute  $\beta(G_{SR} \times H_{SR})$ . To begin with we recall the following well-known result of König and Egerváry. In this sense, we need more terminology. A *matching* on a graph  $G$  is a set of edges of  $G$  such that no two sets share a vertex in common. A matching is *maximum* if it has the maximum possible cardinality. Moreover, if every vertex of the graph is incident to exactly one edge of the matching, then it is called a *perfect matching*.

**Theorem 5.** [12, 24] *For bipartite graphs the size of a maximum matching equals the size of a minimum vertex cover.*

Our next result deals with graphs which strong resolving graphs are bipartite with a perfect matching. We will use the theorem above as a tool for our purposes.

**Theorem 6.** *Let  $G$  and  $H$  be two connected graphs such that  $H_{SR}$  is bipartite with a perfect matching. Let  $G_i$ ,  $i \in \{1, \dots, k\}$ , be the connected components of  $G_{SR}$ . If for each  $i \in \{1, \dots, k\}$ ,  $G_i$  is Hamiltonian or  $G_i$  has a perfect matching, then*

$$\dim_s(G \square H) = \frac{|\partial(G)| |\partial(H)|}{2}.$$

*Proof.* Since  $H_{SR}$  is bipartite,  $G_{SR} \times H_{SR}$  is bipartite. We show next that  $G_{SR} \times H_{SR}$  has a perfect matching.

Let  $n_i$  be the order of  $G_i$ ,  $i \in \{1, \dots, k\}$ , and let  $\{x_1 y_1, x_2 y_2, \dots, x_{|\partial(H)|/2} y_{|\partial(H)|/2}\} \subset E(H_{SR})$  be a perfect matching of  $H_{SR}$ . We distinguish two cases.

Case 1:  $G_i$  has a perfect matching. If  $\{u_1 v_1, u_2 v_2, \dots, u_{n_i/2} v_{n_i/2}\} \subset E(G_i)$  is a perfect matching of  $G_i$ , then the set of edges

$$\begin{aligned} & \{(u_1, y_1)(v_1, x_1), (v_1, y_1)(u_1, x_1), \dots, (u_{n_i/2}, y_1)(v_{n_i/2}, x_1), (v_{n_i/2}, y_1)(u_{n_i/2}, x_1), \dots, \\ & (u_1, y_2)(v_1, x_2), (v_1, y_2)(u_1, x_2), \dots, (u_{n_i/2}, y_2)(v_{n_i/2}, x_2), (v_{n_i/2}, y_2)(u_{n_i/2}, x_2), \dots, \\ & (u_1, y_{|\partial(H)|/2})(v_1, x_{|\partial(H)|/2}), (v_1, y_{|\partial(H)|/2})(u_1, x_{|\partial(H)|/2}), \dots, \\ & (u_{n_i/2}, y_{|\partial(H)|/2})(v_{n_i/2}, x_{|\partial(H)|/2}), (v_{n_i/2}, y_{|\partial(H)|/2})(u_{n_i/2}, x_{|\partial(H)|/2})\} \end{aligned}$$

is a perfect matching of  $G_i \times H_{SR}$ .

Case 2:  $G_i$  is Hamiltonian. Let  $v_1, v_2, \dots, v_{n_i}, v_1$  be a Hamiltonian cycle of  $G_i$ . If  $n_i$  is even, then  $G_i$  has a perfect matching and this case coincides with Case 1. So we suppose that  $n_i$  is odd. In this case, the set of edges

$$\begin{aligned} & \{(v_1, x_1)(v_2, y_1), (v_2, x_1)(v_3, y_1), \dots, (v_{n_i-1}, x_1)(v_{n_i}, y_1), (v_{n_i}, x_1)(v_1, y_1), \\ & (v_1, x_2)(v_2, y_2), (v_2, x_2)(v_3, y_2), \dots, (v_{n_i-1}, x_2)(v_{n_i}, y_2), (v_{n_i}, x_2)(v_1, y_2), \dots, \\ & (v_1, x_{|\partial(H)|/2})(v_2, y_{|\partial(H)|/2}), (v_2, x_{|\partial(H)|/2})(v_3, y_{|\partial(H)|/2}), \dots, \\ & (v_{n_i-1}, x_{|\partial(H)|/2})(v_{n_i}, y_{|\partial(H)|/2}), (v_{n_i}, x_{|\partial(H)|/2})(v_1, y_{|\partial(H)|/2})\} \end{aligned}$$

is a perfect matching of  $G_i \times H_{SR}$ .

According to Cases 1 and 2 the graph  $\bigcup_{i=1}^k G_i \times H_{SR} = G_{SR} \times H_{SR}$  has a perfect matching. Now, since  $G_{SR} \times H_{SR}$  is bipartite and it has a perfect matching, by Theorem 5 we have  $\beta(G_{SR} \times H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$ . By Corollary 4 the result now follows.  $\square$

Since 2-antipodal graphs have strong resolving graphs that are bipartite with a perfect matching, the next result follows from the theorem previous and Observation 2.

**Corollary 7.** *For any connected 2-antipodal graph  $G$  of order  $n$ , the following statements hold.*

- (a) *If  $H$  is a connected 2-antipodal graph of order  $r$ , then  $\dim_s(G \square H) = \frac{nr}{2}$ .*
- (b) *If  $H$  is a connected graph where  $|\partial(H)| = |\sigma(H)|$ , then  $\dim_s(G \square H) = \frac{n|\sigma(H)|}{2}$ . In particular, for any tree  $T$ ,  $\dim_s(G \square T) = \frac{nl(T)}{2}$ , and for any complete graph  $K_r$ ,  $\dim_s(G \square K_r) = \frac{nr}{2}$ .*

On the other hand, by Observation 2 and Theorem 6, we obtain the following values for  $\dim_s(G \square H)$  for some specific examples of graphs  $G$  and  $H$ .

**Corollary 8.** (a)  $\dim_s(K_n \square P_r) = n$ .

(b) *For any tree  $T$ ,  $\dim_s(T \square P_r) = l(T)$ .*

(c)  $\dim_s(C_n \square P_r) = n$ .

(d)  $\dim_s(K_n \square C_{2k}) = nk$ .

(e) *For any tree  $T$ ,  $\dim_s(T \square C_{2k}) = l(T)k$ .*

(f) [30]  $\dim_s(C_n \square C_{2k}) = nk$ .

Our next tool is a well-known consequence of Hall's marriage theorem.

**Lemma 9.** [15] *Every regular bipartite graph has a perfect matching.*

The result above is particularly useful when we have a graph for which its strong resolving graph is regular and bipartite. Notice that there are several classes of graphs satisfying this property, for instance, paths, cycles of even order, hypercubes, etc.

**Theorem 10.** *Let  $G$  and  $H$  be two connected graphs such that  $G_{SR}$  and  $H_{SR}$  are regular and at least one of them is bipartite. Then*

$$\dim_s(G \square H) = \frac{|\partial(G)||\partial(H)|}{2}.$$

*Proof.* Since  $G_{SR}$  and  $H_{SR}$  are regular graphs and at least one of them is bipartite,  $G_{SR} \times H_{SR}$  is a regular bipartite graph. Hence, by Lemma 9,  $G_{SR} \times H_{SR}$  has a perfect matching. Thus, by Theorem 5,  $\beta(G_{SR} \times H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$ . The result follows by Corollary 4.  $\square$

Note that Corollary 8 can also be deduced from Theorem 10.

Our next results, which is obtained from Theorem 10, is derived from the fact that the strong resolving graph of a distance-regular graph is regular.

**Corollary 11.** *Let  $G$  be a distance-regular graph of order  $n$  and let  $H$  be a connected graph such that  $H_{SR}$  is a regular bipartite graph. Then*

$$\dim_s(G \square H) = \frac{n|\partial(H)|}{2}.$$

*In particular, if  $H$  is a 2-antipodal graph of order  $r$ , then*

$$\dim_s(G \square H) = \frac{nr}{2}.$$

Recall that the largest cardinality of a set of vertices of  $G$ , no two of which are adjacent, is called the *independence number* of  $G$  and is denoted by  $\alpha(G)$ . We refer to a  $\alpha(G)$ -set in a graph  $G$  as an independent set of cardinality  $\alpha(G)$ .

The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph. Such a result will provide us with another very useful tool in this work.

**Theorem 12.** [14] *For any graph  $G$  of order  $n$ ,  $\beta(G) + \alpha(G) = n$ .*

By using this result and Corollary 4 we obtain our mentioned useful tool, which relates  $\dim_s(G \square H)$  with the independence number of  $G_{SR} \times H_{SR}$ .

**Corollary 13.**  $\dim_s(G \square H) = |\partial(G)||\partial(H)| - \alpha(G_{SR} \times H_{SR})$ .

We now state a recent result, from [42], on the independence number of the direct product of graphs, that is useful in establishing the subsequent theorem.

**Lemma 14.** [42] *Let  $G$  and  $H$  be two vertex-transitive graphs of order  $n_1, n_2$ , respectively. Then*

$$\alpha(G \times H) = \max\{n_1\alpha(H), n_2\alpha(G)\}.$$

**Theorem 15.** *Let  $G$  and  $H$  be two connected graphs such that  $G_{SR}$  and  $H_{SR}$  are vertex-transitive graphs. Then*

$$\dim_s(G \square H) = \min\{|\partial(G)|\dim_s(H), |\partial(H)|\dim_s(G)\}.$$

*Proof.* Since  $G_{SR}$  and  $H_{SR}$  are vertex-transitive graphs, it follows from Lemma 14 that  $\alpha(G_{SR} \times H_{SR}) = \max\{|\partial(G)|\alpha(H), |\partial(H)|\alpha(G)\}$ . So, by Corollary 13 we have

$$\begin{aligned} \dim_s(G \square H) &= |\partial(G)||\partial(H)| - \alpha(G_{SR} \times H_{SR}) \\ &= |\partial(G)||\partial(H)| - \max\{|\partial(G)|\alpha(H_{SR}), |\partial(H)|\alpha(G_{SR})\} \\ &= \min\{|\partial(G)|\dim_s(H), |\partial(H)|\dim_s(G)\}. \end{aligned}$$

□

With the theorem above presented, it is possible to see that by using Observation 2 in such a result we deduce the following values of  $\dim_s(G \square H)$  for other specific families of graphs  $G$  and  $H$ .

**Corollary 16.** (a)  $\dim_s(K_n \square K_r) = \min\{n(r-1), r(n-1)\}$ .

(b) For any trees  $T_1$  and  $T_2$ ,  $\dim_s(T_1 \square T_2) = \min\{l(T_1)(l(T_2)-1), l(T_2)(l(T_1)-1)\}$ .

(c) [30]  $\dim_s(C_{2n+1} \square C_{2r+1}) = \min\{(2n+1)(r+1), (2r+1)(n+1)\}$ .

(d)  $\dim_s(K_n \square C_{2r+1}) = \min\{n(r+1), (2r+1)(n-1)\}$ .

(e)  $\dim_s(T \square C_{2r+1}) = \min\{l(T)(r+1), (2r+1)(l(T)-1)\}$ .

(f) For any tree  $T$ ,  $\dim_s(K_n \square T) = \min\{l(T)(n-1), n(l(T)-1)\}$ .

We continue with an easily verified bounds relating the strong metric dimension of a graph with the number of its simplicial vertices and the number of its mutually maximally distant vertices. It is clear that, if both sets (simplicial vertices and mutually maximally distant vertices) are equal, then we have an equality, which is useful for our goal of studying  $\dim_s(G \square H)$ .

**Lemma 17.** For every graph  $G$ ,  $|\sigma(G)| - 1 \leq \dim_s(G) \leq |\partial(G)| - 1$ .

*Proof.* Each simplicial vertex of  $G$  is mutually maximally distant with every other simplicial vertex of  $G$ . So,  $G_{SR}$  has a subgraph isomorphic to  $K_{|\sigma(G)|}$ . Thus,  $\beta(G_{SR}) \geq |\sigma(G)| - 1$ . Hence, by Theorem 1,  $\dim_s(G) \geq |\sigma(G)| - 1$ .

On the other hand, notice that for any graph  $H$ ,  $\beta(H) \leq |V(H)| - 1$ . Since  $V(G_{SR}) = \partial(G)$ , it follows that  $\beta(G_{SR}) \leq |\partial(G)| - 1$ . Thus, by Theorem 1, the upper bound follows. □

Note that if  $\sigma(G) = \partial(G)$ , then by Lemma 17,  $\dim_s(G) = |\partial(G)| - 1$ . Hence, as a particular case of Theorem 15 we obtain the following result.

**Corollary 18.** Let  $G$  and  $H$  be two connected graphs. If  $\partial(G) = \sigma(G)$  and  $\partial(H) = \sigma(H)$ , then

$$\dim_s(G \square H) = \min\{|\partial(G)|(|\partial(H)| - 1), |\partial(H)|(|\partial(G)| - 1)\}.$$

Again we use the matching of a graph to give a result for  $\dim_s(G \square H)$ . In this case, we also give a relationship with the matchings of the graph  $G \times K_2$ . To this end, we consider the *matching number* of a graph  $G$  (a largest matching of  $G$ ), which is denoted by  $\mu(G)$ , and we first present the following useful facts.

**Lemma 19.** *For any non-trivial non-empty graphs  $G$  and  $H$ ,*

$$\beta(G \times H) \geq \mu(H)\beta(G \times K_2) = \mu(H)\mu(G \times K_2) \geq 2\mu(G)\mu(H).$$

*Proof.* We consider a maximum matching  $M = \{u_i v_i : i \in \{1, \dots, k\}\}$  of  $H$ , and a minimum vertex cover  $A$  of  $G \times H$ . Now, for every  $i \in \{1, \dots, k\}$ , let  $A_i = A \cap (V(G) \times \{u_i, v_i\})$ . Notice that  $A_i \neq \emptyset$  for every  $i \in \{1, \dots, k\}$ . Also, since  $A_i \cap A_j \neq \emptyset$ , with  $i \neq j$ , it follows  $|A_1| + |A_2| + \dots + |A_k| \leq |A|$ . Moreover, for every  $i \in \{1, \dots, k\}$  we have that  $A_i$  is a vertex cover of  $G \times \langle \{u_i, v_i\} \rangle \cong G \times K_2$ . Thus  $k \beta(G \times K_2) \leq \sum_{i=1}^k |A_i| \leq |A| = \beta(G \times H)$ . As a result,

$$\beta(G \times H) \geq \mu(H)\beta(G \times K_2).$$

Since  $G \times K_2$  is a bipartite graph, from Theorem 5 it follows that

$$\beta(G \times K_2) = \mu(G \times K_2).$$

Finally, every matching  $\{x_i y_i : i \in \{1, \dots, k'\}\}$  of  $G$  induces a matching  $\{(x_i, a)(y_i, b), (y_i, a)(x_i, b) : i \in \{1, \dots, k'\}\}$  of  $G \times K_2$ , where  $\{a, b\}$  is the vertex set of  $K_2$ . Thus,  $\mu(G \times K_2) \geq 2\mu(G)$ . This completes the proof.  $\square$

**Observation 20.** *Let  $G$  and  $H$  be two graphs of orders  $n_1$  and  $n_2$ , respectively. If  $G$  and  $H$  have perfect matchings and at least one of them is bipartite, then  $G \times H$  is bipartite and*

$$\frac{n_1 n_2}{2} \geq \mu(G \times H) = \beta(G \times H) \geq 2\mu(G)\mu(H) = \frac{n_1 n_2}{2}.$$

Moreover,

$$\beta(G \times H) = \mu(H)\beta(G \times K_2) = \mu(H)\mu(G \times K_2) = 2\mu(G)\mu(H) = \frac{n_1 n_2}{2}.$$

Once described the relations above, we are able to give a bound on  $\dim_s(G \square H)$  relating  $\mu(H_{SR})$ ,  $\dim_s(G \square K_2)$  and  $\mu(G_{SR})$ . Note that this result is obtained from Lemma 19 and Corollary 4.

**Corollary 21.** *Let  $G$  and  $H$  be two connected graphs.*

$$\dim_s(G \square H) \geq \mu(H_{SR})\dim_s(G \square K_2) \geq 2\mu(G_{SR})\mu(H_{SR}).$$

Examples of graphs where  $\dim_s(G \square H) = \mu(H_{SR})\dim_s(G \square K_2) = 2\mu(G_{SR})\mu(H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$  are given in Corollary 8.

## 2.1 Strong metric dimension of Hamming graphs

Now we study a particular case of the Cartesian product of graphs, the so called ‘‘Hamming graphs’’. The *Hamming graph*  $H_{k,n}$  is defined as the Cartesian product of  $k$  copies of the complete graph  $K_n$ , *i.e.*,

$$H_{k,n} = \underbrace{K_n \square K_n \square \dots \square K_n}_{k \text{ times}}$$

The strong metric dimension of Hamming graphs was obtained in [25] where the authors gave a long and complicated proof. Here we give a simple proof for this result, using Theorem 3 and the next result due to Valencia-Pabon and Vera [38].

**Lemma 22.** [38] *For any positive integers,  $n_1, n_2, \dots, n_r$ ,*

$$\alpha(K_{n_1} \times K_{n_2} \times \dots \times K_{n_r}) = \max_{1 \leq i \leq r} \left\{ \frac{n_1 n_2 \dots n_r}{n_i} \right\}.$$

By Theorem 3, it follows that for any positive integers,  $n_1, n_2, \dots, n_r$ ,

$$(K_{n_1} \square K_{n_2} \square \dots \square K_{n_r})_{SR} \cong K_{n_1} \times K_{n_2} \times \dots \times K_{n_r}.$$

Therefore, Corollary 13 and Lemma 22 give the following result.

**Theorem 23.** *For any positive integers,  $n_1, n_2, \dots, n_r$ ,*

$$\dim_s(K_{n_1} \square K_{n_2} \square \dots \square K_{n_r}) = n_1 n_2 \dots n_r - \max_{1 \leq i \leq r} \left\{ \frac{n_1 n_2 \dots n_r}{n_i} \right\}.$$

As a consequence of the above result we obtain an expression for the strong metric dimension of Hamming graphs.

**Corollary 24.** [25] *For any Hamming graph  $H_{k,n}$ ,  $\dim_s(H_{k,n}) = (n-1)n^{k-1}$ .*

## 2.2 The strong metric dimension of the Cartesian product of graphs and that of its factors

In this section we give sharp upper and lower bounds for the strong metric dimension for the Cartesian product of graphs. We begin by stating a useful relationship between the independence numbers of the Cartesian product of two graphs and that of its factors.

**Lemma 25.** [20] *For any graphs  $G$  and  $H$  of orders  $n_1$  and  $n_2$ , respectively,*

$$\alpha(G \times H) \geq \max\{n_2 \alpha(G), n_1 \alpha(H)\}.$$

The next result gives a sharp upper bound for the strong metric dimension of the Cartesian product of graphs in terms of the strong metric dimension of its factors and the size of their extremities.

**Theorem 26.** *For any connected graphs  $G$  and  $H$ ,*

$$\dim_s(G \square H) \leq \min\{\dim_s(G)|\partial(H)|, |\partial(G)|\dim_s(H)\}.$$

*Moreover, this bound is sharp.*

*Proof.* By using Lemma 25 we deduce

$$\alpha(G_{SR} \times H_{SR}) \geq \max\{|\partial(H)|\alpha(G_{SR}), |\partial(G)|\alpha(H_{SR})\}.$$

Thus, by Theorem 12,

$$\beta(G_{SR} \times H_{SR}) \leq \min\{|\partial(H)|\beta(G_{SR}), |\partial(G)|\beta(H_{SR})\}.$$

The result now follows from Corollary 4. Several examples of pairs of graphs where the above bound is attained are given in Corollary 16.  $\square$

To prove a lower bound for the strong metric dimension of the Cartesian product of two graphs we use the following:

**Lemma 27.** [36] *For any graphs  $G$  and  $H$  of orders  $n_1$  and  $n_2$ , respectively,*

$$\alpha(G \times H) \leq n_2\alpha(G) + n_1\alpha(H) - \alpha(G)\alpha(H).$$

**Theorem 28.** *For any connected graphs  $G$  and  $H$ ,*

$$\dim_s(G \square H) \geq \dim_s(G)\dim_s(H).$$

*Proof.* Notice that Lemma 27 leads to

$$\alpha(G_{SR} \times H_{SR}) \leq |\partial(H)|\alpha(G_{SR}) + |\partial(G)|\alpha(H_{SR}) - \alpha(G_{SR})\alpha(H_{SR}).$$

Hence, from Theorem 12,

$$\beta(G_{SR} \times H_{SR}) \geq \beta(G_{SR})\beta(H_{SR}).$$

This inequality together with Corollary 4 gives the desired result.  $\square$

With respect to the sharpness of the lower bound in Theorem 28, it is necessary to observe that this bound is sharp if and only if the bound of Lemma 27 is also sharp. It was shown in [36] that there is a sequence of direct products  $G_n \times H_n$  such that

$$\frac{\alpha(G_n \times H_n)}{|V(H_n)|\alpha(G_n) + |V(G_n)|\alpha(H_n) - \alpha(G_n)\alpha(H_n)} \rightarrow 1$$

as  $n \rightarrow \infty$ . Thus to show that the above bound is asymptotically sharp one needs to find sequences of graphs  $G'_1, G'_2, \dots$  and  $H'_1, H'_2, \dots$  such that  $(G'_n)_{SR} = G_n$  and  $(H'_n)_{SR} = H_n$  for every  $n$ . No specific graph or family of graphs was described in [36] where the bound of Lemma 27 is achieved. It appears to be a non-trivial task to describe such sequences of graphs. We do observe that there is an infinite family of Cartesian products for which the strong dimension is “close” to the bound. In particular,  $\dim_s(K_n \square K_2) = n = \dim_s(K_n)\dim_s(K_2) + 1$ .

## 2.3 The Cartesian product of graphs with strong metric dimension 2

Even though those graphs for which the strong metric dimension is 2 are not yet fully understood, in this section we characterize those pairs of graphs for which the Cartesian product has strong metric dimension 2. Recall that, if  $G$  is a graph and  $u, v \in V(G)$ , then the interval between  $u$  and  $v$ , denoted by  $I_G(u, v)$ , is the collection of all vertices on a shortest  $u$ - $v$  path.

**Lemma 29.** *Let  $a, x, c \in V(G)$  and  $b, y, d \in V(H)$ . Then,  $(a, b) \in I_{G \square H}[(x, y), (c, d)]$  if and only if  $a \in I_G[x, c]$  and  $b \in I_H[y, d]$ .*

*Proof.* Suppose first that  $a \in I_G[x, c]$  and  $b \in I_H[y, d]$ . Then,  $d_G(x, c) = d_G(x, a) + d_G(a, c)$  and  $d_H(y, d) = d_H(y, b) + d_H(b, d)$ . Hence

$$\begin{aligned} d_{G \square H}((x, y), (c, d)) &= d_G(x, c) + d_H(y, d) \\ &= (d_G(x, a) + d_G(a, c)) + (d_H(y, b) + d_H(b, d)) \\ &= (d_G(x, a) + d_H(y, b)) + (d_G(a, c) + d_H(b, d)) \\ &= d_{G \square H}((x, y), (a, b)) + d_{G \square H}((a, b), (c, d)). \end{aligned}$$

Thus  $(a, b) \in I_{G \square H}[(x, y), (c, d)]$ .

Conversely, if  $(a, b) \in I_{G \square H}[(x, y), (c, d)]$ , then

$$\begin{aligned} d_{G \square H}((x, y), (c, d)) &= d_{G \square H}((x, y), (a, b)) + d_{G \square H}((a, b), (c, d)) \\ &= (d_G(x, a) + d_H(y, b)) + (d_G(a, c) + d_H(b, d)) \\ &= (d_G(x, a) + d_G(a, c)) + (d_H(y, b) + d_H(b, d)). \end{aligned}$$

Now, if  $a \notin I_G[x, c]$  or  $b \notin I_H[y, d]$ , then  $d_{G \square H}((x, y), (c, d)) > d_G(x, c) + d_H(y, d)$ , a contradiction.  $\square$

**Proposition 30.** *Let  $G$  and  $H$  be two connected graphs of order at least 2. Then,  $\dim_s(G \square H) = 2$  if and only if  $G$  and  $H$  are both paths.*

*Proof.* If  $G$  and  $H$  are paths, then, by Corollary 4,

$$\dim_s(G \square H) = \beta(K_2 \times K_2) = 2.$$

On the other hand, let  $S = \{(a, x), (b, y)\}$  be a strong metric basis of  $G \square H$ . If  $a \neq b$  and  $x \neq y$ . Let  $c$  be a neighbor of  $b$  on a  $a$ - $b$  path (it might be that  $a = c$ ). Let  $z$  be a neighbor of  $y$  on a  $x$ - $y$  path (notice that could be  $x = z$ ). So, we have  $d_{G \square H}((b, z), (a, x)) = d_G(a, b) + d_H(z, x) = d_G(a, c) + 1 + d_H(x, y) - 1 = d_G(a, c) + d_H(x, y) = d_{G \square H}((c, y), (a, x))$ . Thus,  $(b, z) \notin I_{G \square H}[(c, y), (a, x)]$  and  $(c, y) \notin I_{G \square H}[(b, z), (a, x)]$ . Moreover,  $d_{G \square H}((b, z), (b, y)) = d_H(z, y) = 1 = d_G(b, c) = d_{G \square H}((c, y), (b, y))$ . Thus,  $(b, z) \notin I_{G \square H}[(c, y), (b, y)]$  and  $(c, y) \notin I_{G \square H}[(b, z), (b, y)]$ . Therefore,  $S = \{(a, x), (b, y)\}$  does not strongly resolve  $(b, z)$  and  $(c, y)$ , and so either  $a = b$  or  $x = y$ .

If  $a = b$ , then the projection of  $S$  onto  $G$  is a single vertex. By Lemma 29, the projection of  $S$  onto  $G$  strongly resolves  $G$ . As observed in the Introduction,  $G$  is a path. Similarly, if  $x = y$ , then  $H$  is a path. Therefore either  $G$  or  $H$  is a path. We assume, without loss of generality, that  $G$  is a path. By Corollary 4 and Observation 20 it follows that  $2 = \dim_s(G \square H) = \beta(K_2 \times H_{SR})$ . Thus, either  $H_{SR}$  is isomorphic to  $K_2$  or  $\beta(H_{SR}) = 1$  which implies that  $\dim_s(H) = 1$ . Therefore, as observed in the Introduction,  $H$  is a path.  $\square$

### 3 Strong metric dimension of the direct product of graphs

It is well known (see [19] for instance) that the direct product of two graphs is connected if and only if both factors are connected and at least one factor is non-bipartite. As a result the strong metric dimension is not defined for the direct products  $C_r \times C_t$  with  $r, t$  even,  $P_r \times P_t$ , and  $P_r \times C_t$  with  $t$  even. Our focus is on the problem of finding the strong metric dimension for direct products of odd cycles of the same order and the direct product of a complete graph with either a complete graph, path or cycle. The cases where one factor is an odd cycle and the other is an even cycle or a path or an odd cycle of a different order appears to be computationally quite tedious and is not considered here.

**Lemma 31.** [29] *Let  $G$  and  $H$  be two connected graphs. Then,  $G \square H \cong G \times H$  if and only if  $G \cong H \cong C_{2k+1}$  for some positive integer  $k$ .*

The characterization above, and the results from the previous section, allow us to immediately determine the strong metric dimension of the direct product of such pairs of graphs. Specifically, such a lemma above and Corollary 16 (c) give the following result.

**Corollary 32.** *For any positive integer  $k$ ,  $\dim_s(C_{2k+1} \times C_{2k+1}) = (2k + 1)(k + 1)$ .*

For the remainder of this section we focus on the strong metric dimension of the direct product of two graphs one of which is complete. In Section 2 we showed that the strong resolving graph of the Cartesian product of two graphs is the direct product of the strong resolving graphs of the factors. No such result is known for the direct product of two graphs, but the next result gives a relationship between the strong resolving graph of the direct product of complete graphs and their Cartesian product.

**Lemma 33.** *For any positive integers  $r, t \geq 3$ ,  $(K_r \times K_t)_{SR} \cong K_r \square K_t$ .*

*Proof.* Let  $V_1$  and  $V_2$  be the vertex sets of  $K_r$  and  $K_t$ , respectively. Let  $(u_1, v_1)$  and  $(u_2, v_2)$  be two distinct vertices of  $K_r \times K_t$ . If  $u_1 = u_2$  or  $v_1 = v_2$ , then  $d_{K_r \times K_t}((u_1, v_1), (u_2, v_2)) = 2$ . On the other hand, if  $u_1 \neq u_2$  and  $v_1 \neq v_2$ , then  $d_{K_r \times K_t}((u_1, v_1), (u_2, v_2)) = 1$ . Thus, any two distinct vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are mutually maximally distant in  $K_r \times K_t$  if and only if either  $u_1 = u_2$  or  $v_1 = v_2$ . So, every vertex  $(x, y)$  is adjacent in  $(K_r \times K_t)_{SR}$  to all the vertices of the sets  $\{(x, v_i) : v_i \in V_2 - \{y\}\}$  and  $\{(u_i, y) : u_i \in V_1 - \{x\}\}$  and thus,  $(K_r \times K_t)_{SR}$  is isomorphic to the Cartesian product  $K_r \square K_t$ .  $\square$

A well-known result of Vizing will be used to find the strong metric dimension of the direct product of complete graphs.

**Lemma 34.** ([39], Vizing) *For any graphs  $G$  and  $H$  of order  $r$  and  $t$ , respectively,*

$$\alpha(G)\alpha(H) + \min\{r - \alpha(G), t - \alpha(H)\} \leq \alpha(G \square H) \leq \min\{t\alpha(G), r\alpha(H)\}.$$

**Corollary 35.** *For any positive integers  $r, t \geq 3$ ,  $\dim_s(K_r \times K_t) = \max\{r(t - 1), t(r - 1)\}$ .*

*Proof.* By Theorem 1, Lemma 33, and Theorem 12,  $\dim_s(K_r \times K_t) = rt - \alpha(K_r \square K_t)$ . By Lemma 34,  $\alpha(K_r \square K_t) = \min\{r, t\}$ . Thus  $\dim_s(K_r \times K_t) = rt - \min\{r, t\} = \max\{r(t - 1), t(r - 1)\}$ .  $\square$

We now introduce a well-known class of graphs that will be used in deriving a formula for the strong metric dimension of the direct product of cycles and complete graphs. Let  $\mathbb{Z}_n$  be the additive group of integers modulo  $n$  and let  $M \subset \mathbb{Z}_n$ , such that,  $i \in M$  if and only if  $-i \in M$ . We can construct a graph  $G$  as follows: the vertices of  $V(G)$  are the elements of  $\mathbb{Z}_n$  and  $(i, j)$  is an edge in  $E(G)$  if and only if  $j - i \in M$ . This graph is a *circulant of order  $n$*  and is denoted by  $CR(n, M)$ . With this notation, a cycle is the same as  $CR(n, \{-1, 1\})$  and the complete graph is  $CR(n, \mathbb{Z}_n)$ . In order to simplify the notation we will use  $CR(n, t)$ ,  $0 < t \leq \frac{n}{2}$ , instead of  $CR(n, \{-t, -t + 1, \dots, -1, 1, 2, \dots, t\})$ . This is also the  $t^{\text{th}}$  power of  $C_n$ .

**Lemma 36.** *For any circulant graph  $CR(n, 2)$ ,  $\alpha(CR(n, 2)) = \lfloor \frac{n}{3} \rfloor$ .*

*Proof.* Let  $V(CR(n, 2)) = \{u_0, u_1, \dots, u_{n-1}\}$  be the set of vertices of  $CR(n, 2)$ , where two vertices  $u_i, u_j$  are adjacent if and only if  $i - j \in \{-2, -1, 1, 2\}$ . Notice that every vertex  $u_i$  is adjacent to the vertices  $u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}$ , where the operations with the subindex  $i$  are expressed modulo  $n$ . Let  $S$  be the set of vertices of  $CR(n, 2)$  satisfying the following.

- If  $n \equiv 0 \pmod{3}$ , then  $S = \{u_0, u_3, u_6, \dots, u_{n-6}, u_{n-3}\}$ .
- If  $n \equiv 1 \pmod{3}$ , then  $S = \{u_0, u_3, u_6, \dots, u_{n-7}, u_{n-4}\}$ .
- If  $n \equiv 2 \pmod{3}$ , then  $S = \{u_0, u_3, u_6, \dots, u_{n-8}, u_{n-5}\}$ .

Notice that  $S$  is an independent set. Thus,  $\alpha(CR(n, 2)) \geq |S| = \lfloor \frac{n}{3} \rfloor$ . Now, let us suppose that  $\alpha(CR(n, 2)) > \lfloor \frac{n}{3} \rfloor$  and let  $S'$  be an independent set of maximum cardinality in  $CR(n, 2)$ . Hence there exist two vertices  $u_i, u_j \in S'$  such that either  $i = j + 1$ ,  $i = j - 1$ ,  $i = j + 2$  or  $i = j - 2$ , where the operations with the subindexes  $i, j$  are expressed modulo  $n$ . Thus,  $i - j \in \{-2, -1, 1, 2\}$  and, hence,  $u_i$  and  $u_j$  are adjacent, which is a contradiction. Therefore,  $\alpha(CR(n, 2)) = \lfloor \frac{n}{3} \rfloor$  and the proof is complete.  $\square$

The lemma above is particularly useful for our study, as we can see in the next result, since the strong resolving graph of  $C_r \times K_t$  contains several subgraphs which are isomorphic to a circulant graph.

**Theorem 37.** *For any positive integers  $r \geq 4$  and  $t \geq 3$ ,*

$$\dim_s(C_r \times K_t) = \begin{cases} t(r-1), & \text{if } r \in \{4, 5\}, \\ \frac{tr}{2}, & \text{if } r \text{ is even and } r \geq 6, \\ t(r - \lfloor \frac{r}{3} \rfloor), & \text{otherwise.} \end{cases}$$

*Proof.* Let  $V_1 = \{u_0, u_1, \dots, u_{r-1}\}$  and  $V_2 = \{v_1, v_2, \dots, v_t\}$  be the vertex sets of  $C_r$  and  $K_t$ , respectively. We assume  $C_r : u_0 u_1 u_2 \dots u_{r-1} u_0$ . Hereafter all the operations with the subindex of a vertex  $u_i$  of  $C_r$  are expressed modulo  $r$ . Let  $(u_i, v_j), (u_l, v_k)$  be two distinct vertices of  $C_r \times K_t$ .

Case 1: Let  $r = 4$  or  $5$ .

Subcase 1.1:  $u_i = u_l$ . Hence,  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$ . Since  $(u_i, v_j) \sim (u_{i-1}, v_k)$ , if  $k \neq j$  and  $d_{C_r \times K_t}((u_{i-1}, v_k), (u_l, v_k)) = 3$ , then it follows that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $C_r \times K_t$ .

Subcase 1.2:  $v_j = v_k$ . If  $l = i + 1$  or  $i = l + 1$ , then without loss of generality we suppose  $l = i + 1$  and we have that  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3 = D(C_r \times K_t)$ . Thus,  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $C_r \times K_t$ . On the other hand, if  $l \neq i + 1$  and  $i \neq l + 1$ , then  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$ . Since for every vertex  $(u, v) \in N_{C_r \times K_t}(u_i, v_j)$  we have that  $d_{C_r \times K_t}((u, v), (u_l, v_k)) \leq 2$  and also for every vertex  $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$  we have that  $d_{C_r \times K_t}((u, v), (u_i, v_j)) \leq 2$ , we obtain that  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $C_r \times K_t$ .

Subcase 1.3:  $u_i \neq u_l, v_j \neq v_k$  and  $(u_i, v_j) \sim (u_l, v_k)$ . So, there exists a vertex  $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$  such that  $d_{C_r \times K_t}((u, v), (u_i, v_j)) = 2$  and, as a consequence,  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $C_r \times K_t$ .

Subcase 1.4:  $u_i \neq u_l, v_j \neq v_k$  and  $(u_i, v_j) \not\sim (u_l, v_k)$ . Hence,  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$ . We can suppose, without loss of generality, that  $l = i + 2$ . Since we have

- $(u_i, v_j) \sim (u_{l-1}, v_k)$  and  $(u_l, v_k) \sim (u_{l-1}, v_j)$  and also,
- $d_{C_r \times K_t}((u_i, v_j), (u_{l-1}, v_j)) = 3$  and  $d_{C_r \times K_t}((u_l, v_k), (u_{l-1}, v_k)) = 3$ ,

we obtain that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $C_r \times K_t$ . Hence the strong resolving graph  $(C_r \times K_t)_{SR}$  is isomorphic to  $\bigcup_{i=1}^t K_r$ . Thus, by Theorem 1,

$$\dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^t K_r\right) = \sum_{i=1}^t \beta(K_r) = t(r - 1).$$

Case 2:  $r \geq 6$ . Let  $(u_i, v_j), (u_l, v_k)$  be two different vertices of  $C_r \times K_t$ .

Subcase 2.1:  $u_i = u_l$ . As in Subcase 1.1 it can be shown that  $(u_i, v_j), (u_l, v_k)$  are not mutually maximally distant.

Subcase 2.2:  $v_j = v_k$ . We consider the following further subcases.

- (a)  $l = i + 1$  or  $i = l + 1$ . Without loss of generality we assume  $l = i + 1$ . Hence, it follows  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3$ . Notice that  $N_{C_r \times K_t}(u_i, v_j) = \{u_{i-1}, u_{i+1}\} \times (V_2 - \{v_j\})$  and  $N_{C_r \times K_t}(u_l, v_k) = \{u_i, u_{i+2}\} \times (V_2 - \{v_k\})$ . Thus, for every vertex  $(u, v) \in N_{C_r \times K_t}(u_i, v_j)$  it follows that  $d_{C_r \times K_t}((u, v), (u_l, v_k)) \leq 2$  and for every vertex  $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$  it follows that  $d_{C_r \times K_t}((u, v), (u_i, v_j)) \leq 2$ . Hence,  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $C_r \times K_t$ .
- (b)  $l \neq i + 1, i \neq l + 1$  and  $d_{C_r}(u_i, u_l) < D(C_r)$ . So,  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = \min\{l - i, i - l\}$ . Since  $(u_i, v_j) \sim (u_{i-1}, v_q)$  with  $q \neq j$  and  $d_{C_r \times K_t}((u_{i-1}, v_q), (u_l, v_k)) = \min\{l - i + 1, i - l + 1\}$  we have that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $C_r \times K_t$ .
- (c)  $l \neq i + 1, i \neq l + 1$  and  $d_{C_r}(u_i, u_l) = D(C_r)$ . Thus,  $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = \min\{l - i, i - l\} = D(C_r) = \lfloor \frac{r}{2} \rfloor$  and, as a consequence, we have that  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $C_r \times K_t$ .

Subcase 2.3:  $u_i \neq u_l, v_j \neq v_k$  and  $d_{C_r}(u_i, u_l) < D(C_r)$ . As in Subcase 2.2(b) it can be shown that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $C_r \times K_t$ .

Subcase 2.4:  $u_i \neq u_l, v_j \neq v_k$  and  $d_{C_r}(u_i, u_l) = D(C_r)$ . As in Subcase 2.2(c) it can be shown that  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $C_r \times K_t$ .

From the above cases it follows that the strong resolving graph  $(C_r \times K_t)_{SR}$  has vertex set  $V_1 \times V_2$  and two vertices  $(u_i, v_j), (u_l, v_k)$  are adjacent in this graph if and only if either,  $(\min\{l - i, i - l\} = 1$  and  $j = k)$  or  $(\min\{l - i, i - l\} = D(C_r) = \lfloor \frac{r}{2} \rfloor$  and  $1 \leq j, k \leq t)$ . Next we obtain the vertex cover number of  $(C_r \times K_t)_{SR}$ .

If  $r$  is even, then every vertex  $(u_i, v_j)$  has  $t$  neighbors of type  $(u_{i+r/2}, v_l)$ ,  $1 \leq l \leq t$  and two neighbors  $(u_{i-1}, v_j), (u_{i+1}, v_j)$ . So,  $\beta((C_r \times K_t)_{SR}) \geq t\beta(C_r) = t\frac{r}{2}$ . On the other hand, if we take the set of vertices  $A = \{(u_i, v_j) : i \in \{0, 2, 4, \dots, r-2\}, j \in \{1, \dots, t\}\}$ , then every edge of  $(C_r \times K_t)_{SR}$  is incident to some vertex of  $A$ . So,  $A$  is a vertex cover and  $\beta((C_r \times K_t)_{SR}) \leq |A| = t\frac{r}{2}$ . Hence  $\beta((C_r \times K_t)_{SR}) = t\frac{r}{2}$ . Therefore

$$\dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = t\frac{r}{2}.$$

If  $r$  is odd, then every vertex  $(u_i, v_j)$  has  $t$  neighbors of type  $(u_{i+(r-1)/2}, v_l)$ ,  $t$  neighbors of type  $(u_{i+r/2}, v_l)$ ,  $1 \leq l \leq t$ , and the two neighbors  $(u_{i-1}, v_j), (u_{i+1}, v_j)$ . Thus for every  $k \in \{1, \dots, t\}$  it follows that

$$(u_0, v_k) \sim (u_{\frac{r-1}{2}}, v_k) \sim (u_{r-1}, v_k) \sim (u_{\frac{r-1}{2}-1}, v_k) \sim (u_{r-2}, v_k) \sim \dots \sim (u_1, v_k) \sim (u_{\frac{r+1}{2}}, v_k) \sim (u_0, v_k). \quad (1)$$

Also, since  $(u_0, v_k) \sim (u_1, v_k) \sim \dots \sim (u_{r-1}, v_k) \sim (u_0, v_k)$ , the graph  $G'$  formed from  $t$  disjoint copies of a circulant graph  $CR(r, 2)$  is a subgraph of  $(C_r \times K_t)_{SR}$ . By Lemma 36

$$\beta((C_r \times K_t)_{SR}) \geq t\beta(CR(r, 2)) = t(r - \alpha(CR(r, 2))) = t\left(r - \left\lfloor \frac{r}{3} \right\rfloor\right).$$

Now, we will rename the vertices of  $C_r$  according to the adjacencies in (1), *i.e.*,  $u'_0 = u_0, u'_1 = u_{\frac{r-1}{2}}, u'_2 = u_{r-1}, u'_3 = u_{\frac{r-1}{2}-1}, u'_4 = u_{r-2}, \dots, u'_{r-2} = u_1$  and  $u'_{r-1} = u_{\frac{r+1}{2}}$ . With this notation, we define a set  $B$ , of vertices of  $(C_r \times K_t)_{SR}$ , as follows:

- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, \dots, r-3, r-2\}, j \in \{1, \dots, t\}\}$ , if  $r \equiv 0 \pmod{3}$ .
- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, \dots, r-4, r-3, r-1\}, j \in \{1, \dots, t\}\}$ , if  $r \equiv 1 \pmod{3}$ .
- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, \dots, r-5, r-4, r-2, r-1\}, j \in \{1, \dots, t\}\}$ , if  $r \equiv 2 \pmod{3}$ .

Note that if  $(u, v), (x, y) \notin B$ , then  $(u, v) \not\sim (x, y)$  and, thus  $B$  is a vertex cover of  $(C_r \times K_t)_{SR}$ . Hence,  $\beta((C_r \times K_t)_{SR}) \leq |B| = t(r - \lfloor \frac{r}{3} \rfloor)$ , which leads to  $\beta((C_r \times K_t)_{SR}) = t(r - \lfloor \frac{r}{3} \rfloor)$ . Therefore, we have the following

$$\dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = t\left(r - \left\lfloor \frac{r}{3} \right\rfloor\right).$$

□

We finish the exposition of our results throughout the study of  $\dim_s(P_r \times K_t)$ .

**Theorem 38.** *For any positive integers  $r \geq 2$  and  $t \geq 3$ ,*

$$\dim_s(P_r \times K_t) = t\left\lfloor \frac{r}{2} \right\rfloor.$$

*Proof.* Let  $V_1 = \{u_1, u_2, \dots, u_r\}$  and  $V_2 = \{v_1, v_2, \dots, v_t\}$  be the vertex sets of  $P_r$  and  $K_t$ , respectively. We assume  $u_1 \sim u_2 \sim u_3 \sim \dots \sim u_r$  in  $P_r$ . If  $r = 2$ , then a vertex  $(u_i, v_j)$  in  $P_2 \times K_t$  is mutually maximally distant only with the vertex  $(u_l, v_j)$ , where  $i \neq l$ . So,  $(P_2 \times K_t)_{SR} \cong \bigcup_{m=1}^t K_2$ . Thus, by Theorem 1,

$$\dim_s(P_2 \times K_t) = \beta((P_2 \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^t K_2\right) = \sum_{i=1}^t \beta(K_2) = t.$$

If  $r = 3$ , then a vertex  $(u_i, v_j)$  in  $P_3 \times K_t$  is mutually maximally distant only with those vertices  $(u_l, v_j)$ , where  $i \neq l$ . Thus,  $(P_3 \times K_t)_{SR} \cong \bigcup_{m=1}^t K_3$  and, by Theorem 1,

$$\dim_s(P_3 \times K_t) = \beta((P_3 \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^t K_3\right) = \sum_{i=1}^t \beta(K_3) = t \left\lceil \frac{r}{2} \right\rceil.$$

From now on we suppose  $r \geq 4$ . Let  $(u_i, v_j), (u_l, v_k)$  be two different vertices of  $P_r \times K_t$ . We consider the following cases.

Case 1:  $u_i = u_l$ . Hence, it is satisfied that  $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$ . If  $i \neq 1$ , then  $(u_i, v_j) \sim (u_{i-1}, v_k)$  and  $d_{P_r \times K_t}((u_{i-1}, v_k), (u_l, v_k)) = 3$ . Also, if  $i = 1$ , then  $(u_i, v_j) \sim (u_{i+1}, v_k)$  and  $d_{P_r \times K_t}((u_{i+1}, v_k), (u_l, v_k)) = 3$ . Thus,  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $P_r \times K_t$ .

Case 2:  $v_j = v_k$  and, without loss of generality,  $i < l$ . We have the following cases.

- (a) If  $u_i \sim u_l$  in  $P_r$ , then  $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3$ . Let  $(u_a, v_b)$  be a vertex such that  $(u_i, v_j) \sim (u_a, v_b)$ . So,  $(a = i - 1$  or  $a = l)$  and  $b \neq j$ . Thus, for every  $(u_a, v_b)$  we have that  $d_{P_r \times K_t}((u_a, v_b), (u_l, v_k)) = 2 < 3 = d_{P_r \times K_t}((u_i, v_j), (u_l, v_k))$ . Now, let  $(u_c, v_d)$  be a vertex such that  $(u_l, v_k) \sim (u_c, v_d)$ . So,  $(c = i$  or  $c = l + 1)$  and  $d \neq j$ . Thus, for every  $(u_c, v_d)$  we have that  $d_{P_r \times K_t}((u_c, v_d), (u_i, v_j)) = 2 < 3 = d_{P_r \times K_t}((u_i, v_j), (u_l, v_k))$ . Therefore,  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $P_r \times K_t$ .
- (b) If  $u_i \not\sim u_l$  in  $P_r$ , then  $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = |i - l|$ . Now, if  $u_i \neq u_1$ , then for every vertex  $(u_{i-1}, v_p)$ ,  $p \neq j$ , we have that  $(u_i, v_j) \sim (u_{i-1}, v_p)$  and  $d_{P_r \times K_t}((u_{i-1}, v_p), (u_l, v_k)) = |i - l + 1|$ . Similarly, if  $u_l \neq u_r$ , then for every vertex  $(u_{l+1}, v_p)$ ,  $p \neq j$ , we have that  $(u_l, v_k) \sim (u_{l+1}, v_p)$  and  $d_{P_r \times K_t}((u_{l+1}, v_p), (u_i, v_j)) = |i - l + 1|$ . Thus, we obtain that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $P_r \times K_t$ .
- (c) If  $u_i = u_1$  and  $u_l = u_r$ , then  $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = r - 1 = D(P_r \times K_t)$ . Thus,  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $P_r \times K_t$ .

Case 3:  $u_i \neq u_l, v_j \neq v_k$  and we consider, without loss of generality,  $i < l$ . If  $u_i \neq u_1$  or  $u_l \neq u_r$ , then as in Case 2 (b) it follows that  $(u_i, v_j)$  and  $(u_l, v_k)$  are not mutually maximally distant in  $P_r \times K_t$ . On the other hand, if  $u_i = u_1$  and  $u_l = u_r$ , then as in Case 2 (c) it follows that  $(u_i, v_j)$  and  $(u_l, v_k)$  are mutually maximally distant in  $P_r \times K_t$ .

Therefore,  $(P_r \times K_t)_{SR}$  is isomorphic to a graph with vertex set  $V_1 \times V_2$  and such that two vertices  $(u_i, v_j), (u_l, v_k)$  are adjacent if and only if either,  $(|l - i| = 1$  and  $j = k)$  or  $(|l - i| = r - 1$  and  $1 \leq j, k \leq r)$ . Notice that every vertex  $(u_i, v_j)$ , where  $1 < i < r$ , has only two neighbors

$(u_{i-1}, v_j)$  and  $(u_{i+1}, v_j)$ , while every vertex  $(u_1, u_j)$  has a neighbor  $(u_2, u_j)$  and  $r$  neighbors of type  $(u_r, u_l)$ ,  $1 \leq l \leq t$ . Also, every vertex  $(u_r, u_j)$  has a neighbor  $(u_{r-1}, u_j)$  and  $r$  neighbors of type  $(u_1, u_l)$ ,  $1 \leq l \leq t$ . So,  $(P_r \times K_t)_{SR}$  has a subgraph  $G'$  isomorphic to the disjoint union of  $t$  cycles of order  $r$  and, as a consequence,  $\beta((P_r \times K_t)_{SR}) \geq t\beta(C_r) = t \lceil \frac{r}{2} \rceil$ .

On the other hand, let  $r$  be an even number. If we take the set of vertices  $A = \{(u_i, v_j) : i \in \{1, 3, 5, \dots, r-1\}, j \in \{1, \dots, t\}\}$ , then every edge of  $(P_r \times K_t)_{SR}$  is incident to some vertex of  $A$ . Thus,  $A$  is a vertex cover of  $(P_r \times K_t)_{SR}$  and we have that  $\beta((P_r \times K_t)_{SR}) \leq |A| = t \lceil \frac{r}{2} \rceil$ . Now, suppose  $r$  odd. If we take the set of vertices  $B = \{(u_i, v_j) : i \in \{1, 3, 5, \dots, r\}, j \in \{1, \dots, t\}\}$ , then every edge of  $(P_r \times K_t)_{SR}$  is incident to some vertex of  $B$ . So,  $B$  is a vertex cover of  $(P_r \times K_t)_{SR}$  and thus  $\beta((P_r \times K_t)_{SR}) \leq |B| = t \lceil \frac{r}{2} \rceil$ . Hence  $\beta((P_r \times K_t)_{SR}) = t \lceil \frac{r}{2} \rceil$ . Therefore, from Theorem 1,

$$\dim_s(P_r \times K_t) = \beta((P_r \times K_t)_{SR}) = t \lceil \frac{r}{2} \rceil.$$

□

## 4 Concluding Remarks

Motivated by the fact that the problem of finding the strong metric dimension of a graph is NP-hard, we studied this invariant for the Cartesian product and the direct product of graphs using a transformation of this problem to the vertex cover problem described in [30]. It appears to be an easier problem to find the strong metric dimension of a graph than its metric dimension. There are classes of graphs, such as the distance hereditary graphs for which an efficient process has been described that finds the strong metric dimension but for which no such algorithm is known for finding the metric dimension. However, whether or not there is a class of graphs for which the strong metric dimension can be found in polynomial time but for which metric dimension is NP-hard remains an open problem.

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