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On the impact of sup-compositions in the resolution of multi-adjoint relation equations

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Multi-adjoint relation equations are defined by means of a sup-composition operator involving different conjunctions. Former works reveal that the resolution of a multi-adjoint relation equation is closely related to such conjunctions. This paper presents a first approach on the study of the influence of the selection of a sup-composition operator in different aspects of multi-adjoint relation equations, as their solvability or the extremal elements of their solution set. The obtained results are illustrated by an example concerning multi-adjoint logic programming. Last but not least, a procedure for reducing the computation of the solution set when only the sup-composition operator is modified is also introduced.

KEYWORDS

adjoint triple, concept lattice, fuzzy relation equation, multi-adjoint lattice

MSC CLASSIFICATION

06A15, 08A72, 03E72

1 | INTRODUCTION

Fuzzy relation equations (FRE) are one of the most common fuzzy tools in processes involving approximated reasoning and dealing with uncertainty [1–5]. Its relevance has motivated a deep research on resolution techniques [6–9]. Simultaneously, a large number of applications for FRE have arisen in multiple fields such as logic programming [10], abductive reasoning [11], optimization [12], medical diagnosis [13], and bipolar information [14–16].

The approach presented in Díaz-Moreno and Medina [17] defines FRE within a multi-adjoint framework, called multi-adjoint relation equations (MAREs), in which different conjunctions are involved in the sup-composition operator. Different results on solving MARE are shown in Díaz-Moreno and Medina [18] in terms of a property-oriented concept lattice, establishing a link between formal concept analysis (FCA) and FRE.

The consideration of MARE instead of FRE increases the flexibility of the setting and therefore enlarges the number of problems that can be potentially modeled. A question that naturally arises in the process of modeling a real-world problem is: Once we define a first model of the problem, how to improve the model? In our framework, this implies wondering how the solution set of a MARE can help to improve the model. The present paper aims at providing a first insight into this matter.

Recently, it was shown in Lobo et al. [19] that modifying the sup-composition of a MARE has some effects on its greatest solution and its minimal solutions. In particular, a larger composition entails a smaller greatest solution and vice versa, while their minimal solutions might be incomparable. This paper continues this philosophy complementing these results and giving detailed examples. We will show that a modification of a specific conjunction in the sup-composition of a MARE by a greater or smaller conjunction reverses the ordering in the corresponding component of its greatest solution. As a result, it is possible to either increase or decrease a component of the vector of the greatest solution of a MARE by conveniently modifying the corresponding conjunction in the sup-composition operator. Moreover, we will show that this last property also holds for minimal solutions.

The ample range of applications of MARE makes possible applying these results to various real problems. In order to illustrate this fact, this paper contains a full example where the modeling strategy of an abduction problem is discussed and modified according to some observed values, using the results previously presented. This abduction problem is based on multi-adjoint logic programming (MALP), a framework that is strongly related to MARE [10].

The case of a modification in the sup-composition of a MARE giving rise to an unsolvable MARE is also analyzed in the paper. In such case, the resulting MARE is *approximated* by a solvable MARE according to the optimistic and conservative procedures presented in Cornejo et al. [10]. Finally, we deal with the goal of computing the solution set of a MARE from the solution set of another one with a different composition. Thus, this technique enables a reduction in the number of operations required for the computation of the solution set of the second MARE.

The paper is organized as follows. First, in Section 2, some preliminary definitions and the results that connect FRE with FCA are recalled. In Section 3, the selection of conjunctions in a MARE will be studied, including its effects in greatest and minimal solutions. This section will also include an example showing its usability. In Section 4, a procedure for optimizing the process of resolution of a MARE once its sup-composition operator has been changed will be shown, as well as an example which graphically shows how this procedure works. This paper ends with some conclusions and proposals for future work.

2 | PRELIMINARIES

In this section, we will recall some notions and results concerning FCA [20, 21] and FRE [7, 17, 18]. In both cases, the theory of adjoint pairs plays a key role.

Hájek showed that the semantic interpretation of Modus Ponens can be extended to fuzzy sets through a left-continuous t-norm and its residuated implication [22]. In this paper, we will employ a generalized version of residuated pairs, known as adjoint triples [23, 24].

Definition 1 (Cornejo et al. [24]). Let $(P_1, \preceq_1)$, $(P_2, \preceq_2)$, $(P_3, \preceq_3)$ be posets and $\& : P_1 \times P_2 \to P_3$, $\swarrow : P_3 \times P_2 \to P_1$, $\nwarrow : P_3 \times P_1 \to P_2$ mappings; then, $(\&, \swarrow, \nwarrow)$ is called an adjoint triple with respect to P_1, P_2, P_3 if

$$x \preceq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \preceq_3 z \quad \text{iff} \quad y \preceq_2 z \nwarrow x,$$

for each $x \in P_1$, $y \in P_2$, $z \in P_3$.

Notice that if the conjunction & is conmutative and the residuated implications exist, Definition 1 implies that $\nwarrow = \swarrow$.

A partial order can be defined between two conjunctions $\&_1, \&_2 : P_1 \times P_2 \to P_3$ in a natural way: $\&_1 \preceq \&_2$ if $x \,\&_1 y \preceq_3 x \,\&_2\, y$ for all $x \in P_1, y \in P_2$. If the conjunctions $\&_1$,$\&_2$ belong to adjoint triples, the ordering $\preceq$ is reversed for their residuated implications, as the following result states.

Proposition 2. *Let* $(\&_1, \swarrow^1, \nwarrow_1)$, $(\&_2, \swarrow^2, \nwarrow_2)$ *be adjoint triples such that* $\&_1 \preceq \&_2$. *Then,* $z \swarrow^2 y \preceq_1 z \swarrow^1 y$ *and* $z \nwarrow_2 x \preceq_2 z \nwarrow_1 x$, *for all* $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

Proof. We will prove the inequality concerning $\swarrow^1$ and $\swarrow^2$. Let $y \in P_2$ and $z \in P_3$. By Definition 1, from $z \swarrow^2 y \preceq_1 z \swarrow^2 y$, we obtain that

$$(z \swarrow^2 y) \,\&_2\, y \preceq z$$

Now, since $\&_1 \preceq_3 \&_2$, the following chain holds:

$$(z \swarrow^2 y) \,\&_1\, y \preceq_3 (z \swarrow^2 y) \,\&_2\, y \preceq_3 z.$$

Again, applying Definition 1, the previous is equivalent to the inequality

$$z \swarrow^2 y \preceq_1 z \swarrow^1 y.$$

□

The following example shows different adjoint triples on the unit interval.

Example 3. The Gödel t-norm is part of an adjoint triple $(\&_G, \swarrow^G, \nwarrow_G)$, where $\&_G, \swarrow^G, \nwarrow_G\colon [0,1]\times[0,1] \to [0,1]$ are defined as follows:

$$x \&_G y = \min\{x, y\} \quad z \swarrow^G y = \begin{cases} z & \text{if } z < y \\ 1 & \text{if } y \le z \end{cases} \quad \text{and } \swarrow^G = \nwarrow_G.$$

Also, the product t-norm belongs to an adjoint triple $(\&_P, \swarrow^P, \nwarrow_P)$, where the mappings $\&_P, \swarrow^P, \nwarrow_P\colon [0,1]\times[0,1] \to [0,1]$ are

$$x \&_P y = xy \quad z \swarrow^P y = \begin{cases} \frac{z}{y} & \text{if } z < y \\ 1 & \text{if } y \le z \end{cases} \quad \text{and } \swarrow^P = \nwarrow_P.$$

In the same way, the Łukasiewickz t-norm leads to the adjoint triple $(\&_Ł, \swarrow^Ł, \nwarrow_Ł)$, where $\&_Ł, \swarrow^Ł, \nwarrow_Ł\colon [0,1]\times[0,1] \to [0,1]$ are defined as

$$x \&_Ł y = \max\{0, x + y - 1\} \quad z \swarrow^Ł y = \min\{1 - y + z, 1\} \quad \text{and } \swarrow^Ł = \nwarrow_Ł.$$

Notice that the following chain of inequalities holds [25]:

$$\&_Ł \le \&_P \le \&_G.$$

As a result, by Proposition 2, we have that

$$\swarrow^G \le \swarrow^P \le \swarrow^Ł \text{ and that } \nwarrow_G \le \nwarrow_P \le \nwarrow_Ł.$$

Commutativity is not a necessary property in the conjunction of an adjoint triple. For example, if we consider $\& \colon [0,1]\times[0,1] \to [0,1]$, defined as $x \& y = x^2 y$, it is part of an adjoint triple $(\&, \swarrow, \nwarrow)$ where

$$z \swarrow y = \begin{cases} 1 & \text{if } y = 0 \\ \min\left\{\sqrt{\frac{z}{y}}, 1\right\} & \text{else} \end{cases} \quad \text{and} \quad z \nwarrow x = \begin{cases} 1 & \text{if } x = 0 \\ \min\left\{\frac{z}{x^2}, 1\right\} & \text{else.} \end{cases}$$

□

In the different examples contained in the paper, for the sake of simplicity, we will consider the discretization of adjoint triples presented in Example 3 in a partition of the unit interval.

Namely, the discretization of a t-norm in a partition of the form[1] $[0,1]_m$ can be defined in a general form [26]. Moreover, given an adjoint triple $(\&, \swarrow, \nwarrow)$, the triplet $(\overline{\&}, \overline{\swarrow}, \overline{\nwarrow})$ is an adjoint triple with respect to $[0,1]_n$, $[0,1]_m$ and $[0,1]_k$, as it is shown in Cornejo et al. [27].

The following result shows that the existing ordering on $\&_Ł, \&_P, \&_G$ can be extended to the discretized conjunctions.

Proposition 4. *Let $[0,1]_n$ be the regular partition of the unit interval in n pieces and $\overline{\&_G}$, $\overline{\&_P}$, and $\overline{\&_Ł}$ the discretization of the Gödel, product, and Łukasiewicz t-norms, respectively. It holds that $\overline{\&_Ł} \preceq \overline{\&_P} \preceq \overline{\&_G}$.*

A multi-adjoint property-oriented frame is an algebraic structure formed by two lattices and a poset endowed with a set of adjoint triples.

Definition 5 (Medina [28]). Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be two lattices, $(P, \preceq)$ a poset, and $\left\{(\&_i, \swarrow^i, \nwarrow_i) \mid i \in \{1, \dots, n\}\right\}$ a set of adjoint triples with respect to P, L_2, L_1. The tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \preceq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

[1] $[0,1]_m$ denotes a regular partitions of $[0,1]$ into m pieces, for example, $[0,1]_4 = \{0, 0.25, 0.5, 0.75, 1\}$.

is called multi-adjoint property-oriented frame.

In the particular case in which $L_1 = L_2 = P$, the property-oriented multi-adjoint frame will be denoted as

$$(P, \preceq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n).$$

The notion of context is a formal interpretation of a dataset. It consists of two sets, a fuzzy relation between them, and a mapping that assigns (the index of) an adjoint triple to each pair of elements formed by one element of each set.

Definition 6 (Medina [28]). Let $(L_1, L_2, P, \preceq_1, \preceq_2, \preceq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ be a property-oriented multi-adjoint frame. A *context* is a tuple (A, B, R, σ) where A and B are non-empty sets, $R : A \times B \to P$ is a fuzzy relation, and $\sigma : A \times B \to \{1, \dots, n\}$ is a mapping.

In Definition 6, A is interpreted as a set of attributes and B as a set of objects. Hence, the mapping σ assigns an adjoint triple to each pair attribute-object. In the next definitions, we will fix a property-oriented multi-adjoint frame $(L_1, L_2, P, \preceq_1, \preceq_2, \preceq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ and a context (A, B, R, σ).

Consider the fuzzy subsets of attributes $L_1^A = \{f \mid f : A \to L_1\}$ and the fuzzy subsets of objects $L_2^B = \{g \mid g : B \to L_2\}$. It is possible generalizing the necessity and possibility operators defined in Düntsch and Gediga [29] by means of the mappings ${}^{\uparrow_\pi} : L_2^B \to L_1^A$ and ${}^{\downarrow^N} : L_1^A \to L_2^B$, which are defined as

$$g^{\uparrow_\pi}(a) = \bigvee\nolimits_1 \left\{R(a, b) \ \&_{\sigma(a,b)} g(b) \mid b \in B\right\}, \tag{1}$$

$$f^{\downarrow^N}(b) = \bigwedge\nolimits_2 \left\{f(a) \nwarrow_{\sigma(a,b)} R(a, b) \mid a \in A\right\}, \tag{2}$$

for each $f \in L_1^A$ and $g \in L_2^B$, where $\bigvee_1$ and $\bigwedge_2$ represent the supremum and infimum of subsets of the lattices $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$, respectively. These operators were introduced in Medina [28] together with the main property of the pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$, that is, it forms an isotone Galois connection. This property gives rise to the notion of *multi-adjoint property-oriented concept lattice*. Consider the order relation $\preceq_{\pi N}$ defined as $(g_1, f_1) \preceq_{\pi N} (g_2, f_2)$ if and only if $f_1 \preceq_1 f_2$, or equivalently, if and only if $g_1 \preceq_2 g_2$. The multi-adjoint property-oriented concept lattice associated with the multi-adjoint property-oriented frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ is defined as follows:

$$\mathcal{M}_{\pi N}(A, B, R, \sigma) = \left\{\langle g, f\rangle \in L_2^B \times L_1^A \mid g = f^{\downarrow^N}, f = g^{\uparrow_\pi}\right\}. \tag{3}$$

The set $\mathcal{M}_{\pi N}$ with the order $\preceq_{\pi N}$ is a complete lattice [28]. Given a concept $\langle g, f\rangle$, g is called the *extent* of the concept and f is called the *intent*. The set of all extents will be denoted as $\mathcal{E}(\mathcal{M}_{\pi N})$ and the set of all intents as $\mathcal{I}(\mathcal{M}_{\pi N})$. For further details, we refer the reader to Medina [28].

Next, some notions related to MARE will be recalled, as well as their relationship with FCA. A detailed study of this relation can be found in Díaz-Moreno and Medina [17]. From now on, we will fix a property-oriented multi-adjoint frame:

$$(L_1, L_2, P, \preceq_1, \preceq_2, \preceq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n),$$

three sets U, V, W and three fuzzy relations $R \in P^{U\times V}$, $S \in L_2^{V\times W}$, $T \in L_1^{U\times W}$.

A MARE is an expression of the form $R \odot S = T$, where $\odot$ is a multi-adjoint composition operator and either R or S is unknown. In this work, two different compositions will be considered, having both a multi-adjoint nature.

Definition 7 (Díaz-Moreno and Medina [17]). Given a mapping $\sigma : V \to \{1, \dots, n\}$.

1. The operator $\odot_\sigma : P^{U\times V} \times L_2^{V\times W} \to L_1^{U\times W}$ defined as

$$R \odot_\sigma S(u, w) = \bigvee\nolimits_1 \left\{R(u, v) \&_{\sigma(v)} S(v, w) \mid v \in V\right\} \tag{4}$$

is called *sup-$\&_\sigma$-composition*.

2. The operator $\lhd_\sigma : L_1^{U\times W} \times P^{U\times V} \to L_2^{V\times W}$ defined as

$$T \lhd_\sigma R(v,w) = \bigwedge\nolimits_2 \{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\} \tag{5}$$

is called *inf-*$\nwarrow_\sigma$*-composition*.

Different equations can be defined using the sup-compositions from Definition 7, depending on which is the unknown fuzzy relation.

Definition 8 (Díaz-Moreno and Medina [17]). A MARE with sup-$\&_\sigma$-composition is an equality of the form

$$R \odot_\sigma X = T, \tag{6}$$

or of the form

$$Y \odot_\sigma S = T, \tag{7}$$

where, in both cases, $X \in L_2^{V\times W}, Y \in L_1^{U\times V}$ are unknown fuzzy relation.

Notice that, we need L_1 and L_2 to be complete lattices if we want expressions (4) and (5) to be well-defined.

In this paper, we will restrict our study to MARE of the form (6). Nevertheless, a dual version of the results with respect to MARE (7) can be obtained, considering a similar strategy to the one used in Díaz-Moreno and Medina [17, 18].

Notice that the equation $R \odot_\sigma X = T$ can be written as a set of systems of equations. Specifically, for each $w \in W$, we have the following system:

$$\begin{array}{ccccccc} a_{11} \&_1 x_1 & \vee & \dots & \vee & a_{1m} \&_m x_m & = & t_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} \&_1 x_1 & \vee & \dots & \vee & a_{nm} \&_m x_m & = & t_n \end{array} \tag{8}$$

where $a_{ij} = R(u_i, v_j)$, $X(v_j, w) = x_j$, $T(u_i, w) = t_i$, and $\&_j = \&_{\sigma(v_j)}$, for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.

We say that MARE (6) is *solvable* if there exists at least one solution, that is, a relation $X \in L_2^{V\times W}$ satisfying Equation (6). Otherwise, we say it is *unsolvable*.

If we want to use the procedures that were introduced in Díaz-Moreno and Medina [17], the first step for solving a MARE is establishing a correspondence between a MARE and a multi-adjoint context.

Definition 9 (Díaz-Moreno and Medina [17]). The multi-adjoint context associated with the MARE $R \odot_\sigma X = T$ is the multi-adjoint context (U, V, R, σ).

The relation between a MARE and its associated context entails a necessary and sufficient condition for the solvability of the MARE in terms of the Galois connection $({}^{\uparrow_\pi}, {}^{\downarrow^N})$. Moreover, if it is solvable, the greatest solution of the MARE is also characterized.

Proposition 10 (Díaz-Moreno and Medina [17]). *Let $R \odot_\sigma X = T$ be a MARE and $(\mathcal{M}_{\pi N}, \preceq_{\pi N})$ its associated context. Then, $R \odot_\sigma X = T$ is solvable if and only if $T_w \in \mathcal{I}(\mathcal{M}_{\pi N})$ for all $w \in W$. In that case, $T \lhd_\sigma R \in L_2^{V\times W}$ is the greatest solution, being $(T \lhd_\sigma R)_w = T_w^{\downarrow^N}$ for each $w \in W$.*

Notice that there exists an equivalency between the condition $T_w \in \mathcal{I}(\mathcal{M}_{\pi N})$ and the equality $T_w^{\downarrow^N \uparrow_\pi} = T_w$. The following corollary provides a direct way for computing the greatest solution of a solvable MARE.

Corollary 11 (Díaz-Moreno and Medina [17]). *Let $R \odot_\sigma X = T$ be a solvable MARE. Its greatest solution $\overline{X}$ is given by*

$$\overline{X}(v,w) = \bigwedge\nolimits_2 \{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\}$$

for each $(v,w) \in V \times W$

Let $(L, \preceq)$ be a lattice and $x \in L$. The set of lower bounds of x, i.e, $\{a \in L \mid a \preceq x\}$ will be denoted as $(x]$. In this framework, we are specially interested in the set of predecessors of a concept:

$$\text{Pre}_L(x) = \{x' \in L \mid x' \prec x \text{ and there is no } x'' \in L \text{ such that } x' \prec x'' \prec x\}.$$

The importance of the set of predecessors relies on the strong relation that they have with the solution set of a MARE. The last theorem of this section characterizes the whole solution set of a MARE in terms of its associated concept lattice.

Theorem 12 (Díaz-Moreno and Medina [18]). *Let $R \odot_\sigma X = T$ be a solvable MARE and $(\mathcal{M}_{\pi N}, \preceq_{\pi N})$ its associated context. If U, V are finite sets, the solution set of the MARE is*

$$\left\{X \in L_2^{V\times W} \mid X_w \in \left(T_w^{\downarrow^N}\right] \setminus \bigcup \left\{(g] \mid g \in Pre_{\mathcal{E}(\mathcal{M}_{\pi N})}\left(T_w^{\downarrow^N}\right)\right\} \text{ for each } w \in W\right\}.$$

Theorem 12 enables to obtain the whole solution set of any solvable MARE.

3 | ON THE EFFECTS OF SUP-COMPOSITIONS IN MULTI-ADJOINT RELATION EQUATIONS

The multi-adjoint paradigm provides great flexibility in the selection of the conjunctions involved in a reasoning process. In the literature, this flexibility has been usually used with the aim of expressing preferences in environments related to FCA [21]. In this way, desired attributes or objects are assigned to conjunctions that produce higher values, while the rest are assigned to other conjunctions that produce lower values.

In this section, we will also consider fixed a property-oriented multi-adjoint frame:

$$(L_1, L_2, P, \preceq_1, \preceq_2, \preceq_3, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n),$$

three sets U, V, W and the fuzzy relations R, T.

In the case of MARE, as several conjunctions are involved in the sup-composition operator, they can be conveniently selected following a similar philosophy to FCA. The selection of such composition may have a direct impact in the resolution of the MARE. For instance, not only the solvability but the greatest or minimal solutions could be affected. In order to elucidate the scope of these consequences, we devote this section to study the effects of changing the sup-composition operator of a MARE.

As a first insight into the topic, the results shown in Lobo et al. [19] focus on the impact of a sup-composition alteration in the solution set of a MARE. The first result that is presented in Lobo et al. [19] states that considering larger conjunctions will decrease the greatest solution of a MARE. This occurs because greatest solutions are computed by means of implications, which reverse the ordering of conjunctions (Proposition 2). Analogously, considering lower conjunctions will increase the value of the greatest solution.

In what follows, we define a partial order in the set of sup-composition operators, which is induced by the partial order in the set of conjunctions. The whole structure of adjoint triples is studied in Cornejo et al. [24]. Given two sup-composition operators $\odot_\sigma, \odot_{\sigma'} : P^U \times L_2^V \to L_1^W$, we say that $\odot_\sigma \preceq \odot_{\sigma'}$ if $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$ for all $v \in V$.

Proposition 13 (Lobo et al. [19]). *Let $R \odot_\sigma X = T$ and $R \odot_{\sigma'} X = T$ be two solvable MARE such that $\odot_\sigma \preceq \odot_{\sigma'}$ and let $\overline{X}_\sigma, \overline{X}_{\sigma'}$ be their greatest solutions respectively. Then, $\overline{X}_{\sigma'} \preceq_2 \overline{X}_\sigma$.*

Proposition 13 applies when the sup-composition operators are ordered, what means that all conjunctions are ordered in the same way, that is, $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$ for all $v \in V$ or $\&_{\sigma'(v)} \preceq \&_{\sigma(v)}$ for all $v \in V$. In the following result, this condition will be weakened, so that for each pair of ordered conjunctions, their order is inverted in the corresponding argument in the greatest solution.

Theorem 14. *Let $R \odot_\sigma X = T$ and $R \odot_{\sigma'} X = T$ be two solvable MARE and let $\overline{X}_\sigma, \overline{X}_{\sigma'}$ be their greatest solutions respectively. For all $v \in V, w \in W$*

a. *If $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$, then $\overline{X}_{\sigma'}(v, w) \preceq_2 \overline{X}_\sigma(v, w)$.*
b. *If $\&_{\sigma'(v)} \preceq \&_{\sigma(v)}$, then $\overline{X}_\sigma(v, w) \preceq_2 \overline{X}_{\sigma'}(v, w)$.*

Proof. By Corollary 11, for all $v \in V, w \in W$,

$$\overline{X}_\sigma(v, w) = \bigwedge\nolimits_2 \{T(u, w) \nwarrow_{\sigma(v)} R(u, v) \mid u \in U\},$$
$$\overline{X}_{\sigma'}(v, w) = \bigwedge\nolimits_2 \{T(u, w) \nwarrow_{\sigma'(v)} R(u, v) \mid u \in U\}.$$

Consider fixed $v \in V$, $w \in W$ and suppose that $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$. Applying Proposition 2, for all $u \in U$,

$$T(u,w) \nwarrow_{\sigma'(v)} R(u,v) \preceq_2 T(u,w) \nwarrow_{\sigma(v)} R(u,v).$$

This implies that every lower bound of $\{T(u,w) \nwarrow_{\sigma'(v)} R(u,v) \mid u \in U\}$ is also a lower bound of $\{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\}$. Consequently, because of the definition of infimum,

$$\begin{aligned}\overline{X}_{\sigma'}(v,w) &= \bigwedge\nolimits_2 \{T(u,w) \nwarrow_{\sigma'(v)} R(u,v) \mid u \in U\} \\ &\preceq_2 \bigwedge\nolimits_2 \{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\} = \overline{X}_{\sigma}(v,w).\end{aligned}$$

Hence, as $w \in W$ was fixed, $\overline{X}_{\sigma'}(v,w) \preceq_2 \overline{X}_{\sigma}(v,w)$ for all $w \in W$. The case in which $\&_{\sigma'(v)} \preceq \&_{\sigma(v)}$ is analogous. □

Among others, Theorem 14 can be applied in cases in which the sup-compositions $\odot_\sigma$ and $\odot_{\sigma'}$ are incomparable, but the underlying conjunctions preserve some kind of ordering. The following example illustrates how Theorem 14 can be applied.

Example 15. Consider the sets $U = \{u_1, u_2\}$, $V = \{v_1, v_2, v_3\}$, $W = \{w\}$ and the multi-adjoint frame

$$([0,1]_8, \leq, \overline{\&_Ł}, \overline{\swarrow}^{Ł}, \overline{\nwarrow_Ł}, \overline{\&_P}, \overline{\swarrow}^{P}, \overline{\nwarrow_P}, \overline{\&_G}, \overline{\swarrow}^{G}, \overline{\nwarrow_G}), \tag{9}$$

where $\overline{\&_Ł}, \overline{\&_P}, \overline{\&_G}$ are the discretization of the Łukasiewick t-norm, the product t-norm, and the Gödel t-norm, respectively. Let

$$R = \begin{pmatrix} 0.5 & 0.5 & 0.25 \\ 0.5 & 1 & 0.75 \end{pmatrix} \qquad T = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix},$$

and consider the MARE

$$R \odot_{\sigma_1} X = T, \tag{10}$$

where $\sigma_1(v_1) = \text{P}$, $\sigma_1(v_2) = \text{G}$, and $\sigma_1(v_3) = Ł$.

If it is solved, it is obtained that its greatest solution is

$$\overline{X}_{\sigma_1} = \begin{pmatrix} 0.5 \\ 0.25 \\ 0.75 \end{pmatrix}.$$

If we wanted now to decrease the value of the first and the third variables and to increase the value of the second one, by Theorem 14, it is possible defining $\sigma_2(v_1) = \text{G}$, $\sigma_2(v_3) = \text{G}$, so that they are assigned to greater conjunctions and $\sigma_2(v_2) = Ł$, so that it is assigned to a lower conjunction.

If we consider now the MARE

$$R \odot_{\sigma_2} X = T, \tag{11}$$

it is obtained that its greatest solution is

$$\overline{X}_{\sigma_2} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.5 \end{pmatrix},$$

which verifies the conditions that were required. □

Theorem 14 does not ensure anything when conjunctions are not comparable. This example shows that, if this occurs, the behavior of the variables cannot be predicted.

Example 16. Consider the sets $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, $W = \{w\}$ and the multi-adjoint frame

$$\left([0,1]_{16}, \leq, \overline{\&_P}, \overline{\swarrow}^{P}, \overline{\nwarrow_P}, \overline{\&_2}, \overline{\swarrow}^{2}, \overline{\nwarrow_2}, \overline{\&_3}, \overline{\swarrow}^{3}, \overline{\nwarrow_3}\right), \tag{12}$$

where $\overline{\&_P}$ is the discretization of the product t-norm, $\&_2$ is the discretization of the conjuntion defined as $\&_2(x,y) = x^2\sqrt{y}$, and $\&_3$ is the discretization of the conjuntion defined as $\&_3(x,y) = \sqrt{x}y^2$. It can be easily checked that $\&_P \| \&_2$ and $\&_P \| \&_3$. Let

$$R = \begin{pmatrix} 0.5 & 0.25 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 0.25 \end{pmatrix},$$

and consider the MARE

$$R \odot_{\mathrm{P}} X = T. \tag{13}$$

If we calculate its greatest solution, we obtain the following:

$$\overline{X}_{\mathrm{P}} = \begin{pmatrix} 0.5 \\ 1 \\ 0.25 \end{pmatrix}.$$

If we now define $\sigma'(v_1) = 2$, $\sigma'(v_2) = 3$, $\sigma'(v_3) = 2$ and consider the MARE

$$R \odot_{\sigma'} X = T. \tag{14}$$

It is clear that Theorem 14 cannot be applied, because of the incomparability of the different conjunctions. Its greatest solution is

$$\overline{X}_{\sigma'} = \begin{pmatrix} 1 \\ 0.6875 \\ 0.0625 \end{pmatrix}.$$

As a consequence, it has been shown that nothing can be assured about the order when conjunctions are not comparable. $\&_2$ has increased the first variable, but has decreased the third, while $\&_3$ has decreased the second variable. □

Clearly, Proposition 13 and Theorem 14 only make sense when a solvable MARE remains solvable after the change in the sup-composition operator. However, the solvability of a MARE might be affected by such a modification. To be precise, changing the sup-composition in a solvable MARE could lead to an unsolvable MARE.

In that case, following the approach presented in Cornejo et al. [10], it is possible to recover the solvability of a MARE by modifying its right-hand side. Specifically, we will focus on increasing or decreasing all terms of the right-hand side. Given an unsolvable MARE $R \odot_\sigma X = T$, if the resulting MARE $R \odot_\sigma X = T^*$ is solvable, we say that it is an *approximation* of the unsolvable MARE. Besides, we say that it is an *optimistic* approximation if $T \preceq T^*$ and a *conservative/pessimistic* approximation if $T^* \preceq T$. In Cornejo et al. [10], two different procedures are presented to define conservative and optimistic approximations of an unsolvable MARE.

Given an optimistic or a conservative approximation of an unsolvable MARE, it remains to determine under which conditions an existing ordering in the underlying conjunctions entails an ordering relation in the corresponding component of the greatest solutions. The following result sheds lights on this matter.

Theorem 17. *Let $R \odot_\sigma X = T$ be a solvable MARE, $R \odot_{\sigma'} X = T$ an unsolvable MARE, and let $R \odot_{\sigma'} X = T^*$ be an approximation of $R \odot_{\sigma'} X = T$. Given $\overline{X}_\sigma$ the greatest solution of $R \odot_\sigma X = T$ and $\overline{X}^*_{\sigma'}$ the greatest solution of $R \odot_{\sigma'} X = T^*$, for all $v \in V, w \in W$, the following statements hold:*

a) If $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$ and $R \odot_{\sigma'} X = T^*$ is a conservative approximation, then $\overline{X}^*_{\sigma'}(v, w) \preceq_2 \overline{X}_\sigma(v, w)$.
b) If $\&_{\sigma'(v)} \preceq \&_{\sigma(v)}$ and $R \odot_{\sigma'} X = T^*$ is an optimistic approximation, then $\overline{X}_\sigma(v, w) \preceq_2 \overline{X}^*_{\sigma'}(v, w)$.

Proof. We will only prove a, as b follows analogously. Let $v \in V$, $w \in W$ such that $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$ and suppose that $R \odot_{\sigma'} X = T^*$ is a conservative approximation, that is, $T^*_w \preceq_1 T_w$.

Applying Proposition 2, for all $u \in U$,

$$T^*(u, w) \nwarrow_{\sigma'(v)} R(u, v) \preceq_2 T^*(u, w) \nwarrow_{\sigma(v)} R(u, v).$$

As the mapping $\nwarrow_{\sigma(v)}$ is increasing in the first argument, it also holds that

$$T^*(u, w) \nwarrow_{\sigma(v)} R(u, v) \preceq_2 T(u, w) \nwarrow_{\sigma(v)} R(u, v),$$

so the following inequality holds:

$$T^*(u, w) \nwarrow_{\sigma'(v)} R(u, v) \preceq_2 T(u, w) \nwarrow_{\sigma(v)} R(u, v).$$

This implies that every lower bound of $\{T^*(u,w) \nwarrow_{\sigma'(v)} R(u,v) \mid u \in U\}$ is also a lower bound of $\{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\}$. Consequently, because of the definition of infimum,

$$\overline{X}^*_{\sigma'}(v,w) = \bigwedge\nolimits_2 \{T^*(u,w) \nwarrow_{\sigma'(v)} R(u,v) \mid u \in U\}$$
$$\preceq_2 \bigwedge\nolimits_2 \{T(u,w) \nwarrow_{\sigma(v)} R(u,v) \mid u \in U\} = \overline{X}_\sigma(v,w).$$

□

Notice that the way in which a MARE is approximated has special relevancy in Theorem 17. In some cases, depending on the required effect on the solution set, after the modification of the composition, it may be necessary applying an optimistic or conservative approximation.

In case that the sup-composition operators are ordered, either optimistic approximations or the conservative approximations reverse the order between the sup-compositions in the corresponding greatest solutions.

Corollary 18. *Let $R \odot_\sigma X = T$ be a solvable MARE, $R \odot_{\sigma'} X = T$ an unsolvable MARE, and let $R \odot_{\sigma'} X = T^*$ be an approximation of $R \odot_\sigma X = T$. Given $\overline{X}_\sigma$ the greatest solution of $R \odot_\sigma X = T$ and $\overline{X}^*_{\sigma'}$ the greatest solution of $R \odot_{\sigma'} X = T^*$, the following statements hold:*

a) If $\odot_{\sigma(v)} \preceq \odot_{\sigma'(v)}$ and $R \odot_{\sigma'} X = T^*$ is a conservative approximation, then $\overline{X}^*_{\sigma'} \preceq_2 \overline{X}_\sigma$.
b) If $\odot_{\sigma'(v)} \preceq \odot_{\sigma(v)}$ and $R \odot_{\sigma'} X = T^*$ is an optimistic approximation, then $\overline{X}_\sigma \preceq_2 \overline{X}^*_{\sigma'}$.

Proof. It straightforwardly folllows from Theorem 17. □

The following example illustrates the previous results.

Example 19. Consider the sets $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $W = \{w\}$ and the multi-adjoint frame

$$([0,1]_8, \preceq, \overline{\&_Ł}, \overline{\swarrow}^{Ł}, \overline{\nwarrow_Ł}, \overline{\&_P}, \overline{\swarrow}^{P}, \overline{\nwarrow_P}, \overline{\&_G}, \overline{\swarrow}^{G}, \overline{\nwarrow_G}), \tag{15}$$

as in Example 15. Let

$$R = \begin{pmatrix} 0.5 & 0.25 \\ 1 & 0.5 \end{pmatrix} \qquad T = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix},$$

and consider the MARE

$$R \odot_{\sigma_1} X = T, \tag{16}$$

where $\sigma_1(v_1) = \mathrm{G}$ and $\sigma_1(v_2) = \mathrm{P}$. We can now obtain the greatest solution of this MARE, which is

$$\overline{X}_{\sigma_1} = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix}.$$

If we wanted now to increase both variables, by Theorem 14, it is possible defining $\sigma_2(v_1) = \mathrm{P}$, $\sigma_2(v_2) = Ł$, so that they are assigned to lower conjunctions.

However, if we consider now the MARE

$$R \odot_{\sigma_2} X = T, \tag{17}$$

it is unsolvable. Therefore, it is necessary applying Theorem 17, which states that an optimistic approximation of MARE (17) is necessary. This approximation can be computed by means of the methods given in Cornejo et al. [10], obtaining that

$$T^* = \begin{pmatrix} 0.25 \\ 0.375 \end{pmatrix}.$$

If we now solve the MARE,

$$R \odot_{\sigma_2} X = T^*, \tag{18}$$

we obtain that its greatest solution is

$$\overline{X}^*_{\sigma_2} = \begin{pmatrix} 0.375 \\ 0.875 \end{pmatrix},$$

which verifies that $\overline{X}_{\sigma_1} \preceq \overline{X}^*_{\sigma_2}$. □

In Lobo et al. [19], it was shown that ordered sup-compositions may provide incomparable minimal solutions. However, if they are comparable, only one inequality arises.

Theorem 20 (Lobo et al. [19]). *Let $R \odot_\sigma X = T$ and $R \odot_{\sigma'} X = T$ be two solvable MARE such that $\odot_\sigma \preceq \odot_{\sigma'}$. Given X_σ a minimal solution of $R \odot_\sigma X = T$ and $X_{\sigma'}$ a minimal solution of $R \odot_{\sigma'} X = T$ such that $X_\sigma \not\parallel X_{\sigma'}$, then $X_{\sigma'} \preceq_2 X_\sigma$.*

Proof. Suppose that $X_\sigma \preceq_2 X_{\sigma'}$. By hypothesis, $R \odot_\sigma X_\sigma = T$ and $R \odot_{\sigma'} X_{\sigma'} = T$. Consider fixed $u \in U$ and $w \in W$. Taking into account that $\&_{\sigma(v)} \preceq \&_{\sigma'(v)}$ for all $v \in V$, it holds that

$$R(u,v)\&_{\sigma(v)}X_\sigma(v,w) \preceq_1 R(u,v)\&_{\sigma'(v)}X_\sigma(v,w),$$

for all $v \in V$. This implies that every upper bound of $\{R(u,v)\&_{\sigma'(v)}X_\sigma(v,w) \mid v \in V\}$ is also a upper bound of $\{R(u,v)\&_{\sigma(v)}X_\sigma(v,w) \mid v \in V\}$. Consequently, because of the definition of suprema,

$$\begin{aligned} R \odot_\sigma X_\sigma(u,v) &= \bigvee\{R(u,v)\&_{\sigma(v)}X_\sigma(v,w) \mid v \in V\} \\ &\preceq_1 \bigvee\{R(u,v)\&_{\sigma'(v)}X_\sigma(v,w) \mid v \in V\} = R \odot_{\sigma'} X(u,v). \end{aligned}$$

As u, w were fixed, $R \odot_\sigma X_\sigma(u,w) \preceq_1 R \odot_{\sigma'} X_\sigma(u,w)$ for all (u,w) in $U \times W$, and hence,

$$R \odot_\sigma X_\sigma \preceq_1 R \odot_{\sigma'} X_\sigma.$$

Now, as $\&_{\sigma'(v)}$ is increasing on the second argument for all $v \in V$ and $X_\sigma \preceq_2 X_{\sigma'}$, it can be similarly proved that

$$R \odot_{\sigma'} X_\sigma \preceq_1 R \odot_{\sigma'} X_{\sigma'}.$$

Consequently, the following chain of inequalities holds:

$$T = R \odot_\sigma X_\sigma \preceq_1 R \odot_{\sigma'} X_\sigma \preceq_1 R \odot_{\sigma'} X_{\sigma'} = T.$$

Therefore, $R \odot_{\sigma'} X_\sigma = T$, so X_σ is a solution of $R \odot_{\sigma'} X = T$ verifying that $X_\sigma \preceq_2 X_{\sigma'}$.

As $X_{\sigma'}$ is a minimal solution of $R \odot_{\sigma'} X = T$, it must be verified that $X_\sigma = X_{\sigma'}$. We conclude that necessarily $X_{\sigma'} \preceq_2 X_\sigma$. □

The following example illustrates that, in general, comparable sup-compositions do not give rise to comparable minimal solutions. Moreover, if $\check{M}_1$ and $\check{M}_2$ are the sets of minimal solutions of the original and the modified MARE, respectively, it may happen that, for some $\check{m}_1 \in \check{M}_1$, for all $\check{m}_2 \in \check{M}_2$, it holds that $\check{m}_1 \| \check{m}_2$. Therefore, the hypothesis of comparability cannot be removed from Theorem 20.

Example 21. Consider the sets $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, $W = \{w\}$ and the multi-adjoint frame

$$\left([0,1]_8, \leq, \overline{\&_Ł}, \overline{\swarrow}^{Ł}, \overline{\nwarrow_Ł}, \overline{\&_p}, \overline{\swarrow}^{p}, \overline{\nwarrow_p}\right), \tag{19}$$

where $\overline{\&_Ł}$, $\overline{\&_p}$ are the discretization of the Łukasiewick t-norm and the product t-norm, respectively. Let

$$R = \begin{pmatrix} 0.25 & 0.75 & 0.875 & 0.5 \\ 0.5 & 0.25 & 0.75 & 0.125 \\ 0.25 & 0.25 & 0.125 & 0.25 \\ 0.5 & 0.125 & 0.375 & 0.75 \end{pmatrix} \qquad T = \begin{pmatrix} 0.375 \\ 0.375 \\ 0.125 \\ 0.375 \end{pmatrix},$$

and consider the MARE

$$R \odot_{Ł} X = T, \tag{20}$$

and the MARE

$$R \odot_{P} X = T, \tag{21}$$

which are both solvable. Applying Proposition 10 and Theorem 12, we obtain that MARE (20) has a greatest solution

$$\overline{X}_{Ł} = \begin{pmatrix} 0.875 \\ 0.625 \\ 0.5 \\ 0.625 \end{pmatrix}$$

and two minimal solutions

$$X_{Ł,1} = \begin{pmatrix} 0.875 \\ 0.625 \\ 0 \\ 0 \end{pmatrix} \quad X_{Ł,2} = \begin{pmatrix} 0.875 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}.$$

On the other hand, MARE (21) has a greatest solution

$$\overline{X}_{P} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.375 \\ 0.5 \end{pmatrix},$$

and a smallest solution

$$X_{P,1} = \begin{pmatrix} 0 \\ 0 \\ 0.375 \\ 0.375 \end{pmatrix}.$$

Clearly, $X_{P,1} \| X_{Ł,1}$ and $X_{P,1} \| X_{Ł,2}$, but, by Proposition 4, we have that $\odot_{Ł} \preceq \odot_{P}$. Figures 1 and 2 show the sublattices that allow the computation of these solutions by Theorem 12.

The following result introduces the analogous result of Corollary 18 for minimal solutions, taking into account that the minimal solutions could be incomparable. If two minimal solutions are comparable, considering a greater

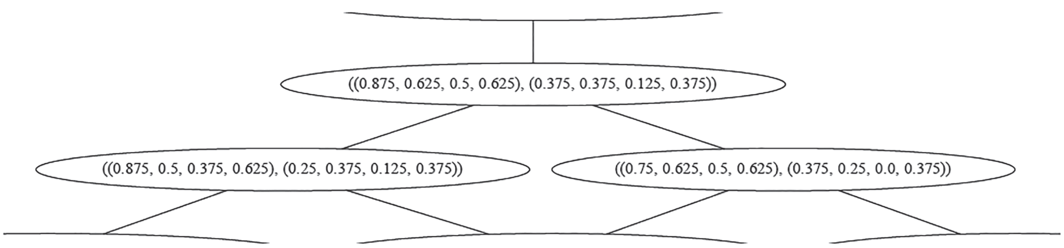

FIGURE 1 Sublattice of the concept lattice associated with MARE (20).

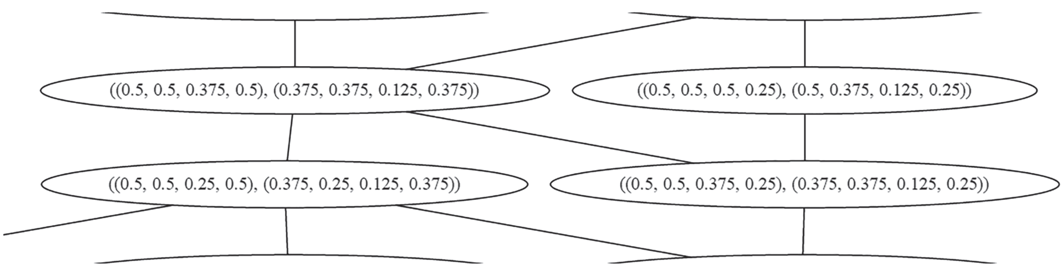

FIGURE 2 Sublattice of the concept lattice associated with MARE (21).

sup-composition and a conservative approximation ensures lower minimal solutions, while lower sup-compositions with optimistic approximations entail greater minimal solutions.

Theorem 22. *Let $R \odot_\sigma X = T$ be a solvable MARE, $R \odot_{\sigma'} X = T$ an unsolvable MARE, and let $R \odot_{\sigma'} X = T^*$ be an approximation of $R \odot_{\sigma'} X = T$. Given X_σ a minimal solution of $R \odot_\sigma X = T$ and $X^*_{\sigma'}$ a minimal solution of $R \odot_{\sigma'} X = T^*$, it holds that*

a) If $\odot_\sigma \preceq \odot_{\sigma'}$ and $R \odot_{\sigma'} X = T^*$ is a conservative approximation, then $X^*_{\sigma'} \preceq_2 X_\sigma$ or $X_\sigma \| X^*_{\sigma'}$.
b) If $\odot_{\sigma'} \preceq \odot_\sigma$ and $R \odot_{\sigma'} X = T^*$ is an optimistic approximation, then $X_\sigma \preceq_2 X^*_{\sigma'}$ or $X_\sigma \| X^*_{\sigma'}$.

Proof. We will prove a. Suppose that $X_\sigma \preceq_2 X^*_{\sigma'}$. Following an analogous procedure as in the proof of Theorem 20, we obtain that

$$R \odot_{\sigma'} X_\sigma \preceq_1 R \odot_{\sigma'} X^*_{\sigma'},$$

which leads to the following chain:

$$T = R \odot_\sigma X_\sigma \preceq_1 R \odot_{\sigma'} X_\sigma \preceq_1 R \odot_{\sigma'} X^*_{\sigma'} = T^*.$$

Therefore, $T \preceq_1 T^*$ and by hypothesis $T^* \preceq_1 T$. Consequently, $T = T^*$ what contradicts the unsolvability of $R \odot_{\sigma'} X = T$. This proves that necessarily $X^*_{\sigma'} \preceq_2 X_\sigma$. □

In the following example, we show the usability of the results presented along Section 3. This will be done in a MALP framework, in which an abduction problem will be considered.

A multi-adjoint logic program is a set of rules with associated weights, where each rule has a body (antecedent) and a head (consequent) [30]. An abduction problem is the process of looking for possible antecedents when consequents and weights are known. This process in a MALP is equivalent to a MARE [11]. Because of this connection between MALP and MARE, the possible antecedents of a situation are described by the solution set of a MARE. It may occur that, after some particular cases, it is determined that this solution set is not coherent with the reality under study. The presented results will be used for modifying this solution set so that the new solutions are coherent with these values that have been observed. For instance, if the greatest solution is lower than expected, we will increase it until the expected values are part of the solution set.

Additionally, the logic program that leads to the abduction process will have the same requirements as the one presented in Lobo et al. [11], that is:

1. There are no cyclic dependencies in the rules of the program.
2. Bodies of the rules can only contain one propositional symbol.
3. For the sake of simplicity, we will assign the same residuated implication to all rules with the same body.

Example 23. Consider the following set of propositional symbols:

$$\Omega = \{\text{high\_engine\_temperature, low\_tyres\_pressure, low\_oil\_level, high\_fuel\_consumption, engine\_failure}\},$$

and the multi-adjoint frame

$$([0,1]_8, [0,1]_4, [0,1]_8, \leq, \overline{\&_Ł}, \overline{\swarrow}^{Ł}, \overline{\nwarrow_Ł}, \overline{\&_P}, \overline{\swarrow}^{P}, \overline{\nwarrow_P}, \overline{\&_G}, \overline{\swarrow}^{G}, \overline{\nwarrow_G}).$$

As weights are elements of $[0,1]_4$, we can consider that a rule is irrelevant (0), bad (0.25), acceptable (0.5), good (0.75), and excellent (1). Remember that each rule containing a different body can be associated with a different implication.

In this environment, the following multi-adjoint logic program can be considered:

⟨engine_failure	$\nwarrow_P$	high_engine_temperature	; 1⟩
⟨high_fuel_consumption	$\nwarrow_P$	high_engine_temperature	; 0.5⟩
⟨high_fuel_consumption	$\nwarrow_G$	low_tyres_pressure	; 0.75⟩
⟨engine_failure	$\nwarrow_Ł$	low_oil_level	; 0.75⟩
⟨high_fuel_consumption	$\nwarrow_Ł$	low_oil_level	; 0.25⟩

This multi-adjoint logic program can be used in order to perform an abduction process, in which the observed values of engine failure and high fuel consumption are used to estimate the values of the other propositional symbols. As described in Lobo et al. [11], the abduction process is modeled by the following MARE:

$$\begin{aligned} 1\,\&_P x_{h\_e\_t} \ \vee\ 0.75\,\&_Ł x_{l\_o\_l} &= t_{e\_f}, \\ 0.25\,\&_P x_{h\_e\_t} \ \vee\ 0.5\,\&_Ł x_{l\_o\_l} \ \vee\ 0.75\,\&_G x_{l\_t\_p} &= t_{h\_f\_c}, \end{aligned} \tag{22}$$

whose resolution allows the calculation of the hypothetical values associated with the propositional symbols in the bodies of the rules. If we define $\sigma(\text{h\_e\_t}) = \text{P}$, $\sigma(\text{l\_o\_l}) = \text{Ł}$ and $\sigma(\text{l\_t\_p}) = \text{G}$, MARE (22) is expressed as

$$R \odot_\sigma X = T,$$

where

$$R = \begin{pmatrix} 1 & 0.75 & 0 \\ 0.25 & 0.5 & 0.75 \end{pmatrix} \qquad X = \begin{pmatrix} x_{h\_e\_t} \\ x_{l\_o\_l} \\ x_{l\_t\_p} \end{pmatrix} \qquad T = \begin{pmatrix} t_{e\_f} \\ t_{h\_f\_c} \end{pmatrix},$$

and T must be substituted by the observed values that will be used in the abduction process.

Suppose now that we have observed an engine has failure with certain value 0.5 and high fuel consumption with certain value 0.5. The resolution process of the MARE leads to a greatest solution

$$\overline{X} = \begin{pmatrix} 0.5 \\ 0.75 \\ 0.5 \end{pmatrix}$$

and two minimal solutions

$$X_1 = \begin{pmatrix} 0 \\ 0.75 \\ 0.5 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \end{pmatrix}.$$

The greatest solution represents the higher possible hypothetical values for the antecedents, which are $x_{h\_e\_t} = 0.5$, $x_{l\_o\_l} = 0.75$, and $x_{l\_t\_p} = 0.5$, while the minimal solutions are lower bounds of the possible values of the propositional symbols in the antecedents of the rules (hypothesis).

Suppose now that, after the abduction process, we measure engine temperature, tire pressure, and oil level, obtaining $x_{h\_e\_t} = 0.5$, $x_{l\_o\_l} = 0.5$, and $x_{l\_t\_p} = 0.75$. As these values are not in the solution set, we can assert that the logic program is not properly modeled.

Since we only want to increase one variable and decrease other, by Theorem 14, it is enough with defining $\sigma'(\text{h\_e\_t}) = \text{P}$, $\sigma'(\text{l\_o\_l}) = \text{G}$, and $\sigma'(\text{l\_t\_p}) = \text{Ł}$ and solving now $R \odot_{\sigma'} X = T$. The new greatest solution is

$$\overline{X}_{\sigma'} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.75 \end{pmatrix},$$

what is compatible with the values obtained in the measurement and, hence, coherent with the reality.

Something similar may had occurred if we had measured that $x_{h\_e\_t} = 0$, $x_{l\_o\_l} = 0.5$, and $x_{l\_t\_p} = 0.25$, because these values are lower than one of the minimal solutions of the MARE. In this case, by Theorem 20, we can define $\sigma'(\text{h\_e\_t}) = \text{P}$, $\sigma'(\text{l\_o\_l}) = \text{G}$, and $\sigma'(\text{l\_t\_p}) = \text{G}$, what turns the minimal solution into

$$X_{\sigma',1} = \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix},$$

what is coherent with the situation that was obtained in the measurements.

4 | COMPUTING THE SOLUTION SET OF A MARE FROM ANOTHER ONE

The results introduced in Section 3 allow to perform a change on the sup-composition operator of a MARE in order to increase or decrease its most relevant solutions, resulting in an enlargement or decrement of its solution set. Although the usability of this change has been illustrated in Example 23, it implies the resolution of a new MARE and the computation of a new concept lattice.

It is well-known that the computational cost of this procedure scales quickly when the number of variables increases. This is because, as stated in Belohlávek et al. [31], "the problem of computing all formal concepts reduces to the problem of computing all fixpoints of an operator," what entails a large number of operations. Moreover, in some cases, a single change is not enough to turn an unsolvable MARE into a solvable one, but the sup-composition operator may have to be modified several times, with its subsequent cost. For the sake of improving this process, this section presents a first approach to the use of the concept lattice associated with a MARE in the resolution process of another MARE in which only the sup-composition operator has been modified.

As Theorem 12 states that the solution set of a MARE is fully determined by a concept and its predecessors in a property-oriented concept lattice, the procedures introduced in this section will try to reduce the number of candidates for being predecessors in that concept lattice. The first result of this section gives a way for easily locating some of the needed predecessors.

In order to improve readability, in the following results, we will differentiate between the concept lattice and the Galois connection associated with the MARE $R \odot_1 X = T$ and the ones associated with the MARE $R \odot_2 X = T$ by means of subscripts.

Proposition 24. *Let $R \odot_1 X = T$ and $R \odot_2 X = T$ be two solvable MARE and $w \in W$. For each predecessor $\langle g_{p,1}, f_{p,1}\rangle \in \mathcal{M}_{\pi N,1}$ of $\left\langle T_w^{\downarrow_1^N}, T_w\right\rangle$, there exists a predecessor $\langle g_{p,2}, f_{p,2}\rangle \in \mathcal{M}_{\pi N,2}$ of $\left\langle T_w^{\downarrow_2^N}, T_w\right\rangle$ such that*

$$\left\langle f_{p,1}^{\downarrow_2^N}, f_{p,1}^{\downarrow_2^N\uparrow_\pi^2}\right\rangle \preceq \langle g_{p,2}, f_{p,2}\rangle \prec \left\langle T_w^{\downarrow_2^N}, T_w\right\rangle .$$

Proof. The result follows immediately from the fact that

$$f_{p,1}^{\downarrow_2^N\uparrow_\pi^2} \preceq f_{p,1} \preceq T_w,$$

what implies that or $\left\langle f_{p,1}^{\downarrow_2^N}, f_{p,1}^{\downarrow_2^N\uparrow_\pi^2}\right\rangle$ is a predecessor or there must exist a predecessor $\langle g_{p,2}, f_{p,2}\rangle$ verifying that

$$\left\langle f_{p,1}^{\downarrow_2^N}, f_{p,1}^{\downarrow_2^N\uparrow_\pi^2}\right\rangle \prec \langle g_{p,2}, f_{p,2}\rangle \prec \left\langle T_w^{\downarrow_2^N}, T_w\right\rangle .$$

□

Proposition 24 can be used to compute a subset of the set of lower bounds of a concept in which at least a predecessor must be contained. The following result uses all the predecessors located by Proposition 24 in order to reduce the number of candidates for being predecessors of a concept. As a consequence, the computational cost of the calculation of the set of predecessors of a concept is reduced.

Proposition 25. *Let $P_1 \subseteq \mathcal{M}_{\pi N,1}$ be the set of predecessors of $\left\langle T_w^{\downarrow_1^N}, T_w\right\rangle$ in $\mathcal{M}_{\pi N,1}$ and $P_2 \subseteq \mathcal{M}_{\pi N,2}$ the set of predecessors of $\left\langle T_w^{\downarrow_2^N}, T_w\right\rangle$ in $\mathcal{M}_{\pi N,2}$. It holds that*

$$P_2 \subseteq \left(\langle T_w^{\downarrow_2^N}, T_w\rangle\right]\!\!\right] \setminus \bigcup \left\{ \left(\langle f_p^{\downarrow_2^N}, f_p^{\downarrow_2^N\uparrow_\pi^2}\rangle\right]\!\!\right] \text{ for each } \langle g_p, f_p\rangle \in P_1 \right\}$$

where $(\langle g, f\rangle]\!]$ denotes the set $\left\{(g_i, f_i) \in L_2^U \times L_1^V \mid (g_i, f_i) \prec \langle g, f\rangle\right\}$.

Proof. By definition, the predecessors of a concept are lower bounds of it, so it holds that

$$P_2 \subseteq \left(\langle T_w^{\downarrow_2^N}, T_w\rangle\right]\!\!\right]$$

Moreover, for each $\langle g_p, f_p\rangle \in P_1$, it is satisfied that, by Theorem 24, there exists a predecessor $\langle g_{p,2}, f_{p,2}\rangle$ verifying that

$$\left\langle f_p^{\downarrow_2^N}, f_p^{\downarrow_2^N\uparrow_\pi^2}\right\rangle \preceq \langle g_{p,2}, f_{p,2}\rangle \prec \left\langle T_w^{\downarrow_2^N}, T_w\right\rangle.$$

Therefore, we can conclude that, by definition of predecessor, the predecessors of $\left\langle T_w^{\downarrow_2^N}, T_w\right\rangle$ are not in any set of the form $\left(\left\langle f_p^{\downarrow_2^N}, f_p^{\downarrow_2^N\uparrow_\pi^2}\right\rangle\right]$. Consequently,

$$P_2 \cap \bigcup \left\{ (\langle f_p^{\downarrow_2^N}, f_p^{\downarrow_2^N\uparrow_\pi^2}\rangle] \text{ for each } \langle g_p, f_p\rangle \in P_1 \right\} = \varnothing$$

and it holds that

$$P_2 \subseteq (\langle T_w^{\downarrow_2^N}, T_w\rangle] \setminus \bigcup \left\{ (\langle f_p^{\downarrow_2^N}, f_p^{\downarrow_2^N\uparrow_\pi^2}\rangle] \text{ for each } \langle g_p, f_p\rangle \in P_1 \right\}$$

□

The following example illustrates the results introduced in Section 4.

Example 26. Consider the MARE

$$R \odot_{\sigma_1} X = T, \tag{23}$$

defined over the multi-adjoint frame

$$([0,1]_4, [0,1]_4, [0,1]_4, \&_P, \&_G),$$

where $\sigma_1(v_1) = G$, $\sigma_1(v_2) = P$ and

$$R = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.75 \end{pmatrix}, \quad T = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

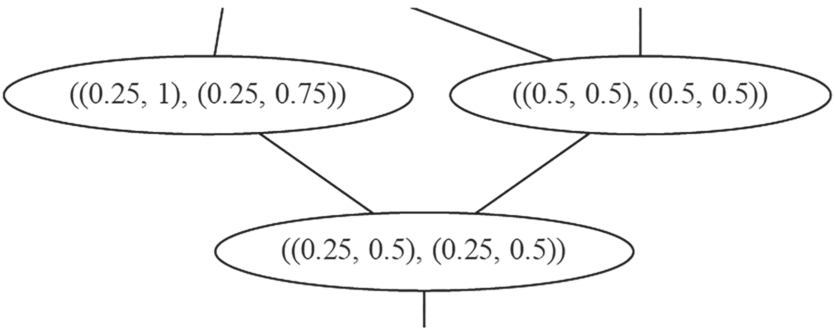

FIGURE 3 Sublattice of the concept lattice associated with MARE (23).

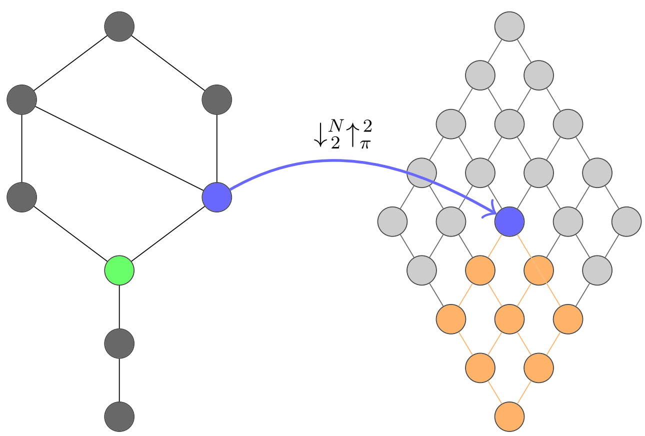

FIGURE 4 Representation of the application of Proposition 25. [Colour figure can be viewed at wileyonlinelibrary.com]

Applying the results presented in Díaz-Moreno and Medina [17, 18], MARE (23) can be solved by means of its associated concept lattice. Among others, the computation of the sublattice shown in Figure 3 is required to determine its solution set.

Suppose now that the sup-composition operator of MARE (23) is modified, resulting in MARE (24) given by

$$R \odot_{\sigma_2} X = T, \tag{24}$$

where

$$\sigma_2(v_1) = \text{P} \quad \text{and} \quad \sigma_2(v_2) = \text{G}.$$

According to Proposition 25, it is possible using the sublattice shown in Figure 3 in order to compute the solution set of MARE (24). Firstly, applying Proposition 10, it can be checked that MARE (24) is solvable, as

$$T^{\downarrow_2^N\uparrow_\pi^2} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}^{\downarrow_2^N\uparrow_\pi^2} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}^{\uparrow_\pi^2} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = T.$$

Additionally, the greatest solution of MARE (24) is

$$T^{\downarrow_2^N} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

By Theorem 12, the computation of the whole solution set of MARE (24) depends on the predecessors of the concept $\langle T^{\downarrow_2^N}, T\rangle$. Applying Proposition 25, they belong to the set of lower bounds of $\langle T^{\downarrow_2^N}, T\rangle$ that are no lower bounds of

$$\langle (0.25, 0.5)^{\downarrow_2^N}, (0.25, 0.5)^{\downarrow_2^N\uparrow_\pi^2}\rangle = \langle (0.25, 0.5), (0.25.0.5)\rangle.$$

As the complete lattices $\mathcal{M}_{\pi N,2}$ and $\mathcal{I}(\mathcal{M}_{\pi N,2})$ are isomorph, we can graphically see how Proposition 25 works. The left side of Figure 4 represents the complete lattice $\mathcal{I}(\mathcal{M}_{\pi N,1})$ and the right side is associated with any mapping from U to $[0,1]_4$, that is, the lattice $([0,1]_4^2, \leq)$, where the order is defined pointwise. Applying $\downarrow_2^N\uparrow_\pi^2$ to T (in blue) and to the intent of its predecessor (in green), we can see at the right which are the lower bounds of $T^{\downarrow_2^N\uparrow_\pi^2}$ in the lattice $([0,1]_4^2, \leq)$ (in orange).

Now, applying $\downarrow_2^N\uparrow_\pi^2$ to the predecessor of T, we can discard all the lower bounds of this mapping (Proposition 25), what is shown in red in Figure 5. As a consequence, only three possibilities for being the predecessors of T in $\mathcal{I}(\mathcal{M}_{\pi N,2})$ remain, which can be seen in the right side of Figure 5.

Making the corresponding computations, it is obtained that

$$\langle (0.25, 0.5), (0.25, 0.5)\rangle \quad \text{and} \quad \langle (0.5, 0.25), (0.5, 0.25)\rangle$$

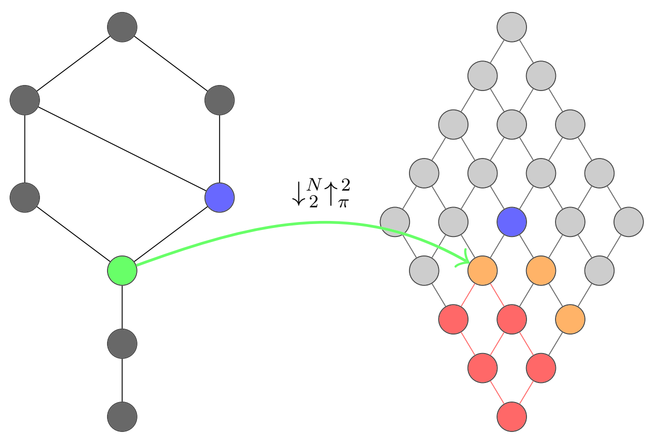

FIGURE 5 Representation of the application of Proposition 25. [Colour figure can be viewed at wileyonlinelibrary.com]

are the predecessors of $\langle(0.5, 0.5), (0.5, 0.5)\rangle$ in $\mathcal{M}_{\pi N,2}$. These predecessors allow the computation of the whole solution set.

The relevancy of this reduction is greater in examples in which a large number of variables is involved.

5 | CONCLUSIONS AND FUTURE WORK

This work has introduced a study about the consequences of modifying the sup-composition operator of a MARE. As a result, it is shown how an intentional modification of the sup-composition operator enables to manipulate the MARE in order to model better a reality under study. An example involving logic programming illustrates this fact. Furthermore, a procedure has been detailed for showing how it is possible to take advantage of the computation of the solutions of a MARE to solve a variant in which the composition has been modified.

The presented results have been applied to theoretical examples and an abduction problem, but they are potentially applicable in many other problems such as control theory, bipolar reasoning, and optimization, which will be studied in the near future. In addition, the introduced study will be applied to real examples, such as in Digital Forensics in the framework of the COST Action DigForAsp. Moreover, more properties will be studied to reuse the computations of the whole set of solutions of a MARE and reduce its complexity.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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