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Exact solutions to a family of position-dependent mass damped oscillators from variational λ-symmetries

Adrián Ruiz Serván | **María Concepción Muriel Patino**

Departamento de Matemáticas,
Universidad de Cádiz, Puerto Real, Spain

Correspondence
Adrián Ruiz Serván, Facultad de Ciencias,
Departamento de Matemáticas, Av.
República Saharaui sn, Puerto Real 11510,
Spain.
Email: adrian.ruiz@uca.es

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A wide family of position-dependent mass damped oscillators affected by an external potential is investigated. First, a Lagrangian formulation is introduced for the corresponding problem. The Lagrangian function is time-dependent, and the problem cannot be approached with classical procedures because it lacks variational symmetries. Therefore, the variational λ-symmetry method is applied to find exact solutions. Variational λ-symmetries are determined for a family of potential functions, which lead to a one-parameter family of exact solutions. The results are applied to particular examples corresponding to some interesting mass functions reported in the previous literature.

KEYWORDS

exact solutions, Liénard equation, position-dependent mass, variational λ-symmetry

MSC CLASSIFICATION

34A34, 34A05, 34A26

1 | INTRODUCTION

The interest of both classical and quantum particles with effective position-dependent mass has increased in the recent years motivated by their wide range of applications [1–14]. For instance, the position-dependent mass formalism can be applied to the movement of an object immersed into a dense fluid [9, 10], as well as to micro production techniques like molecular-beam epitaxy and nanolithography [7, 8].

In this work, we aim to obtain exact solutions for a family of damped position-dependent mass oscillators affected by an external potential. Throughout the work, we will denote by x and t the position and time, respectively; derivatives with respect to the time t will be indicated by overdot, while prime will denote derivative with respect to the position x. If $p = p(x)$ stands for the linear momentum of the oscillator and $m = m(x)$ is the corresponding position-dependent mass function, then we have that

$$p(x) = m(x)\dot{x}. \tag{1}$$

Newton's second law implies that

$$\dot{p} = \mathcal{F}(t, x, \dot{x}), \tag{2}$$

where $\mathcal{F}$ is the total force of the system. Apart from the external force $F_E(x) = -V'(x)$ produced by a potential function $V = V(x)$, in the position-dependent mass formalism it has been proved the existence of an internally by-produced force

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R_{PDM} given by [13, 14]

$$R_{PDM}(x,\dot{x}) = \frac{1}{2}m'(x)\dot{x}^2.$$

If we also consider a frictional force F_R, then the total force splits into

$$\mathcal{F} = F_E + R_{PDM} + F_R,$$

which, by (2) and (1), yields

$$\ddot{x}m(x) + \frac{1}{2}m'(x)\dot{x}^2 = -V'(x) + F_R. \tag{3}$$

The value of the viscous friction depends on the design and the shape of the oscillator, as well as on the viscous coefficient γ of the material under which the oscillator is immersed. In the case of constant mass m, it is assumed that $F_R = -\gamma m\dot{x}$ [15]; therefore, in our case, we consider that

$$F_R = -\gamma m(x)\dot{x}.$$

Thus, by (3), the motion equation of the oscillator under study reads as follows:

$$\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \gamma\dot{x} + \frac{V'(x)}{m(x)} = 0. \tag{4}$$

In general, Equation (4) only admits the vector field $\mathbf{v} = \partial_t$ as Lie point symmetry [16–20]. This implies that Equation (4) can be reduced to a first-order equation, by using the classical Lie method of reduction. It can be checked that by means of the change $y = x$ and $w = \dot{x}$, Equation (4) is transformed into the Abel equation

$$w_y w + \frac{m'(y)}{2m(y)}w^2 + \gamma w + \frac{V'(y)}{m(y)} = 0, \tag{5}$$

for which the determination of some exact solutions is, in general, a difficult task. Some exact solutions of (5) for specific forms of the arbitrary functions that appear in (5) can be found in Polyanin and Zaitsev [21]. However, without its general solution, the classical Lie method cannot be applied to solve completely Equation (4).

Equation (4) is a special case of the Liénard II equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x)\dot{x} + h(x) = 0 \tag{6}$$

for

$$f(x) = \frac{m'(x)}{2m(x)}, \qquad g(x) = \gamma, \qquad h(x) = \frac{V'(x)}{m(x)}.$$

The determination of Lagrangians for Liénard II equations of the type (6) has been intensively investigated in the recent years. For instance:

- In previous works [22–24], Lagrangian functions for some families of the Liénard II equation were found by using specific ansätze.
- The Jacobi last multiplier method was applied to obtain a Lagrangian formulation for several types of the Liénard II and I type equation in [25] and [26], respectively.
- In Kudryashov and Sinelshchikov [27], new types of time-independent Lagrangians were obtained for some families of Equation (6) by using nonlocal transformations [28–31].
- For the undamped case ($\gamma = 0$), it is well known [3, 10] that Equation (4) admits the first integral

$$I(x,\dot{x}) = \dot{x}^2 m(x) + 2V(x). \tag{7}$$

- As a consequence of proposition 3.1 in Cieśliński and Nikiciuk [24], it can be checked that when the potential function $V = V(x)$ satisfies the condition

$$V''(x) = \frac{2}{9}\gamma^2 m(x) + \frac{V'(x)m'(x)}{2m(x)}, \tag{8}$$

then Equation (4) admits the time-independent Lagrangian

$$L(x, \dot{x}) = \frac{\gamma\sqrt{m(x)}}{\gamma m(x)\dot{x} + 3V'(x)}.$$

In this case, the symmetry $\mathbf{v} = \partial_t$ is variational and Noether's theorem [32] provides the following first integral:

$$J(x, \dot{x}) = -\frac{\gamma\sqrt{m(x)}\left(2\gamma m(x)\dot{x} + 3V'(x)\right)}{\left(\gamma m(x)\dot{x} + 3V'(x)\right)^2}.$$

From now on, we assume that $\gamma \neq 0$. In the general case, that is, for arbitrary $m = m(x)$, $V = V(x)$ and γ, a Lagrangian function for the family of equations (4) can be deduced by following proposition 2.1 in Cieśliński and Nikiciuk [24] (see also the original work of Jacobi and Euler [25] and references therein):

$$L(t, x, \dot{x}) = e^{\gamma t} m(x)\dot{x}^2 + \dot{x}\left(\frac{2}{\gamma}e^{\gamma t}V'(x) + \int G_x(x, t)dt + h(x)\right) + G(t, x), \tag{9}$$

where $G = G(t, x)$ and $h = h(x)$ are arbitrary smooth functions. Since variational symmetries are inherited as Lie point symmetries of the corresponding Euler–Lagrange equation [16], in the general case the symmetry $\mathbf{v} = \partial_t$ is the unique candidate to be a variational symmetry of the Lagrangian (9). However,

$$\partial_t(L) = \gamma e^{\gamma t} m(x)\dot{x}^2 + \dot{x}\left(2e^{\gamma t}V'(x) + G_x\right) + G_t \neq 0$$

for $\gamma \neq 0$. Therefore, the Lie point symmetry ∂_t is not variational and hence Noether's theorem cannot be applied to obtain directly a first integral of the equation.

In this work, we address this problem by using the variational λ-symmetry method, with the ultimate goal of obtaining exact solutions for Equation (4). Variational λ-symmetries were firstly introduced in Muriel et al. [33] as a generalization of the classical notion of variational symmetry. An extension of variational λ-symmetries for variational problems with several independent variables was introduced in Cicogna et al. [34]. These generalizations have turned out to be very effective to find exact solutions of variational problems, specially in the absence of standard variational symmetries [35–37].

For ease of reading, in Section 2, we recall the notion of variational λ-symmetry, as well as the associated procedure to reduce the order of Euler–Lagrange equations, adapted to the case of first-order Lagrangians. In Section 3 we investigate the existence of variational λ-symmetries whose canonical representative [38] is defined by a function λ that is linear in the first-order derivative. From this analysis, we obtain a family of potential functions, depending on an arbitrary smooth function, which determines a wide family of equations of the form (4) that can be addressed by the variational λ-symmetry method.

In Section 4, we use the variational λ-symmetry to reduce the original problem to an algebraic equation with trivial solutions. After solving a first-order separable auxiliary ODE, a one-parameter family of exact solutions to the original second-order equation is obtained. Remarkably, as a by-product of the presented approach, some particular solutions to the underlying Abel equation (5) are also derived. Finally, in Section 5 we consider different particular examples by using position-dependent mass functions that have been previously studied in the literature.

2 | VARIATIONAL λ-SYMMETRIES FOR FIRST-ORDER LAGRANGIANS

In this section, we recall the notion of variational λ-symmetry, which was introduced in Muriel et al. [33] (see also previous works [35, 36]) as a generalization of the classical concept of variational symmetry. Due to the nature of the problem considered in this paper, we will focus on the case of non-degenerate first-order Lagrangians whose associated

Euler–Lagrange equation becomes a second-order ordinary differential equation. We would like to emphasize that the results presented in this section are also valid for a general scalar nth-order Lagrangian with associated Euler–Lagrange equation of order $2n$.

Variational λ-symmetries are based on a deformed prolongation of a vector field which uses a smooth function λ as an auxiliary object (see definition 2.1 in Muriel and Romero [39]):

Definition 1. Let $\mathbf{v} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ be a smooth vector field defined on an open set of the coordinate space (t,x) and $\lambda = \lambda(t,x,\dot{x})$ a smooth function. The first-order λ-prolongation of $\mathbf{v}$, denoted by $\mathbf{v}^{[\lambda,(1)]}$, is defined as the vector field

$$\mathbf{v}^{[\lambda,(1)]} = \xi(t,x)\partial_t + \eta(t,x)\partial_x + \eta^{[\lambda,(1)]}(t,x,\dot{x})\partial_{\dot{x}}, \tag{10}$$

where

$$\eta^{[\lambda,(1)]} = (\mathbf{D}_t + \lambda)\eta - \dot{x}(\mathbf{D}_t + \lambda)\xi \tag{11}$$

and $\mathbf{D}_t = \partial_t + \dot{x}\partial_x + \ddot{x}\partial_{\dot{x}} + \cdots$ stands for the total derivative with respect to t.

The concepts of variational symmetry and divergence symmetry [16] can be extended when λ-prolongations are considered (see definition 2.1 in Muriel et al. [33]):

Definition 2. A pair $(\mathbf{v}, \lambda)$, where $\mathbf{v} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is a smooth vector field and $\lambda = \lambda(t,x,\dot{x})$ is a smooth function, is a variational λ-symmetry of a first-order Lagrangian $L = L(t,x,\dot{x})$ if:

$$\mathbf{v}^{[\lambda,(1)]}(L) + L(\mathbf{D}_t + \lambda)\xi = (\mathbf{D}_t + \lambda)(B), \tag{12}$$

for some smooth function $B = B(t,x)$.

Observe that the classical notion of variational symmetry [16] is included in Definition 2 as the particular case when $\lambda = 0$. If $B = 0$, then the pair $(\mathbf{v}, \lambda)$ is called a *strict* variational λ-symmetry. When a variational λ-symmetry of a nth-order Lagrangian is known, it is possible to construct a reduced Lagrangian, of order $n-1$, whose associated Euler–Lagrange equation (of order $2n-2$) provides a $(2n-1)$-parameter family of exact solutions for the original $2n$th-order Euler–Lagrange equation (see theorem 1 in Muriel et al. [33]).

Since the problem addressed in this work corresponds to a first-order Lagrangian, in what follows we briefly sketch the reduction method associated to variational λ-symmetries for the particular case when $n = 1$. For further details of the reduction method associated to nth-order Lagrangians, the reader is referred to the literature [33, 35].

Step 1: In an open neighborhood where $\mathbf{v}$ does not vanish, it is possible to introduce rectifying coordinates (y, α) such that $\mathbf{v} = \partial_\alpha$. In order to simplify the notation, we keep denoting such coordinates as (t, x).
Step 2: Consider the Lagrangian $\hat{L} = L + \mathbf{D}_t(A)$, where $B = -A_x$. Thus, the Lagrangian $\widehat{L}$ is equivalent to L and $(\mathbf{v}, \lambda)$ becomes a strict variational λ-symmetry for $\widehat{L}$ (proposition 2.1 in Muriel et al. [33]).
Step 3: Calculate a function $w = w(t,x,\dot{x})$ such that $\mathbf{v}^{[\lambda,(1)]}(w) = 0$.
Step 4: The Lagrangian $\widehat{L}$ can be expressed in terms of the coordinates (t, w). Therefore, a reduced zero-order Lagrangian $\tilde{L} = \tilde{L}(t, w)$ is obtained.
Step 5: The corresponding reduced Euler–Lagrange equation turns out to be $\tilde{L}_w = 0$, which is related to the original Euler–Lagrange equation by means of the formula (see eq. (29) in Muriel et al. [33]):

$$E\left[\widehat{L}\right] = (\mathbf{D}_t + \lambda)\left(-w_{\dot{x}}\tilde{L}_w\right), \tag{13}$$

where $\tilde{L}_w$ is assumed to be expressed in terms of the original coordinates $(t, x, \dot{x})$.
Step 6: Assuming that it is possible to obtain an expression $w = H(t)$ from $\tilde{L}_w = 0$, consider the first-order equation $w(t,x,\dot{x}) = H(t)$ and denote its general solution by $x = h(t;k)$ where $k \in \mathbb{R}$. Then the right-hand side of (13) evaluated on $x = h(t;k)$, assuming that is well defined, must vanish. Therefore, $x = h(t;k)$ provides a family of exact solutions, depending on the parameter k, to the original second-order Euler–Lagrange equation.

In the next section, we apply this procedure to the Lagrangians (9) in order to obtain exact solutions to the motion equation (4).

3 | VARIATIONAL λ-SYMMETRIES FOR POSITION-DEPENDENT MASS DAMPED OSCILLATORS

Once a Lagrangian formulation for the motion Equation (4) has been obtained, in terms of the Lagrangians given in (9), in this section we search for a variational λ-symmetry $(\mathbf{v}, \lambda)$ for the Lagrangian (9) with the goal of obtaining exact solutions to Equation (4).

Since the first step of the associated reduction procedure requires rectifying coordinates for the vector field $\mathbf{v}$, we can assume, for simplicity, that $\mathbf{v} = \partial_x$. In order to ease the search of the corresponding function λ, we can assume that it is of the form

$$\lambda = \alpha(x)\dot{x} + \beta(x). \tag{14}$$

By using (11), it can be checked that the first-order λ-prolongation of $\mathbf{v} = \partial_x$, by using a function λ of the form (14), becomes

$$\mathbf{v}^{[\lambda,(1)]} = \partial_x + (\alpha(x)\dot{x} + \beta(x))\,\partial_{\dot{x}}.$$

Strict variational λ-symmetries of the form (14) arise from condition (15) with $B = 0$:

$$\mathbf{v}^{[\lambda,(1)]}(L) = L_x + (\alpha(x)\dot{x} + \beta(x))\,L_{\dot{x}} = 0, \tag{15}$$

where L is given by (9). Since $\alpha = \alpha(x)$ and $\beta = \beta(x)$ do not depend on $\dot{x}$, (15) splits into the following system of determining equations for α and β :

$$\begin{aligned}
&\text{Equation 1: } 2\alpha(x)m(x) + m'(x) = 0,\\
&\text{Equation 2: } 2e^{\gamma t}\left(\gamma m(x)\beta(x) + \alpha(x)V'(x) + V''(x)\right)\\
&\qquad + \gamma\left(\alpha(x)h(x) + \alpha(x)\int G_x(t,x)dt + h'(x) + \int G_{xx}(t,x)dt\right) = 0,\\
&\text{Equation 3: } 2e^{\gamma t}\beta(x)V'(x) + \gamma\left(\beta(x)\left(h(x) + \int G_x(t,x)dt\right) + G_x(t,x)\right) = 0.
\end{aligned} \tag{16}$$

From Equation (1) in (16), we get

$$\alpha(x) = -\frac{m'(x)}{2m(x)}. \tag{17}$$

In order to continue with the analysis of system (16), we distinguish the following cases:

3.1 | Case 1: $\beta(x) = 0$

By setting $\beta(x) = 0$ in Equation (3) in (16), we obtain

$$G_x(x,t) = 0. \tag{18}$$

Therefore, taking (17) and (18) into account, Equation (2) in (16), with $\beta(x) = 0$, becomes

$$2e^{\gamma t}\left(2V''(x)m(x) - V'(x)m'(x)\right) = \gamma\left(h(x)m'(x) - 2m(x)h'(x)\right). \tag{19}$$

Since the right-hand side of (19) does not depend on t and $\gamma \neq 0$, then we deduce that

$$2V''(x)m(x) - V'(x)m'(x) = 0 \quad \text{and} \quad h(x)m'(x) - 2m(x)h'(x) = 0,$$

which yields

$$V(x) = C_2 + C_1 \int \sqrt{m(x)}dx, \qquad h(x) = C_3\sqrt{m(x)}, \qquad C_1, C_2, C_3 \in \mathbb{R}. \tag{20}$$

The family of potential functions $V = V(x)$ given in (20) provides, through (4), the following motion equation:

$$\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \gamma\dot{x} + \frac{C_1}{\sqrt{m(x)}} = 0. \tag{21}$$

It can be checked that Equation (21) admits several Lie point symmetries that could be used to integrate completely the equation [16, 19, 20]. Alternatively, a family of integrating factors that only depends on t and x can be found by solving the determining equations (3.365a)–(3.369b) presented in Bluman and Anco (p. 194) [19]. Such integrating factors become $\mu(t, x) = \left(c_1 + c_2e^{\gamma t}\right)\sqrt{m(x)}$, for $c_1, c_2 \in \mathbb{R}$. We can choose, for instance, $c_1 = 0$ and $c_2 = 1$, and use the integrating factor $\mu(t, x) = e^{\gamma t}\sqrt{m(x)}$ to write Equation (21) in the form:

$$e^{\gamma t}\sqrt{m(x)}\left(\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \gamma\dot{x} + \frac{C_1}{\sqrt{m(x)}}\right) = \mathbf{D}_t\left(e^{\gamma t}\left(\frac{C_1}{\gamma} + \sqrt{m(x)}\dot{x}\right)\right) = 0.$$

From

$$e^{\gamma t}\left(\frac{C_1}{\gamma} + \sqrt{m(x)}\dot{x}\right) = k_1, \qquad k_1 \in \mathbb{R},$$

we get the general solution to Equation (21) in implicit form:

$$k_1e^{-\gamma t} + C_1t + \gamma\int\sqrt{m(x)}dx = k_2, \qquad k_1, k_2 \in \mathbb{R}.$$

3.2 | Case 2: $\beta(x) \neq 0$

Solving Equation (3) in (16) for $V'(x)$ yields:

$$V'(x) = -\frac{\gamma e^{-\gamma t}\left(\beta(x)\left(h(x) + \int G_x(t, x)dt\right) + G_x(t, x)\right)}{2\beta(x)}. \tag{22}$$

Since the potential function V does not depend on t, the right-hand side of Equation (22) cannot depend on t neither. This fact restricts the possible choices of the functions $h = h(x)$ and $G = G(t, x)$ that appear in the Lagrangian (9). If we set, for instance,

$$h(x) = 0, \qquad G(t, x) = e^{\gamma t}q(x) + g(t), \tag{23}$$

where $q = q(x)$ and $g = g(t)$ are arbitrary smooth functions, then Equation (22) becomes

$$V'(x) = -\frac{q'(x)(\gamma + \beta(x))}{2\beta(x)}. \tag{24}$$

After substituting (17) and (24) into Equation (2) in (16), the following first-order Bernoulli-type differential equation for β is obtained:

$$-2m(x)q'(x)\beta'(x) + \left(2m(x)q''(x) - q'(x)m'(x)\right)\beta(x) - 4m(x)^2\beta(x)^3 = 0. \tag{25}$$

It can be checked that, by means of the standard change of dependent variable $\overline{\beta} = \beta^{-2}$, Equation (25) is mapped into the following first-order linear equation:

$$m(x)q'(x)\overline{\beta}'(x) - \left(2m(x)q''(x) - q'(x)m'(x)\right)\overline{\beta}(x) - 4m(x)^2 = 0. \tag{26}$$

Observe that if $q'(x) = 0$ Equation (26) implies that $m(x) = 0$; therefore, we assume from this point on that $q = q(x)$ is a non-constant function. It can be checked that general solution to (26) becomes

$$\overline{\beta}(x) = \frac{m(x)(4q(x) + C)}{q'(x)^2}, \qquad C \in \mathbb{R}.$$

Since $\overline{\beta} = \beta^{-2}$, the general solution to Equation (25), assuming that $4q(x) + C > 0$, becomes

$$\beta(x) = \pm \frac{q'(x)}{\sqrt{m(x)\,(4q(x) + C)}}, \; C \in \mathbb{R}. \tag{27}$$

The corresponding potential functions $V = V(x)$ arise from (24), where β is given by (27):

$$V'(x) = -\frac{1}{2}\left(q'(x) \pm \gamma \sqrt{m(x)\,(4q(x) + C)}\right). \tag{28}$$

Therefore, the family of position-dependent mass damped oscillators (4) whose potential functions $V = V(x)$ satisfy (28) becomes

$$\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \gamma \dot{x} - \frac{q'(x) \pm \gamma \sqrt{m(x)\,(4q(x) + C)}}{2m(x)} = 0. \tag{29}$$

A corresponding Lagrangian formulation is obtained by substituting (23) and (28) into (9), which produces

$$L = e^{\gamma t}\left(m(x)\dot{x}^2 \mp x_1 \sqrt{m(x)\,(4q(x) + C)} + q(x)\right) + g(t). \tag{30}$$

Previous discussion proves the following theorem.

Theorem 1. *The family of position-dependent mass damped oscillators modeled by Equation (29) can be written as the Euler–Lagrange equation of the Lagrangian function (30), and it admits the variational λ-symmetry given by the pair*

$$\left(\partial_x, -\frac{m'(x)}{2m(x)}\dot{x} \pm \frac{q'(x)}{\sqrt{m(x)\,(4q(x) + C)}}\right), \; C \in \mathbb{R}, \tag{31}$$

where $q = q(x)$ is a non-constant smooth function and the condition $4q(x) + C > 0$ is satisfied.

4 | EXACT SOLUTIONS DERIVED FROM THE VARIATIONAL λ-SYMMETRY

Once the variational λ-symmetry (31) for the Lagrangian function (30) has been obtained, we can use the procedure described at the end of Section 2 to obtain exact solutions for the family of equations (29).

According to (11), the first-order λ-prolongation for (31) becomes

$$\mathbf{v}^{[\lambda,(1)]} = \partial_x + \left(-\frac{m'(x)}{2m(x)}x_1 \pm \frac{q'(x)}{\sqrt{m(x)\,(4q(x) + C)}}\right)\partial_{\dot{x}}. \tag{32}$$

It can be checked that $\{t, w\}$ where

$$w = x_1 \sqrt{m(x)} \mp \frac{1}{2}\sqrt{4q(x) + C} \tag{33}$$

is a complete set of differential invariants for $\mathbf{v}^{[\lambda,(1)]}$. The Lagrangian (30) can be expressed in terms of the invariants $\{t, w\}$ as the following zero-order reduced Lagrangian:

$$\tilde{L} = e^{\gamma t}\left(w^2 - \frac{C}{4}\right) + g(t).$$

The associated Euler–Lagrange equation (of zero order) becomes

$$\frac{\partial \tilde{L}}{\partial w} = 2e^{\gamma t} w = 0,$$

with trivial solution $w = 0$.

Therefore, by using the expression (33) in $w = 0$, a one-parameter family of exact solutions to the family (29) can be retrieved by solving the auxiliary first-order equation

$$x_1 \sqrt{m(x)} \mp \frac{1}{2}\sqrt{4q(x) + C} = 0.$$

This is a separable equation whose general solution is implicitly defined by

$$t \mp \int \frac{2\sqrt{m(x)}}{\sqrt{4q(x) + C}} + k = 0, \ \ k \in \mathbb{R}.$$

As a result of the previous discussion, the following theorem has been proved:

Theorem 2. *Consider the family of position-dependent mass damped oscillators given by*

$$\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \gamma \dot{x} - \frac{q'(x) \pm \gamma \sqrt{m(x)\,(4q(x) + C)}}{2m(x)} = 0, \tag{34}$$

where $q = q(x)$ is a non-constant function, C is a constant and the condition $4q(x) + C > 0$ is satisfied. A one-parameter family of exact solutions, associated to the variational λ-symmetry (31), is given, in implicit form, by

$$t \mp \int \frac{2\sqrt{m(x)}}{\sqrt{4q(x) + C}} dx + k = 0, \qquad k \in \mathbb{R}. \tag{35}$$

As an indirect by-product, the one-parameter family of solutions (35) can be used to obtain a particular solution to a subfamily of the underlying reduced Abel equation (5) associated to (34), as stated in the following corollary:

Corollary 1. *A particular exact solution for the family of Abel equations*

$$w_y w + \frac{m'(y)}{2m(y)} w^2 + \gamma w - \frac{q'(y) \pm \gamma \sqrt{m(y)\,(4q(y) + C)}}{2m(y)} = 0, \tag{36}$$

where $q = q(y)$ is a non-constant function, C is a constant and the condition $4q(y) + C > 0$ is satisfied, is given by

$$w = \pm \frac{\sqrt{4q(y) + C}}{2\sqrt{m(y)}}. \tag{37}$$

Proof. The family of Abel equations (36) has been obtained from (29) through the transformation $y = x$ and $w = \dot{x}$. Implicit differentiation of (35) with respect to t yields

$$1 \mp \frac{2\sqrt{m(x)}}{\sqrt{4q(x) + C}} \dot{x} = 0.$$

By setting $y = x$ and $w = \dot{x}$ in this expression, we obtain the one-parameter family of solutions (37), which proves the result. □

5 | PARTICULAR EXAMPLES

In this section, we consider several equations of the family given in Theorem 2, that correspond to mass functions that have been previously studied in the literature. Graphical representations of the corresponding applied external forces, together with the corresponding exact solutions, are also plotted in Figures 1–7 for particular values of the integration constant k that appears in (35).

5.1 | Constant mass and cubic external force

For the case of constant mass function $m(x) = m_0$, $C = 0$ and $q(x) = (ax^2 + bx + c)^2$, with $a, b, c \in \mathbb{R}$, $a \neq 0$, the family of equations (29) becomes

$$\ddot{x} + \gamma\dot{x} + \alpha x^3 + \beta x^2 + \mu x + \delta = 0, \tag{38}$$

where

$$\alpha = -\frac{2a^2}{m_0}, \quad \beta = -\frac{a\left(\gamma\sqrt{m_0} + 3b\right)}{m_0}, \quad \mu = -\frac{b\gamma\sqrt{m_0} + 2ac + b^2}{m_0}, \quad \delta = -\frac{c\left(\gamma\sqrt{m_0} + b\right)}{m_0}. \tag{39}$$

Physically, Equation (38) models a usual damped oscillator affected by a quartic potential that produces a cubic external force given by

$$F_E = -m_0\left(\alpha x^3 + \beta x^2 + \mu x + \gamma\right). \tag{40}$$

Equations of the form (38) are known in the literature as damped Duffing equations with cubic nonlinearities, which have been widely investigated in the recent literature from both analytical and numerical methods. For instance, in Demina and Sinelshchikov [40], the case of linear damping and cubic nonlinearity was studied by means of linearizability

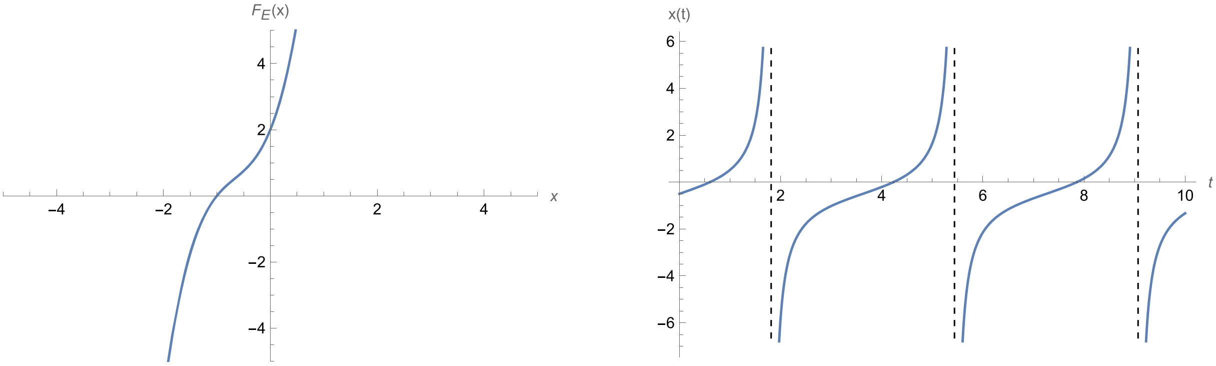

FIGURE 1 External force (40) and exact solution (41) of Equation (38) for $a = b = c = m_0 = 1$ and $k = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

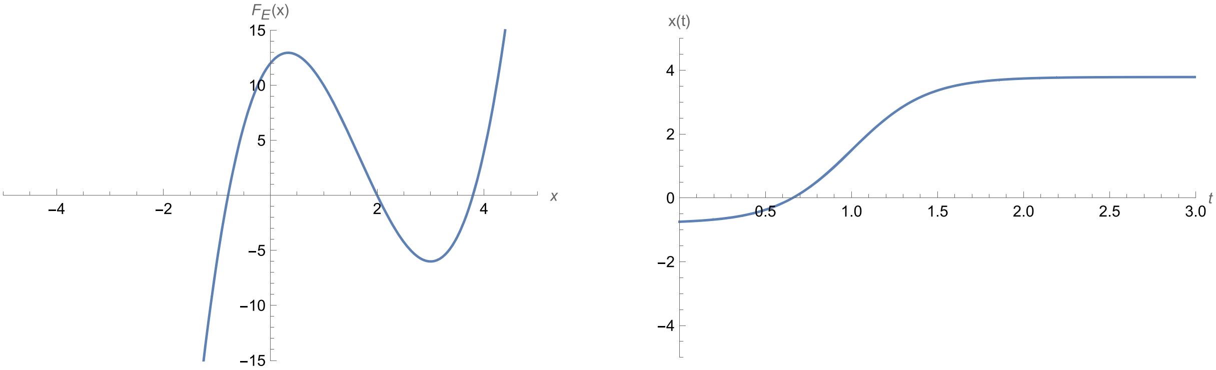

FIGURE 2 External force (40) and exact solution (42) of Equation (38) for $a = -1$, $b = c = 3$, $m_0 = 1$ and $k = -1$. [Colour figure can be viewed at wileyonlinelibrary.com]

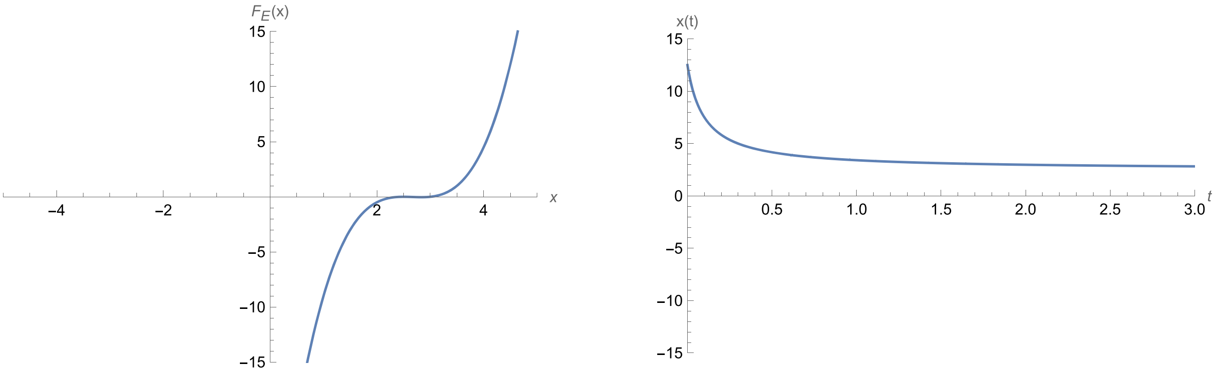

FIGURE 3 External force (40) and exact solution (43) of Equation (38) for $a = -1$, $b = 5$, $m_0 = 1$ and $k = 0.1$. [Colour figure can be viewed at wileyonlinelibrary.com]

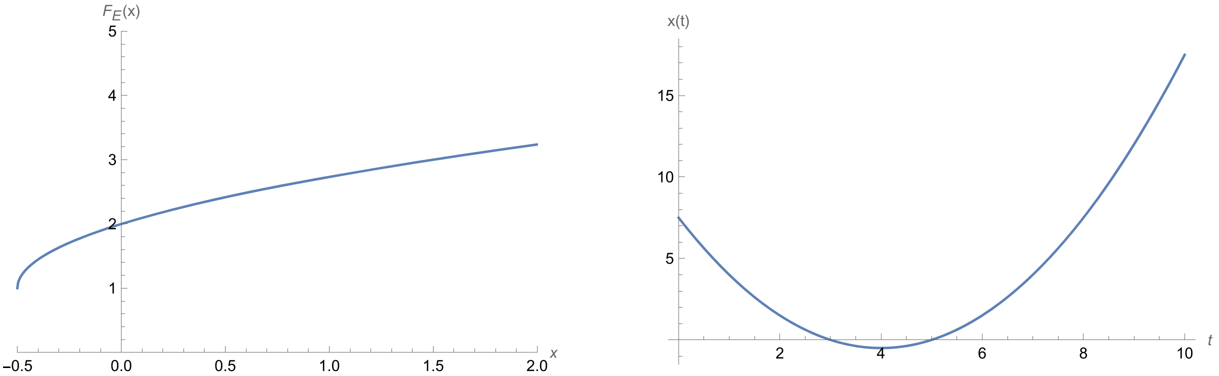

FIGURE 4 External force and exact solution (47) of Equation (46), respectively, for $\gamma = 1$, $a = 2$, $m_0 = 1$, $C = 4$ and $k = -4$. [Colour figure can be viewed at wileyonlinelibrary.com]

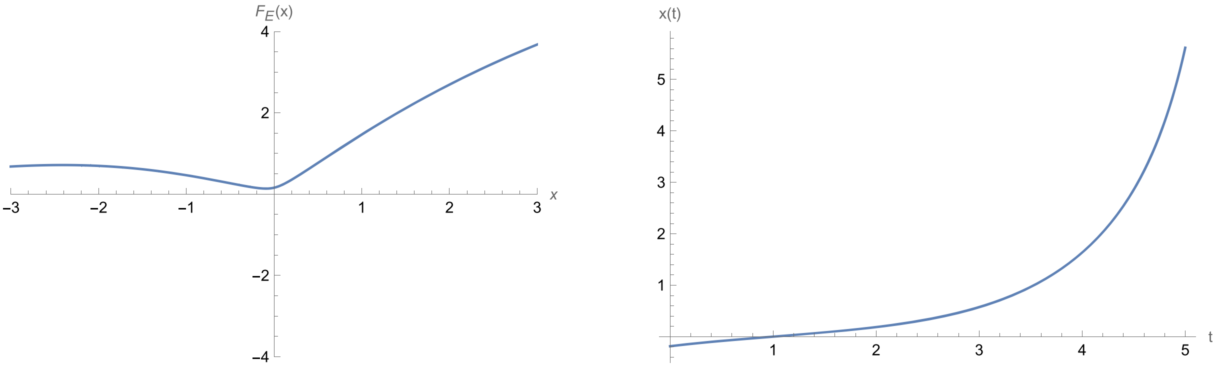

FIGURE 5 External force and exact solution (51) of Equation (50), respectively, for $k = -1$, $m_0 = 1$, $l = C = 1/10$ and $\gamma = 1$. [Colour figure can be viewed at wileyonlinelibrary.com]

via nonlocal transformations, whereas an approximate solution was obtained in Pirbodaghis et al. [41] for the undamped case and for cubic-quintic nonlinearities.

By Theorem 2, a one-parameter family of explicit solutions can be obtained from (35):

- If $4ac - b^2 > 0$:

$$x(t) = \frac{\sqrt{4ac-b^2}\,\tan\left(\frac{1}{2}\sqrt{\frac{4ac-b^2}{m_0}}(t+k)\right) - b}{2a}. \tag{41}$$

- If $4ac - b^2 < 0$:

$$x(t) = \frac{-\sqrt{b^2-4ac}\,\tanh\left(\frac{1}{2}\sqrt{\frac{b^2-4ac}{m_0}}(t+k)\right) - b}{2a}. \tag{42}$$

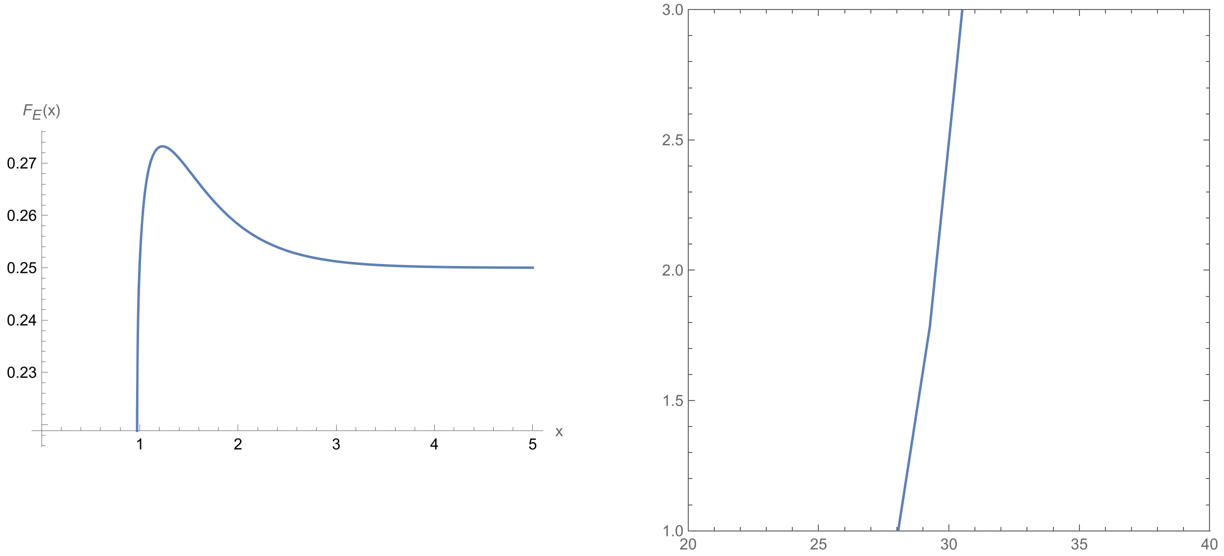

FIGURE 6 External force and exact solution (56) of Equation (55), respectively, for $C = -3$ and $\lambda = \gamma = 1$, $m_0 = 1/4$ and $k = -28$. [Colour figure can be viewed at wileyonlinelibrary.com]

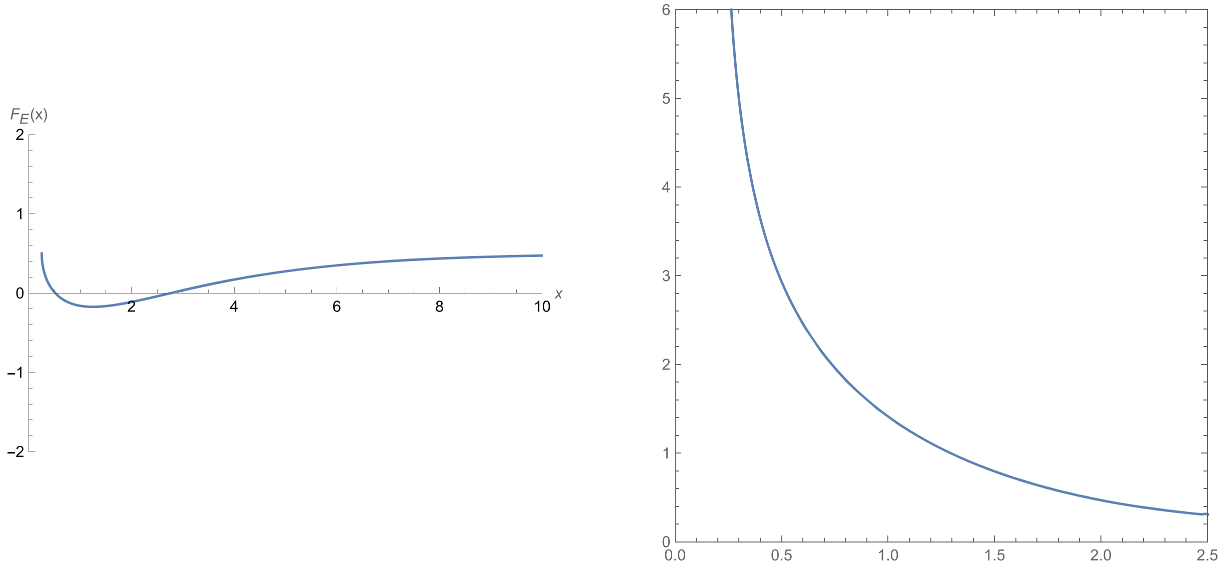

FIGURE 7 External force and exact solution (60) of Equation (59), respectively, for $C = -1$, $\gamma = l = m_0 = 1$ and $k = -3$. [Colour figure can be viewed at wileyonlinelibrary.com]

- If $4ac - b^2 = 0$:

$$x(t) = -\frac{b(t+k) + 2\sqrt{m_0}}{2a(t+k)}. \tag{43}$$

In this particular example, the reduced Abel equation (5) becomes

$$w_y w + \gamma w + \alpha y^3 + \beta y^2 + \mu y + \delta = 0, \tag{44}$$

where the coefficients α, β, μ and δ are given in (39). As a by-product of the present approach, according to Corollary 1, the following particular solution to Equation (44) is obtained:

- If $b^2 - 4ac \neq 0 : w = \frac{1}{\sqrt{m_0}} \left(ay^2 + by + c \right)$.
- If $b^2 - 4ac = 0$: $w = \frac{1}{4a\sqrt{m_0}}(2ay + b)^2$.

5.2 | Constant mass and an external force with square root nonlinearity

In this case, we consider a constant mass function, that is, $m(x) = m_0$ and a function q of the form $q(x) = ax$, with $a \in \mathbb{R}$, $a \neq 0$. This situation corresponds to a classical damped oscillator affected by an external force modelled by the function

$$F_E(x) = \frac{a + \gamma\sqrt{m_0\,(4ax + C)}}{2}, \qquad C \in \mathbb{R}, \tag{45}$$

defined for $x < -\frac{C}{4a}$ if $a < 0$ and for $x > -\frac{C}{4a}$ if $a > 0$. The associated motion equation (29) becomes

$$\ddot{x} + \gamma\dot{x} - \frac{a + \gamma\sqrt{m_0\,(4ax + C)}}{2m_0} = 0. \tag{46}$$

The corresponding solution (35), derived in Theorem 2 by the variational λ-symmetry approach, provides the following one-parameter family of exact solutions to Equation (46):

$$x(t) = \frac{a^2(t + k)^2 - m_0 C}{4am_0}, \qquad k \in \mathbb{R}. \tag{47}$$

5.3 | Mathews–Lakshmanan type position-dependent mass function

Mathews and Lakshmanan [3] derived interesting dynamical properties of a nonlinear oscillator that corresponds to the nonlinear equation (4) for the case $\gamma = 0$, a position-dependent mass function given by

$$m(x) = \frac{m_0}{1 + lx^2}, \qquad l > 0, \tag{48}$$

and a quadratic oscillator potential function.

In this example, we consider the motion Equation (4) with the same mass function (48). By setting, for instance $q(x) = x^2$, which produces a external force given by

$$F_E(x) = \frac{1}{2(1 + lx^2)} \left(\gamma\sqrt{m_0(4x^2 + C)(1 + lx^2)} + lx^3 + x \right), \tag{49}$$

it can be checked that the corresponding motion equation (29) becomes

$$\ddot{x} - \frac{lx}{1 + lx^2}\dot{x}^2 + \gamma\dot{x} - \frac{\gamma\sqrt{m_0\left(4x^2 + C\right)\left(1 + lx^2\right)} + 2x(1 + lx^2)}{2m_0} = 0. \tag{50}$$

The external function (49), as well as the motion Equation (50), are well defined for any value of the dependent variable x if $C \geq 0$ or for $|x| > \sqrt{-C}/2$ if $C < 0$.

For this equation, Theorem 2 provides the following one-parameter family of exact solutions that can be obtained from (35):

- For $C > 0$ and $lC < 4$,

$$x(t) = \frac{\sqrt{C}}{2}\mathrm{sc}\left(\frac{t + k}{\sqrt{m_0}}; \sqrt{1 - \frac{lC}{4}} \right), \qquad k \in \mathbb{R}, \tag{51}$$

where sc denotes the Jacobi sc elliptic function [42].

- For $C < 0$,

$$x(t) = \frac{\sqrt{-C}}{2}\mathrm{sc}\left(\frac{t+k}{\sqrt{m_0}}; \sqrt{1-\frac{lC}{4}}\right), \qquad k \in \mathbb{R}.$$

- For $C = 0$,

$$x(t) = \pm\frac{\sqrt{-l\left(\tanh^2\left(\frac{k+t}{\sqrt{m_0}}\right)-1\right)}}{l\tanh\left(\frac{k+t}{\sqrt{m_0}}\right)}, \qquad k \in \mathbb{R}. \tag{52}$$

5.4 | Hyperbolic type position-dependent mass

Cruz et al. [4] considered the position-dependent mass function

$$m(x) = m_0\tanh^2(lx), \qquad l \neq 0, \tag{53}$$

in the framework of the classical position-dependent mass harmonic oscillator. By setting $q(x) = \tanh(lx)$, we obtain the following external force:

$$F_E(x) = \frac{1}{2}\tanh(lx)\left(\gamma\sqrt{m_0\,(4\tanh(lx)+C)} - l\tanh(lx)\right) + \frac{1}{2}l, \tag{54}$$

which is well defined provided that $4\tanh(lx) + C > 0$. The corresponding motion Equation (29), defined also for $4\tanh(lx) + C > 0$, turns out to be

$$\ddot{x} - \frac{l(\tanh^2(lx)-1)}{\tanh(lx)}\dot{x}^2 + \gamma\dot{x} - \frac{\tanh(lx)\left(\gamma\sqrt{m_0\,(4\tanh(lx)+C)} - l\tanh(lx)\right) + l}{2m_0\tanh^2(lx)} = 0. \tag{55}$$

According to (35), a one-parameter family of exact solutions to (55) in implicit form becomes

- For $C \neq \pm 4$,

$$l(t+k)\sqrt{C^2-16} = 2\sqrt{m_0(C+4)}\,\mathrm{arctanh}\left(\sqrt{\frac{4\tanh(lx)+C}{C-4}}\right) + 2\sqrt{m_0(C-4)}\,\mathrm{arctanh}\left(\sqrt{\frac{4\tanh(lx)+C}{C+4}}\right), \tag{56}$$

  where $k \in \mathbb{R}$.

- For $C = 4$,

$$t+k = \frac{\sqrt{m_0}\left(\sqrt{2(\tanh(lx)+1)}\,\mathrm{arctanh}\left(\frac{1}{2}\sqrt{2(\tanh(lx)+1)}\right)+2\right)}{2l\sqrt{\tanh(lx)+1}},$$

  where $k \in \mathbb{R}$.

- For $C = -4$,

$$t+k = \frac{\sqrt{m_0}\left(\sqrt{2(\tanh(lx)-1)}\,\mathrm{arctanh}\left(\frac{1}{2}\sqrt{2(\tanh(lx)-1)}\right)-2\right)}{2l\sqrt{\tanh(lx)-1}},$$

  where $k \in \mathbb{R}$.

5.5 | Position-dependent mass function in exponential form

Dong et al. [5] considered the mass function

$$m(x) = \frac{lm_0}{1 - e^{-l}} e^{-lx} \tag{57}$$

in the context of Schrödinger equations with position-dependent mass. The authors analyzed the solutions of the corresponding Schrödinger equation by considering different external potential functions such as a confined hard-all potential and a harmonic oscillator, among others. On the other hand, potential functions of exponential-type play an important role in quantum chemical calculations since they are used in the theoretical study of vibration properties for diatomic molecules [5].

The position-dependent mass function (57) along with $q(x) = x$ leads, through (28), to the following external force:

$$F_E(x) = \frac{\sqrt{1 - e^{-l}} - \gamma \sqrt{lm_0(4x + C)}e^{-\frac{1}{2}lx}}{2\sqrt{1 - e^{-l}}}, \tag{58}$$

and, by means of (29), to the following family of equations:

$$\ddot{x} - \frac{1}{2}l\dot{x}^2 + \gamma\dot{x} + \frac{\gamma\sqrt{lm_0(4x + C)(1 - e^{-l})} + e^{-l} - 1}{2lm_0e^{-lx}} = 0. \tag{59}$$

Both expressions (58) and (59) are defined for $x > -C/4$. According to Theorem 2, a one-parameter family of exact solutions to (59) is given in implicit form by

$$t\sqrt{1 - e^{-l}} + \sqrt{2\pi m_0}e^{\frac{1}{8}lC}\Psi\left(\frac{\sqrt{2l(4x + C)}}{4}\right) + k\sqrt{1 - e^{-l}} = 0, \tag{60}$$

where the function $\Psi = \Psi(z)$ is such that

$$\Psi'(z) = \frac{2e^{-z^2}}{\sqrt{\pi}}.$$

6 | CONCLUDING REMARKS

In this work, the variational λ-symmetry method has been applied to a family of second-order ordinary differential equations that models the motion equation of position-dependent mass damped oscillators. The considered family of equations only admits one Lie point symmetry and its corresponding reduced equation turns out to be an Abel equation for which the determination of some exact solutions seems, in general, a difficult task. Indeed, the admitted Lie point symmetry is not variational, and hence, Noether's theorem cannot be applied to obtain any solution of the considered equations.

In this work, the existence of variational λ-symmetries whose canonical representative is defined by a function λ that is linear in $\dot{x}$ has been investigated. As a consequence, we have determined a family of potential functions for which the variational λ-symmetry method can be applied. The procedure provides a one-parameter family of exact solutions for a wide class of the considered equations, involving an arbitrary smooth function and an arbitrary parameter.

These results have been applied to construct exact solutions for some particular equations, corresponding to different choices of the mass functions that have been considered in the previous literature.

AUTHOR CONTRIBUTIONS

Adrián Ruiz Serván: Conceptualization; investigation; writing—original draft; methodology; validation; visualization; writing—review and editing; software; formal analysis; data curation; supervision. **María Concepción Muriel**

Patino: Writing—original draft; methodology; validation; visualization; writing—review and editing; formal analysis; supervision; conceptualization.

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CONFLICTS OF INTEREST STATEMENT

This work does not have any conflicts of interest.

ORCID

Adrián Ruiz Serván https://orcid.org/0000-0002-5417-4773
María Concepción Muriel Patino https://orcid.org/0000-0003-4478-4524

REFERENCES

1. G. Bastard, *Wave mechanics applied to semiconductor heterostructure*, Les Editions de Physique, Les Ulis, France, 1988.
2. O. Von Roos and H. Mavromatis, *Position-dependent effective masses in semiconductor theory. II*, Phys. Rev. B **31** (1985), no. 4, 2294–2298.
3. P. M. Mathews and M. Lakshmanan, *On a unique nonlinear oscillator*, Q. Appl. Math. **32** (1974), 215–218.
4. S. C. y Cruz, J. Negro, and L. M. Nieto, *On position-dependent mass harmonic oscillators*, J. Phy. Conf. Ser. **128** (2008), 12053–12065.
5. S. H. Dong, W. H. Huang, P. Sedaghatnia, and H. Hassanabadi, *Exact solutions of an exponential type position dependent mass problem*, Results Phys. **34** (2022), 105294–105298.
6. T. Q. Dai and Y. F. Cheng, *Bound state solutions of the Klein-Gordon equation with position-dependent mass for the inversely linear potential*, Phys. Scr. **79** (2009), no. 1, 15007–15011.
7. C. Quesne and V. M. Tkachuk, *Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem*, J. Phys. A: Math. Gen. **37** (2004), no. 14, 4267–4281.
8. S. Zare and H. Hassanabadi, *Properties of quasi-oscillator in position-dependent mass formalism*, Adv. High Energy Phys. **37** (2004), no. 14, 4267–4281.
9. R. L. Norton, *Cam design and manufacturing handbook*, Instrustrial Press, U.S., 2009.
10. A. G. Choudhury and P. Guha, *Quantum Liénard II equation and Jacobi's last multiplier*, Sur. Math. Appl. **10** (2015), 1–21.
11. B. G. Da Costa and E. P. Borges, *A position-dependent mass harmonic oscillator and deformed space*, J. Math. Phys. **59** (2018), 42101.
12. S. H. Mazharimousavi and O. Mustafa, *Classical and quantum quasi-free position-dependent mass: Pöschl-Teller and ordering ambiguity*, Phys. Scr. **87** (2013), 55008–55015.
13. O. Mustafa, *n-dimensional pdm-damped harmonic oscillators: linearizability, and exact solvability*, Phys. Scr. **96** (2021), 65205–65217.
14. O. Mustafa, *Comment on 'nonlinear dynamics of a position-dependent mass-driven duffing-type oscillator'*, J. Phys. A: Math. Theor. **46** (2013), no. 36, 368001.
15. R. M. Eisberg and L. S. Lerner, *Physics: foundations and applications, volume 1*, McGraw-Hill Inc., USA, 1981.
16. P. J. Olver, *Introduction of lie groups to differential equations*, Springer-Verlag, New York, 1993.
17. P. J. Olver, *Equivalence, invariants and symmetry*, Cambridge University Press, New York, 1995.
18. I. S. Krasil'shckik and A. M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, American Mathematical Society, Providence, 1999.
19. G. W. Bluman and S. Anco, *Symmetry and integration methods for differential equations*, Springer-Verlag, New York, 2002.
20. G. W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New York, 1989.
21. A. D. Polyanin and V. F. Zaitsev, *Handbook of exact solutions for ordinary differential equations*, Chapman & Hall/CRC, 2003.
22. Z. E. Musielak, *Standard and non-standard Lagragians for dissipative dynamical systems with variable coefficients*, J. Phys. A: Math. Theor. **41** (2008), 55205–55222.
23. Z. E. Musielak, *General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems*, Chaos, Solit. Fractals **42** (2009), 2645–2652.

24. J. L. Cieśliński and T. Nikiciuk, *A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients*, J. Phys. A: Math. Theor. **43** (2010), 175205–175220.
25. M. C. Nucci and K. M. Tamizhmani, *Lagrangians for dissipative nonlinear oscillators: the method of Jacobi last multiplier*, J. Nonlinear Math. Phys. **17** (2010), 167–178.
26. G. G. Polat and O. Teoman, *New conservation laws, Lagrangian forms and exact solutions of modified Emden equation*, J. Comput. Nonlinear Dyn. **12** (2017), no. 4, 41001.
27. N. A. Kudryashov and D. I. Sinelshchikov, *New non-standard Lagrangians for the Liénard-type equations*, Appl. Math. Lett. **63** (2017), 124–129.
28. K. S. Govinder and P. G. L. Leach, *On the determination of non-local symmetries*, J. Phys. A: Math. Gen. **28** (1995), no. 18, 5349–5359.
29. N. A. Kudryashov and D. I. Sinelshchikov, *On the criteria for integrability of the Liénard equation*, Appl. Math. Lett. **57** (2016), 114–120.
30. N. A. Kudryashov and D. I. Sinelshchikov, *On connections of the Liénard equation with some equations of Painlevé–Gambier type*, J. Math. Appl. **449** (2017), 1570–1580.
31. N. A. Kudryashov and D. I. Sinelshchikov, *On the integrability conditions for a family of Liénard-type equations*, Regul. Chaotic Dyn. **21** (2016), 548–555.
32. E. Noether, *Invariante variationsprobleme*, Nachr Konig Gessell Wissen Gottingen Mathphys Kl. **1918** (1918), 235–257. see Transport Theory and Stat. Phys. 1 (1971) 186–207 for an English translation.
33. C. Muriel, J. L. Romero, and P. J. Olver, *Variational C^∞-symmetries and Euler-Lagrange equations*, J. Differ. Equ. **222** (2006), no. 1, 164–184.
34. C. Cicogna and G. Gaeta, *Noether theorem for μ-symmetries*, J. Phys. A: Math. Gen. **39** (2007), 11899–11921.
35. A. Ruiz, C. Muriel, and P. J. Olver, *On the commutator of C^∞-prolonged vector fields and the reduction of Euler-Lagrange equations*, J. Phys. A: Math. Theor. **51** (2018), no. 14, 145202–145223.
36. A. Ruiz and C. Muriel, *Variational λ-symmetries and exact solutions to Euler-Lagrange equations lacking standard symmetries*, Math. Methods Appl. Sci. **2022** (2022), 1–13.
37. M. Nadjafikhah, S. Dodangeh, and P. Kabi-Nejad, *On the variational problems without having desired variational symmetries*, J. Math. **2013** (2013), 4 pages.
38. C. Muriel, J. L. Romero, and A. Ruiz, *λ-symmetries and integrability by quadratures*, IMA J. Appl. Math. **82** (2017), no. 5, 1061–1087.
39. C. Muriel and J. L. Romero, *New methods of reduction for ordinary differential equations*, IMA J. Appl. Math. **66** (2001), 111–125.
40. M. Demina and D. Sinelshchikov, *Integrability properties of cubic Lienard oscillators with linear damping*, Symmetry **11** (2019), no. 11, 1378–1388.
41. T. Pirbodaghis, S. H. Hoseinim, M. T. Ahmadiang, and G. H. Farrahi, *Duffing equations with cubic and quintic nonlinearities*, Comput. Math. Appl. **57** (2009), 500–506.
42. F. W. K. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, 2010.

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