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Optimization of partially monotonic functions subject to bipolar fuzzy relation equations

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ABSTRACT

A method to solve a latticed optimization problem constrained by a bipolar fuzzy relation equation is presented in this paper, under the hypothesis of a partially monotonic objective function. The solving strategy consists of transforming the problem into optimizing an order-preserving function in all arguments subject to another bipolar fuzzy relation equation. As a result, all the solutions of the original optimization problem can be deduced from the extremal elements of the feasible domain of the transformed problem. The presented approach embraces the particular case of linear optimization constrained by bipolar fuzzy relation equations.

1. Introduction

Optimization problems subject to fuzzy relation equations (FRE) were firstly studied in [16]. The investigation of this topic has attracted numerous researchers since then, and several works can be found in the literature. The most common approach is based on transforming the optimization problem into a 0-1 integer linear programming problem, which is then solved by means of a branch-and-bound method [16,36]. This also applies to optimization problems subject to a bipolar fuzzy relation equation (BFRE) [27,38]. In these equations, the variables can be positive and negative in order to capture the bipolarity in human reasoning, and two different relations A^+ and A^-, without any originally required value restriction or relation between them, determine their respective impact on the system. Diverse kinds of bipolarity have been studied and analyzed in the literature, as it was extensively discussed in [15].

Another common technique to solve optimization problems constrained by FRE (or BFRE) relies on modifying the classical genetic algorithm [22]. The philosophy of modified genetic algorithms has been present in the literature from the beginning of the topic [30,31] to the most recent researches [20,33]. In both strategies, the result is an approximation of a solution of the optimization problem, which may result in major drawbacks, depending on the context. For instance:

- Formally, the optimization problem can be non-solvable if the feasible domain is not bounded. The existing works in the literature cannot capture this fact.
- Approximating only one solution of the optimization problem implies that there is only one form to optimize the objective function, what can be inadequate from an applied point of view.

In this paper, we present a method to solve optimization problems subject to BFRE from a different perspective, pursuing all the exact solutions of the problem. The foundations of the method were first outlined in [17], considering the specific case of linear

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optimization problems constrained by a max-min bipolar fuzzy relation equation with the standard negation. The underlying idea can be described as follows. A given optimization problem is transformed into an equivalent optimization problem whose objective function is strictly order-preserving in all arguments. The monotonicity of the objective function implies that the optimal solutions of the new problem are either maximal or minimal elements of its feasible domain. Therefore, it is sufficient to evaluate the extremal solutions of the BFRE constraint of the transformed optimization problem in order to obtain the optimal solutions of the original problem. To this aim, there are several works in the literature devoted to the resolution of BFRE and the characterization of its extremal solutions [3,7,26,35,37].

Our contribution focuses on generalizing the approach taken in [17], presenting a transformation procedure which only requires two conditions: The objective function is partially monotonic, i.e. its partial mappings are either order-preserving or order-reversing, and the negation operator in the BFRE constraint is involutive. Notice that, among others, the required conditions hold for linear optimization in the unit interval constrained by FRE or by BFRE with the standard negation, which includes the problems considered in [17,23–25,28]. Above and beyond that, the presented approach enables the optimization of non-linear functions in general algebraic structures, such as lattices endowed with an involutive negation.

The rest of the paper is organized as follows. Section 2 shows the formal definition of BFRE and recalls the resolution approach presented in [7]. Section 3 introduces optimization problems constrained by BFRE and illustrates some of the main difficulties of the topic, as the lack of solutions. The resolution method is shown first for optimization problems subject to FRE and then generalized to the case of BFRE constraint. Different examples are provided to highlight the main advantages of the procedure. The paper concludes with some conclusions and prospects for future work.

2. Bipolar fuzzy relation equations

In this section, we present the algebraic framework and the formal definition of bipolar fuzzy relation equations. As usual in the literature of the topic, we consider equations defined over a finite set, so we will use matrix notation to represent relations. Then, we will recall the main results shown in [7] concerning the resolution of a general family of bipolar fuzzy relation equations.

2.1. Algebraic structure and definition of bipolar fuzzy relation equations

Bipolar fuzzy relation equations (BFRE) are expressions of the form

$$(A^+ \circ x) \vee (A^- \circ \neg x) = b \tag{1}$$

where A^+ and A^- are fuzzy relations, b is a vector, $\circ$ is a sup-composition operator and $\neg$ is an order-reversing mapping, commonly a negation operator. It needs to be stressed that A^+ and A^- are independent from each other. Generally, BFRE are written as systems of bipolar sup-$*$ equations, where $*$ is a mapping associated with the sup-composition $\circ$.

In its most simple and fundamental form, a bipolar sup-$*$ equation can be defined as a join of bipolar products in a join-semilattice, that is, in a partially ordered set that has a join (a least upper bound) for any nonempty finite subset. First of all, we recall the notion of join-semilattice [8].

Definition 1. A *join-semilattice* is a partially ordered set $(L, \preceq)$ such that $x \vee y \in L$, for all $x, y \in L$.

Next, the definition of bipolar sup-$*$ equation on a join-semilattice is formally included.

Definition 2. Let $(L, \preceq)$ be a join-semilattice, $*\colon L \times L \to L$ a binary mapping, $\neg\colon L \to L$ an order-reversing mapping and $a_j^+, a_j^-, b \in L$ with $j \in \{1, \dots, m\}$. A *bipolar sup-$*$ equation* is an expression of the form:

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) = b$$

in the unknowns $x_1, \dots, x_m \in L$.

Denoting the elements of the relations A^+ and A^- as a_{ij}^+ and a_{ij}^-, respectively, with $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, and $b = (b_1, \dots, b_n)$, we can rewrite the BFRE (1) as the system of bipolar sup-$*$ equations:

$$\bigvee_{j \in \{1,\dots,m\}} (a_{ij}^+ * x_j) \vee (a_{ij}^- * \neg x_j) = b_i, \qquad i \in \{1, \dots, n\}$$

Notice that, the solution set of a system of bipolar sup-$*$ equations is the intersection of the solution sets of the individual bipolar sup-$*$ equations.

A practical interpretation of the symbols involved in the BFRE can be found in Section 4 in [3], where the behavior of a motor is modeled based on diverse variables. One of these variables is the "lack of oil". In this case, it is assumed that the motor overheating is directly proportional to the "lack of oil" (x), being the proportionality constant $a^+ = 0.2$. In addition, an overheating of $a^- = 0.1$ occurs when the "oil exceeds" ($\neg x$) the permitted limit. As a consequence, the overheating caused by the oil level is modeled by the expression $(0.2 * x) \vee (0.1 * \neg x)$. The whole example can be seen in [3].

2.2. Resolution of bipolar sup-$*$ equations

In what follows, we recall the main results of the study carried out in [7], where a complete resolution of a bipolar sup-$*$ equation is presented under some general requirements. Such results will be useful in order to illustrate the contribution given in this paper by means of different examples.

The next definitions concern some basic notions related to lattice theory. For more details, the reader is referred to [1,8,18].

Definition 3. A *lattice* is a partially ordered set $(L,\preceq)$ which is a join-semilattice and a meet-semilattice, that is, $x\vee y\in L$ and $x\wedge y\in L$, for all $x,y\in L$. Besides, if $\bigvee S\in L$ and $\bigwedge S\in L$ for all $S\subseteq L$, we say that $(L,\preceq)$ is a *complete lattice*.

Definition 4. A *lattice* $(L,\preceq)$ is called *distributive* if

$$x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$$

for all $x,y,z\in L$.

The notion of residuated lattice was introduced in [13] from a purely algebraic perspective as a lattice endowed with a residuated pair. Later, Hájek showed in [19] that residuated pairs enable to generalize the philosophy of the Modus Ponens inference.

Definition 5. A tuple $(L,\preceq,*,\rightarrow)$ is a *residuated lattice* if $(L,\preceq)$ is a lattice with top element $\top$, $(L,*,\top)$ is a commutative monoid and $(*,\rightarrow)$ is a *residuated pair*, that is, for all $x,y,z\in L$:

$$x*y\preceq z\qquad\text{if and only if}\qquad x\preceq y\rightarrow z\tag{2}$$

The condition (2) is usually referred to as *residuated property*, whilst the mapping $\rightarrow$ is called the *residuated implication* of $*$.

A residuated lattice with bottom element $\bot$ is called *bounded*. In that case, we can define negation operators as follows.

Definition 6. Let $(L,\preceq)$ be a bounded residuated lattice. A *negation* is an order-reversing mapping $\neg\colon L\to L$ such that $\neg\top=\bot$ and $\neg\bot=\top$. Besides, if $\neg\neg x=x$ for all $x\in L$, we say that $\neg$ is an *involutive negation*.

The concept of symmetric residuated lattice was firstly introduced in [2] and recently characterized in [7]. For the sake of simplicity, its definition is directly shown here in terms of the characterization.

Definition 7. A tuple $(L,\preceq,*,\rightarrow,\neg)$ is a symmetric residuated lattice if $(L,\preceq,*,\rightarrow)$ is a residuated lattice and $\neg\colon L\to L$ is an involutive negation.

The study presented in [7] analyzes the resolution of bipolar sup-$*$ equations in a complete distributive symmetric residuated lattice (CDSRL). Additionally, two extra conditions are required: the right-hand side of the equation is join-irreducible and the operator $*$ is a homomorphism of the underlying lattice. Let us recall the formal definition of these concepts.

Definition 8. Let $(L,\preceq)$ be a complete lattice.

- An element $x\in L$ is *join-irreducible* if $x\neq\bot$ and $x=a\vee b$ implies $x=a$ or $x=b$, for all $a,b\in L$.
- A mapping $f\colon L\to L$ is a *(lattice) homomorphism* if it preserves the supremum and the infimum of any nonempty subset, i.e. $f(\bigvee A)=\bigvee f(A)$ and $f(\bigwedge A)=\bigwedge f(A)$, for all $A\in\mathcal{P}(L)\setminus\{\varnothing\}$, where $\mathcal{P}(L)$ is the powerset of L.

We fix now some notation in order to recall two results introduced in [7], which characterize the solution set and the solvability of a sup-$*$ equation, respectively.

Firstly, two mappings related to the triplet $(L,\preceq,*)$ can be highlighted, which were firstly introduced in [14]. On the one hand, according to the residuated property (2), if the residuated implication of $*$ exists, then it is given by

$$a\rightarrow b=\max\{x\in L\mid a*x\preceq b\}$$

Additionally, as shown in [9], the mapping $\rightsquigarrow$ defined as

$$a\rightsquigarrow b=\inf\{x\in L\mid b\preceq a*x\}$$

is especially useful to describe the solutions of a sup-$*$ equation, and so it is for the case of bipolar sup-$*$ equations.

Assume that the negation $\neg$ and the supremum $\vee$ and infimum $\wedge$ operators are naturally extended from L to L^m. Given a bipolar sup-$*$ equation

$$\bigvee_{j\in\{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) = b \tag{3}$$

the following terms are fixed:

- The tuple $s^{j+} \in L^m$ is defined as $s^{j+} = (\bot, \dots, \bot, a_j^+ \rightsquigarrow b, \bot, \dots, \bot)$, for each $j \in \{1, \dots, m\}$, and

$$S^+ = \{s^{j+} \in L^m \mid b \preceq a_j^+, a_j^+ \rightsquigarrow b \preceq a_j^+ \to b\}$$

- Similarly, we define $s^{j-} = (\bot, \dots, \bot, a_j^- \rightsquigarrow b, \bot, \dots, \bot)$ and

$$S^- = \{s^{j-} \in L^m \mid b \preceq a_j^-, a_j^- \rightsquigarrow b \preceq a_j^- \to b\}$$

- The tuples $g^+, g^- \in L^m$ are respectively given by

$$g^+ = (a_1^+ \to b, \dots, a_m^+ \to b)$$
$$g^- = (a_1^- \to b, \dots, a_m^- \to b)$$

The solution set of a solvable bipolar sup-$*$ equation can be analytically described as follows. Recall that, the partial mappings of $*$ are the unary mappings $*_x, *_y \colon L \to L$ given by $*_x(y) = x * y$ and $*_y(x) = x * y$, for all $x, y \in L$.

Theorem 9 *([7]). Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. If the bipolar sup-$*$ equation (3) is solvable, then its solution set is equal to*

$$D = \Big(\bigcup_{s\in S^+} [s \vee \neg g^-, g^+]\Big) \cup \Big(\bigcup_{s\in S^-} [\neg g^-, g^+ \wedge \neg s]\Big)$$

The next result provides a simple characterization of the solvability of a bipolar sup-$*$ equation.

Theorem 10 *([7]). Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1, \dots, m\}$, and b a join-irreducible element of L. The bipolar sup-$*$ equation (3) is solvable if and only if either g^+ or $\neg g^-$ is a solution.*

Despite the fact that Theorems 9 and 10 are stated for bipolar sup-$*$ equations, they can straightforwardly be used for systems of bipolar sup-$*$ equations, and thus for BFRE. For example, the solution set of a system of bipolar sup-$*$ equations can be obtained as the intersection of the solution sets of the individual equations, which can be computed applying Theorem 9. Similarly, if a system of bipolar sup-$*$ equations is solvable, then all of its equations are solvable, i.e. the necessary and sufficient condition in Theorem 10 holds for all equations in the system.

3. Optimization problems subject to bipolar fuzzy relation equations

In this section, a bounded lattice $(L, \preceq)$, a sup-composition $\circ$ based on an operator $* \colon L \times L \to L$, and a negation operator $\neg \colon L \to L$, will be fixed. An optimization problem constrained by a bipolar fuzzy relation equation

$$(A^+ \circ x) \vee (A^- \circ \neg x) = b \tag{4}$$

can be formulated as one of the following expressions:

$$\begin{array}{ll} \text{Maximize} & f(x) \\ \text{s.t.} & (A^+ \circ x) \vee (A^- \circ \neg x) = b \end{array} \tag{5}$$

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{s.t.} & (A^+ \circ x) \vee (A^- \circ \neg x) = b \end{array} \tag{6}$$

where $f \colon L^m \to L$ is an (objective) mapping, $A^+, A^- \in \mathcal{M}_{n\times m}(L)$ are matrices, $b \in L^m$ is a vector. Usually, we denote the elements of A^+ as a_{ij}^+ and we write $A^+ = (a_{ij}^+)$, where i, j refer to the i-th row and the j-th column of A^+. Similarly, the elements of A^- are denoted as a_{ij}^- and we write $A^- = (a_{ij}^-)$. Notice that, the *feasible domain* of an optimization problem of the form (5) or (6) coincides with the solution set of (4).

This section is devoted to the resolution of optimization problems constrained by FRE or BFRE. It needs to be emphasized that the methods and the results presented here are entirely novel, as well as the different examples that illustrate some of the main difficulties of the topic.

From here on, we will focus on the monotonicity of the objective function in terms of its partial mappings. In this context, given $j \in \{1, \dots, m\}$ and fixing $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m \in L$, a *$j$-th partial mapping* of f is a unary mapping $f_j \colon L \to L$ defined, for all $x \in L$, as

$$f_j(x) = f(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_m)$$

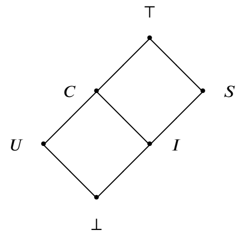

Fig. 1. Hasse diagram of the lattice $(L, \preceq)$ in Example 11.

We say that f is *(strictly) order-preserving (resp. order-reversing) in the j-th argument* if all j-th partial mappings of f are (strictly) order-preserving (resp. order-reversing), i.e. the mapping f_j is (strictly) order-preserving (resp. order-reversing) for all $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m \in L$.

The case of f being a strictly order-preserving mapping in all arguments, or a strictly order-reversing mapping in all arguments, has already been analyzed in [25] for linear optimization problems subject to FRE with continuous sup-t-norm composition. Previously, a similar result was presented in [16] for the specific case of linear optimization with max-min FRE constraints. Basically, the idea behind these results is that the solution set of a FRE with continuous sup-t-norm composition in the unit interval is completely determined by its greatest solution and a set of minimal solutions, thus forming a *roots system* [10]. Therefore, depending on the monotonicity of f and the optimization criterion (maximize or minimize), either the greatest solution or one of the minimal solutions is an optimal solution of the problem.

The assumption of a FRE defined on a bounded lattice instead of the unit interval, and a sup-$*$ composition where $*$ is not necessarily a t-norm nor a continuous mapping, entails a wider range of situations. The following example aims at illustrating the sort of situations that require the consideration of finite lattices, and which may result in problems modeled by FRE defined on a bounded lattice.

Example 11. A teacher needs to evaluate an algebra subject through two academic works, where the mark of every work is based on the quality of the content and the soundness of the mathematical results. To this aim, the teacher rises the following statements:

- A work with good content and high soundness has the highest mark ($\top$). Similarly, bad content and low soundness mean that the work is bad ($\bot$).
- When the content of a work is average, the soundness of the work determines its mark, resulting in a sound work (S) or in an imprecise work (I).
- If the mathematical soundness of a work is acceptable but not brilliant, the teacher decides its mark depending on the content, as a complete work (C) or as an unfinished (U) work.

Since all of the above marks are purely qualitative, one requires the use of a finite domain to represent the set of marks, like

$$L = (\bot, U, I, C, S, \top)$$

In order to define an order relation $\preceq$ to compare different grades, the teacher states that:

- The next chains of inequalities straightforwardly hold:

$$\begin{array}{ccccccc} \bot & \preceq & U & \preceq & C & \preceq & \top \\ \bot & \preceq & I & \preceq & S & \preceq & \top \end{array}$$

- It is not possible to decide, in general, whether a complete work is better or worse than a sound work, as well as unfinished works and imprecise works are incomparable, which is denoted as $C||S$ and $U||I$, respectively.
- On the contrary, complete works are clearly better than imprecise works, as mathematical mistakes penalize a lot in algebra. That is, $I \preceq C$.
- Finally, there is no general agreement on whether sound works are better than unfinished works, as the complexity of the topics need to be taken into account to reach a decision. Hence, $U||S$.

According to the previous information about the partial order $\preceq$, we can represent the set of marks of the academic works as the finite lattice $(L, \preceq)$ shown in Fig. 1.

After evaluating the two works of a student, his/her final grade is computed applying the operator $*$ given in Table 1. Notice that, for instance, if a student obtains C in the first work and S in the second work, the teacher decides a final grade of I, since there is a clear lack of soundness in his/her first work.

Now, provided a class of three students, the teacher may wonder if there exists some student that, although not brilliant, can be asserted to be rigorous. Making use of the supremum operator, the query can be represented as an expression of the form

$$(t_{11} * t_{12}) \vee (t_{21} * t_{22}) \vee (t_{31} * t_{32}) = S$$

Table 1
Operator $*$ in Example 11.

$*$	$\bot$	U	I	C	S	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
U	$\bot$	U	$\bot$	U	$\bot$	U
I	$\bot$	$\bot$	$\bot$	$\bot$	I	I
C	$\bot$	U	$\bot$	U	I	C
S	$\bot$	$\bot$	I	I	S	S
$\top$	$\bot$	U	I	C	S	$\top$

where t_{ij} is the grade of the j-th work of the i-th student.

Finally, if the first work has already been corrected, the query results in a FRE. For instance, if $t_{11} = C$, $t_{21} = \top$ and $t_{31} = S$, we obtain the FRE given by

$$(C * t_{12}) \vee (\top * t_{22}) \vee (S * t_{32}) = S \tag{7}$$

The solutions of Equation (7) model the possible marks of the second work of the students that allow to assert that some student has been rigorous in his/her academic works. □

Naturally, the fixed general algebraic structure is fundamental in the resolution of FRE and BFRE. For instance, the existence of minimal solutions cannot be assured when infinite lattices are taken into account (see Example 3 in [11]) and the set of solutions depend on the considered adjoint pairs [29]. Furthermore, the inclusion of a negation operator, i.e. the consideration of a BFRE, implies the possible existence of maximal solutions instead of a greatest solution (see Example 19 in [6]).

Following the philosophy of the results shown in [16] and [25], a necessary condition for solving optimization problems subject to BFRE (Equation (4)) can be deduced when f is strictly order-preserving in all arguments.

Proposition 12. *Let $f : L^m \to L$ be a strictly order-preserving mapping in all arguments and $x \in L^m$. The following statements hold:*

- *If x is an optimal solution of Problem (5), then x is a maximal solution of Equation (4).*
- *If x is an optimal solution of Problem (6), then x is a minimal solution of Equation (4).*

Proof. We will show the proof of the first item of Proposition 12. The second item can be proved similarly.

Given $x \in L$ an optimal solution of Problem (5), we suppose that x is not a maximal solution of Equation (4), that is, there exists $x' \in L^m$ such that x' is a solution of Equation (4) with $x \prec x'$. Since f is a strictly order-preserving mapping in all arguments, then $f(x) \prec f(x')$, in contradiction with the optimality of x. □

A dual necessary condition holds when f is strictly order-reversing.

Proposition 13. *Let $f : L^m \to L$ be a strictly order-reversing mapping in all arguments and $x \in L^m$. The following statements hold:*

- *If x is an optimal solution of Problem (5), then x is a minimal solution of Equation (4).*
- *If x is an optimal solution of Problem (6), then x is a maximal solution of Equation (4).*

Proof. It is dual to the proof of Proposition 12. □

The following toy example illustrates the necessity of requiring strict monotonicity of the objective function in the hypothesis of Proposition 12 and Proposition 13.

Example 14. Consider the product t-norm $*$ and the optimization problem in the lattice $L = [0, 1]$ given by

$$\begin{array}{ll} \text{Maximize} & f(x_1, x_2) = \max\{x_1 - 0.5, x_2\} \\ \text{s.t.} & A \circ_P (x_1, x_2) = b \end{array}$$

being $\circ_P$ the sup-composition operator based on the product t-norm and

$$A = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix}, \qquad b = [0.6]$$

In this optimization problem, the objective mapping is order-preserving but not strictly order-preserving. The feasible domain of the problem, i.e. the solution set of the FRE constraint, is the set $[0, 1] \times \{0.75\}$, which has a greatest element $(1, 0.75)$. As a result, due to the order-preserving monotonicity of the objective function, its optimal value is

$$f(1, 0.75) = \max\{1 - 0.5, 0.75\} = 0.75$$

Notice that, for instance, the tuple $(0.5, 0.75)$ is not a maximal element of the feasible domain, but it is an optimal solution of the optimization problem, since

$$f(0.5, 0.75) = \max\{0.5 - 0.5, 0.75\} = 0.75 \quad \square$$

On the basis of the computations shown in Example 14, it seems that any optimal solution is bounded by a maximal/minimal element of the feasible domain which also optimizes the problem. In that case, a possible alternative of Proposition 12 for arbitrary order-preserving mappings could be stated as follows:

Let $f : L^m \to L$ be an order-preserving mapping in all arguments and $x \in L^m$. The following statements hold:

1. *If x is an optimal solution of Problem (5), there exists $x' \in L^m$ with $x \preceq x'$ such that x' is a maximal solution of Equation (4) and an optimal solution of Problem (5).*
2. *If x is an optimal solution of Problem (6), there exists $x' \in L^m$ with $x' \preceq x$ such that x' is a minimal solution of Equation (4) and an optimal solution of Problem (6).*

The next counterexample shows that the previous conjecture does not hold, in general.

Example 15. Let $*$ be the product t-norm and $\neg_P : [0,1] \to [0,1]$ the product negation operator, defined as $\neg_P(0) = 1$ and $\neg_P(x) = 0$ if $x \in (0,1]$. Consider the minimization of the non-strictly increasing function

$$f(x_1, x_2, x_3) = x_2$$

constrained by the bipolar max-product FRE

$$A^+ \circ_P (x_1, x_2, x_3) \vee A^- \circ_P \neg_P (x_1, x_2, x_3) = b$$

being $\circ_P$ the sup-composition operator based on the product t-norm and

$$A^+ = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.5 \end{bmatrix}, \qquad A^- = \begin{bmatrix} 0.7 \\ 0.1 \\ 0.2 \end{bmatrix}, \qquad b = [0.3]$$

Observe that, the BFRE constraint can be equivalently written as

$$(0.4 * x_1) \vee (0.7 * \neg_P(x_1)) \vee (0.2 * x_2) \vee (0.1 * \neg_P(x_2)) \vee (0.5 * x_3) \vee (0.2 * \neg_P(x_3)) = 0.3$$

The BFRE constraint was already solved in Example 18 in [4], from which the feasible domain of the problem is

$$D = \big[(0.75, 0, 0), (0.75, 1, 0.6)\big] \ \cup \ D_2$$

with $D_2 = \{(x, y, 0.6) \mid 0 < x \le 0.75, 0 \le y \le 1\}$.

Consequently, the tuple $(0.2, 0, 0.6)$ is an optimal solution of the problem, as it belongs to $D_2 \subseteq D$ and its second argument is 0. Hence, $f(0.2, 0, 0.6) = 0$. However, the set D_2 has no smallest element, which implies that there is no $x' \in D$ with $x' \prec x$ such that x' is a minimal element of D and an optimal solution of the problem. In other words, there exists a semi-open interval of optimal solutions given by: $\{(x, 0, 0.6) \mid 0 < x \le 0.75\}$. Besides, the rest of optimal solutions of the problem forms a closed interval: $\{(0.75, 0, z) \mid 0 \le z \le 0.6\}$. $\square$

According to Propositions 12 and 13, and in order to solve Problems (5) and (6), one might be tempted to think that it is sufficient to compute the maximal/minimal solutions of Equation (4) and then evaluate the objective function in the obtained extrema. Nevertheless, as the next example shows, the existence of maximal/minimal solutions of the BFRE does not guarantee the solvability of the optimization problem.

Example 16. Considering the algebraic structure of Example 15, we take into account now the optimization problem consisting of minimizing the function

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

constrained by the bipolar max-product FRE already considered in Example 15, that is,

$$A^+ \circ_P (x_1, x_2, x_3) \vee A^- \circ_P \neg_P (x_1, x_2, x_3) = b$$

with

$$A^+ = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.5 \end{bmatrix}, \qquad A^- = \begin{bmatrix} 0.7 \\ 0.1 \\ 0.2 \end{bmatrix}, \qquad b = [0.3]$$

As shown in Example 15, the solution set of the BFRE constraint is given by $D = D_1 \cup D_2$, where

$$\begin{array}{rcl} D_1 & = & \big[(0.75,0,0),(0.75,1,0.6)\big] \\ D_2 & = & \{(x,y,0.6) \mid 0 < x \le 0.75, 0 \le y \le 1\} \end{array}$$

Clearly, the minimum element of D_1, i.e. the tuple $(0.75,0,0)$, is a minimal element of D, which is the unique minimal solution of the BFRE constraint. Therefore, since f is a strictly order-preserving mapping in all arguments, Proposition 12 implies that $(0.75,0,0)$ is the unique possible solution of the optimization problem. Nevertheless, we can easily see that the tuple $(0.1,0,0.6)$ is also a solution of the BFRE but

$$f(0.1,0,0.6) = 0.7 < 0.75 = f(0.75,0,0)$$

Hence, the optimization problem is unsolvable. The reason behind the unsolvability is that the feasible domain of the optimization problem, that is, the solution set of the BFRE, does not have a minimum element, but an infimum, because of D_2 provides solutions of the optimization problem less than the only existing minimal element in D_1, but D_2 is a left-open interval. This implies that the BFRE constraint does not have a minimum solution.

As a result, the set of possible values of the objective function is

$$\begin{array}{rcl} f(D) & = & \{f(x,y,z) \mid (x,y,z) \in D\} \\ & = & \{f(x,y,z) \mid (x,y,z) \in D_1\} \cup \{f(x,y,z) \mid (x,y,z) \in D_2\} \\ & = & \big[f(0.75,0,0), f(0.75,1,0.6)\big] \cup \big]f(0,0,0.6), f(0.75,1,0.6)\big] \\ & = & \big[0.75, 2.35\big] \cup \big]0.6, 2.35\big] \\ & = & \big]0.6, 2.35\big] \end{array}$$

Since the set $f(D)$ does not have a minimum but an infimum element, the optimization problem is unsolvable, although an approximate value can be obtained. □

Example 16 proves that the existence of maximal/minimal solutions of the BFRE constraint is not sufficient for the solvability of Problems (5) and (6). On the contrary, the existence of a greatest/smallest solution guarantees the solvability of the corresponding problems, even in case of non-strict monotonicity of the objective function.

Proposition 17. *Let $f : L^m \to L$ be an order-preserving mapping in all arguments. The following statements hold:*

- *If Equation (4) has a greatest solution $\hat{x} \in L^m$, then $\hat{x}$ is an optimal solution of Problem (5).*
- *If Equation (4) has a smallest solution $\check{x} \in L^m$, then $\check{x}$ is an optimal solution of Problem (6).*

Proof. We prove here the first item. The second item can be proved dually.

Let $\hat{x}$ be the greatest solution of Equation (4). As a consequence, if D is the feasible domain of Problem (5), then $x \preceq \hat{x}$ for all $x \in D$. Since f is order-preserving, this implies that $f(x) \preceq f(\hat{x})$ for all $x \in D$. In other words, $\hat{x}$ is an optimal solution of Problem (5). □

Dually to Proposition 17, the following result holds for order-reversing objective functions.

Proposition 18. *Let $f : L^m \to L$ be an order-reversing mapping in all arguments. The following statements hold:*

- *If Equation (4) has a smallest solution $\check{x} \in L^m$, then $\check{x}$ is an optimal solution of Problem (5).*
- *If Equation (4) has a greatest solution $\hat{x} \in L^m$, then $\hat{x}$ is an optimal solution of Problem (6).*

Proof. It is dual to the proof of Proposition 17. □

The previous results and examples aim at emphasizing the complexity of the problem under study. Having said this, the situation shown in Examples 14 and 15 can be simply avoided by requiring strict monotonicity on the objective function. Furthermore, whenever the solution set of the BFRE constraint is known, we can conclude whether situations like those of Examples 14, 15 or 16 may occur or not. To this aim, we can make use of Theorem 9. Last but not least, the presented scenarios cannot take place if the underlying algebraic structure is a finite lattice.

3.1. Optimization of partially strictly monotonic functions subject to fuzzy relation equations

In this section, we will describe a procedure to solve an optimization problem of a partially strictly monotonic objective function constrained by a FRE in a bounded lattice endowed with an involutive negation. The foundations of such procedure rely on the fact that the involutive negation of a strictly order-reversing mapping is a strictly order-preserving mapping, and viceversa. As a consequence, an equivalent optimization problem can be obtained, with the special feature of having a strictly order-preserving objective function in all arguments, which can be solved applying Proposition 12.

In the rest of the paper, the negation $\neg\colon L \to L$ will be involutive and the mapping $*\colon L \times L \to L$ will satisfy the boundary condition with the bottom element of the bounded lattice $(L, \preceq)$ on the right argument, that is, $x * \bot = \bot$ for all $x \in L$. For the sake of readability, we will adopt the terminology employed in [5]:

Definition 19. A *(strict) extended aggregator* $f\colon L^m \to L$ is a partially (strictly) monotonic function

$$f(x_1, \dots, x_m) = f(x_1, \dots, x_k; x_{k+1}, \dots, x_m)$$

where f is (strictly) order-preserving in the first k arguments and (strictly) order-reversing in the remaining arguments.

Let an optimization problem of the form

$$\begin{array}{ll} \text{Maximize/minimize} & f(x_1, \dots, x_k; x_{k+1}, \dots, x_m) \\ \text{s.t.} \quad A \circ (x_1, \dots, x_m) = b & \end{array} \tag{8}$$

Since the negation $\neg$ is involutive, in particular, it is a bijective mapping. Therefore, for each $x_j \in L$, there exists $y_j \in L$ such that $x_j = \neg y_j$. This implies that optimizing f is equivalent to optimizing the mapping $g_f\colon L^m \to L$ defined as

$$g_f(x_1, \dots, x_k, y_{k+1}, \dots, y_m) = f(x_1, \dots, x_k; \neg y_{k+1}, \dots, \neg y_m)$$

Notice that, g_f is an order-preserving mapping in all of the arguments, and therefore the hypothesis of Proposition 12 hold. It remains to see how exchanging f by g_f affects to the FRE constraint of Problem (8). Clearly, the expression

$$A \circ (x_1, \dots, x_m) = b$$

is equivalent to

$$A \circ (x_1, \dots, x_k, \neg y_{k+1}, \dots, \neg y_m) = b$$

Now, the previous expression can be seen as a BFRE of the form

$$(A^+ \circ (x_1, \dots, x_k, y_{k+1}, \dots, y_m)) \vee (A^- \circ \neg (x_1, \dots, x_k, y_{k+1}, \dots, y_m)) = b$$

where $A^+ = (a^+_{ij})$ and $A^- = (a^-_{ij})$ are $n \times m$ matrices defined as

$$a^+_{ij} = \begin{cases} a_{ij} & \text{if} \quad j \leq k \\ \bot & \text{if} \quad j \geq k+1 \end{cases}, \qquad a^-_{ij} = \begin{cases} \bot & \text{if} \quad j \leq k \\ a_{ij} & \text{if} \quad j \geq k+1 \end{cases}$$

Then, the resulting optimization problem can be written as

$$\begin{array}{ll} \text{Maximize/minimize} & g_f(x) \\ \text{s.t.} \quad (A^+ \circ x) \vee (A^- \circ \neg x) = b & \end{array} \tag{9}$$

The equivalence of Problems (8) and (9) is guaranteed by the following results. First of all, a preliminary lemma is presented below, which states an equivalence between the feasible domain of both problems.

Lemma 20. *A tuple $(x_1, \dots, x_m) \in L^m$ belongs to the feasible domain of Problem (8) if and only if $(x_1, \dots, x_k, \neg x_{k+1}, \dots, \neg x_m)$ belongs to the feasible domain of Problem (9).*

Proof. Let $(x_1, \dots, x_m) \in L^m$. Consider the expression $A \circ (x_1, \dots, x_m)$ written as a tuple of elements of the form

$$\bigvee_{j \in \{1, \dots, m\}} (a_{ij} * x_j)$$

with $i \in \{1, \dots, n\}$. Clearly, we can split the previous expression into two parts for every $i \in \{1, \dots, n\}$:

$$\bigvee_{j \in \{1, \dots, m\}} (a_{ij} * x_j) = \bigvee_{j \in \{1, \dots, k\}} (a_{ij} * x_j) \vee \bigvee_{j \in \{k+1, \dots, m\}} (a_{ij} * x_j) \tag{10}$$

Since $\bot * x = \bot$ and $x \vee \bot = \bot \vee x = x$ for each $x \in L$, the following equalities hold, for each $i \in \{1, \dots, n\}$:

$$\begin{aligned} \bigvee_{j \in \{1, \dots, k\}} (a_{ij} * x_j) &= \bigvee_{j \in \{1, \dots, k\}} (a_{ij} * x_j) \vee \bigvee_{j \in \{1, \dots, k\}} (\bot * \neg x_j) \\ \bigvee_{j \in \{k+1, \dots, m\}} (a_{ij} * x_j) &= \bigvee_{j \in \{1, \dots, k\}} (\bot * \neg x_j) \vee \bigvee_{j \in \{1, \dots, k\}} (a_{ij} * x_j) \end{aligned}$$

In other words, for every $i \in \{1, \dots, n\}$, the element in the i-th argument of the tuple

$$A^+ \circ (x_1, \dots, x_k, \neg x_{k+1}, \dots, \neg x_m) \tag{11}$$

is equivalent to $\bigvee_{j\in\{1,\dots,k\}}(a_{ij} * x_j)$, and the element in the i-th argument of the tuple

$$A^{-}\circ(\neg x_1,\dots,\neg x_k,x_{k+1},\dots,x_m) \tag{12}$$

is equivalent to $\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij} * x_j)$.

Now, since $\neg$ is an involutive negation, we can rewrite (12) as

$$A^{-}\circ(\neg x_1,\dots,\neg x_k,\neg\neg x_{k+1},\dots,\neg\neg x_m)$$

or equivalently

$$A^{-}\circ\neg(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m) \tag{13}$$

Hence, from (10), (11) and (13) we obtain that

$$A\circ(x_1,\dots,x_m)=A^{+}\circ(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)\vee A^{-}\circ\neg(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$$

which concludes the proof of Lemma 20. □

The following theorem shows that any solution of Problem (8) is associated with a solution of Problem (9).

Theorem 21. *Let $f : L^m \to L$ be a strict extended aggregator. A tuple $(x_1,\dots,x_m)\in L^m$ is an optimal solution of Problem (8) if and only if $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is an optimal solution of Problem (9).*

Proof. We will assume that Problems (8) and (9) consist of maximizing a function f. The case of minimization problems proceeds dually. The proof will be given by the contraposition of the equivalence of both problems. Thus, by the contraposition law, we will have that a tuple $(x_1,\dots,x_m)\in L^m$ is not an optimal solution of Problem (8) if and only if the tuple $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is not an optimal solution of Problem (9).

Suppose that a tuple $(x_1,\dots,x_m)\in L^m$ is not an optimal solution of Problem (8) and we will see that the tuple $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is not an optimal solution of Problem (9).

Let $D_{(8)}$ and $D_{(9)}$ be the feasible domains of Problems (8) and (9), respectively. Clearly, the case $(x_1,\dots,x_m)\notin D_{(8)}$ is straightforward, because Lemma 20 implies that $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)\notin D_{(9)}$. Consider then $(x_1,\dots,x_m)\in D_{(8)}$, and therefore $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)\in D_{(9)}$.

Since $(x_1,\dots,x_m)$ is not an optimal solution of Problem (8), there exists $(z_1,\dots,z_m)\in D_{(8)}$ such that

$$f(x_1,\dots,x_k;x_{k+1},\dots,x_m)<f(z_1,\dots,z_k;z_{k+1},\dots,z_m) \tag{14}$$

Because of the involutivity of $\neg$, (14) can be rewritten as

$$f(x_1,\dots,x_k;\neg(\neg x_{k+1}),\dots,\neg(\neg x_m))<f(z_1,\dots,z_k;\neg(\neg z_{k+1}),\dots,\neg(\neg z_m))$$

or equivalently, by the definition of the mapping g_f:

$$g_f(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)<g_f(z_1,\dots,z_k,\neg z_{k+1},\dots,\neg z_m)$$

Notice that, $(z_1,\dots,z_k,\neg z_{k+1},\dots,\neg z_m)\in D_{(9)}$ by Lemma 20. Consequently, we conclude that $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is not an optimal solution of Problem (9).

Following a dual reasoning, we can see that if $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is not an optimal solution of Problem (9), then $(x_1,\dots,x_m)$ is not an optimal solution of Problem (8). □

Notice that, as $\neg$ is a bijective mapping, the relation provided by Theorem 21 is one to one. Besides, due to the involutivity of $\neg$, the following result holds.

Corollary 22. *Let $f : L^m \to L$ be a strict extended aggregator. A tuple $(x_1,\dots,x_m)\in L^m$ is an optimal solution of Problem (9) if and only if $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ is an optimal solution of Problem (8).*

Proof. Given $(x_1,\dots,x_m)\in L^m$, since the negation $\neg$ is involutive, there exists $z_j\in L$ such that $x_j=\neg z_j$ for all $j\in\{k+1,\dots,m\}$. Additionally, $\neg x_j=z_j$ for all $j\in\{k+1,\dots,m\}$. Hence, Corollary 22 is equivalent to the following statement, which holds as a consequence of Theorem 21:

The tuple $(x_1,\dots,x_k,\neg z_{k+1},\dots,\neg z_m)\in L^m$ is an optimal solution of Problem (9) if and only if $(x_1,\dots,x_k,z_{k+1},\dots,z_m)$ is an optimal solution of Problem (8). □

The usefulness of transforming the optimization problem (8) into the optimization problem (9) is highlighted in the following example.

Example 23. Let $*$ be the product t-norm and $\neg_S : [0,1] \to [0,1]$ the standard negation defined as $\neg_S x = 1 - x$, for all $x \in [0,1]$. Consider the optimization problem

$$\begin{aligned} &\text{Minimize} \quad e^{x_1 - x_2} + cos(x_3 + x_4) \\ &\text{s.t.} \quad A \circ (x_1, x_2, x_3, x_4) = b \end{aligned} \tag{15}$$

being

$$A = \begin{bmatrix} 0.2 & 0.8 & 0.4 & 0.8 \\ 0.4 & 1 & 0.5 & 0.2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}$$

The objective function of (15) is a strictly order-preserving mapping in x_1 and a strictly order-reversing mapping in the last three arguments. Therefore, we can write the optimization criterion as a strict extended aggregator

$$\text{Minimize} \quad f(x_1; x_2, x_3, x_4) = e^{x_1 - x_2} + cos(x_3 + x_4)$$

By definition of sup-$*$ composition, the FRE constraint of Problem (15) can be written as the system of sup-$*$ equations given by

$$\begin{aligned} (0.2 * x_1) \vee (0.8 * x_2) \vee (0.4 * x_3) \vee (0.8 * x_4) &= 0.5 \\ (0.4 * x_1) \vee (1 * x_2) \vee (0.5 * x_3) \vee (0.2 * x_4) &= 0.2 \end{aligned}$$

Thus, according to the procedure detailed in this section, Problem (15) can be transformed into the problem

$$\text{Minimize} \quad g_f(x_1, x_2, x_3, x_4) = e^{x_1 + x_2 - 1} + cos(2 - x_3 - x_4) \tag{16}$$

constrained by the BFRE given by the system of sup-$*$ equations

$$\begin{aligned} (0.2 * x_1) \vee (0.8 * (1 - x_2)) \vee (0.4 * (1 - x_3)) \vee (0.8 * (1 - x_4)) &= 0.5 \\ (0.4 * x_1) \vee (1 * (1 - x_2)) \vee (0.5 * (1 - x_3)) \vee (0.2 * (1 - x_4)) &= 0.2 \end{aligned} \tag{17}$$

Applying Theorem 9 and carrying out the corresponding computations, we obtain that the solution set of the first equation of (17) is

$$D_1 = \left[(0, 0.375, 0, 0.375), (1, 1, 1, 0.375)\right] \cup \left[(0, 0.375, 0, 0.375), (1, 0.375, 1, 1)\right]$$

Concerning the second equation of (17), its solution set turns out to be

$$D_2 = \left[(0.5, 0.8, 0.6, 0), (0.5, 1, 1, 1)\right] \cup \left[(0, 0.8, 0.6, 0), (0.5, 0.8, 1, 1)\right] \cup \left[(0, 0.8, 0.6, 0), (0.5, 1, 0.6, 1)\right] \cup \left[(0, 0.8, 0.6, 0), (0.5, 1, 1, 0)\right]$$

Hence, the solution set of (17) coincides with the intersection of D_1 and D_2:

$$D = D_1 \cap D_2 = \left[(0.5, 0.8, 0.6, 0.375), (0.5, 1, 1, 0.375)\right] \cup \left[(0, 0.8, 0.6, 0.375), (0.5, 0.8, 1, 0.375)\right] \cup \left[(0, 0.8, 0.6, 0.375), (0.5, 1, 0.6, 0.375)\right]$$

As the smallest element of the set D is the tuple $(0, 0.8, 0.6, 0.375)$, taking into account Proposition 17, it is an optimal solution of Problem (16). Furthermore, applying Proposition 12, we can assert that $(0, 0.8, 0.6, 0.375)$ is the unique optimal solution of Problem (16).

Finally, applying Corollary 22, we conclude that the unique optimal solution of Problem (15) is the tuple

$$(0, 1 - 0.8, 1 - 0.6, 1 - 0.375)$$

that is, the tuple

$$(0, 0.2, 0.4, 0.625)$$

which gives rise to the value $f(0, 0.2, 0.4, 0.625) \approx 1.37$.

The utility of the proposed approach is emphasized when we try to directly solve Problem (15) from its feasible domain, since the optimal solution of Problem (15) is "in the middle" of the feasible domain instead of in its extremal elements. Namely, following the study presented in [12], we obtain that the FRE constraint in Problem (15) has a greatest solution $(0.5, 0.2, 0.4, 0.625)$ and a smallest solution $(0, 0.2, 0, 0.625)$. Nevertheless, the optimal solution of Problem (15) is not any of these two tuples. Indeed, the images of these tuples by f are strictly greater than 1.37:

$$\begin{aligned} f(0.5, 0.2, 0.4, 0.625) &\approx 1.87 \\ f(0, 0.2, 0, 0.625) &\approx 1.63 \end{aligned} \qquad \square$$

A well-known approach for solving optimization problems with FRE constraints consists of separating the problem into two subproblems [21,25,34], in case that the objective function is a separate function, i.e. an extended aggregator that can be written as a linear combination of its partial mappings. Notice that, for instance, this does not hold for the objective function considered in Example 23. Therefore, the procedure presented in this section enables to solve a wider range of optimization problems.

3.2. Optimization of partially strictly monotonic functions subject to bipolar fuzzy relation equations

The strategy followed in Section 3.1 can be adapted to transform the optimization of a partially strictly monotonic function constrained by a BFRE into a similar problem with a strictly order-preserving objective function, as we show next.

Consider an optimization problem of the form

$$\begin{array}{l}\text{Maximize/minimize} \quad f(x_1,\dots,x_k;x_{k+1},\dots,x_m)\\ \text{s.t.} \quad (A^+\circ(x_1,\dots,x_m))\vee(A^-\circ(\neg x_1,\dots,\neg x_m))=b\end{array} \tag{18}$$

As discussed in Section 3.1, optimizing f is equivalent to optimizing the strictly order-preserving mapping $g_f : L^m \to L$ defined as

$$g_f(x_1,\dots,x_k,y_{k+1},\dots,y_m)=f(x_1,\dots,x_k;\neg y_{k+1},\dots,\neg y_m)$$

where $x_j=\neg y_j$ for all $j\in\{k+1,\dots,m\}$.

In what regards to the BFRE constraint of Problem (18), as $x_j=\neg y_j$ and $\neg$ is an involutive mapping, it is equivalent to the equality

$$(A^+\circ(x_1,\dots,x_k,\neg y_{k+1},\dots,\neg y_m))\vee(A^-\circ(\neg x_1,\dots,\neg x_k,y_{k+1},\dots,y_m))=b$$

Hence, we obtain a BFRE that can be written as

$$(A^{+-}\circ(x_1,\dots,x_k,y_{k+1},\dots,y_m))\vee(A^{-+}\circ(\neg x_1,\dots,\neg x_k,\neg y_{k+1},\dots,\neg y_m))=b$$

where $A^{+-}=(\alpha_{ij}^{+-})$ and $A^{-+}=(\alpha_{ij}^{-+})$ are $n\times m$ matrices such that

$$\alpha_{ij}^{+-}=\begin{cases} a_{ij}^+ & \text{if} \quad j\le k\\ a_{ij}^- & \text{if} \quad j\ge k+1\end{cases} \qquad \alpha_{ij}^{-+}=\begin{cases} a_{ij}^- & \text{if} \quad j\le k\\ a_{ij}^+ & \text{if} \quad j\ge k+1\end{cases}$$

We conclude then that the optimization problem (18) can be transformed into the problem

$$\begin{array}{l}\text{Maximize/minimize} \quad g_f(x)\\ \text{s.t.} \quad (A^{+-}\circ x)\vee(A^{-+}\circ\neg x)=b\end{array} \tag{19}$$

The results below show that, as one would expect, both optimization problems are equivalent. Again, we introduce a preliminary lemma for the equivalence of their feasible domains.

Lemma 24. *Let $f : L^m \to L$ be a strict extended aggregator. A tuple $(x_1,\dots,x_m)\in L^m$ belongs to the feasible domain of Problem (18) if and only if $(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m)$ belongs to the feasible domain of Problem (19).*

Proof. Given $(x_1,\dots,x_m)\in L^m$, the expression

$$(A^+\circ(x_1,\dots,x_m))\vee(A^-\circ(\neg x_1,\dots,\neg x_m)) \tag{20}$$

can be written as a set of elements of the form

$$\bigvee_{j\in\{1,\dots,m\}}(a_{ij}^+ * x_j)\vee\bigvee_{j\in\{1,\dots,m\}}(a_{ij}^- * \neg x_j),\quad i\in\{1,\dots,n\}$$

Now, for each $i\in\{1,\dots,n\}$, both suprema involved in the previous expression can be split into two parts, resulting in:

$$\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^+ * x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^+ * x_j)\ \vee\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^- * \neg x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^- * \neg x_j)$$

Due to the associativity of the supremum operator, we can reorder the terms above as:

$$\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^+ * x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^- * \neg x_j)\ \vee\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^- * \neg x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^+ * x_j)$$

Since $x_j=\neg\neg x_j$ for all $j\in\{k+1,\dots,m\}$, modifying the last supremum we obtain:

$$\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^+ * x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^- * \neg x_j)\ \vee\bigvee_{j\in\{1,\dots,k\}}(a_{ij}^- * \neg x_j)\ \vee\bigvee_{j\in\{k+1,\dots,m\}}(a_{ij}^+ * \neg\neg x_j)$$

That is, by the definition of the matrices A^{+-} and A^-+, the previous chain of suprema is equivalent to:

$$(A^{+-}\circ(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m))\vee(A^{-+}\circ\neg(x_1,\dots,x_k,\neg x_{k+1},\dots,\neg x_m))$$

Since the last expression coincides with (20), then Lemma 24 holds. □

Once the one-to-one correspondence between the feasible domains of Problems (18) and (19) has been guaranteed by Lemma 24, the following theorem demonstrates the equivalence of both problems.

Theorem 25. *Let $f : L^m \to L$ be a strict extended aggregator. A tuple $(x_1, \dots, x_m) \in L^m$ is an optimal solution of Problem (18) if and only if $(x_1, \dots, x_k, \neg x_{k+1}, \dots, \neg x_m)$ is an optimal solution of Problem (19).*

Proof. It is analogous to the proof of Theorem 21, employing Lemma 24 instead of Lemma 20. □

Corollary 26. *Let $f : L^m \to L$ be a strict extended aggregator. A tuple $(x_1, \dots, x_m) \in L^m$ is an optimal solution of Problem (19) if and only if $(x_1, \dots, x_k, \neg x_{k+1}, \dots, \neg x_m)$ is an optimal solution of Problem (18).*

Proof. It is analogous to the proof of Corollary 22. □

Notice that, the procedure shown in Section 3.1 can be seen as a particular case for optimization problems of the form (18) being A^- the null matrix.

The following example illustrates the results shown in this section, highlighting the capacity of the presented procedure to find all optimal solutions of an optimization problem subject to a BFRE.

Example 27. Consider the following optimization problem with BFRE constraint defined with the max-product composition $\circ$ and the standard negation $\neg_S$:

$$\begin{aligned} &\text{Maximize} \quad \frac{x_1 x_3}{5} + 2\,\frac{x_2}{{x_5}^2} + \frac{1}{\log(x_4)} \\ &\text{s.t.} \quad (A^+ \circ (x_1, x_2, x_3, x_4, x_5)) \vee (A^- \circ \neg_S (x_1, x_2, x_3, x_4, x_5)) = b \end{aligned} \tag{21}$$

being

$$A^+ = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.2 & 1 \\ 0.2 & 0 & 0 & 0.6 & 0.3 \end{bmatrix}, \quad A^- = \begin{bmatrix} 0.8 & 1 & 0.8 & 0.2 & 0.5 \\ 0.4 & 0.6 & 0.4 & 0.4 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}$$

Since the mapping

$$f(x_1, x_2, x_3; x_4, x_5) = \frac{x_1 x_3}{5} + 2\,\frac{x_2}{{x_5}^2} + \frac{1}{\log(x_4)}$$

is strictly order-preserving in the first three arguments and strictly order-reversing in the last two arguments, we can transform Problem (21) into the problem

$$\text{Maximize} \quad g_f(x_1, x_2, x_3, x_4) = \frac{x_1 x_3}{5} + 2\,\frac{x_2}{(1 - x_5)^2} + \frac{1}{\log(1 - x_4)} \tag{22}$$

constrained by the BFRE

$$(A^{+-} \circ x) \vee (A^{-+} \circ \neg x) = b \tag{23}$$

where

$$A^{+-} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.2 & 0.5 \\ 0.2 & 0 & 0 & 0.4 & 0.5 \end{bmatrix}, \quad A^{-+} = \begin{bmatrix} 0.8 & 1 & 0.8 & 0.2 & 1 \\ 0.4 & 0.6 & 0.4 & 0.6 & 0.3 \end{bmatrix}$$

Now, applying Theorem 9 to the two sup-equations related to (23) and intersecting their solution sets, we obtain that the solution set of (23) is

$$\begin{aligned} D \;=\; & \big[(0.25, 0.5, 0.25, 0.5, 0.4), (0.25, 1, 1, 0.75, 0.6)\big] \\ \cup\; & \big[(0.25, 0.5, 0.25, 0.5, 0.4), (1, 1, 0.25, 0.75, 0.6)\big] \\ \cup\; & \big[(0.25, 0.5, 0.25, 0.75, 0.4), (1, 1, 1, 0.75, 0.4)\big] \\ \cup\; & \big[(0.25, 0.5, 0.25, 0.5, 0.4), (0.25, 1, 1, 0.5, 0.4)\big] \\ \cup\; & \big[(0.25, 0.5, 0.25, 0.5, 0.4), (1, 0.5, 1, 0.75, 0.4)\big] \end{aligned}$$

Notice that, the BFRE (23) has three maximal solutions: $(0.25, 1, 1, 0.75, 0.6)$, $(1, 1, 0.25, 0.75, 0.6)$ and $(1, 1, 1, 0.75, 0.4)$. As a result, Proposition 12 implies that the unique possible optimal solutions of (23) are the three mentioned tuples. Having said this, the evaluation of the mapping g results in the values

$$\begin{aligned} g(0.25, 1, 1, 0.75, 0.6) &\approx 11.8287 \\ g(1, 1, 0.25, 0.75, 0.6) &\approx 11.8287 \\ g(1, 1, 1, 0.75, 0.4) &\approx 5.03421 \end{aligned}$$

Hence, we conclude that $(0.25, 1, 1, 0.75, 0.6)$ and $(1, 1, 0.25, 0.75, 0.6)$ are the optimal solutions of Problem (22), and by Corollary 26 the optimal solutions of Problem (21) are the tuples $(0.25, 1, 1, 0.25, 0.4)$ and $(1, 1, 0.25, 0.25, 0.4)$. □

4. Conclusions

A novel procedure to optimize a partially monotonic function constrained by a BFRE has been presented, basing on transforming the problem into optimizing an order-preserving mapping in all arguments subject to a BFRE. As a result, this method can be applied to an ample range of optimization problems constrained to BFRE, including linear optimization problems. One of the main characteristics of the presented procedure is the possibility to find all optimal solutions of this kind of problems.

The computational effort of the procedure relies on determining the extremal solutions of a BFRE, as the transformation of Problem (18) into Problem (19) is straightforward. It remains as a future work the development of efficient procedures to analytically determine such extremal solutions, and a comparison of the efficiency of these procedures with the existing algorithms in the literature. In this sense, it needs to be stressed that, to the best of our knowledge, the existing methods focus on the computation of only one optimal solution. Furthermore, real-world applications of the presented approach are one of the main challenges for the immediate future.

The consideration of specific families of BFRE is also an appealing prospect for future work. For instance, in expression (1), the relation matrices A^+ and A^- are assumed to be independent from each other. But the requirement of extra properties between A^+ and A^-, or between the values in the matrices, may give rise to interesting kinds of BFRE like, for instance, intuitionistic bipolar fuzzy relation equations.

Additionally, the consideration of other kind of constraints is another proposal for future work. One possibility is the study of problems constrained by BFRE and bipolar fuzzy relation inequalities simultaneously, in a similar way to the non-bipolar case considered in [32].

CRediT authorship contribution statement

In this work, all the authors have contributed in equally to the development of the contents shown in this publication; results, proofs, examples, etc.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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