

Article

Decision Making in Fuzzy Rough Set Theory

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Abstract: Decision rules are powerful tools to manage information and to provide descriptions of data sets; as a consequence, they can acquire a useful role in decision-making processes where fuzzy rough set theory is applied. This paper focuses on the study of different methods to classify new objects, which are not considered in the starting data set, in order to determine the best possible decision for them. The classification methods are supported by the relevance indicators associated with decision rules, such as support, certainty, and credibility. Specifically, the first one is based on how the new object matches decision rules that describe the data set, while the second one also takes into account the representativeness of these rules. Finally, the third and fourth methods take into account the credibility of the rules compared with the new object. Moreover, we have shown that these methods are richer alternatives or generalize other approaches given in the literature.

Keywords: fuzzy rough set theory; decision rules; classification methods

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1. Introduction

Fuzzy rough set theory (FRST) [1,2] arises as an extension of the theoretical approach introduced by Pawlak [3,4] in order to analyze relational data sets with imprecise or incomplete information. One of the most flexible approaches considers the multi-adjoint paradigm [5–7]. In FRST, data sets are represented by using decision tables containing information about objects of interest characterized by some attributes. Rule generation is an interesting task in FRST since it allows us to classify, simplify, and acquire knowledge from the data set. Decision rules, together with their relevance indicators, that is, the support, the certainty, the credibility, etc., have been widely studied from a Boolean perspective, such as in the classical framework [8,9], in incomplete decision contexts [10], taking into account bireducts [11], and in three-way rough set theory [12]. However, the use of relevance indicators on decision rules in FRST is really limited [13]. In this paper, we are interested in exploiting the powerful advantages provided by decision rules in FRST for decision-making methods.

A fundamental goal in decision-making methods is the discovery of patterns and the application of this knowledge to new scenarios, which have not been considered in the starting data set, in order to determine their behavior. This task has already been addressed in different fields such as artificial intelligence for dynamic classification decision making [14], for rule-driven algorithms [15], for multi-criteria decision-making methods [16,17], for clinical practice [18], and for psychotherapy [19]. In particular, the classification of new data has also been studied by the rough set theory framework due to its potential to deal with imprecise information. For instance, in [20,21], the new objects to classify are compared with the antecedents of the considered decision rules. This philosophy is similar to the clustering in machine learning [22] as it considers each rule as a cluster and studies the belonging of the new object to each of them. In [20], a vote is made among the different

decisions to be made for the new object. For that purpose, the support of the decision rules whose values in the condition attributes are exactly the same those in the new object is taken into account. However, this restrictive consideration of matching exact values makes it difficult to apply this method in real data sets. In [21], this issue is solved, since, thanks to the use of tolerance relations, a value in the unit interval is obtained, which means how much the new object resembles each of the rules. Thus, the best decision to be made for the new object is given by the highest value obtained from the tolerance relation. In addition, in [21], the support of the rules is also taken into account when it is difficult to make a decision. Note that the method presented in [23] provides a similar classification to the ones given in [20,21]. This aforementioned paper considers two kinds of attributes, categorical and numerical attributes, and it requires an exact match in the comparison between the new object and the decision rules for categorical attributes and the use of the Euclidean distance for numerical attributes. Nevertheless, in the fuzzy case, tolerance relations can be considered instead of similarity ones, and, with respect to the categorical attributes of the objects, they do not need to be boolean, but they can have a truth degree and take into account tolerance relations instead of exact matches. In this paper, we will incorporate the added benefits of considering separable tolerance relations to compare new objects with decision rules. Furthermore, the support of the rules will enrich this comparison, which will provide the possibility of obtaining extra knowledge for making a final selection when the values of the possible decisions are very close.

On the other hand, Ref. [24] presents the notion of credibility of a decision rule in terms of t-norms and implications and a method to classify new objects based on that credibility. In this paper, we will extend that notion of credibility, taking advantage of the multi-adjoint framework, that is, taking adjoint pairs into consideration. As a consequence, our classification method based on this notion will be also more general. Another notion of credibility is used in [25] based on particular operators, whose relationship with the previous notion is not clear. Hence, we will show that this notion of credibility is a particular case of the one given in [24], taking into account the Gödel t-norm and a particular fuzzy implication operator. Therefore, we will perform a thorough comparison of the methods given in [24,25] with the classification methods presented in this paper based on the credibility notion.

Specifically, this paper presents four different original methods to classify new objects, after a previous analysis of the data set from a FRST perspective. The two first methods are supported by the notions introduced in [13], which are the (1) degree of satisfiability, to measure how much the new object resembles the antecedent of decision rules extracted from the decision table, and the (2) support in the fuzzy setting, to take into account the representativeness of the rules whose antecedents are more similar to the new object. A third method, based on the notion of credibility [23,25], and a fourth one, consisting of a combination of the three previous ones, are also proposed. In summary, the main benefits of our contribution will be the definition of these methods in FRST, taking advantage of the great level of flexibility provided by the multi-adjoint framework. Furthermore, an example is worked through after presenting each method in order to illustrate them and to analyze the obtained results. We will also compare our first proposed method with the one given in [21] and the third one with the one given in [24,25], showing that our contribution presents relevant advantages, such as the flexibility inherent in the multi-adjoint framework.

The outline followed in the paper is shown below. Section 2 recalls some preliminary notions of FRST, together with an illustrative example. Section 3 introduces the four methods devoted to classifying new objects in FRST. In Section 4, we make a detailed comparison between the third classification method proposed in this paper and the ones presented in [24,25]. Next, in Section 5, a real case is studied to expose the effectiveness of each method. Finally, Section 6 presents some conclusions and prospects for future works.

2. Decision Rules in Fuzzy Rough Set Theory

In this section, important notions of FRST [5,13] are recalled. First of all, the multi-adjoint framework [5] allows us to set preferences among the objects and attributes of the decision table in order to give them more or less importance. These different considerations are based on adjoint pairs.

Definition 1 ([5]). Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$ be partially ordered sets (posets) and $\&: P_1 \times P_2 \rightarrow P_3, I: P_2 \times P_3 \rightarrow P_1$ be mappings. Then, $(\&, I)$ is an adjoint pair with respect to P_1, P_2 , and P_3 if these mappings satisfy the adjoint property; that is,

$$\&(x, y) \leq_3 z \quad \text{iff} \quad x \leq_1 I(y, z)$$

for all $x \in P_1, y \in P_2, z \in P_3$.

Some examples of adjoint pairs are the Gödel, product, and Łukasiewicz t-norms, together with their corresponding residuated implications [26]. Next, the notion of a multi-adjoint property-oriented frame is presented.

Definition 2 ([5]). The tuple $(P_1, L_2, L_3, \&_1, \dots, \&_n)$ is a multi-adjoint property-oriented frame where $(P_1, \leq_1)$ is a poset, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$ are two complete lattices, and $(\&_i, I_i)$ are adjoint pairs with respect to P_1, L_2, L_3 , for all $i \in \{1, \dots, n\}$.

In the following definition, we recall the notion of context.

Definition 3 ([5]). Let A and B be non-empty sets and $(P_1, L_2, L_3, \&_1, \dots, \&_n)$ be a multi-adjoint property-oriented frame. A context is a tuple (A, B, R, τ) , where R is a P_1 -fuzzy relation $R: A \times B \rightarrow P_1$ and $\tau: A \times B \rightarrow \{1, \dots, n\}$ is a mapping that associates any pair of elements in $A \times B$ with a particular adjoint pair in the frame.

Now, we introduce the generalization of the notion of upper and lower approximation of FRST given in [5]. Before that, we remark that, given two sets L and X , L^X will denote the set of mappings from X to L , that is, $L^X = \{h \mid h: X \rightarrow L\}$.

Definition 4 ([5]). Let $(P_1, L_2, L_3, \&_1, \dots, \&_n)$ be a multi-adjoint property-oriented frame and (A, B, R, τ) be a context. Given $g \in L_2^B$ and $f \in L_3^A$, we define the possibility and necessity operators, $\uparrow^\pi: L_2^B \rightarrow L_3^A$ and $\downarrow^N: L_3^A \rightarrow L_2^B$, respectively, as,

$$\begin{aligned} g^{\uparrow^\pi}(a) &= \sup\{\&_{\tau(a,b)}(R(a,b), g(b)) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \inf\{I_{\tau(a,b)}(R(a,b), f(a)) \mid a \in A\} \end{aligned}$$

where $g^{\uparrow^\pi}$ generalizes the classical notion of upper approximation in RST. Similarly, $f^{\downarrow^N}$ generalizes the notion of lower approximation in RST.

From now on, we will focus on a multi-adjoint property-oriented frame $([0, 1], [0, 1], [0, 1], \&_1, \dots, \&_n)$ and a context (U, U, R_A, τ) . In a proper rough set framework, the related data set is interpreted as an information or decision table instead of a formal context. In this paper, we will consider decision tables.

Definition 5 ([13]). Let U and $\mathcal{A}$ be non-empty sets of objects and attributes, respectively. A decision table is a tuple $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}}_d)$ such that $\mathcal{A}_d = \mathcal{A} \cup \{d\}$ with $d \notin \mathcal{A}$, $\mathcal{V}_{\mathcal{A}_d} = \{V_a \mid a \in \mathcal{A}_d\}$, where V_a is the set of values associated with the attribute a over U , and $\overline{\mathcal{A}}_d = \{\bar{a} \mid a \in \mathcal{A}_d, \bar{a}: U \rightarrow V_a\}$. In this case, the attributes of $\mathcal{A}$ are called condition attributes and d is called a decision attribute.

From a decision table, an indiscernibility relation $R_A: U \times U \rightarrow [0, 1]$ is defined on A and a highlighted relation $R_d: U \times U \rightarrow [0, 1]$, which shows how two objects are related with respect to the attribute d . Finally, we recall the notion of a positive region in the fuzzy environment when R_d only takes boolean values, which was also introduced in [5].

Definition 6 ([5]). Let $(U, A_d, \mathcal{V}_{A_d}, \overline{A_d})$ be a decision table and R_d be a boolean relation. The multi-adjoint fuzzy A -positive region is defined, for each $y \in U$, as

$$POS_A^f(y) = (R_d y)^{\downarrow^N}(y) = \inf\{I_{\tau(x,y)}(R_A(x,y), (R_d y)(x)) \mid x \in U\}$$

where $R_d y: U \rightarrow \{0, 1\}$ is defined as $(R_d y)(x) = R_d(y, x)$ and $I_{\tau(x,y)}$ is the residuated fuzzy implication of $\&_{\tau(x,y)}$ associated with the pair of objects x, y .

It is also necessary to recall a particular type of tolerance relation [27], which will be used in the rest of this work.

Definition 7 ([27]). Let $S = (U, A_d, \mathcal{V}_{A_d}, \overline{A_d})$ be a decision table, and let $a \in A$ and $T_a: V_a \times V_a \rightarrow [0, 1]$ be a tolerance relation. If $T_a(v, w) = 1$ implies $v = w$ for each $v, w \in V_a$, then T_a is called a separable tolerance relation.

Thanks to the reflexivity of tolerance relations, we can conclude that a separable tolerance relation T_a satisfies that $T_a(v, w) = 1$ if and only if $v = w$ for all $v, w \in V_a$.

Now, we recall the notions related to decision rules in the fuzzy framework [13] required for this paper. The first one presents the formulas, which are indispensable to express the information contained in decision tables in logical terms. Moreover, this notion will also allow us to compute how much an object satisfies a formula.

Definition 8 ([13]). Let $S = (U, A_d, \mathcal{V}_{A_d}, \overline{A_d})$ be a decision table and $B \subseteq A_d$. The set of formulas associated with B , denoted as $For(B)$, is built from attribute-value pairs (a, v) , where $a \in B$ and $v \in V_a$, by means of the conjunction logical connective $\wedge$.

Given $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in A_d\}$, a family of separable $[0, 1]$ -fuzzy tolerance relations, the mapping $\|\cdot\|_S^T: For(B) \rightarrow [0, 1]^U$ is inductively defined as

$$\|\Phi\|_S^T(x) = T_a(\bar{a}(x), v)$$

for all $x \in U$ and $\Phi = (a, v)$, where $a \in B$ and $v \in V_a$. Therefore, $\|\Phi\|_S^T(x)$ is the degree of satisfiability to the formula Φ of the object x , through the relationships between the values of the attributes in the object x and the values of the attributes in the formula Φ . Moreover, we will need the α -cuts of these fuzzy sets, which are defined as

$$(\|\Phi\|_S^T)_\alpha = \{x_i \in U \mid \|\Phi\|_S^T(x_i) > \alpha\}$$

Finally, for every $\Phi, \Psi \in For(B)$, the conjunction of formulas is defined, for each $x \in U$, as follows:

$$\|\Phi \wedge \Psi\|_S^T(x) = \min\{\|\Phi\|_S^T(x), \|\Psi\|_S^T(x)\}$$

Now, we introduce the notion of a decision rule, which is supported by Definition 8.

Definition 9 ([13]). Let $S = (U, A_d, \mathcal{V}_{A_d}, \overline{A_d})$ be a decision table, $B \subseteq A$ and $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in A_d\}$ be a family of separable $[0, 1]$ -fuzzy tolerance relations. A decision rule in S is an expression $\Phi \rightarrow \Psi$, with $\Phi \in For(B)$, $\Psi \in For(\{d\})$, where Φ and Ψ are the antecedent and the consequent of the decision rule, respectively.

In addition, we will say that an object $x \in U$ satisfies the decision rule $\Phi \rightarrow \Psi$ if $\|\Phi \wedge \Psi\|_S^T(x) = 1$. In this way, we can say that the object x induces the decision rule $\Phi \rightarrow \Psi$, which can be denoted as $\Phi_x \rightarrow \Psi_x$.

Decision rules are very important to summarize the information contained in decision tables. However, they must be analyzed in order to be interpreted. With this purpose, we will describe them by using some measures. First of all, we must recall the cardinal of a fuzzy set.

Definition 10 ([28]). Given a universe U and $V \subseteq U$, a fuzzy set characterized by a membership function $g: U \rightarrow [0, 1]$, the cardinal of V is defined as

$$\text{card}_F(V) = \sum_{x \in U} g(x)$$

Now, we introduce a pair of measures used to study the decision rules. They determine the representativeness of the rules in the considered decision table.

Definition 11. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\text{Dec}(S)$ be a set of decision rules, and $\Phi \rightarrow \Psi \in \text{Dec}(S)$ and $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in \mathcal{A}_d\}$ be a family of separable $[0, 1]$ -fuzzy tolerance relations. We call

- T -support of the decision rule $\Phi \rightarrow \Psi$ to the value

$$\text{supp}_S^T(\Phi, \Psi) = \text{card}_F(\|\Phi \wedge \Psi\|_S^T)$$

- Normalized T -support (NT-support, in short) of the decision rule $\Phi \rightarrow \Psi$ to the value

$$\text{supp}_S^{NT}(\Phi, \Psi) = \frac{\text{supp}_S^T(\Phi, \Psi)}{\max\{\text{supp}_S^T(\Phi, \Psi) \mid \Phi \rightarrow \Psi \in \text{Dec}(S)\}}$$

Another important measure to describe decision rules is the credibility, already defined in [24,25,29]. The credibility of a decision rule can be seen as the degree to which, if an object satisfies the antecedent of the rule, or the antecedent of similar rules, then it also satisfies the consequent of the given rule. In this way, it provides the consistency of the rule in the table, so that the higher the credibility, the more reliable the decision according to the conditions. Now, we present the extension of this notion to the multi-adjoint framework.

Definition 12. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $\text{Dec}(S)$ be a set of decision rules, $\Phi_x \rightarrow \Psi_x \in \text{Dec}(S)$ be the decision rule induced by an object $x \in U$, and $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in \mathcal{A}_d\}$ be a family of separable $[0, 1]$ -fuzzy tolerance relations. Given the multi-adjoint property oriented frame $([0, 1], [0, 1], [0, 1], \&_1, \dots, \&_n)$, where $\&_i$ is an associative conjunctive for all $i \in \{1, \dots, n\}$; we call α - T -credibility of the decision rule $\Phi_x \rightarrow \Psi_x$ to the value

$$\mu_{S,\tau}^\alpha(\Phi_x, \Psi_x) = \&_i\{I_{\tau(x,y)}(\|\Phi_x\|_S^T(y), \mu_{\Psi_{x,S,\tau}}(y)) \mid y \in (\|\Phi_x\|_S^T)_\alpha\}$$

with

$$\mu_{\Psi_{x,S,\tau}}(y) = \&_{j_y}\{I_{\tau(y,y')}(\|\Phi_y\|_S^T(y'), \|\Psi_x\|_S^T(y')) \mid y' \in (\|\Phi_y\|_S^T)_\alpha \text{ and } \Phi_y \rightarrow \Psi_y \in \text{Dec}(S)\}$$

where $i, j_y \in \{1, \dots, n\}$ and $\Phi_y \rightarrow \Psi_y \in \text{Dec}(S)$ is a decision rule induced by $y \in (\|\Phi_x\|_S^T)_\alpha$, if it exists in $\text{Dec}(S)$.

Note that the conjunctors $\&_{j_y}$ can be the adjoints of the implications $I_{\tau(x,y)}$ considered in the computation of $\mu_{S,\tau}^\alpha(\Phi_x, \Psi_x)$. It is important to emphasize that the α -cuts $(\|\Phi\|_S^T)_\alpha$ are considered in order to compare only the objects most similar between them given α . Moreover, the multi-adjoint framework allows us to consider different adjoint pairs for each pair of objects of the table, providing a higher level of flexibility in the studies.

In the next example, we sequentially illustrate the notions presented in this section.

Example 1. We consider the decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ represented in Table 1, where the set of objects is $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, the set of attributes is $\mathcal{A} = \{a_1, a_2, a_3\}$, and $V_a = [0, 1]$ for all $a \in \mathcal{A}$ and $V_d = \{0, 1\}$; that is, d is a boolean attribute.

Table 1. Table associated with $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 1.

	a_1	a_2	a_3	d
x_1	0.34	0.31	0.75	0
x_2	0.21	0.71	0.5	1
x_3	0.52	0.92	1	0
x_4	0.85	0.65	1	1
x_5	0.43	0.89	0.5	0
x_6	0.21	0.47	0.25	1
x_7	0.09	0.93	0.25	0

In order to define decision rules to describe S , we take into account the conjunction of all the attributes of $\mathcal{A}$ for the antecedents of these rules, since they will provide us with more accurate information and will not imply many computations given that $\text{card}(\mathcal{A}) = 3$. On the other hand, the consequent obviously takes into account the only decision attribute, d . As a consequence, the next decision rules are obtained.

$$\begin{aligned}
 r_1 &: (a_1, 0.34) \wedge (a_2, 0.31) \wedge (a_3, 0.75) \rightarrow (d, 0) \\
 r_2 &: (a_1, 0.21) \wedge (a_2, 0.71) \wedge (a_3, 0.5) \rightarrow (d, 1) \\
 r_3 &: (a_1, 0.52) \wedge (a_2, 0.92) \wedge (a_3, 1) \rightarrow (d, 0) \\
 r_4 &: (a_1, 0.85) \wedge (a_2, 0.65) \wedge (a_3, 1) \rightarrow (d, 1) \\
 r_5 &: (a_1, 0.43) \wedge (a_2, 0.89) \wedge (a_3, 0.5) \rightarrow (d, 0) \\
 r_6 &: (a_1, 0.21) \wedge (a_2, 0.47) \wedge (a_3, 0.25) \rightarrow (d, 1) \\
 r_7 &: (a_1, 0.09) \wedge (a_2, 0.93) \wedge (a_3, 0.25) \rightarrow (d, 0)
 \end{aligned}$$

Although each decision rule is induced by an object, we denote them as $\Phi_i \rightarrow \Psi_i$ for each $i \in \{1, \dots, 7\}$ instead of $\Phi_{x_i} \rightarrow \Psi_{x_i}$ to simplify the notation. Now, $\|\Phi_i\|_S^T(x_j)$ with $i, j \in \{1, 2, \dots, 7\}$ is computed, that is, the degree of satisfiability to each antecedent of each object. For that purpose, we consider the family of separable $[0, 1]$ -fuzzy tolerance relation $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in \mathcal{A}_d\}$ given $T_a(\bar{a}(x), v) = 1 - |\bar{a}(x) - v|$, for all $a \in \mathcal{A}_d$, $x \in U$ and $v \in [0, 1]$. All the results are shown in Table 2.

Table 2. Degree of satisfiability of each antecedent of the decision rules of Example 1.

	$\ \Phi_1\ _S^T$	$\ \Phi_2\ _S^T$	$\ \Phi_3\ _S^T$	$\ \Phi_4\ _S^T$	$\ \Phi_5\ _S^T$	$\ \Phi_6\ _S^T$	$\ \Phi_7\ _S^T$
x_1	1	0.6	0.39	0.49	0.42	0.5	0.38
x_2	0.6	1	0.5	0.36	0.78	0.75	0.75
x_3	0.39	0.5	1	0.67	0.5	0.25	0.25
x_4	0.49	0.36	0.67	1	0.5	0.25	0.24
x_5	0.42	0.78	0.5	0.5	1	0.58	0.66
x_6	0.5	0.75	0.25	0.25	0.58	1	0.54
x_7	0.38	0.75	0.25	0.24	0.66	0.54	1

On the other hand, since d is a Boolean attribute, we deduce that

$$\|\Psi_i\|_S^T(x_j) = \|(d, v_i)\|_S^T(x_j) = \begin{cases} 1 & \text{if } \bar{d}(x_j) = v_i \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

In order to compute the α -T-credibility of the previous decision rules, we set the multi-adjoint property-oriented frame $([0, 1], [0, 1], [0, 1], \&_L, \&_P, \&_G)$, where $(\&_L, I_L)$, $(\&_P, I_P)$ and $(\&_G, I_G)$ are the Łukasiewicz, product and Gödel adjoint pairs, respectively, and the context (U, U, R_A, τ) with $\tau: U \times U \rightarrow \{L, P, G\}$ defined for each $x, y \in U$ as

$$\tau(x, y) = \begin{cases} L & \text{if } \bar{d}(x) = \bar{d}(y) = 1 \\ P & \text{if } \bar{d}(x) = \bar{d}(y) = 0 \\ G & \text{otherwise} \end{cases}$$

We have considered this mapping to reinforce the computation of the credibility when the objects have the same decision (implication I_G is smaller than I_P and I_L) and model the preferences of the user, who is more confident with the decision 1 than with the decision 0 (implication I_P is smaller than I_L). On the other hand, we consider the adjoint pair $(\&_L, I_L)$ and the 0.76 cut in order to compute the 0.76-T-credibility of each decision rule belonging to $\text{Dec}(S)$. This threshold has been chosen to minimize the number of objects related via the antecedent but not via the consequent in the sets $(\|\Phi\|_S^T)_\alpha$ with $\Phi \rightarrow \Psi \in \text{Dec}(S)$, while some flexibility is allowed in the comparisons. In this way, thanks to this choice of α in Table 2, we deduce that the only inconsistencies in these sets are caused by the objects x_2 and x_5 .

By using Table 2, Equation (1), and Definitions 11 and 12, we compute the T support, NT support, and 0.76-T-credibility of each decision rule. All these measures are shown in Table 3.

Table 3. T support, NT support, and 0.76-T-credibility of the decision rules of Example 1.

Rule	supp_S^T	supp_S^{NT}	$\mu_{S,\tau}^{0.76}$
r_1	2.19	0.85	1
r_2	2.11	0.82	0
r_3	2.14	0.83	1
r_4	1.61	0.62	1
r_5	2.58	1	0
r_6	2	0.78	1
r_7	2.29	0.89	1

The decision rules $\Phi \rightarrow \Psi \in \text{Dec}(S)$ with 0.76-T-credibility equal to 1 satisfy that $(\|\Phi\|_S^T)_{0.76} = \{x\}$, where $x \in U$ is the object that induces them. On the other hand, $\mu_{S,\tau}^{0.76}(\Phi_2, \Psi_2) = \mu_{S,\tau}^{0.76}(\Phi_5, \Psi_5) = 0$ since $\|\Phi_2\|_S^T(x_5) = \|\Phi_5\|_S^T(x_2) = 0.78 > 0.76$; that is, $(\|\Phi_2\|_S^T)_{0.76} = (\|\Phi_5\|_S^T)_{0.76} = \{x_2, x_5\}$, whereas $\|\Psi_2\|_S^T(x_5) = \|\Psi_5\|_S^T(x_2) = 0$. Therefore, rules r_2 and r_5 are very related in the antecedent but not at all in the consequent. As a consequence, these rules do not have any credibility.

In conclusion, in this section, we have recalled important notions to study data sets in the FRST framework. In particular, we must emphasize the relevance of the notions related to decision rules, since they will be indispensable to classify objects outside of the data set. This task is carried out in the following section.

3. Classification of New Objects

In this section, we analyze different methods to make decisions in the FRST framework, and study their properties and which method is the most suitable, depending on the case. With this purpose, we consider a set of decision rules obtained from a decision table

that represents it, and we will try to classify new objects not belonging to that table. For that purpose, we will know the values of these objects based on the condition attributes and we will detect the most suitable value of the decision attribute, which is originally unknown. Hence, we are considering an extended universe $\mathcal{U}$ where we have two different classes of objects, the objects in U from which the decision value is known and the objects in $\mathcal{U} \setminus U = U^c$ from which the decision value is unknown. That is, the main goal is to determine the value of the decision attribute in these objects in order to make predictions about them. For that purpose, we will study each possible decision that can be made for the new objects by comparing the values given in the condition attributes of the new objects and the values given in the rules of the set considered. From this study, a truth value is obtained for each possible decision that can be made. Finally, the recommended decision will be determined by paying attention to the highest truth value among all of them. We will study four classification methods with different features.

3.1. Suitability Method

This section is devoted to presenting the first method we will use to make a decision for the new objects, which extends the one given in [21]. The comparison with this method will be given at the end of the section. This classification method will take into account Definition 8, since we will compute how much the new objects satisfy the antecedents of the rules considered in the study. In particular, the next notion focuses only on the maximum degree of satisfiability of the antecedents for each possible consequent so that a truth value is provided for all the possible values of the decision attribute.

Definition 13. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $Dec(S)$ be a set of decision rules. For each new object $x \in U^c$ and $v \in V_d$, the degree of suitability of the decision $\vec{d}(x) = v$ according to $Dec(S)$ is given via the mapping $\sigma_{Dec(S)}: U^c \times V_d \rightarrow [0, 1]$, defined as

$$\sigma_{Dec(S)}(x, v) = \max\{\|\Phi\|_S^T(x) \mid \Phi \rightarrow \Psi \in Dec(S), \Psi = (d, v)\}$$

Notice that, in Definition 8, $\|\Phi\|_S^T$ is evaluated in objects $x \in U$ to compute how much each object of U satisfies the formula Φ , while in Definition 13, $\|\Phi\|_S^T(x)$ is calculated with $x \in U^c$. However, this computation is carried out to compare new objects with those present in the original decision table. Moreover, the degree of suitability is a truth value, since $0 \leq \sigma_{Dec(S)}(x, v) \leq 1$ for all new objects x , decisions $\vec{d}(x) = v$, and sets of decision rules $Dec(S)$ according to Definition 8.

Once the degree of suitability of a given decision has been defined, we present the method used to classify new objects according to Definition 13, which will be called the *suitability method* or *S-method* for short.

- S1. First of all, given a new object to study $x \in U^c$, we compute the degrees of suitability of each possible decision $\vec{d}(x) = v$, with $v \in V_d$.
- S2. Then, all the obtained degrees of suitability are compared. If the degree of suitability of a decision $\vec{d}(x) = v$ is high and different enough from the degrees of suitability of the rest of the possible decisions, then we will consider the decision $\vec{d}(x) = v$ for the object x .
- S3. If there exist some values of the decision attribute $v \in V \subset V_d$ with similar degrees of suitability, then we must consider more degrees of satisfiability for these decisions instead of only the maximum depending on a threshold $\beta \in [0, 1]$. In particular, denoting as $\mathcal{F}_x^\beta(v) = \{\Phi \in For(\mathcal{A}) \mid \Phi \rightarrow \Psi \in Dec(S), \beta \leq \|\Phi\|_S^T(x), \Psi = (d, v)\}$, we will consider for each decision $\vec{d}(x) = v$, with $v \in V$, the value

$$\mathcal{M}_{\mathcal{F}_x^\beta(v)} = \sum_{\Phi \in \mathcal{F}_x^\beta(v)} \frac{\|\Phi\|_S^T(x)}{\text{card}(\mathcal{F}_x^\beta(v))}$$

Once this computation is carried out for each $v \in V$, the obtained values are compared, and we come back to the second step.

- S4. If it is not possible to make a decision according to the second step, then the set $\mathcal{F}_x^\beta(v)$ is modified, either by changing the value β or by considering another aggregation operator instead of the arithmetic mean.

We illustrate this first method in the following example.

Example 2. Coming back to the environment of Example 1, we will study three new objects, x_8 , x_9 , and x_{10} , in order to determine the best decision for them. In Table 4, we show these objects together with the objects previously presented in Table 1.

Table 4. Table 1 together with the new objects to classify x_8 , x_9 , and x_{10} .

	a_1	a_2	a_3	d
x_1	0.34	0.31	0.75	0
x_2	0.21	0.71	0.5	1
x_3	0.52	0.92	1	0
x_4	0.85	0.65	1	1
x_5	0.43	0.89	0.5	0
x_6	0.21	0.47	0.25	1
x_7	0.09	0.93	0.25	0
x_8	0.05	0.95	0.1	
x_9	0.89	0.5	0.95	
x_{10}	0.4	0.67	0.31	

First of all, we compute the degree of satisfiability to each antecedent of the decision rules of Example 1 of each new object in Table 4 considering again the $[0, 1]$ -fuzzy tolerance relation given as $T_a(\bar{a}(x), v) = 1 - |\bar{a}(x) - v|$ for each $a \in \mathcal{A}$. These results are presented in Table 5.

Table 5. Degree of satisfiability of each antecedent of decision rules from Example 1.

	$\ \Phi_1\ _S^T$	$\ \Phi_2\ _S^T$	$\ \Phi_3\ _S^T$	$\ \Phi_4\ _S^T$	$\ \Phi_5\ _S^T$	$\ \Phi_6\ _S^T$	$\ \Phi_7\ _S^T$
x_8	0.35	0.6	0.1	0.1	0.6	0.52	0.85
x_9	0.45	0.32	0.58	0.85	0.54	0.3	0.2
x_{10}	0.56	0.81	0.31	0.31	0.78	0.8	0.69

Now, we follow the method presented above to make a decision for the objects x_8 , x_9 and x_{10} .

- S1. We begin computing the degrees of suitability of each possible decision for these objects. Then, using Definition 13, we obtain that

$$\sigma_{Dec(S)}(x_8, 0) = \|\Phi_7\|_S^T(x_8) = 0.85$$

$$\sigma_{Dec(S)}(x_8, 1) = \|\Phi_2\|_S^T(x_8) = 0.6$$

$$\sigma_{Dec(S)}(x_9, 0) = \|\Phi_3\|_S^T(x_9) = 0.58$$

$$\sigma_{Dec(S)}(x_9, 1) = \|\Phi_4\|_S^T(x_9) = 0.85$$

$$\sigma_{Dec(S)}(x_{10}, 0) = \|\Phi_5\|_S^T(x_{10}) = 0.78$$

$$\sigma_{Dec(S)}(x_{10}, 1) = \|\Phi_2\|_S^T(x_{10}) = 0.81$$

- S2. Regarding to the object x_8 , there is a remarkable difference between the values $\sigma_{Dec(S)}(x_8, 0) = 0.85$ and $\sigma_{Dec(S)}(x_8, 1) = 0.6$. Consequently, we consider the decision $\bar{d}(x_8) = 0$ for the object x_8 with a degree of suitability of 0.85.

The object x_9 is also easy to decide according to this first method, since both degrees of suitability are significantly different, considering that $\sigma_{Dec(S)}(x_9, 0) = 0.58$ and $\sigma_{Dec(S)}(x_9, 1) = 0.85$. Hence, we consider the decision $\bar{d}(x_9) = 1$ for the object x_9 with a degree of suitability of 0.85. Finally, with respect to the object x_{10} , it is not possible to make a decision for it due to both degrees of suitability, namely $\sigma_{Dec(S)}(x_{10}, 0) = 0.78$ and $\sigma_{Dec(S)}(x_{10}, 1) = 0.81$, are not sufficiently different. As a consequence, we need to carry out more computations to be able to determine the best decision for this object.

- S3. First of all, we set the threshold $\beta = 0.7$ and we compute the set $\mathcal{F}_{x_{10}}^{0.7}(v)$ for $v \in V_d = \{0, 1\}$. Based on Table 5, we have that $\mathcal{F}_{x_{10}}^{0.7}(0) = \{\Phi_5\}$ and $\mathcal{F}_{x_{10}}^{0.7}(1) = \{\Phi_2, \Phi_6\}$. Thus, we obtain that

$$\begin{aligned}\mathcal{M}_{\mathcal{F}_{x_{10}}^{0.7}(0)} &= \|\Phi_5\|_S^T(x_{10}) \\ &= 0.78 \\ \mathcal{M}_{\mathcal{F}_{x_{10}}^{0.7}(1)} &= \frac{\|\Phi_2\|_S^T(x_{10}) + \|\Phi_6\|_S^T(x_{10})}{2} = \frac{0.81 + 0.8}{2} \\ &= 0.81\end{aligned}$$

We can conclude that it is still difficult to make a decision for the object x_{10} . Therefore, we must continue to study this object.

- S4. Now, we set the threshold $\beta = 0.5$. Hence, it is deduced that $\mathcal{F}_{x_{10}}^{0.5}(0) = \{\Phi_1, \Phi_5, \Phi_7\}$ and $\mathcal{F}_{x_{10}}^{0.5}(1) = \{\Phi_2, \Phi_6\}$. Thus, we obtain that

$$\begin{aligned}\mathcal{M}_{\mathcal{F}_{x_{10}}^{0.5}(0)} &= \frac{\|\Phi_1\|_S^T(x_{10}) + \|\Phi_5\|_S^T(x_{10}) + \|\Phi_7\|_S^T(x_{10})}{3} = \frac{0.56 + 0.78 + 0.69}{3} \\ &= 0.68 \\ \mathcal{M}_{\mathcal{F}_{x_{10}}^{0.5}(1)} &= \frac{\|\Phi_2\|_S^T(x_{10}) + \|\Phi_6\|_S^T(x_{10})}{2} = \frac{0.81 + 0.8}{2} \\ &= 0.81\end{aligned}$$

It is easy to check that the difference between both values has increased. As a result, following this method, we will adopt the decision $\bar{d}(x_{10}) = 1$ for the object x_{10} with a degree of suitability of 0.81. However, we should take into account that we have notably decreased the threshold β .

Thus, we must remark that, while the decision $\bar{d}(x_8) = 0$ and $\bar{d}(x_9) = 1$ for the objects x_8 and x_9 , respectively, seems quite clear, the object x_{10} has been more difficult to classify, needing more steps of the presented method to be applied. However, even then, the decision for the object x_{10} is not as clear as for the objects x_8 and x_9 .

Hence, the difficulty in classifying the object x_{10} in the previous example has shown that it is necessary to find another approach to the decision-making process to obtain a more confident result, which is presented in the next section.

It is important to mention that the two first steps of this classification method are similar to the ones carried out in [21]. However, we have presented this method in the fuzzy framework by using separable $[0, 1]$ fuzzy tolerance relations to compute the degree of satisfiability to antecedents of decision rules of the objects. We have chosen this kind of relation because we consider it indispensable that a degree of satisfiability to an antecedent is 1 only in the cases that the values taken by the attributes in the new object are the same as those present in that antecedent, because otherwise, we would be losing information. On the other hand, in [21], tolerance relations are considered for attributes $a \in \mathcal{A}$ such that $V_a \subseteq \mathbb{R}$ and separable tolerance relations for the rest of attributes. Moreover, they consider a discordance relation to make a penalty in the degree of satisfiability to the considered antecedent of the new object due to the existence of some attributes in that antecedent with a very different value to that taken by these attributes in the new object. In our work, we

have discarded this penalization because it is already implicit in the degree of satisfiability. Furthermore, the inclusion of the two last steps of the classification method provides more flexibility to the study, allowing us to make decisions in those cases where the method in [21] is unable to offer a clear decision.

3.2. ε -Representativeness Method

As we have shown in the previous section, the suitability method cannot provide good enough results in some cases, such as in the study of the object x_{10} in Example 2. However, in that method, all the rules have the same importance, and it is not taken into account how many objects satisfy them (that is, the T -support). Therefore, the method presented in this section will be based on both the degrees of satisfiability to the antecedents of the rules compared with the new object and the T -support of those rules.

Definition 14. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $Dec(S)$ be a set of decision rules. Given $\varepsilon \in [0, 1]$, for each new object $x \in U^c$ and $v \in V_d$ the degree of ε -representativeness of the decision $\vec{d}(x) = v$ according to $Dec(S)$ is given by the mapping $\rho_{Dec(S)}^\varepsilon : U^c \times V_d \rightarrow [0, 1]$ defined as

$$\rho_{Dec(S)}^\varepsilon(x, v) = \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x), \Psi = (d, v)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)}$$

If $\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x)\} = \emptyset$, the object x is ε -undecidable.

It is easy to check that $0 \leq \rho_{Dec(S)}^\varepsilon(x, v) \leq 1$, regardless of the object x to classify, the decision $\vec{d}(x) = v$ studied, and the set of decision rules $Dec(S)$ considered, so it is a truth value. We must emphasize the importance of the threshold ε in Definition 14. If a high value is fixed, then only the most similar antecedents to the new object are considered, so a more restrictive study is carried out. On the other hand, if a low value is considered, then many antecedents are compared with the new object, so the classification is less accurate. As a consequence, different values of ε can be considered depending on the flexibility chosen by the user. In addition, we have deemed indispensable the inclusion of the factor $\|\Phi\|_S^T(x)$ to weigh the T -supports of the decision rules and thus give more importance to the rules with the most similar antecedents to the new object x .

The next result shows a pair of interesting properties relating to Definitions 13 and 14.

Proposition 1. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $Dec(S)$ be a set of decision rules, $x \in U^c$ be a new object, and $\varepsilon \in [0, 1]$. Then

- $\sum_{v \in V_d} \rho_{Dec(S)}^\varepsilon(x, v) = 1$.
- $\max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d\} < \varepsilon$ if and only if the object x is ε -undecidable.

Proof. The first item is immediate since

$$\begin{aligned} \sum_{v \in V_d} \rho_{Dec(S)}^\varepsilon(x, v) &= \sum_{v \in V_d} \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x), \Psi = (d, v)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &\stackrel{(1)}{=} \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | \varepsilon \leq \|\Phi\|_S^T(x)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)} = 1 \end{aligned}$$

where (1) is obtained due to

$$\sum_{v \in V_d} \sum_{\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x), \Psi = (d, v)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi) = \sum_{\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x)\}} \|\Phi\|_S^T(x) \cdot \text{supp}_S^T(\Phi, \Psi)$$

For the second item, we suppose that $\max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d\} < \varepsilon$. Then, we have that $\sigma_{Dec(S)}(x, v) < \varepsilon$ for each $v \in V_d$. Hence, $\|\Phi\|_S^T(x) < \varepsilon$ for each $\Phi \rightarrow \Psi \in Dec(S)$. Therefore,

$$\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x)\} = \emptyset$$

Consequently, the object x is ε -undecidable.

Now, we suppose that the object x is ε -undecidable. Using reductio ad absurdum, we assume that $\varepsilon \leq \max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d\}$. Therefore, there exists $v \in V_d$ such that $\varepsilon \leq \sigma_{Dec(S)}(x, v)$. As a consequence, by Definition 13, there exists $\Phi \rightarrow \Psi \in Dec(S)$ with $\Psi = (d, v)$ such that $\varepsilon \leq \|\Phi\|_S^T(x)$, which leads to a contradiction, given that x is ε -undecidable. $\square$

From the first item of Proposition 1, it is possible to clearly differentiate a possible decision from the others by choosing an appropriate threshold ε , as the next result shows.

Lemma 1. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $Dec(S)$ be a set of decision rules, and $x \in U^c$ be a new object. If there exists $v_k \in V_d$ and $\varepsilon \in [0, 1]$ satisfying

$$\max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d \setminus \{v_k\}\} < \varepsilon \leq \sigma_{Dec(S)}(x, v_k)$$

then

$$\begin{aligned} \rho_{Dec(S)}^\varepsilon(x, v_k) &= 1 \\ \rho_{Dec(S)}^\varepsilon(x, v_j) &= 0 \end{aligned}$$

for all $v_j \in V_d \setminus \{v_k\}$.

Proof. First of all, since $\varepsilon \leq \sigma_{Dec(S)}(x, v_k)$, via Proposition 1, we deduce that x is not ε -undecidable. Consequently,

$$\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x)\} \neq \emptyset$$

On the other hand, given that $\max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d \setminus \{v_k\}\} < \varepsilon$, we obtain that $\sigma_{Dec(S)}(x, v_j) < \varepsilon$ for all $v_j \in V_d \setminus \{v_k\}$. Thus, using Definition 13, we deduce that $\{\Phi \rightarrow \Psi \in Dec(S) \mid \varepsilon \leq \|\Phi\|_S^T(x), \Psi = (d, v_j)\} = \emptyset$ for all $v_j \in V_d \setminus \{v_k\}$. Therefore, using Definition 14, for each $v_j \in V_d \setminus \{v_k\}$, we conclude that $\rho_{Dec(S)}^\varepsilon(x, v_j) = 0$. Finally, using Proposition 1, it is obtained that $\rho_{Dec(S)}^\varepsilon(x, v_k) = 1 - \sum_{v \in V_d \setminus \{v_k\}} \rho_{Dec(S)}^\varepsilon(x, v) = 1$. $\square$

This result presents a great advance with respect to the suitability method, since it allows one to select a decision above the rest by choosing an adequate value of ε . However, this value may not be satisfactory for the user since it may be too high. Therefore, we present a classification method based on Definition 14, called the ε -representativeness method, or the R -method for short, in which the threshold ε can be chosen by the user.

- R1. Given a new object to classify $x \in U^c$ and once the threshold ε is fixed, we compute the degrees of ε -representativeness $\rho_{Dec(S)}^\varepsilon(x, v)$ for all $v \in V_d$.
- R2. After that, all the obtained degrees of ε -representativeness are compared. If the degree of ε -representativeness of a decision $\bar{d}(x) = v$ is high and different enough from the

rest of the degrees of ε -representativeness, then we will consider the decision $\bar{d}(x) = v$ for the object x .

- R3. If there exist some values of the decision attribute with similar degrees of ε -representativeness, then the value ε must be changed to be more or less flexible in the study, depending on the case.

It is important to emphasize that if an object x is ε -undecidable, then it is impossible to make a decision for it following this method, given that $\rho_{Dec(S)}^\varepsilon(x, v)$ cannot be computed for any $v \in V_d$. In order to solve this problem, the value ε must be changed.

In the following example, we classify the new objects in Table 4 by using Definition 14, Proposition 1 and Lemma 1.

Example 3. Coming back to the environment of Example 1 and taking into account the degrees of suitability obtained in Example 2, we will determine the best value to be taken by the decision attribute in the new objects x_8 , x_9 and x_{10} according to the ε -representativeness method. Since this method has been introduced to improve the first one, it is expected that it will classify x_8 and x_9 as the suitability method and determine a clearer decision for x_{10} .

- R1. First of all, we set the value $\varepsilon = 0.7$. Notice that it is close to 1, so the antecedents taken into account in the study are significantly similar to the new object, and it allows some flexibility in the comparatives. Notice that, for the object x_8 , the value $v = 0$ and $\varepsilon = 0.7$ verifies the condition of Lemma 1, that is

$$\max\{\sigma_{Dec(S)}(x, v) \mid v \in V_d \setminus \{0\}\} = \sigma_{Dec(S)}(x_8, 1) = 0.6 < \varepsilon \leq 0.85 = \sigma_{Dec(S)}(x_8, 0)$$

As a consequence, $\rho_{Dec(S)}^{0.7}(x_8, 1) = 0$ and $\rho_{Dec(S)}^{0.7}(x_8, 0) = 1$.

In a similar way, for the object x_9 , we have that

$$\sigma_{Dec(S)}(x_9, 0) = 0.58 < \varepsilon \leq 0.85 = \sigma_{Dec(S)}(x_9, 1)$$

Thus, from Lemma 1, $\rho_{Dec(S)}^{0.7}(x_9, 0) = 0$ and $\rho_{Dec(S)}^{0.7}(x_9, 1) = 1$.

Finally, we study the classification of the object x_{10} . From Definition 14 and taking into account Tables 3 and 5, we obtain that

$$\begin{aligned} \rho_{Dec(S)}^{0.7}(x_{10}, 0) &= \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) \mid 0.7 \leq \|\Phi\|_S^T(x_{10}), \Psi = (d, 0)\}} \|\Phi\|_S^T(x_{10}) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) \mid 0.7 \leq \|\Phi\|_S^T(x_{10})\}} \|\Phi\|_S^T(x_{10}) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{\|\Phi_5\|_S^T(x_{10}) \cdot \text{supp}_S^T(\Phi_5, \Psi_5)}{\sum_{\Phi \rightarrow \Psi \in \{\Phi_2 \rightarrow \Psi_2, \Phi_5 \rightarrow \Psi_5, \Phi_6 \rightarrow \Psi_6\}} \|\Phi\|_S^T(x_{10}) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{0.78 \cdot 2.58}{0.81 \cdot 2.11 + 0.78 \cdot 2.58 + 0.8 \cdot 2} = \frac{2.01}{1.71 + 2.01 + 1.6} \\ &= \frac{2.01}{5.32} = 0.38 \end{aligned}$$

On the other hand, from Proposition 1, we have that

$$\rho_{Dec(S)}^{0.7}(x_{10}, 0) + \rho_{Dec(S)}^{0.7}(x_{10}, 1) = 1$$

As a result, we deduce that

$$\rho_{Dec(S)}^{0.7}(x_{10}, 1) = 1 - 0.38 = 0.62$$

- R2. With respect to the object x_8 , the decision to be made is clear, given that $\rho_{Dec(S)}^{0.7}(x_8, 1) = 0$ and $\rho_{Dec(S)}^{0.7}(x_8, 0) = 1$. In this way, the recommended decision for the object x_8 following the ε -representativeness method is $\bar{d}(x_8) = 0$ with a degree of 0.7 representativeness of 1.

For the object x_9 , we obtain similar conclusions. Since $\rho_{Dec(S)}^{0.7}(x_9, 0) = 1$ and $\rho_{Dec(S)}^{0.7}(x_9, 1) = 0$, this method recommends the decision $\bar{d}(x_9) = 1$ for the object x_9 with a degree of 0.7-representativeness of 1.

Finally, regarding the object x_{10} , we obtained the results that $\rho_{Dec(S)}^{0.7}(x_{10}, 1) = 0.62$ and $\rho_{Dec(S)}^{0.7}(x_{10}, 0) = 0.38$. Although the decision for the object x_{10} is not as clear as for the other objects, this method recommends the decision $\bar{d}(x_{10}) = 1$ with a degree of 0.7 representativeness of 0.62.

Now, we present some remarks about the classification of these objects, taking advantage of Proposition 1 and Lemma 1. First of all, for the object x_8 , if $\varepsilon \in (\sigma_{Dec(S)}(x_8, 1), \sigma_{Dec(S)}(x_8, 0)] = (0.6, 0.85]$ from Lemma 1, we have that $\rho_{Dec(S)}^{0.7}(x_8, 1) = 0$ and $\rho_{Dec(S)}^{0.7}(x_8, 0) = 1$. As a consequence, for many values ε , we have a clear classification for it. On the other hand, from Proposition 1, if we consider a value ε such that

$$\max\{\sigma_{Dec(S)}(x_8, v) \mid v \in \{0, 1\}\} = \sigma_{Dec(S)}(x_8, 0) = 0.85 < \varepsilon$$

then the object x_8 is ε -undecidable. Finally, we have studied the degree of ε -representativeness of both possible decisions for each $\varepsilon \in [0, 0.85]$, obtaining that

$$\rho_{Dec(S)}^{\varepsilon}(x_8, 1) < \rho_{Dec(S)}^{\varepsilon}(x_8, 0) \quad \text{if } \varepsilon \leq 0.85$$

As a consequence, for any $\varepsilon \in [0, 0.85]$ the most recommended decision for the object x_8 is $\bar{d}(x_8) = 0$.

Now, for the object x_9 , if $\varepsilon \in (\sigma_{Dec(S)}(x_9, 0), \sigma_{Dec(S)}(x_9, 1)] = (0.58, 0.85]$ from Lemma 1, $\rho_{Dec(S)}^{0.7}(x_9, 0) = 0$ and $\rho_{Dec(S)}^{0.7}(x_9, 1) = 1$. Once again, if

$$\max\{\sigma_{Dec(S)}(x_9, v) \mid v \in \{0, 1\}\} = \sigma_{Dec(S)}(x_9, 1) = 0.85 < \varepsilon$$

the object x_9 is ε -undecidable according to Proposition 1. Moreover, it can be checked that

$$\begin{aligned} \rho_{Dec(S)}^{\varepsilon}(x_9, 1) &< \rho_{Dec(S)}^{\varepsilon}(x_9, 0) & \text{if } 0 \leq \varepsilon \leq 0.54 \\ \rho_{Dec(S)}^{\varepsilon}(x_9, 0) &< \rho_{Dec(S)}^{\varepsilon}(x_9, 1) & \text{if } 0.54 < \varepsilon \leq 0.85 \end{aligned}$$

Although the degree of ε -representativeness changes with respect to the value ε , it is clear that the decision $\bar{d}(x_9) = 1$ should be made, since the degree of ε -representativeness of this decision is higher when higher values of ε are considered and, as we mentioned above, the study is more accurate in these cases.

Finally, for the object x_{10} , if $\varepsilon \in (\sigma_{Dec(S)}(x_{10}, 0), \sigma_{Dec(S)}(x_{10}, 1)] = (0.78, 0.81]$ then by Lemma 1 $\rho_{Dec(S)}^{0.7}(x_{10}, 0) = 0$ and $\rho_{Dec(S)}^{0.7}(x_{10}, 1) = 1$. Moreover, from Proposition 1, the object x_{10} is ε -undecidable if

$$\max\{\sigma_{Dec(S)}(x_{10}, v) \mid v \in \{0, 1\}\} = \sigma_{Dec(S)}(x_{10}, 1) = 0.81 < \varepsilon$$

which is a lower threshold than the present in the previous objects, allowing us to choose fewer values of ε to classify this object. On the other hand, it can be checked that

$$\begin{aligned} \rho_{Dec(S)}^{\varepsilon}(x_{10}, 1) &< \rho_{Dec(S)}^{\varepsilon}(x_{10}, 0) & \text{if } 0 \leq \varepsilon \leq 0.69 \\ \rho_{Dec(S)}^{\varepsilon}(x_{10}, 0) &< \rho_{Dec(S)}^{\varepsilon}(x_{10}, 1) & \text{if } 0.69 < \varepsilon \leq 0.81 \end{aligned} \quad (2)$$

In this case, the recommended decision for the object x_{10} changes with a higher value of ε and the object is ε -undecidable with a lower ε . This fact also exposes the higher difficulty of classifying the object x_{10} because Equation (2) takes place for fewer values ε than in the object x_9 . Notice that this problem of classification was also reflected in Example 2, when we had to apply the third step of the suitability method to be able to decide for this object. Nevertheless, the decision provided by the

ε -representativeness method is more accurate when ε is higher, as we commented above. In this way, the decision $\vec{d}(x_{10}) = 1$ is more confident since, in these cases, $\rho_{Dec(S)}^\varepsilon(x_{10}, 0) < \rho_{Dec(S)}^\varepsilon(x_{10}, 1)$.

Notice that we have obtained more conclusive results than in Example 2 thanks to the consideration of the T support of the decision rules. This is due to the fact that the difference between the degrees of 0.7 representativeness of both possible decisions for each new object is greater than the corresponding differences between the degrees of suitability.

3.3. α -Credibility Method

Some authors [24,25] have considered the measure of the credibility to classify new objects to be important. In this section, we will propose a method in which the α - T -credibility plays an indispensable role. The corresponding method will be based on the following definition.

Definition 15. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $Dec(S)$ be a set of decision rules, $\alpha \in [0, 1)$, and $(\&, I)$ be an adjoint pair. For each new object $x \in U^c$ and $v \in V_d$, the degree of α -credibility of the decision $\vec{d}(x) = v$ according to $Dec(S)$ is given by the mapping $\theta_{Dec(S)}^\alpha : U^c \times V_d \rightarrow [0, 1]$, defined as

$$\theta_{Dec(S)}^\alpha(x, v) = \max\{\&(\|\Phi\|_S^T(x), \mu_{S,\tau}^\alpha(\Phi, \Psi)) \mid \Phi \rightarrow \Psi \in Dec(S), \Psi = (d, v)\}$$

As in the previous notions, the degree of α -credibility is a truth value satisfying that $0 \leq \theta_{Dec(S)}^\alpha(x, v) \leq 1$ for all new objects x , decisions $\vec{d}(x) = v$, $\alpha \in [0, 1)$ and sets of decision rules $Dec(S)$. The third proposed classification method, called α -credibility method, or C -method for short, which is based on Definition 15, is presented below.

- C1. Given a new object to classify $x \in U^c$, we compute the degrees of α -credibility $\theta_{Dec(S)}^\alpha(x, v)$ for all $v \in V_d$.
- C2. Next, we compare all the obtained degrees of α -credibility. If there exists a decision $w \in V_d$ such that $\theta_{Dec(S)}^\alpha(x, w)$ is high and different enough from $\theta_{Dec(S)}^\alpha(x, v)$, for each $v \in V_d \setminus \{w\}$, then the decision $\vec{d}(x) = w$ will be chosen for the object x .
- C3. If there exist some values of the decision attribute with similar degrees of α -credibility, then we must re-examine the relationship between the variables involved in the computation. Then, it is possible to select another adjoint pair $(\&, I)$ from the considered multi-adjoint property framework in order to aggregate the variables in a better way.

Now, we classify the new objects for Example 2 by using this method.

Example 4. Coming back to the environment of Example 2, we study the best decision to make for the objects x_8 , x_9 , and x_{10} according to the α -credibility method.

- C1. First of all, we compute the degrees of 0.76 credibility of each decision for the new objects. With this purpose, we consider the Łukasiewicz adjoint pair $(\&_L, I_L)$, since we assume that the degree of satisfiability and the α - T -credibility are practically independent. For the object x_8 and decision $\vec{d}(x_8) = 0$, via Tables 3 and 5 and Definition 15, we obtain that

$$\begin{aligned} \theta_{Dec(S)}^{0.76}(x_8, 0) &= \max\{\&_L(\|\Phi_1\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_1, \Psi_1)), \&_L(\|\Phi_3\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3)), \\ &\quad \&_L(\|\Phi_5\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_5, \Psi_5)), \&_L(\|\Phi_7\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_7, \Psi_7))\} \\ &= \max\{\&_L(0.35, 1), \&_L(0.1, 1), \&_L(0.6, 0), \&_L(0.85, 1)\} \\ &= \max\{0.35, 0.1, 0, 0.85\} = 0.85 \end{aligned}$$

Analogously, the rest of the degrees of 0.76 credibility are computed as follows:

$$\begin{aligned}\theta_{Dec(S)}^{0.76}(x_8, 1) &= \&_L(\|\Phi_6\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_6, \Psi_6)) = 0.52 \\ \theta_{Dec(S)}^{0.76}(x_9, 0) &= \&_L(\|\Phi_3\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3)) = 0.58 \\ \theta_{Dec(S)}^{0.76}(x_9, 1) &= \&_L(\|\Phi_4\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_4, \Psi_4)) = 0.85 \\ \theta_{Dec(S)}^{0.76}(x_{10}, 0) &= \&_L(\|\Phi_7\|_S^T(x_{10}), \mu_{S,\tau}^{0.76}(\Phi_7, \Psi_7)) = 0.69 \\ \theta_{Dec(S)}^{0.76}(x_{10}, 1) &= \&_L(\|\Phi_6\|_S^T(x_{10}), \mu_{S,\tau}^{0.76}(\Phi_6, \Psi_6)) = 0.8\end{aligned}$$

- C2. For the object x_8 , there is a big difference between the degrees of 0.76 credibility of both possible decisions. Therefore, the α -credibility method recommends the decision $\bar{d}(x_8) = 0$ for the object x_8 with a degree of 0.76-credibility of 0.85.

On the other hand, the case of the object x_9 is very similar. Since $\theta_{Dec(S)}^{0.76}(x_9, 0) = 0.58$ and $\theta_{Dec(S)}^{0.76}(x_9, 1) = 0.85$, the recommended decision for the object x_9 according to the α -credibility method is $\bar{d}(x_9) = 1$ with a degree of 0.76 credibility of 0.85.

Finally, with respect to the object x_{10} , the decision is less clear than in the objects x_8 and x_9 , as the degrees of 0.76 credibility of both possible decisions are more similar. Nevertheless, we consider that the difference between both degrees of 0.76-credibility is sufficient to conclude that the best decision for the object x_{10} is $\bar{d}(x_{10}) = 1$ with a degree of 0.76 credibility of 0.8. Anyway, the support decision system incorporating this procedure will show that the other decision has a 0.76 credibility with a degree of 0.69.

Now, we make some remarks about the results obtained. First of all, notice that the decisions suggested by the α -credibility method are the same as those recommended by the methods presented in Sections 3.1 and 3.2. In fact, the degrees of 0.76 credibility of the decisions $\bar{d}(x_8) = 0$ and $\bar{d}(x_9) = 1$ in Example 4 coincide with their degrees of suitability in Example 2. This is due to the fact that the decision rules involved in the computation of the degrees of suitability of these decisions, which are r_7 and r_4 , respectively, have a 0.76-T-credibility equal to 1. Finally, the decision for the object x_{10} has been slightly more difficult to make, as in Example 2, where the T support of the rules was also not taken into account. Therefore, we can conclude that the inclusion of this measure helps to classify new objects, so it is an aspect to take into account in the last method that we will present in this paper.

The main advantage of this method is the inclusion of the α -T-credibility, which is useful for checking the reliability of the rules. In this way, the decision given by this method will be supported by a decision rule with a high α -T-credibility, which strengthens the decision to be made. However, a disadvantage of this method is that, as we have mentioned above, we have not taken into account the T support of the rules, which is also a relevant measure. In this way, the decision may be driven by a decision rule with a low representativeness in the decision table. This problem is addressed with the method proposed in the next section.

3.4. Support-Driven α -Credibility Method

In this section, we present the last proposed classification method. It is very similar to the one presented in Section 3.3, including the NT-support. The corresponding method is based on the following definition.

Definition 16. Let $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table, $Dec(S)$ be a set of decision rules, $\alpha \in [0, 1]$, and $(\&, I)$ be an adjoint pair, where $\&$ is an associative conjunctive. For each new object $x \in U^c$ and $v \in V_d$, the degree of support-driven α -credibility of the decision $\bar{d}(x) = v$ according to $Dec(S)$ is given by the mapping $\delta_{Dec(S)}^\alpha: U^c \times V_d \rightarrow [0, 1]$ defined as

$$\delta_{Dec(S)}^\alpha(x, v) = \max\{\&(\|\Phi\|_S^T(x), \mu_{S,\tau}^\alpha(\Phi, \Psi), \text{supp}_S^{NT}(\Phi, \Psi)) \mid \Phi \rightarrow \Psi \in Dec(S), \Psi = (d, v)\}$$

We must emphasize that $0 \leq \delta_{Dec(S)}^\alpha(x, v) \leq 1$ for all new objects x , decisions $\bar{d}(x) = v$, $\alpha \in [0, 1)$ and sets of decision rules $Dec(S)$, so $\delta_{Dec(S)}^\alpha$ is a membership function of the new object on the decision value set V_d . The steps of the fourth proposed classification method, called *support-driven α -credibility method* or *D-method* for short, are the following:

- D1. First of all, given a new object to classify $x \in U^c$, we compute the degrees of support-driven α -credibility $\delta_{Dec(S)}^\alpha(x, v)$ for all $v \in V_d$.
- D2. After that, all the obtained degrees of support-driven α -credibility are compared. If there exists a decision $w \in V_d$ such that $\delta_{Dec(S)}^\alpha(x, w)$ is high and sufficiently different from $\delta_{Dec(S)}^\alpha(x, v)$, for each $v \in V_d \setminus \{w\}$, then the decision $\bar{d}(x) = w$ will be chosen for the object x .
- D3. If there exist some values of the decision attribute with similar degrees of support-driven α -credibility, then we will check the relationship between the variables involved in the computation. Then, another adjoint pair $(\&, I)$ from the considered multi-adjoint property framework can be chosen to represent more appropriately the relationship and to obtain a better aggregation of the corresponding values.

In the next example, we will use this method to classify the new objects considered in Table 4.

Example 5. Coming back to the environment of Example 2, the goal is to determine a decision for the objects x_8 , x_9 , and x_{10} according to the support-driven α -credibility method.

- D1. We begin computing the degrees of support-driven 0.76 credibility of each decision for the new objects. Now, we suppose that there exists a close relationship between the considered variables, so we use the Gödel adjoint pair $(\&_G, I_G)$. Using Tables 3 and 5 and Definition 16 for the object x_8 and decision $\bar{d}(x_8) = 0$, we obtain that

$$\begin{aligned} \delta_{Dec(S)}^{0.76}(x_8, 0) &= \max\{\&_G(\|\Phi_1\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_1, \Psi_1), \text{supp}_S^{NT}(\Phi_1, \Psi_1)), \\ &\quad \&_G(\|\Phi_3\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3), \text{supp}_S^{NT}(\Phi_3, \Psi_3)), \\ &\quad \&_G(\|\Phi_5\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_5, \Psi_5), \text{supp}_S^{NT}(\Phi_5, \Psi_5)), \\ &\quad \&_G(\|\Phi_7\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_7, \Psi_7), \text{supp}_S^{NT}(\Phi_7, \Psi_7))\} \\ &= \max\{\&_G(0.35, 1, 0.85), \&_G(0.1, 1, 0.83), \\ &\quad \&_G(0.6, 0, 1), \&_G(0.85, 1, 0.89)\} \\ &= \max\{0.35, 0.1, 0, 0.85\} = 0.85 \end{aligned}$$

In a similar way, we compute the rest of the degrees of support-driven 0.76 credibility, whose results are the following ones:

$$\begin{aligned} \delta_{Dec(S)}^{0.76}(x_8, 0) &= \&_G(\|\Phi_7\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_7, \Psi_7), \text{supp}_S^{NT}(\Phi_7, \Psi_7)) = 0.85 \\ \delta_{Dec(S)}^{0.76}(x_8, 1) &= \&_G(\|\Phi_6\|_S^T(x_8), \mu_{S,\tau}^{0.76}(\Phi_6, \Psi_6), \text{supp}_S^{NT}(\Phi_6, \Psi_6)) = 0.52 \\ \delta_{Dec(S)}^{0.76}(x_9, 0) &= \&_G(\|\Phi_3\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3), \text{supp}_S^{NT}(\Phi_3, \Psi_3)) = 0.58 \\ \delta_{Dec(S)}^{0.76}(x_9, 1) &= \&_G(\|\Phi_4\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_4, \Psi_4), \text{supp}_S^{NT}(\Phi_4, \Psi_4)) = 0.62 \\ \delta_{Dec(S)}^{0.76}(x_{10}, 0) &= \&_G(\|\Phi_7\|_S^T(x_{10}), \mu_{S,\tau}^{0.76}(\Phi_7, \Psi_7), \text{supp}_S^{NT}(\Phi_7, \Psi_7)) = 0.69 \\ \delta_{Dec(S)}^{0.76}(x_{10}, 1) &= \&_G(\|\Phi_6\|_S^T(x_{10}), \mu_{S,\tau}^{0.76}(\Phi_6, \Psi_6), \text{supp}_S^{NT}(\Phi_6, \Psi_6)) = 0.78 \end{aligned}$$

- D2. First of all, notice that the values obtained for the object x_8 are the same as in Example 4. Consequently, the support-driven α -credibility method also recommends the decision $\bar{d}(x_8) = 0$ with a degree of support-driven 0.76 credibility of 0.85. However, with respect to the object x_9 , the degrees of support-driven 0.76 credibility of both decisions are quite similar, making it difficult to decide about this object. In order to solve this,

we will use other adjoint pairs in the next step of the method that consider the three values in the computation.

Finally, the case of the object x_{10} is very similar to the one in Example 4, recommending once again the decision $\bar{d}(x_{10}) = 1$ with a degree of support-driven 0.76 credibility of 0.78.

- D3. Now, assuming that there is less of a relationship between the variables, we compute the degree of support-driven 0.76 credibility of the decisions $\bar{d}(x_9) = 0$ and $\bar{d}(x_9) = 1$ by using the adjoint pair $(\&_P, I_P)$, which multiplies the three values equitably. We obtained

$$\begin{aligned}\delta_{Dec(S)}^{0.76}(x_9, 0) &= \&_P(\|\Phi_3\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3), \text{supp}_S^{NT}(\Phi_3, \Psi_3)) = 0.48 \\ \delta_{Dec(S)}^{0.76}(x_9, 1) &= \&_P(\|\Phi_4\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_4, \Psi_4), \text{supp}_S^{NT}(\Phi_4, \Psi_4)) = 0.53\end{aligned}$$

We can observe similar difficulties in classifying the object x_9 . Finally, we suppose that there is almost no relationship between the considered variables. Hence, we consider the adjoint pair $(\&_L, I_L)$, obtaining

$$\begin{aligned}\delta_{Dec(S)}^{0.76}(x_9, 0) &= \&_L(\|\Phi_3\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_3, \Psi_3), \text{supp}_S^{NT}(\Phi_3, \Psi_3)) = 0.41 \\ \delta_{Dec(S)}^{0.76}(x_9, 1) &= \&_L(\|\Phi_4\|_S^T(x_9), \mu_{S,\tau}^{0.76}(\Phi_4, \Psi_4), \text{supp}_S^{NT}(\Phi_4, \Psi_4)) = 0.47\end{aligned}$$

We conclude that, independently of the adjoint pair considered in the multi-adjoint property framework $([0, 1], [0, 1], [0, 1], \&_L, \&_P, \&_G)$, it is difficult to make a decision for the object x_9 by using the support-driven α -credibility method. Nevertheless, since in all the cases we have obtained that $\delta_{Dec(S)}^{0.76}(x_9, 0) < \delta_{Dec(S)}^{0.76}(x_9, 1)$, the support-driven α -credibility method recommends the decision $\bar{d}(x_9) = 1$ for the object x_9 .

This difficult classification for the object x_9 has been caused by the inclusion of the NT support in Definition 16. As shown in Table 3, the NT support of the decision rules whose consequent is $(d, 1)$ is lower than the NT support of the rules with consequent $(d, 0)$.

We must emphasize that all the methods presented in this paper are important since each one approaches the decision-making process from a different point of view. Hence, they should be considered together in order to make decisions. In particular, the first one focuses on the degree of satisfiability with respect to the antecedents of the rules of the new object to classify in order to discard rules whose antecedents are very different from the new object. With respect to the ε -representativeness method, it is presented as an alternative to the suitability method to solve some situations in which this method is unable to provide a satisfactory decision. For that purpose, the ε -representativeness method also considers the T -support of the decision rules, given that it shows the representativeness of the rule in the decision table. On the other hand, the α -credibility method is proposed since the credibility of the decision rules has been considered in other studies [20,25] to classify new objects. The main novelty of our contribution with this third method is the consideration of the multi-adjoint framework. In addition, the support-driven α -credibility method is based on the third one but also considers the NT support of the rules compared with the new object for the same reasons as the ε -representativeness method was introduced. Furthermore, it is also important to remark again that these classification methods are not mutually exclusive but can complement each other, as we have mentioned above, to classify objects when one of the methods does not give a sufficiently conclusive result.

Finally, Table 6 presents a summary of the different notions considered in the methods proposed in this paper. In this table, “satisfiability” represents whether the method takes into account in the computation the degree to which the new object satisfies the given decision rules. It is clear that this degree is very relevant to classifying new objects, and so it is considered in every method.

Table 6. Summary of the notions considered in each method.

Method	Satisfiability	Support	Credibility
Suitability	X		
ε -representativeness	X	X	
α -credibility	X		X
Support-driven α -credibility	X	X	X

4. Comparison with Other Methods

In the introduction section, the differences between the introduced methods and the ones given in [20,21,23] were enumerated. This section will be focused on the methods introduced in [24,25], whose comparison is not trivial.

A detailed study on the classification of new data is carried out in [25]. The authors sequentially present several notions, such as the upper and lower approximations of sets of attributes or the degree of credibility of a decision rule. They play an indispensable role in the classification method proposed, which is given as an algorithm. Since we have detected some similarities between our α -credibility method and the one given in [25], we have considered it important to compare both studies.

First of all, in [25], the following specific fuzzy tolerance relation is considered to compare pairs of objects according to each attribute $a \in \mathcal{A}_d$:

$$R_a(x, y) = \frac{\max\{0, \min\{\bar{a}(x), \bar{a}(y)\} + k_a - \max\{\bar{a}(x), \bar{a}(y)\}\}}{k_a}$$

where $k_a > 0$ for all $a \in \mathcal{A}_d$. Notice that it is also a separable relation. In addition, if $k_a = 1$ and $V_a = [0, 1]$ for each $a \in \mathcal{A}_d$, then the expression of the relation R_a coincides with one of the mappings T_a considered in Example 1 since

$$\begin{aligned} R_a(x, y) &= \frac{\max\{0, \min\{\bar{a}(x), \bar{a}(y)\} + 1 - \max\{\bar{a}(x), \bar{a}(y)\}\}}{1} \\ &= \min\{\bar{a}(x), \bar{a}(y)\} + 1 - \max\{\bar{a}(x), \bar{a}(y)\} \\ &= 1 - (\max\{\bar{a}(x), \bar{a}(y)\} - \min\{\bar{a}(x), \bar{a}(y)\}) \\ &= 1 - |\bar{a}(x) - \bar{a}(y)| \\ &= T_a(\bar{a}(x), \bar{a}(y)) \end{aligned}$$

Moreover, in [25], the relation between objects according to the total set of attributes $\mathcal{A}$ is defined for each $x, y \in U$ as

$$R_{\mathcal{A}}(x, y) = \min\{R_a(x, y) \mid a \in \mathcal{A}\}$$

Notice that $R_{\mathcal{A}}$ coincides with the degree of satisfiability to a conjunction of formulas of Definition 8. Indeed, given $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\Phi = \bigwedge_{i=1}^n (a_i, v_i) \in \text{For}(\mathcal{A})$ induced by an object $y \in U$, that is, $v_i = \bar{a}_i(y)$ for all $i \in \{1, \dots, n\}$, by Definition 8, we have

$$\begin{aligned} \|\Phi\|_S^T(x) &= \left\| \bigwedge_{i=1}^n (a_i, v_i) \right\|_S^T(x) = \min\{\|(a_i, v_i)\|_S^T(x) \mid a_i \in \mathcal{A}\} \\ &= \min\{\|(a_i, \bar{a}_i(y))\|_S^T(x) \mid a_i \in \mathcal{A}\} = \min\{T_{a_i}(\bar{a}_i(x), \bar{a}_i(y)) \mid a_i \in \mathcal{A}\} \\ &= \min\{R_{a_i}(x, y) \mid a_i \in \mathcal{A}\} \\ &= R_{\mathcal{A}}(x, y) \end{aligned}$$

This relation is used to compare each new object with the antecedents of the rules of the decision table, as we have considered in Section 3.1. In order to carry out the classification of new objects, Ref. [25] also takes into account the credibility of the decision rule $r_i : \Phi_i \rightarrow \Psi_i$

compared with the new object and whose antecedent is composed of all the condition attributes of the table, defined as

$$\mu_{\mathcal{A}}(\Phi_i, \Psi_i) = \min\{\max\{1 - R_{\mathcal{A}}(x, x_i), \mu_{\bar{d}(x_i)\mathcal{A}}(x)\} \mid x \in R_{\mathcal{A}}(x_i)\}$$

where $x_i \in U$ is the object that induces the rule r_i and

$$\begin{aligned}\mu_{\bar{d}(x_i)\mathcal{A}}(x) &= \min\{\max\{1 - R_{\mathcal{A}}(x, y), R_d(x_i, y)\} \mid y \in R_{\mathcal{A}}(x)\} \\ R_{\mathcal{A}}(x_i) &= \{x \in U \mid R_{\mathcal{A}}(x, x_i) > 0\}\end{aligned}$$

In our terms, these notions are given as

$$\begin{aligned}\mu_{S,\tau}^0(\Phi_i, \Psi_i) &= \min\{\max\{1 - \|\Phi_i\|_S^T(x), \mu_{\Psi_{i,S,\tau}}(x)\} \mid x \in (\|\Phi_i\|_S^T)_0\} \\ \mu_{\Psi_{i,S,\tau}}(x) &= \min\{\max\{1 - \|\Phi\|_S^T(y), \|\Psi_i\|_S^T(y)\} \mid y \in (\|\Phi\|_S^T)_0\} \\ (\|\Phi_i\|_S^T)_0 &= \{x \in U \mid \|\Phi_i\|_S^T(x) > 0\}\end{aligned}$$

where $\Phi \rightarrow \Psi \in Dec(S)$ is the decision rule induced by x considering all the condition attributes for the antecedent. Notice that $\mu_{\mathcal{A}}(\Phi_i, \Psi_i)$ can be expressed as

$$\mu_{S,\tau}^0(\Phi_i, \Psi_i) = \&_G(I(\|\Phi_i\|_S^T(x), \&_G(I(\|\Phi\|_S^T(y), \|\Psi_i\|_S^T(y)))))$$

where $x \in (\|\Phi_i\|_S^T)_0$, $y \in (\|\Phi\|_S^T)_0$, $\&_G$ is the Gödel t-norm and the I is implication residuated of the conjunctive $\&$, which are both defined for each $x, y \in [0, 1]$ as

$$\begin{aligned}\&(x, y) &= \begin{cases} 0 & \text{if } x \leq 1 - y \\ x & \text{otherwise} \end{cases} \\ I(x, y) &= \max\{1 - x, y\}\end{aligned}$$

On the other hand, it is also possible to obtain a relationship between the credibility and Definition 4. First of all, notice that

$$\&_G(\{I(R_{\mathcal{A}}(x, y), R_d(x_i, y)) \mid y \in R_{\mathcal{A}}(x)\}) = \&_G(\{I(R_{\mathcal{A}}(x, y), R_d(x_i, y)) \mid y \in U\})$$

holds because if $y \notin R_{\mathcal{A}}(x)$ then

$$\begin{aligned}I(R_{\mathcal{A}}(x, y), R_d(x_i, y)) &= \max\{1 - R_{\mathcal{A}}(x, y), R_d(x_i, y)\} \\ &= \max\{1 - 0, R_d(x_i, y)\} \\ &= 1\end{aligned}$$

and $\&_G(v, 1) = v$ for each $v \in [0, 1]$ due to the boundary property satisfied by the t-norms. As a consequence, we deduce that

$$\begin{aligned}\mu_{\bar{d}(x_i)\mathcal{A}}(x) &= \&_G(\{I(R_{\mathcal{A}}(x, y), R_d(x_i, y)) \mid y \in R_{\mathcal{A}}(x)\}) \\ &= \min\{I(R_{\mathcal{A}}(x, y), R_d(x_i, y)) \mid y \in U\} \\ &= (R_d x_i)^{\downarrow N}(x) \\ \mu_{\bar{d}(x_i)\mathcal{A}}(x_i) &= (R_d x_i)^{\downarrow N}(x_i) = \text{POS}_{\mathcal{A}}^f(x_i) \\ \mu_{\mathcal{A}}(\Phi_i, \Psi_i) &= \&_G(\{I(R_{\mathcal{A}}(x, x_i), \mu_{\bar{d}(x_i)\mathcal{A}}(x)) \mid x \in R_{\mathcal{A}}(x_i)\}) \\ &= \min\{I(R_{\mathcal{A}}(x, x_i), \mu_{\bar{d}(x_i)\mathcal{A}}(x)) \mid x \in U\} \\ &= (\mu_{\bar{d}(x_i)\mathcal{A}})^{\downarrow N}(x_i)\end{aligned}$$

In conclusion, we must emphasize that the credibility $\mu_{\mathcal{A}}(\Phi_i, \Psi_i)$ is similar to the one given in Definition 12. However, in that definition, we have considered different

adjoint pairs for each pair of objects of the decision table, extending the definition given in [25] and allowing preferences to be set among the objects. Moreover, in the definition of $(\|\Phi\|_S^T)_\alpha$, we have set an α -cut $\|\Phi\|_S^T(x_i) > \alpha$ that can be changed depending on the study, providing a greater level of flexibility, while in [25], the threshold is $\|\Phi\|_S^T(x_i) > 0$. This main difference is caused by the definition of the tolerance relations used to compare the objects. We have considered tolerance relations T_a in general, while in [25] a specific one R_a is provided, in which a value k_a is included to distinguish objects. This choice is justified by the fact that, if $k_a \leq |\bar{a}(x) - \bar{a}(y)|$, then it is assumed that the objects x, y are not related according to a , obtaining $R_a(x, y) = 0$. Nevertheless, in this paper, we assume that a pair of objects x, y are not related if and only if $|\bar{a}(x) - \bar{a}(y)| = 1$, which is the motivation for the introduction of the α -cut. Despite this difference between both approaches, it is possible to relate them, so that $y \in R_A(x)$ if and only if $y \in (\|\Phi\|_S^T)_\alpha$, where $\Phi \rightarrow \Psi$ is the decision rule induced by x with $\Phi \in \text{For}(\mathcal{A})$. Indeed, defining $\alpha = 1 - \min\{k_a \mid a \in \mathcal{A}\}$ when $T_a(\bar{a}(x), \bar{a}(y)) = 1 - |\bar{a}(x) - \bar{a}(y)|$, we deduce that, if $\|\Phi\|_S^T(y) > \alpha$, then from Definition 8, $\min\{T_a(\bar{a}(x), \bar{a}(y)) \mid a \in \mathcal{A}\} > \alpha$. Hence, $|\bar{a}(x) - \bar{a}(y)| < 1 - \alpha = \min\{k_a \mid a \in \mathcal{A}\}$ for all $a \in \mathcal{A}$. In conclusion, the objects x, y are related in both environments.

Finally, the decision $\bar{d}(x) = v$ to be made for the new object x according to [25] is given as

$$v = \arg \max \{ \min \{ \|\Phi\|_S^T(x), \mu_{\mathcal{A}}(\Phi, \Psi) \mid \Phi \rightarrow \Psi \in \text{Dec}(S) \} \mid v_d \in V_d, \Psi = (d, v_d) \}$$

where $\arg \max$ is the argument of the maximum of a function; that is, given two sets X, Y and a mapping $f: X \rightarrow Y$, the *argument of the maximum of f in X* is defined as $\arg \max f = \{x \in X \mid f(y) \leq f(x) \text{ for all } y \in X\}$. Notice that the α -credibility method extends this one since Definition 12 generalizes the credibility used in [25] due to the use of adjoint pairs, the consideration of a family of separable $[0, 1]$ -fuzzy tolerance relations to define the degree of satisfiability of a formula, and the replacement of the minimum with an adjoint conjunctive in general.

With respect to the comparison between the α -credibility method and the given in [24], there are also some similarities. On the one hand, Definition 12 generalizes the credibility presented in [24], since we can consider different adjoint pairs depending on the objects compared, but they consider a t-norm and an implication for all the objects of the table. On the other hand, with the classification method, they consider a t-conorm to obtain the values to be compared to make a decision. This choice is due to the fact that they make a preprocess to discard those decision rules whose credibility or similarity of their antecedents to the new object studied is low. In this paper, we have considered the maximum from the beginning in order to avoid the scenario in which the user has to make a preprocess in which a threshold has to be fixed. Furthermore, they determine the best possible decision for the new object by means of the maximum. However, we believe that it is essential to compare all the values obtained so that the decision to be made is not only inferred from the maximum, but its value must be sufficiently different from the others obtained to provide a solid classification. Otherwise, as stated in the third step of all the methods presented, we must continue applying the corresponding method.

Thus, the introduced methods generalize the given one in the literature, such as [24,25], as we have detailed above, and [20,21,23], which were commented on in the introduction.

5. A Toy Example

This section is devoted to studying the classification of a new object in a real case following the different strategies presented in this paper. In particular, the considered data set concerns eight project evaluations described using the attributes construction expense (CE), financial income (FI), strategy benefit (SB), and external influence (EI) in order to decide whether to make an investment (I), a delayed investment (DI) or no investment (NI) in each project. This data set was introduced in [30] and is represented in Table 7, where every unit corresponds to 10,000 yuan.

Table 7. Knowledge system on water conservancy project investment decision making.

	CE	FI	SB	EI	Decision (d)
Project 1 (p_1)	500	200	690	150	DI
Project 2 (p_2)	700	470	650	440	NI
Project 3 (p_3)	300	410	500	280	NI
Project 4 (p_4)	200	455	550	290	I
Project 5 (p_5)	250	260	490	105	DI
Project 6 (p_6)	510	380	430	130	DI
Project 7 (p_7)	350	550	255	145	I
Project 8 (p_8)	650	600	570	120	NI

This data set was analyzed from an RST perspective in [30] and was studied in the fuzzy framework in [13]. Now, we use that study to obtain the decision rules considered to decide on the investment in a new project. First of all, the previous knowledge system is formally represented as the decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$, where the set of objects is $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$, the set of attributes is $\mathcal{A} = \{CE, FI, SB, EI\}$, $\mathcal{V}_a = [0, 1]$, for all $a \in \mathcal{A}$, $\mathcal{V}_d = \{I, NI, DI\}$, and $\overline{\mathcal{A}_d}$ is represented in Table 8. Notice that Table 8 provides a normalization of the values in Table 7, taking into account the possible range of values for each condition attribute, which we have considered to be 1000 for the attributes CE, FI, and SB and 500 for the attribute EI.

Table 8. Decision table $S = (U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ extracted from Table 7.

	CE	FI	SB	EI	d
p_1	0.5	0.2	0.69	0.3	DI
p_2	0.7	0.47	0.65	0.88	NI
p_3	0.3	0.41	0.5	0.56	NI
p_4	0.2	0.455	0.55	0.58	I
p_5	0.25	0.26	0.49	0.21	DI
p_6	0.51	0.38	0.43	0.26	DI
p_7	0.35	0.55	0.255	0.29	I
p_8	0.65	0.6	0.57	0.24	NI

The decision rules $r_i: \Phi_i \rightarrow \Psi_i$ for each $i \in \{1, \dots, 8\}$ used to classify the new project are the following:

$$\begin{aligned}
 r_1 &: (CE, 0.5) \wedge (FI, 0.2) \wedge (SB, 0.69) \wedge (EI, 0.3) \rightarrow (d, DI) \\
 r_2 &: (CE, 0.7) \wedge (FI, 0.47) \wedge (SB, 0.65) \wedge (EI, 0.88) \rightarrow (d, NI) \\
 r_3 &: (CE, 0.3) \wedge (FI, 0.41) \wedge (SB, 0.5) \wedge (EI, 0.56) \rightarrow (d, NI) \\
 r_4 &: (CE, 0.2) \wedge (FI, 0.455) \wedge (SB, 0.55) \wedge (EI, 0.58) \rightarrow (d, I) \\
 r_5 &: (CE, 0.25) \wedge (FI, 0.26) \wedge (SB, 0.49) \wedge (EI, 0.21) \rightarrow (d, DI) \\
 r_6 &: (CE, 0.51) \wedge (FI, 0.38) \wedge (SB, 0.43) \wedge (EI, 0.26) \rightarrow (d, DI) \\
 r_7 &: (CE, 0.35) \wedge (FI, 0.55) \wedge (SB, 0.255) \wedge (EI, 0.29) \rightarrow (d, I) \\
 r_8 &: (CE, 0.65) \wedge (FI, 0.6) \wedge (SB, 0.57) \wedge (EI, 0.24) \rightarrow (d, NI)
 \end{aligned}$$

We denote this set of decision rules as $Dec(S)$. In order to compute the measures of Definition 11 for each rule of $Dec(S)$, we consider the family of separable $[0, 1]$ -fuzzy

tolerance relations $T = \{T_a: V_a \times V_a \rightarrow [0, 1] \mid a \in \mathcal{A}_d\}$ defined as $T_a(v, w) = 1 - |v - w|$, for all $a \in \mathcal{A}$, $v, w \in [0, 1]$ and

$$T_d(v, w) = \begin{cases} 1 & \text{if } v = w \\ 0.5 & \text{if } v = \text{DI and } w \neq \text{DI} \\ 0 & \text{otherwise} \end{cases}$$

since we consider that DI is an intermediate decision between I and NI. For this study, we will consider the multi-adjoint property-oriented frame $([0, 1], [0, 1], [0, 1], \&_L, \&_P, \&_G)$ and the context $(U, U, R_{\mathcal{A}}, \tau)$, where $\tau: U \times U \rightarrow \{L, P, G\}$ is defined for each $x, y \in U$ as

$$\tau(x, y) = \begin{cases} L & \text{if } \bar{d}(x) = \bar{d}(y) \\ P & \text{if } \bar{d}(x) = \text{DI and } \bar{d}(y) \neq \text{DI} \\ G & \text{otherwise} \end{cases}$$

As in Example 1, we have considered this mapping to incorporate some preferences among the objects. Specifically, we have also reinforced the computation of the credibility when the objects have the same decision (implication I_L is greater than I_P and I_G). Moreover, we will assign more relevance to the objects with the intermediate decision DI (implication I_P is greater than I_G).

Furthermore, we will use the adjoint pair $(\&_L, I_L)$, and we will fix the α cut to be equal to 0.79 in order to compute the 0.79- T -credibility of each decision rule belonging to $Dec(S)$ to reduce the number of inconsistencies in the sets $(\|\Phi\|_S^T)_\alpha$ with $\Phi \rightarrow \Psi \in Dec(S)$, while some flexibility is allowed in the comparisons.

Now, the T -support, NT -support, and 0.79- T -credibility of each decision rule are computed by using Definitions 11 and 12. The corresponding results are shown in Table 9.

Table 9. T support, NT support, and 0.79- T -credibility of each rule in $Dec(S)$.

Rule	$supp_S^T$	$supp_S^{NT}$	$\mu_{S,\tau}^{0.79}$
r_1	4.91	1	1
r_2	3.09	0.63	1
r_3	3.75	0.76	0
r_4	3.21	0.65	0
r_5	4.82	0.98	1
r_6	4.86	0.99	0.37
r_7	3.21	0.65	0.37
r_8	3.51	0.71	1

As in Example 1, the decision rules $\Phi \rightarrow \Psi \in Dec(S)$ with 0.79 T -credibility equal to 1 satisfy $(\|\Phi\|_S^T)_{0.79} = \{x\}$ where $x \in U$, the objects that induce them. Analogously, $\mu_{S,\tau}^{0.79}(\Phi_3, \Psi_3) = \mu_{S,\tau}^{0.79}(\Phi_4, \Psi_4) = 0$ since $\|\Phi_3\|_S^T(x_4) = \|\Phi_4\|_S^T(x_3) = 0.9 > 0.79$ and $\|\Psi_3\|_S^T(x_4) = \|\Psi_4\|_S^T(x_3) = 0$. The case of r_6 and r_7 is similar, but $\|\Phi_6\|_S^T(x_7) = \|\Phi_7\|_S^T(x_6) = 0.83$, whereas $\|\Psi_6\|_S^T(x_7) = \|\Psi_7\|_S^T(x_6) = 0.5$; that is, both rules are less related in the antecedent and more related in the consequent than the rules r_3 and r_4 , making the 0.79 T -credibility higher.

Now, we use these rules and each method presented in this paper to classify the new project p_9 given in Table 10.

Table 10. New project p_9 to evaluate.

	CE	FI	SB	EI
p_9	500	300	800	300

As can be seen in Table 10, the project p_9 presents a low financial income, a high strategy benefit, and intermediate construction expense and external influence. In order to decide whether or not to invest in it, we normalize these values according to the criteria established above. In Table 11, the project p_9 is shown together with the rest of the projects.

Table 11. Table 8 together with the new project to evaluate p_9 .

	CE	FI	SB	EI	d
p_1	0.5	0.2	0.69	0.3	DI
p_2	0.7	0.47	0.65	0.88	NI
p_3	0.3	0.41	0.5	0.56	NI
p_4	0.2	0.455	0.55	0.58	I
p_5	0.25	0.26	0.49	0.21	DI
p_6	0.51	0.38	0.43	0.26	DI
p_7	0.35	0.55	0.255	0.29	I
p_8	0.65	0.6	0.57	0.24	NI
p_9	0.5	0.3	0.8	0.6	

We begin computing the degree of satisfiability for each antecedent of the decision rules of $Dec(S)$ of the new project p_9 . These results are shown in Table 12.

Table 12. Degree of satisfiability for each antecedent of decision rules of the new project p_9 .

	$\ \Phi_1\ _S^T$	$\ \Phi_2\ _S^T$	$\ \Phi_3\ _S^T$	$\ \Phi_4\ _S^T$	$\ \Phi_5\ _S^T$	$\ \Phi_6\ _S^T$	$\ \Phi_7\ _S^T$	$\ \Phi_8\ _S^T$
p_9	0.7	0.72	0.7	0.7	0.61	0.63	0.455	0.64

Notice that, by Definitions 13–16, the degree of satisfiability to antecedents for the new object plays a key role in each of the presented methods. Taking into account that, from Table 12, most degrees of satisfiability are quite similar, we can anticipate relevant difficulties to provide an accurate classification for the project p_9 .

Next, we are going to apply the classification methods presented in Section 3 to determine the best possible decision for the project p_9 according to the set of decision rules $Dec(S)$.

5.1. Application of the Suitability Method

We begin studying the new project p_9 by means of the suitability method.

- S1. First of all, we compute the degree of suitability of each possible decision to be made. Thus, from Definition 13, we obtain that

$$\begin{aligned}\sigma_{Dec(S)}(p_9, I) &= \|\Phi_4\|_S^T(p_9) = 0.7 \\ \sigma_{Dec(S)}(p_9, NI) &= \|\Phi_2\|_S^T(p_9) = 0.72 \\ \sigma_{Dec(S)}(p_9, DI) &= \|\Phi_1\|_S^T(p_9) = 0.7\end{aligned}$$

- S2. We can immediately check that all the degrees of suitability are very similar. As a result, we cannot make a decision for the project p_9 . To solve that, we apply the third step of the method.

- S3. Since many degrees of satisfiability are very similar, we will consider all of them in the computations by setting the threshold $\beta = 0.4$. Consequently, we obtain $\mathcal{F}_{p_9}^{0.4}(I) = \{\Phi_4, \Phi_7\}$, $\mathcal{F}_{p_9}^{0.4}(NI) = \{\Phi_2, \Phi_3, \Phi_8\}$ and $\mathcal{F}_{p_9}^{0.4}(DI) = \{\Phi_1, \Phi_5, \Phi_6\}$. Therefore, it is

$$\begin{aligned}\mathcal{M}_{\mathcal{F}_{p_9}^{0.4}(I)} &= \frac{\|\Phi_4\|_S^T(p_9) + \|\Phi_7\|_S^T(p_9)}{2} = \frac{0.7 + 0.455}{2} \\ &= 0.58 \\ \mathcal{M}_{\mathcal{F}_{p_9}^{0.4}(NI)} &= \frac{\|\Phi_2\|_S^T(p_9) + \|\Phi_3\|_S^T(p_9) + \|\Phi_8\|_S^T(p_9)}{3} = \frac{0.72 + 0.7 + 0.64}{3} \\ &= 0.69 \\ \mathcal{M}_{\mathcal{F}_{p_9}^{0.4}(DI)} &= \frac{\|\Phi_1\|_S^T(p_9) + \|\Phi_5\|_S^T(p_9) + \|\Phi_6\|_S^T(p_9)}{3} = \frac{0.7 + 0.61 + 0.63}{3} \\ &= 0.65\end{aligned}$$

Notice that the values obtained for the decisions $\bar{d}(p_9) = NI$ and $\bar{d}(p_9) = DI$ are quite similar. As a consequence, we cannot ensure which is the best decision for the new project according to the suitability method, resulting in the slight recommendation not to invest in it. In any case, we can discard the decision $\bar{d}(p_9) = I$ because the value obtained for it is significantly lower than for the other decisions.

Since we were unable to make a decision for project p_9 according to the suitability method, we must address this classification problem by using other methods proposed in this paper.

5.2. Application of the ε -Representativeness Method

Now, we will try to make a satisfactory decision for the project p_9 by applying the ε -representativeness method.

- R1. Once again, we set the threshold $\varepsilon = 0.7$ for the reasons presented in Example 3. Hence, by Definition 14 and Proposition 1, and taking into account Tables 9 and 12, we have that

$$\begin{aligned}\rho_{Dec(S)}^{0.7}(p_9, I) &= \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | 0.7 \leq \|\Phi\|_S^T(p_9), \Psi = (d, I)\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | 0.7 \leq \|\Phi\|_S^T(p_9)\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{\|\Phi_4\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi_4, \Psi_4)}{\sum_{\Phi \rightarrow \Psi \in \{\Phi_1 \rightarrow \Psi_1, \Phi_2 \rightarrow \Psi_2, \Phi_3 \rightarrow \Psi_3, \Phi_4 \rightarrow \Psi_4\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{0.7 \times 3.21}{0.7 \times 4.91 + 0.72 \times 3.09 + 0.7 \times 3.75 + 0.7 \times 3.21} \\ &= \frac{2.24}{10.53} = 0.21 \\ \rho_{Dec(S)}^{0.7}(p_9, NI) &= \frac{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | 0.7 \leq \|\Phi\|_S^T(p_9), \Psi = (d, NI)\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)}{\sum_{\{\Phi \rightarrow \Psi \in Dec(S) | 0.7 \leq \|\Phi\|_S^T(p_9)\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{\|\Phi_2\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi_2, \Psi_2) + \|\Phi_3\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi_3, \Psi_3)}{\sum_{\Phi \rightarrow \Psi \in \{\Phi_1 \rightarrow \Psi_1, \Phi_2 \rightarrow \Psi_2, \Phi_3 \rightarrow \Psi_3, \Phi_4 \rightarrow \Psi_4\}} \|\Phi\|_S^T(p_9) \cdot \text{supp}_S^T(\Phi, \Psi)} \\ &= \frac{0.72 \times 3.09 + 0.7 \times 3.75}{0.7 \times 4.91 + 0.72 \times 3.09 + 0.7 \times 3.75 + 0.7 \times 3.21} \\ &= \frac{4.85}{10.53} = 0.46 \\ \rho_{Dec(S)}^{0.7}(p_9, DI) &= 1 - \rho_{Dec(S)}^{0.7}(p_9, I) - \rho_{Dec(S)}^{0.7}(p_9, NI) \\ &= 1 - 0.21 - 0.46 = 0.33\end{aligned}$$

- R2. Since relevance differences exist between all the degrees of 0.7 representativeness, we conclude that this method recommends the decision $\bar{d}(p_9) = \text{NI}$ with a degree of 0.7 representativeness of 0.46; that is, it is suggested that this new project is not invested in. Therefore, the ε -representativeness method solves some situations in which the first one is not conclusive enough for the decision to be taken. Notice that, by definition, the values are “complementary”, so they are usually low but should be different.

Moreover, we can extract some more conclusions. From Lemma 1, we obtain the result that if the value ε satisfies

$$\max\{\sigma_{Dec(S)}(p_9, v) \mid v \in V_d \setminus \{\text{NI}\}\} = 0.7 < \varepsilon \leq \sigma_{Dec(S)}(p_9, \text{NI}) = 0.72$$

then $\rho_{Dec(S)}^\varepsilon(p_9, \text{I}) = \rho_{Dec(S)}^\varepsilon(p_9, \text{DI}) = 0$ and $\rho_{Dec(S)}^\varepsilon(p_9, \text{NI}) = 1$. Hence, few values of the threshold ε provide a clear classification for the project p_9 , showing the existing difficulty of deciding about it explained above. In addition, from Proposition 1 the project p_9 is ε -undecidable if ε satisfies

$$\max\{\sigma_{Dec(S)}(p_9, v) \mid v \in \{\text{I}, \text{NI}, \text{DI}\}\} = \max\{\sigma_{Dec(S)}(p_9, \text{NI})\} < \varepsilon$$

Finally, it can be checked that

$$\begin{aligned} \rho_{Dec(S)}^\varepsilon(p_9, \text{I}) < \rho_{Dec(S)}^\varepsilon(p_9, \text{NI}) < \rho_{Dec(S)}^\varepsilon(p_9, \text{DI}) & \quad \text{if } 0 \leq \varepsilon \leq 0.61 \\ \rho_{Dec(S)}^\varepsilon(p_9, \text{I}) < \rho_{Dec(S)}^\varepsilon(p_9, \text{DI}) < \rho_{Dec(S)}^\varepsilon(p_9, \text{NI}) & \quad \text{if } 0.61 < \varepsilon \leq 0.72 \end{aligned}$$

Therefore, given that the study is more accurate when ε is higher, these inequalities strengthen the idea that the best decision for the project p_9 is $\bar{d}(p_9) = \text{NI}$.

Although we have provided a solid decision for project p_9 , we consider it important to continue testing the other two methods to identify potential benefits and drawbacks.

5.3. Application of the α -Credibility Method

Next, the α -credibility method will be applied.

- C1. We assume the variables are related somehow. Hence, we consider the adjoint pair $(\&_P, I_P)$ to compute the degree of 0.79 credibility of each possible decision. In this way, from Definition 15 and Tables 9 and 12, for the decision $\bar{d}(p_9) = \text{I}$, it is obtained that

$$\begin{aligned} \theta_{Dec(S)}^{0.79}(p_9, \text{I}) &= \max\{\&_P(\|\Phi_4\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_4, \Psi_4)), \\ &\quad \&_P(\|\Phi_7\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_7, \Psi_7))\} \\ &= \max\{\&_P(0.7, 0), \&_P(0.455, 0.37)\} \\ &= 0.17 \end{aligned}$$

In a similar way, the degree of 0.79 credibility of the decisions $\bar{d}(p_9) = \text{NI}$ and $\bar{d}(p_9) = \text{DI}$ are computed.

$$\begin{aligned} \theta_{Dec(S)}^{0.79}(p_9, \text{NI}) &= \&_P(\|\Phi_2\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_2, \Psi_2)) = 0.72 \\ \theta_{Dec(S)}^{0.79}(p_9, \text{DI}) &= \&_P(\|\Phi_1\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_1, \Psi_1)) = 0.7 \end{aligned}$$

- C2. As a result, since $\theta_{Dec(S)}^{0.79}(p_9, \text{NI})$ and $\theta_{Dec(S)}^{0.79}(p_9, \text{DI})$ are very similar, we cannot make any decision for the project p_9 according to the α -credibility method. Moreover, it can be shown that the same degrees of 0.79 credibility of both decisions are obtained if the adjoint pairs $(\&_G, I_G)$ and $(\&_L, I_L)$ are considered, since from Table 3, $\mu_{S,\tau}^{0.79}(\Phi_2, \Psi_2) = \mu_{S,\tau}^{0.79}(\Phi_1, \Psi_1) = 1$. In conclusion, following this method, we can only discard the decision $\bar{d}(p_9) = \text{I}$ since its degree of 0.79 credibility is much lower than the others.

Consequently, we cannot make a decision for project p_9 by using the α -credibility method due to the existing similarity between the degrees of α -credibility of the two decisions. Therefore, a further study is needed.

5.4. Application of Support-Driven α -Credibility Method

Finally, we conclude the analysis of the classification of the new project p_9 by applying the support-driven α -credibility method.

- D1. First of all, we compute the degree of support-driven 0.79 credibility of the decisions $\bar{d}(p_9) = I$, $\bar{d}(p_9) = NI$ and $\bar{d}(p_9) = DI$. As in Example 5, we first suppose that the variables are related. Therefore, we will use the adjoint pair $(\&_G, I_G)$. Consequently, from Definition 16 and Tables 9 and 12, we obtain for the decision $\bar{d}(p_9) = I$ that

$$\begin{aligned}\delta_{Dec(S)}^{0.79}(p_9, I) &= \max\{\&_G(\|\Phi_4\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_4, \Psi_4), supp_S^{NT}(\Phi_4, \Psi_4)), \\ &\quad \&_G(\|\Phi_7\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_7, \Psi_7), supp_S^{NT}(\Phi_7, \Psi_7))\} \\ &= \max\{\&_G(0.7, 0, 0.65), \&_G(0.455, 0.37, 0.65)\} \\ &= 0.37\end{aligned}$$

Analogously, we compute the degree of support-driven 0.79 credibility of the decisions $\bar{d}(p_9) = NI$ and $\bar{d}(p_9) = DI$, which are the following:

$$\begin{aligned}\delta_{Dec(S)}^{0.79}(p_9, NI) &= \&_G(\|\Phi_8\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_8, \Psi_8), supp_S^{NT}(\Phi_8, \Psi_8)) = 0.64 \\ \delta_{Dec(S)}^{0.79}(p_9, DI) &= \&_G(\|\Phi_1\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_1, \Psi_1), supp_S^{NT}(\Phi_1, \Psi_1)) = 0.7\end{aligned}$$

- D2. Following the support-driven α -credibility method, although the decision to be made for the project p_9 is not very clear, the suggested decision contradicts the previous methods, since $\delta_{Dec(S)}^{0.79}(p_9, NI) < \delta_{Dec(S)}^{0.79}(p_9, DI)$, meaning that a delayed investment for the project p_9 is now recommended. In order to examine this change in more detail, we will assume that the variables are less related. As a consequence, we will consider another adjoint pair taking into account all the input values in the computation to obtain the degree of support-driven 0.79 credibility of these two decisions.
- D3. Now, following the same procedure as in Example 5, we suppose that the variables are partially related. With this purpose, we use the adjoint pair $(\&_P, I_P)$, obtaining

$$\begin{aligned}\delta_{Dec(S)}^{0.79}(p_9, NI) &= \&_P(\|\Phi_8\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_8, \Psi_8), supp_S^{NT}(\Phi_8, \Psi_8)) = 0.45 \\ \delta_{Dec(S)}^{0.79}(p_9, DI) &= \&_P(\|\Phi_1\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_1, \Psi_1), supp_S^{NT}(\Phi_1, \Psi_1)) = 0.7\end{aligned}$$

Now, the difference between $\delta_{Dec(S)}^{0.79}(p_9, NI)$ and $\delta_{Dec(S)}^{0.79}(p_9, DI)$ has increased notably, allowing us to conclude that the support-driven α -credibility method suggests the decision $\bar{d}(p_9) = DI$ for the project p_9 with a degree of support-driven 0.79 credibility of 0.7. Nevertheless, for a more detailed study, we will also study the classification of project p_9 by means of the adjoint pair $(\&_L, I_L)$, assuming that the variables are very independent. Thus, we obtain

$$\begin{aligned}\delta_{Dec(S)}^{0.79}(p_9, NI) &= \&_L(\|\Phi_8\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_8, \Psi_8), supp_S^{NT}(\Phi_8, \Psi_8)) = 0.35 \\ \delta_{Dec(S)}^{0.79}(p_9, DI) &= \&_L(\|\Phi_1\|_S^T(p_9), \mu_{S,\tau}^{0.79}(\Phi_1, \Psi_1), supp_S^{NT}(\Phi_1, \Psi_1)) = 0.7\end{aligned}$$

The difference between both degrees of support-driven 0.79 credibility has increased even more so that the decision $\bar{d}(p_9) = DI$ is the best to be made for the project p_9 . This fact is due to the fact that decision rules with higher NT support take the decision DI in the consequent due to the tolerance relation T_d , as can be checked in Table 9.

Consequently, and taking into account that similar conclusions were extracted in Example 5, we have to pay special attention to the introduction of the NT support in the

third classification method. For instance, in Definition 14, we included it in a weighted average that also depends on a threshold ε , giving more satisfactory results.

On the other hand, the α - T -credibility is a great notion for detecting inconsistencies in the rules and thus in the decision table, since it measures how reliable a decision rule is by comparing the objects similar to the antecedent of this rule with the objects similar to the first ones. Nevertheless, its consideration in the study of new objects may not be useful in decision tables in which the condition attributes are more granulated than the decision attributes, since small changes in the α cut may lead to drastic changes in the α - T -credibility. We will keep studying this notion in decision tables with a fuzzy decision attribute in the future.

6. Conclusions and Future Work

In this paper, we have faced the decision-making problem from the FRST framework. We have presented four different methods to classify new objects, each of them based on a novel notion, such as the degree of suitability of a decision, the degree of ε -representativeness of a decision, the degree of α -credibility of a decision, and the degree of support-driven α -credibility of a decision. We have introduced a detailed discussion about the advantages and disadvantages of each method, showing that each one solves problems of the previous method and that they can be combined and applied to the same data set. We have provided examples for illustrating each classification method and the developed theory, as well as carrying out a comparison between the proposed methods and the ones given in the literature. In particular, this comparison allows us to ensure that the introduced classification methods are richer alternatives or generalize the ones considered in [20,21,23–25]. Finally, we have studied the classification of a new object in a real case associated with a knowledge system for decision making on investments in a water conservancy project.

As future work, we will continue studying the decision-making process in the multi-adjoint framework, given its usefulness in studying complex data sets and proposing new methods to classify objects. Furthermore, we will study the notion of the credibility of a decision rule in decision tables in which all the attributes are fuzzy, including the decision attribute d . We will also study the credibility of decision rules considering fuzzy reducts [5] in their antecedents, given their relevant properties, allowing us to analyze possible relationships between the credibility and these reducts. Finally, we are also interested in studying this classification problem in other frameworks, such as Formal Concept Analysis.

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