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Non-degenerate evolution algebras ☆,☆☆

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ABSTRACT

In this paper we introduce a new invariant for a non-degenerate evolution algebra, which consists of an ordered sequence of evolution algebras of lower dimension, belonging all of them to a specific family. We use this invariant to propose a method to classify non-degenerate evolution algebras, and we apply it up to dimension 3. We also use it to describe the derivations of some families of evolution algebras and the degenerations of the variety of evolution algebras with square not greater than 1.

Contents

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0. Introduction

It seems that the notion of evolution algebras was first introduced in 2004 in the thesis of Tian (see also, his first paper about elementary algebraic properties of evolution algebras published together with Vojtěchovský in 2006 [30] and a book published in 2008 [29]). Evolution algebras are a new type of genetic algebras that describe the functioning of the non-mendelian genetic, including, for example, the asexual propagation or the asexual inheritance, as well as physical phenomena, both dynamic and kinematic. In geometry, evolution algebras can describe the motions of particles in a graph embedded in a 3-manifold. Also, some stochastic processes, such as Markov chains on countable state spaces, can be modelled using evolution algebras. Relations of evolution algebras with other structures and the history of studies of them in the first decade are given in a recently published survey [14].

In recent years, many different aspects of the evolution algebras have been considered by many authors [1–14,16–19,22–30]. There are works in which its structure is studied [6,7,9], a description of the evolution algebras up to dimension three [10], a classification of the simple evolution algebras up to dimension three [8] and a classification of the nilpotent evolution algebras up to dimension five [17]. Also, there are studies of some types of derivations of these algebras [11], different representations using graphs [10,17], and even applications to certain specific population problems, among many others.

The main purpose of this paper is to introduce an invariant for non-degenerate evolution algebras and to show how it can be used to construct a general method to classify non-degenerate evolution algebras and also, to provide an example of the application of this method for dimension two and three. It should be mentioned that all known (particular) classifications of 3-dimensional evolution algebras are given without defining non-isomorphic algebras [10,15]. The present classification gives a classification of non-isomorphic 3-dimensional non-degenerate evolution algebras. Additionally, we show some relations between this invariant and the derivations of the algebra, and, in particular, we describe the derivations of some families of evolution algebras. Our starting point is the decomposition of a finite-dimensional evolution algebra as a direct sum of adequate linear subspaces given in [1].

1. Definitions and preliminary results

Unlike other classes of non-associative algebras, the class of evolution algebras is not defined by a set of identities, instead, they are defined in the following way:

Definition 1. Let $\mathbb{F}$ be a field. An *evolution algebra* is an algebra $\mathbb{A}$ provided with a basis $\mathbf{B} = \{e_i \mid i \in \Lambda\}$ such that $e_i e_j = 0$ whenever $i \neq j$. Such a basis $\mathbf{B}$ is called a *natural basis*. Fixed a natural basis $\mathbf{B}$ in $\mathbb{A}$, the scalars $w_{ki} \in \mathbb{F}$ such that $e_i^2 = \sum_{k\in\Lambda} w_{ki} e_k$ will be called the *structure constants* of $\mathbb{A}$ *relative to* $\mathbf{B}$ and the matrix $\mathbf{M_B}(\mathbb{A}) := (w_{ki})$ is said to be the *structure matrix of* $\mathbb{A}$ *relative to* $\mathbf{B}$. Thus, every evolution algebra is uniquely determined by its structure matrix.

Any algebra in this paper will be finite-dimensional and over an algebraically closed base field $\mathbb{F}$ of characteristic different from 2. We will denote by $\mathbb{S}_n$ the symmetric group of all permutations of the set $\{1, \ldots, n\}$. We continue with the introduction of some definitions and results.

Definition 2. A *natural vector* in an evolution algebra $\mathbb{A}$ is a vector that can be extended into a natural basis. A *natural set* is a set of linearly independent vectors that can be extended into a natural basis of $\mathbb{A}$. Any (linear) subspace $\mathbb{E}$ of $\mathbb{A}$ generated by a natural set will be called an *extending evolution subspace* of $\mathbb{A}$ and the natural set will be called an *extending natural basis* of $\mathbb{E}$.

Definition 3. Given an algebra $\mathbb{A}$, the annihilator of the algebra is the ideal

$$\mathrm{Ann}(\mathbb{A}) = \{x \in \mathbb{A} : x\mathbb{A} + \mathbb{A}x = 0\}$$

We say an algebra is non-degenerate if it has a trivial annihilator.

The following remark is a well-known fact about non-degenerate evolution algebras.

Remark 4. Given a non-degenerate evolution algebra, then for every natural vector e we have $e^2 \neq 0$. Therefore, the structure matrix on any natural basis does not contain zero columns.

Theorem 5 *([10]). Let $\mathbb{A}$ be an evolution algebra and let $\mathbf{B} = \{e_1, \ldots, e_n\}$ be a natural basis of $\mathbb{A}$ with structure matrix $\mathbf{M_B}(\mathbb{A})$. Then for another natural basis $\mathbf{B}' = \{f_1, \ldots, f_n\}$ of $\mathbb{A}$ and $\mathbf{C} = (c_{ij})$ change of basis between the natural bases $\mathbf{B}'$ and $\mathbf{B}$, i.e., $f_j = \sum_i c_{ij} e_i$, then*

$$\mathbf{M_{B'}}(\mathbb{A}) = \mathbf{C}^{-1}\mathbf{M_B}(\mathbb{A})\mathbf{C}^{(2)},$$

where $\mathbf{C}^{(2)} = (c_{ij}^2)$.

Our classifying method parts from the decomposition below.

Theorem 6 *([1], Theorem 2.11). Let $\mathbb{A}$ be an evolution algebra. Then*

$$\mathbb{A} = \mathrm{Ann}(\mathbb{A}) \oplus \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r,$$

where $\mathbb{E}_1, \ldots, \mathbb{E}_r$ are extending evolution subspaces of $\mathbb{A}$ satisfying

$$\dim(\mathbb{E}_i^2) = 1,\ \mathbb{E}_i\mathbb{E}_j = 0 \text{ and } \dim(\mathbb{E}_i^2 + \mathbb{E}_j^2) = 2 \text{ for } i \neq j.$$

Moreover, if $\mathbb{A}$ is non-degenerate, the decomposition is unique (up to the permutation of the linear subspaces).

Definition 7. Let $\mathbb{A}$ be a non-degenerate evolution algebra. The unique, (up to permutation of the linear subspaces), decomposition

$$\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r,$$

with $\mathbb{E}_1, \ldots, \mathbb{E}_r$ extending evolution subspaces of $\mathbb{A}$ satisfying conditions in Theorem 6 will be called the standard extending evolution subspaces decomposition of $\mathbb{A}$.

1.1. The δ-index of a non-degenerate evolution algebra

In this subsection, we are going to define the first invariant that will be used in our classification method. It is a consequence of the following lemma.

Lemma 8. *Let $\mathbb{A}$, $\mathbb{A}'$ be two non-degenerate evolution algebras with standard extending evolution decompositions $\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r$ and $\mathbb{A}' = \mathbb{E}'_1 \oplus \ldots \oplus \mathbb{E}'_s$ respectively. If $\mathbb{A} \cong \mathbb{A}'$, then $r = s$ and $\phi(\mathbb{E}_i) = \mathbb{E}'_{\sigma(i)}$ for a permutation $\sigma \in \mathbb{S}_r$.*

Proof. Denote by $\phi : \mathbb{A} \to \mathbb{A}'$ an isomorphism. Then $\phi(\mathbb{E}_i)^2 = \phi(\mathbb{E}_i^2)$ has dimension 1. Also, $\phi(\mathbb{E}_i)^2 + \phi(\mathbb{E}_j)^2 = \phi(\mathbb{E}_i^2 + \mathbb{E}_j^2)$ for $i \neq j$ has dimension 2. Moreover, since $\phi(\mathbb{E}_i)\phi(\mathbb{E}_j) = \phi(\mathbb{E}_i\mathbb{E}_j) = 0$ for $i \neq j$ then $\mathbb{A}' = \phi(\mathbb{E}_1) \oplus \ldots \oplus \phi(\mathbb{E}_r)$ is a standard extending evolution decomposition of $\mathbb{A}'$. By the uniqueness of this decomposition we get $r = s$ and $\phi(\mathbb{E}_i) = \mathbb{E}'_{\sigma(i)}$ for some permutation $\sigma \in \mathbb{S}_r$. $\square$

By Theorem 6, the following index is well-defined.

Definition 9. Let $\mathbb{A}$ be a non-degenerate evolution algebra with standard extending subspaces decomposition $\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r$. Then we define the δ-index of $\mathbb{A}$ as the sequence

$$\delta(\mathbb{A}) := \big(\dim(\mathbb{E}_1), \dim(\mathbb{E}_2), \ldots, \dim(\mathbb{E}_r)\big)$$

with $\dim(\mathbb{E}_i) \geq \dim(\mathbb{E}_{i+1})$.

Corollary 10. *Let $\mathbb{A}$, $\mathbb{A}'$ be two isomorphic non-degenerate evolution algebras. Then $\delta(\mathbb{A}) = \delta(\mathbb{A}')$.*

The converse of Corollary 10 does not hold as the next example shows.

Example 11. Consider the non-degenerate evolution algebras defined by

$$\mathbb{A} : \; e_1^2 = e_1, \; e_2^2 = e_2 \text{ and } \mathbb{A}' : \; e_1^2 = e_2, \; e_2^2 = e_1.$$

We have that $\delta(\mathbb{A}) = \delta(\mathbb{A}') = (2, 1)$, but $\mathbb{A}$ is not isomorphic to $\mathbb{A}'$.

Note that every perfect evolution algebra has δ-index equal to $(1, \ldots, 1)$. Recall that every simple algebra is perfect. Regarding the changes of natural bases, we can observe the following remark.

Remark 12. Given a non-degenerate evolution algebra $\mathbb{A}$ with $\delta(\mathbb{A}) = (\delta_1, \ldots, \delta_r)$, and two natural bases $\mathbf{B} = \mathbf{B_1} \cup \mathbf{B_2} \cup \ldots \cup \mathbf{B_r}$ and $\mathbf{B}' = \mathbf{B}'_\mathbf{1} \cup \mathbf{B}'_\mathbf{2} \cup \ldots \cup \mathbf{B}'_\mathbf{r}$ of $\mathbb{A}$ given by the standard extending subspaces decomposition of $\mathbb{A}$. Then the change of basis matrix has the following block form:

$$\begin{pmatrix} \mathbf{D}_1 & 0 & 0 & 0 \\ 0 & \mathbf{D}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{D}_r \end{pmatrix} \cdot \mathbf{P}$$

where any $\mathbf{D}_i$ is a $\delta_i \times \delta_i$ block of natural vectors of $\mathbb{A}$ (as columns) and $\mathbf{P}$ is a permutation matrix.

Observe that if in the above remark, we have that $\delta_i = 1$ for every $1 \leq i \leq r$, then the only changes of natural basis are products of diagonal matrices by permutations, which is indeed a group with the composition that has been previously studied in [10].

Theorem 13. *Let $\mathbb{A}$ be a non-degenerate evolution algebra with $\delta(\mathbb{A}) = (\delta_1, \dots, \delta_r)$. Then there is a natural basis $\mathbf{B} = \{e_1, \dots, e_n\}$ of $\mathbb{A}$ such that the structure matrix $\mathbf{M_B}(\mathbb{A})$ has the columns form*

$$\mathbf{M_B}(\mathbb{A}) = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r),$$

in such a way that $\dim(\mathbb{F}v_i + \mathbb{F}v_j) = 2$ for $i \neq j$.

Proof. Let $\mathbb{A} = \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_r$ be the standard extending subspaces decomposition of $\mathbb{A}$ and suppose $\dim(\mathbb{E}_i) \geq \dim(\mathbb{E}_{i+1})$. For any $i \in \{1, \dots, r\}$ consider $\mathbf{B}_i = \left\{e^i_1, \dots, e^i_{\delta_i}\right\}$ an extending natural basis of $\mathbb{E}_i$. Then $\mathbf{B} = \cup_{i\in\{1,\dots,r\}} \mathbf{B}_i$ is a natural basis of $\mathbb{A}$. For this basis we have the following structure matrix of $\mathbb{A}$ by columns:

$$\mathbf{M_B}(\mathbb{A}) = (v_1 \;\; \lambda_{1,2}v_1 \dots \lambda_{1,\delta_1}v_1 | v_2 \;\; \lambda_{2,2}v_2 \dots \lambda_{2,\delta_2}v_2 | \dots\dots | v_r \;\; \lambda_{r,2}v_r \dots \lambda_{r,\delta_r}v_r)$$

with any $\lambda_{r,s} \in \mathbb{F}$ and such that $\dim(\mathbb{F}v_i + \mathbb{F}v_j) = 2$ for $i \neq j$.

Now, consider a new set $\mathbf{B}'$ containing the following vectors:

- $f^i_1 = e^i_1$ for every $1 \leq i \leq r$.
- $f^i_j = \lambda_{i,j}^{-\frac{1}{2}} e^i_j$ for every $1 \leq i \leq r$ and $2 \leq j \leq \delta_i$.

From, Remark 12 the set $\mathbf{B}'$ is a natural basis of $\mathbb{A}$, and from Theorem 5 we have

$$\mathbf{M_{B'}}(\mathbb{A}) = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r),$$

which completes the proof. □

By Theorem 13, fixed a basis $\mathbf{B}$, every non-degenerate evolution algebra $\mathbb{A}$ with $\delta(\mathbb{A}) = (\delta_1, \dots, \delta_r)$ is isomorphic to an algebra $\mathbb{A}'$ with structure matrix columns

$$\mathbf{M_B}(\mathbb{A}') = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r),$$

satisfying that $\dim(\mathbb{F}v_i + \mathbb{F}v_j) = 2$ for $i \neq j$. Therefore, to obtain a classification of non-degenerate evolution algebras, it will be enough to consider those with a structure matrix of the above form. Moreover, the changes of natural basis that preserve this shape are those with $\mathbf{D}_i^{\mathrm{T}}\mathbf{D}_i = \lambda_i \mathbf{1}_{\delta_i}$ from Remark 12, for some $\lambda_i \in \mathbb{F}$ and where $\mathbf{1}_{\delta_i}$ denotes the $\delta_i \times \delta_i$ identity matrix.

1.2. The Δ-trace of a non-degenerate evolution algebra

Let $\mathbb{A}$ be a non-degenerate evolution algebra with standard extending subspaces decomposition $\mathbb{A} = \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_r$. Our next aim is to introduce a sequence of r evolution algebras associated to this decomposition. For any $i \in \{1, \dots, r\}$, denote by

$$\pi_{\mathbb{E}_i} : \mathbb{A} \to \mathbb{E}_i$$

the projection map onto $\mathbb{E}_i$. Then we consider the algebra $\mathbb{A}_i$ defined over the vector space $\mathbb{E}_i$ by the product

$$\mathbb{A}_i : x \cdot_{\mathbb{A}_i} y = \pi_{\mathbb{E}_i}(x \cdot_{\mathbb{A}} y)$$

for any $x, y \in \mathbb{E}_i$. We call to $\mathbb{A}_i$ the *algebra associated to* $\mathbb{E}_i$.

Remark 14. Let $\mathbb{A}$ be a non-degenerate evolution algebra with standard extending subspaces decomposition $\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r$. Then, for any $i \in \{1, \ldots, r\}$, the algebra $\mathbb{A}_i$ associated to $\mathbb{E}_i$ satisfies:

(1) $\mathbb{A}_i$ is an evolution algebra.
(2) $\dim \mathbb{A}_i^2 \leq 1$.
(3) The structure matrix of $\mathbb{A}_i$ in any natural basis has proportional columns (see Theorem 13).
(4) $\mathbb{A}_i$ is either the zero product algebra $\mathbf{O_{n_i}}$ or a non-degenerate evolution algebra with $\dim(\mathbb{A}_i^2) = 1$.

Definition 15. Let $\mathbb{A}$ be a non-degenerate evolution algebra with standard extending subspaces decomposition $\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r$, and consider for any $\mathbb{E}_i$ its evolution associated algebra $\mathbb{A}_i$. We call to the sequence of algebras

$$(\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_r),$$

(unique up to permutation of the algebras), the standard sequence of algebras of $\mathbb{A}$.

Example 16. Let $\mathbb{A}$ be an evolution algebra and let $\mathbf{B} = \{e_1, \ldots, e_5\}$ be a natural basis of $\mathbb{A}$ such that the structure matrix of $\mathbb{A}$ relative to the basis $\mathbf{B}$ is:

$$\mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 \\ 2 & 4 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The unique decomposition of $\mathbb{A}$ into extending evolution subspaces is $\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$, where $\mathbb{E}_1 = \langle e_1, e_3 \rangle$, $\mathbb{E}_2 = \langle e_2, e_5 \rangle$ and $\mathbb{E}_3 = \langle e_4 \rangle$. Therefore, $\delta(\mathbb{A}) = (2, 2, 1)$. Moreover, the standard sequence of algebras $(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$ of $\mathbb{A}$ is if formed by the evolution algebras with structure matrices:

$$\mathbf{M}_{\mathbf{B}_{\mathbb{E}_1}}(\mathbb{A}_1) = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, \qquad \mathbf{M}_{\mathbf{B}_{\mathbb{E}_2}}(\mathbb{A}_2) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_{\mathbf{B}_{\mathbb{E}_3}}(\mathbb{A}_3) = (0),$$

where $\mathbf{B}_{\mathbb{E}_1} = \{e_1, e_3\}$, $\mathbf{B}_{\mathbb{E}_2} = \{e_2, e_5\}$ and $\mathbf{B}_{\mathbb{E}_3} = \{e_4\}$.

By Remark 14, we are interested in knowing non-degenerate evolution algebras $\mathbb{A}$ such that $\dim(\mathbb{A}^2) = 1$. For any natural basis $\mathbf{B}$ of such an algebra $\mathbb{A}$, the structure matrix of $\mathbb{A}$ has the columns form

$$\mathbf{M_B}(\mathbb{A}) = (v \quad \lambda_2 v \quad \ldots \quad \lambda_n v),$$

where v is a non-zero column and any $\lambda_r \in \mathbb{F}$, also being $\delta(\mathbb{A}) = (n)$.

The classification of the n-dimensional non-degenerate evolution algebras with one-dimensional square was given in [2]. We provide a brief proof with the explicit construction of the algebras in this class in a convenient basis.

Theorem 17. *Let $\mathbb{A}$ be an n-dimensional non-degenerate evolution algebra with $\dim(\mathbb{A}^2) = 1$. We have*

(1) *if $n = 1$ then $\mathbb{A}$ is isomorphic to $\mathbf{E}_1 : e_1^2 = e_1$.*
(2) *if $n \geq 2$ then:*
  (a) *if $(\mathbb{A}^2)^2 \neq 0$, then $\mathbb{A}$ is isomorphic to $\mathbf{E}_n : e_1^2 = e_2^2 = \ldots = e_n^2 = e_1$.*
  (b) *if $(\mathbb{A}^2)^2 = 0$, then $\mathbb{A}$ is isomorphic to $\mathbf{I}_n : e_1^2 = e_2^2 = \ldots = e_n^2 = e_1 + \mathbf{i}e_2$.*

Proof. Let $\mathbb{A}$ be a non-degenerate evolution algebra with $\dim(\mathbb{A}^2) = 1$. The case $n = 1$ is trivial, therefore we will suppose $n \geq 2$. Since $\mathbb{F}$ is algebraically closed, there is a natural basis of $\mathbb{A}$, $\mathbf{B} = \{e_1, \dots, e_n\}$, such that $e_i^2 = v = v_1 e_1 + \dots + v_n e_n$ by Theorem 13.

(a) Suppose $(\mathbb{A}^2)^2 \neq 0$. Now, if $\lambda = \sum \lambda_i e_i$ is an idempotent, then $\lambda^2 = \sum \lambda_i^2 e_i^2 = \sum \lambda_i^2 v = \sum \lambda_i e_i$. Matching the scalars we have the following system of equations:

$$(\sum \lambda_i^2) v_1 = \lambda_1, \quad (\sum \lambda_i^2) v_2 = \lambda_2, \quad \dots, \quad (\sum \lambda_i^2) v_n = \lambda_n,$$

which has solution if and only if $v^2 \neq 0$, and the solution is $\lambda_i = \frac{v_i}{\sum v_i^2}$. Since $(\mathbb{A}^2)^2 \neq 0$ implies $v^2 \neq 0$, we have found a natural idempotent λ. From here, constructing the change of basis is straightforward.

(b) Suppose $(\mathbb{A}^2)^2 = 0$, then $v^2 = 0$. Let $u \in \mathbb{A}$ be such that $uv = v$ and define a vector $w := -\frac{1}{2}u^2 + u$. Clearly $wv = v$ and $w^2 = 0$. Since $\dim\big(\ker(l_v)\big) = n - 1$, where l_v denotes left multiplication by v, there exists a set $\{\lambda_3, \dots, \lambda_n\}$ of linearly independent orthogonal vectors such that $\lambda_i v = 0$. Assume without loss of generality that $\lambda_i^2 = v$. Define $z_i := \lambda_i - \lambda_i w$, then $\{z_3, \dots, z_n\}$ is a set of linearly independent orthogonal vectors such that $z_i^2 = v$ and $z_i v = z_i w = 0$. Finally, construct a linear map $\phi : \mathbb{A} \to \mathbf{I}_n$ such that

$$\begin{aligned} \phi(v) &= e_1 + \mathbf{i} e_2, \\ \phi(w) &= \tfrac{1}{2} e_1 - \tfrac{\mathbf{i}}{2} e_2, \\ \phi(z_i) &= e_i, \text{ for } i = 3, \dots, n. \end{aligned}$$

It is easy to verify that this is the isomorphism that we are looking for. □

Theorem 17 allows us to introduce an order relation in the class of non-degenerate evolution algebras with $\dim(\mathbb{A}^2) \leq 1$ as follows.

Definition 18. Let $\mathbb{A}$ and $\mathbb{A}'$ be two non-degenerate evolution algebras such that $\dim(\mathbb{A}^2) \leq 1$, $\dim((\mathbb{A}')^2) \leq 1$. We will say that $\mathbb{A} \geq_* \mathbb{A}'$ if:

- Either $\dim(\mathbb{A}) \geq \dim(\mathbb{A}')$ or
- $\dim(\mathbb{A}) = \dim(\mathbb{A}')$ and
  - (i) $\mathbb{A}$ is isomorphic to $\mathbf{O}_n$ and $\mathbb{A}'$ is isomorphic to $\mathbf{O}_n$ or to $\mathbf{E}_n$ or to $\mathbf{I}_n$.
  - (ii) $\mathbb{A}$ is isomorphic to $\mathbf{E}_n$ and $\mathbb{A}'$ is isomorphic to $\mathbf{E}_n$ or to $\mathbf{I}_n$.
  - (iii) $\mathbb{A}$ is isomorphic to $\mathbf{I}_n$ and $\mathbb{A}'$ is also isomorphic to $\mathbf{I}_n$.

Theorem 17 also allows us to introduce the next notation:

Notation 19. Let $\mathbb{A}$ be an n-dimensional non-degenerate or zero evolution algebra such that $\dim(\mathbb{A}^2) \leq 1$. We denote by

$$\overline{\mathbb{A}} = \begin{cases} \mathbf{O}_n & \text{if } \mathbb{A} \text{ is isomorphic to } \mathbf{O}_n \\ \mathbf{E}_n & \text{if } \mathbb{A} \text{ is isomorphic to } \mathbf{E}_n \\ \mathbf{I}_n & \text{if } \mathbb{A} \text{ is isomorphic to } \mathbf{I}_n \end{cases}$$

Let us now introduce the main invariant in our development.

Definition 20. Let $\mathbb{A}$ be a non-degenerate evolution algebra with standard sequence of algebras $(\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_r)$ in such a way that $\mathbb{A}_i \geq_* \mathbb{A}_{i+1}$. Then we define the Δ-trace of $\mathbb{A}$ as

$$\Delta(\mathbb{A}) := (\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r}).$$

Example 21. Let $\mathbb{A}$ be the evolution algebra in Example 16. Its standard sequence of algebras was $(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$ with

$$\mathbf{M}_{\mathbf{B}_{\mathbb{E}_1}}(\mathbb{A}_1) = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, \qquad \mathbf{M}_{\mathbf{B}_{\mathbb{E}_2}}(\mathbb{A}_2) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_{\mathbf{B}_{\mathbb{E}_3}}(\mathbb{A}_3) = (0).$$

Since $\mathbb{A}_1$ and $\mathbb{A}_2$ are both isomorphic to $\mathbf{E}_2$ and $\mathbb{A}_3$ is isomorphic to $\mathbf{O_1}$, we have that

$$\Delta(\mathbb{A}) := (\mathbf{E}_2, \mathbf{E}_2, \mathbf{O_1}).$$

This notion generalizes in some sense the notion of the diagonal subspace of an evolution algebra with a unique natural basis (for example, see [8]) into the context of non-degenerate evolution algebras. Note that evolution algebras with a unique natural basis are precisely the non-degenerate evolution algebras with δ-index a sequence of ones, sometimes called 2LI (see [1]). The following result shows that Δ is an invariant up to isomorphisms.

Theorem 22. *Let $\mathbb{A}$ and $\mathbb{A}'$ be two isomorphic non-degenerate evolution algebras. Then $\Delta(\mathbb{A}) = \Delta(\mathbb{A}')$.*

Proof. Let us denote by $\phi : \mathbb{A} \to \mathbb{A}'$ the isomorphism between these algebras and write by

$$\mathbb{A} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_r$$

the standard extending subspaces decomposition $\mathbb{A}$. By Theorem 8 we have

$$\mathbb{A}' = \phi(\mathbb{E}_1) \oplus \ldots \oplus \phi(\mathbb{E}_r).$$

Consider the algebras $\mathbb{A}_i$ and $\mathbb{A}'_i$, where $\mathbb{A}'_i : x \cdot_{\mathbb{A}'_i} y = \pi_{\phi(\mathbb{E}_i)}(x \cdot_{\mathbb{A}'} y)$. Note that they have the same dimension δ_i. Now, since $x \cdot_{\mathbb{A}} y = \phi^{-1}(\phi(x) \cdot_{\mathbb{A}'} \phi(y))$ for $x, y \in \mathbb{E}_i$, we have

$$x \cdot_{\mathbb{A}_i} y = \pi_{\mathbb{E}_i}(x \cdot_{\mathbb{A}} y) = \pi_{\mathbb{E}_i}\big(\phi^{-1}(\phi(x) \cdot_{\mathbb{A}'} \phi(y))\big) = \phi^{-1}\big(\pi_{\phi(\mathbb{E}_i)}(\phi(x) \cdot_{\mathbb{A}'} \phi(y))\big) = \phi^{-1}\big(\phi(x) \cdot_{\mathbb{A}'_i} \phi(y)\big).$$

Then, $\mathbb{A}_i^2 = 0$ if and only if $(\mathbb{A}'_i)^2 = 0$, and they are the algebra $\mathbf{O}_{\delta_i}$. Moreover, for $x, y, z, w \in \mathbb{E}_i$, we obtain

$$\begin{aligned}
(x \cdot_{\mathbb{A}_i} y) \cdot_{\mathbb{A}_i} (z \cdot_{\mathbb{A}_i} w) &= \pi_{\mathbb{E}_i}(\pi_{\mathbb{E}_i}(x \cdot_{\mathbb{A}} y) \cdot_{\mathbb{A}} \pi_{\mathbb{E}_i}(z \cdot_{\mathbb{A}} w)) \\
&= \pi_{\mathbb{E}_i}(\phi^{-1}(\phi(\pi_{\mathbb{E}_i}(\phi^{-1}(\phi(x) \cdot_{\mathbb{A}'} \phi(y)))) \cdot_{\mathbb{A}'} \phi(\pi_{\mathbb{E}_i}(\phi^{-1}(\phi(z) \cdot_{\mathbb{A}'} \phi(w)))))) \\
&= \phi^{-1}(\pi_{\phi(\mathbb{E}_i)}(\phi(\phi^{-1}(\pi_{\phi(\mathbb{E}_i)}(\phi(x) \cdot_{\mathbb{A}'} \phi(y)))) \cdot_{\mathbb{A}'} \phi(\phi^{-1}(\pi_{\phi(\mathbb{E}_i)}(\phi(z) \cdot_{\mathbb{A}'} \phi(w)))))) \\
&= \phi^{-1}(\pi_{\phi(\mathbb{E}_i)}(\pi_{\phi(\mathbb{E}_i)}(\phi(x) \cdot_{\mathbb{A}'} \phi(y)) \cdot_{\mathbb{A}'} \pi_{\phi(\mathbb{E}_i)}(\phi(z) \cdot_{\mathbb{A}'} \phi(w)))) \\
&= \phi^{-1}((\phi(x) \cdot_{\mathbb{A}'_i} \phi(y)) \cdot_{\mathbb{A}'_i} (\phi(z) \cdot_{\mathbb{A}'_i} \phi(w))).
\end{aligned}$$

Then, $(\mathbb{A}_i^2)^2 = 0$ if and only if $((\mathbb{A}'_i)^2)^2 = 0$, and they are isomorphic to the algebra $\mathbf{I}_{\delta_i}$ from Theorem 17. Otherwise, they are isomorphic to $\mathbf{E}_{\delta_i}$. □

From now on, if we give the Δ-trace of an evolution algebra, we are assuming that it is non-degenerate.

2. The classification method

In the previous section, we introduced the Δ-trace of a non-degenerate evolution algebra. In this section, we are going to construct a classification method of non-degenerate evolution algebras based on this invariant. To achieve this, the following result is essential.

Theorem 23. *Let $\mathbb{A}$ be a non-degenerate evolution algebra over an algebraically closed field $\mathbb{F}$ with $\Delta(\mathbb{A}) := (\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$. Then there exists a natural basis $\mathbf{B}$ of $\mathbb{A}$ such that the structure matrix of $\mathbb{A}$ respect to $\mathbf{B}$ has the following diagonal blocks:*

$$\mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} \overline{\mathbb{A}_1} & * & * & * \\ * & \overline{\mathbb{A}_2} & * & * \\ * & * & \ddots & * \\ * & * & * & \overline{\mathbb{A}_r} \end{pmatrix}.$$

Also, $\mathbf{M_B}(\mathbb{A})$ has the columns form

$$\mathbf{M_B}(\mathbb{A}) = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r).$$

Proof. Consider $\mathbb{A} = \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_r$, the extending evolution subspaces decomposition of $\mathbb{A}$. By Theorem 13, for any $i \in \{1, \dots, r\}$ there is a natural basis $\mathbf{B}'_i = \{e_{i,1}, \dots, e_{i,k_i}\}$ of $\mathbb{E}_i$ such that the structure matrix of $\mathbb{A}$ respect to the basis $\mathbf{B}' := \cup_{i=1}^r \mathbf{B}'_i$ has the columns form

$$\mathbf{M_{B'}}(\mathbb{A}) = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r)$$

with $\dim(\mathbb{F}v_i + \mathbb{F}v_j) = 2$ when $i \neq j$.

Since, for any $i \in \{1, \dots, r\}$, the algebra $\mathbb{A}_i$ associated to the linear subspace $\mathbb{E}_i$ is isomorphic to an algebra $\mathbf{C}_i \in \{\mathbf{O}_{k_i}, \mathbf{E}_{k_i}, \mathbf{I}_{k_i}\}$, there exists a change of natural basis ϕ_i from $\mathbb{A}_i$ to $\mathbf{C}_i$ as given in Theorem 17) when $\mathbf{C}_i \in \{\mathbf{E}_{k_i}, \mathbf{I}_{k_i}\}$. In case $\mathbf{C}_i = \mathbf{O}_{k_i}$ we define $\phi_i = \mathbf{1}_{k_i}$. From here, we get for any $i \in \{1, \dots, r\}$, a basis $\mathbf{B}_i = \{\phi(e_{i,1}), \dots, \phi(e_{i,k_i})\}$ of $\mathbf{E}_i$.

Consider now the basis $\mathbf{B}$ of $\mathbb{A}$ given by $\mathbf{B} = \cup_{i=1}^r \mathbf{B}_i$. This is a natural basis of $\mathbb{A}$ in satisfying that the structure matrix of $\mathbb{A}$ with respect to it is

$$\mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} \overline{\mathbb{A}_1} & * & * & * \\ * & \overline{\mathbb{A}_2} & * & * \\ * & * & \ddots & * \\ * & * & * & \overline{\mathbb{A}_r} \end{pmatrix}.$$

Also, we get as in Theorem 13 that $\mathbf{M_B}(\mathbb{A})$ has the columns form

$$\mathbf{M_B}(\mathbb{A}) = (v_1 \dots v_1 | v_2 \dots v_2 | \dots\dots | v_r \dots v_r). \quad \square$$

At this point, fixed a basis $\mathbf{B}$, to obtain a classification of the non-degenerate evolution algebras of a given dimension n it is enough to classify the evolution algebras $\mathbb{A}$ with structure matrices as in Theorem 23.

Definition 24. We denote by $\mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$ the set of non-degenerate evolution algebras $\mathbb{A}$ with standard sequence of algebras $(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$.

Remark 25. The following assertions extend the [10, Remark 3.1]:

(i) Any isomorphism between two algebras $\mathbb{A}$ and $\mathbb{A}'$ in $\mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$ is of the form in Remark 12 in such a way that $\mathbf{D}_i$ is an automorphism of $\overline{\mathbb{A}_i}$ with $\mathbf{D}_i^{\mathrm{T}} \mathbf{D}_i = \lambda_i \mathbf{1}_{\delta_i}$, and $\mathbf{P}$ permutes blocks i and j of the diagonal if and only if $\overline{\mathbb{A}_i} = \overline{\mathbb{A}_j}$.

Observe that this isomorphism does not depend on the fixed algebras $\mathbb{A}$ and $\mathbb{A}'$ in $\mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$.

(ii) The set of all these linear automorphisms that induce isomorphisms between two algebras in $\mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$ is a group with the composition denoted by

$$\mathcal{G}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r}).$$

This group has the subgroups

$$\mathcal{G}_{\mathbf{D}}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$$

of all the block diagonal (by means of $\mathbf{D}_i$), matrices, and

$$\mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$$

of all the permutations of blocks i and j when $\overline{\mathbb{A}_i} = \overline{\mathbb{A}_j}$.

Any element in $\mathcal{G}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$ is the composition of one element in $\mathcal{G}_{\mathbf{D}}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$ and one element in $\mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$.

(iii) We have the action

$$\mathcal{G}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r}) \times \mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r}) \to \mathcal{E}(\overline{\mathbb{A}_1}, \dots, \overline{\mathbb{A}_r})$$
$$(\phi, \mathbf{M_B}(\mathbb{A})) \mapsto \phi^{-1}\mathbf{M_B}(\mathbb{A})\phi^{(2)}.$$

At this point, we can build a method to classify the non-degenerate evolution algebras. However, this method will require the group of automorphisms of the algebras $\mathbf{E_n}$ and $\mathbf{I}_n$.

Theorem 26. *The automorphisms group of* $\mathbf{E}_n$ *consists of the elements* ϕ *of the form:*

$$\phi = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & v_{22} & v_{23} & \dots & v_{2n} \\ 0 & v_{32} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_{n2} & v_{n3} & \dots & v_{nn} \end{pmatrix},$$

where $\sum_{k=2}^{n} v_{ki}^2 = 1$ *and* $\sum_{k=2}^{n} v_{ki}v_{kj} = 0$ *for* $i \neq j$. *That is* $\phi^{\mathrm{T}}\phi = \mathbf{1}_n$.

Proof. Let $\phi \in \mathrm{Aut}(\mathbf{E}_n)$, the group of automorphisms of $\mathbf{E}_n$, and denote $\phi(e_i) = v_{1i}e_1 + \dots + v_{ni}e_n$, then

$$\phi(e_i)\phi(e_i) = (v_{1i}^2 + \dots + v_{ni}^2)e_1.$$

On the one hand, for $i = 1$:

$$v_{11}e_1 + \dots + v_{n1}e_n = \phi(e_1) = \phi(e_1^2) = \phi(e_1)\phi(e_1) = (v_{11}^2 + \dots + v_{n1}^2)e_1.$$

Therefore $v_{i1} = 0$ for $i > 1$ and $v_{11}^2 = v_{11}$, then $\phi(e_1) = e_1$.

On the other hand, $\phi(e_i)\phi(e_i) = \phi(e_i^2) = \phi(e_1) = e_1$ implies $(v_{1i}^2 + \dots + v_{ni}^2)e_1 = e_1$, hence

$$(v_{1i}^2 + \dots + v_{ni}^2) = 1.$$

Finally, since $\phi(e_1)\phi(e_i) = e_1\phi(e_i) = 0$, then $v_{1i} = 0$. The converse is a straightforward verification. □

Theorem 27. *The automorphisms group of* $\mathbf{I}_n$ *consists of the elements of the form:*

$$\phi = \begin{pmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\ \mathbf{i}v_{11} - \mathbf{i} & \mathbf{i}v_{12} + 1 & v_{23} & \dots & v_{2n} \\ v_{31} & \mathbf{i}v_{31} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & \mathbf{i}v_{n1} & v_{n3} & \dots & v_{nn} \end{pmatrix},$$

where $v_{1i}^2 + \ldots + v_{ni}^2 = v_{11} + \mathbf{i}v_{12}$*, and* $\sum_{k=1}^{n} v_{ki}v_{kj} = 0$ *for* $i \neq j$*. That is* $\phi^{\mathrm{T}}\phi = (v_{11} + \mathbf{i}v_{12})\,\mathbf{1}_n$*.*

Proof. Suppose $\phi \in \mathrm{Aut}(\mathbf{I}_n)$ and denote $\phi(e_i) = v_{1i}e_1 + \ldots + v_{ni}e_n$, then

$$\phi(e_i)\phi(e_i) = (v_{1i}^2 + \ldots + v_{ni}^2)(e_1 + \mathbf{i}e_2).$$

We have $\phi(e_i)\phi(e_i) = \phi(e_i^2) = \phi(e_1 + \mathbf{i}e_2) = \phi(e_1) + \mathbf{i}\phi(e_2)$, hence

$$v_{k2} = \mathbf{i}v_{k1}, \text{ for } k \geq 3.$$

Also, $(v_{1i}^2 + \ldots + v_{ni}^2)(e_1 + \mathbf{i}e_2) = (v_{11} + \mathbf{i}v_{12})e_1 + (v_{21} + \mathbf{i}v_{22})e_2$. From here, we can write

$$(v_{1i}^2 + \ldots + v_{ni}^2) = v_{11} + \mathbf{i}v_{12} \quad \text{ and } \quad \mathbf{i}(v_{1i}^2 + \ldots + v_{ni}^2) = v_{21} + \mathbf{i}v_{22}$$

Then, it follows

$$\mathbf{i}v_{11} - v_{12} = v_{21} + \mathbf{i}v_{22} \text{ and } v_{1i}^2 + \ldots + v_{ni}^2 = v_{11} + \mathbf{i}v_{12}.$$

For $i = 1, 2$, we obtain the following system of equations:

$$\begin{aligned} v_{11}^2 + v_{21}^2 + v_{31}^2 + \ldots + v_{n1}^2 &= v_{11} + \mathbf{i}v_{12}, \\ v_{12}^2 + v_{22}^2 - v_{31}^2 - \ldots - v_{n1}^2 &= v_{11} + \mathbf{i}v_{12}, \\ \mathbf{i}v_{11} - v_{12} &= v_{21} + \mathbf{i}v_{22}. \end{aligned}$$

From where $v_{21} = \mathbf{i}v_{11} - \mathbf{i}$ and $v_{22} = \mathbf{i}v_{12} + 1$. The converse is a straightforward verification. □

In order to describe the classes of non-degenerate evolution algebras, we are going to use the notation introduced in [17], since it allows us to study the action

$$\mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r}) \times \mathcal{E}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r}) \to \mathcal{E}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$$

in a comfortable way.

Following [17], we recall that given an n-dimensional evolution algebra $\mathbb{A}$ with structure matrix $\mathbf{M}_{\mathbf{B}}(\mathbb{A}) := (w_{ki})$ respect to the natural basis $\mathbf{B}$, the graph (V, E) with $V = \{1, 2, \ldots, n\}$ and $E = \{(i, j) \in V \times V : w_{i,j} \neq 0\}$ is called *the graph associated to* $\mathbb{A}$ *respect to* ***B***. If we label the graph (V, E) with the map $\omega : E \to \mathbb{F}$ given by $\omega((i, j)) = w_{i,j}$ we get the so called *the weighted graph associated to* $\mathbb{A}$ *respect to* ***B***.

Observe that given a permutation $\mathbf{P} \in \mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$, we have that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$ and $\mathbf{P}^{(2)} = \mathbf{P}$. Therefore, the group $\mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$ acts on $\mathcal{E}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$ by conjugation. Recall that the class of a graph modulo isomorphisms contains all permutations of the vertices and it can be seen as a representative graph removing the vertices labels. Note that graph isomorphisms act on the adjacency matrix by conjugation. So, it is enough to consider the graph associated to $\mathbb{A}$ respect to $\mathbf{B}$ without labeled vertices, to get a representation of the algebra $\mathbb{A}$ module the action of $\mathcal{G}_{\mathbf{P}}(\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$.

We have obtained a method to classify n-dimensional non-degenerate evolution algebras $\mathbb{A}$. This consists in following the next steps:

(1) Fix the possible values of $\delta(\mathbb{A})$ (expressions of n as sum of natural numbers) and $\Delta(\mathbb{A})$.
(2) For any value $\mathfrak{s} = \Delta(\mathbb{A})$, construct the set $\mathcal{E}(\mathfrak{s})$ which elements are all of the possible $\mathbb{A}$ such that $\Delta(\mathbb{A}) = \mathfrak{s}$.
(3) For any $\mathfrak{s}$ and $\mathcal{E}(\mathfrak{s})$.
   (a) Compute the group $\mathcal{G}_{\mathbf{D}}(\Delta(\mathbb{A}))$.
   (b) Calculate the orbits of the action

$$\mathcal{G}_{\mathbf{D}}(\Delta(\mathbb{A})) \times \mathcal{E}(\Delta(\mathbb{A})) \to \mathcal{E}(\Delta(\mathbb{A})),$$

   (see Remark 25-(c)).
   (c) (Optional) For every orbit, construct the weighted graph with an adjacency matrix the structure matrix corresponding to one representative of the family. Remove the vertices labels from the graph. In case we obtain multiple families of orbits for the same $\Delta(\mathbb{A})$, there might be a family that contains others for certain values of the weights. Combine them to reduce the number of families.

3. Classification of 2-dimensional complex non-degenerate evolution algebras

As an application of the method developed in the previous section, we classify the 2-dimensional non-degenerate evolution algebras in this section. Fix a basis $\mathbf{B} = \{e_1, e_2\}$. For the sake of simplicity, we may refer to an algebra $\mathbb{A}$ by simply writing its structure matrix in the basis $\mathbf{B}$. Also, we may assume that the characteristic of the ground field is zero in the upcoming technical sections. Now, in order to classify the 2-dimensional non-degenerate evolution algebras, we have to study the following cases.

3.1. Case $\delta(\mathbb{A}) = (1, 1)$

Suppose the standard extending evolution subspaces decomposition is given by $\mathbb{E}_1 = \langle e_1 \rangle$, $\mathbb{E}_2 = \langle e_2 \rangle$. We have 3 subcases.

3.1.1. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}, \text{ where } a_{12}a_{21} \neq 0 \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{O}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \text{ where } x_1 x_2 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 0 & a_{12}x_2^2x_1^{-1} \\ a_{21}x_1^2x_2^{-1} & 0 \end{pmatrix}.$$

By choosing $x_1 = (a_{12}a_{21}^2)^{-\frac{1}{3}}$ and $x_2 = (a_{12}^2a_{21})^{-\frac{1}{3}}$, we obtain a $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{O}_1)$-orbit with representative $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here, the group $\mathcal{G}_{\mathbf{P}}(\mathbf{O}_1, \mathbf{O}_1) = \mathbb{S}_2$ sends our representative to itself.

3.1.2. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_1, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 1 \end{pmatrix}, \text{ where } a_{12}a_{21} \neq 0 \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \text{ where } x_1 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_1, \mathbf{E}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 0 & a_{12}x_1^{-1} \\ a_{21}x_1^2 & 1 \end{pmatrix}.$$

If $x_1 = a_{21}^{-\frac{1}{2}}$, we obtain the representatives $\begin{pmatrix} 0 & \alpha \\ 1 & 1 \end{pmatrix}$ for $\alpha \in \mathbb{F}^*$. Here, the group $\mathcal{G}_{\mathbf{P}}(\mathbf{O}_1, \mathbf{E}_1)$ is trivial.

3.1.3. Subcase $\Delta(\mathbb{A}) = (\mathbf{E}_1, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix}, \text{ where } a_{12}a_{21} \neq 1 \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{E}_1, \mathbf{E}_1)$ is trivial and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1)$ sends it to itself. Then, we have one $\mathcal{G}_{\mathbf{D}}(\mathbf{E}_1, \mathbf{E}_1)$-orbit for every element in $\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1)$. Now, since the group $\mathcal{G}_{\mathbf{P}}(\mathbf{E}_1, \mathbf{E}_1) = \mathbb{S}_2$, we obtain the representatives $\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ for $(\alpha, \beta) \in \mathbb{F}^2$ with $\alpha\beta \neq 1$, where (α, β) and (β, α) produce isomorphic algebras.

3.2. Case $\delta(\mathbb{A}) = (2)$

By Theorem 17, $\mathbb{A}$ is either isomorphic to $\mathbf{E}_2$ or to $\mathbf{I}_2$.

Summing up this section, we have the following classification theorem.

Theorem 28. *Given a two-dimensional non-degenerate evolution algebra $\mathbb{A}$ over an algebraically closed field of characteristic zero $\mathbb{F}$, then it is isomorphic to only one of the following algebras:*

- *If $\delta(\mathbb{A}) = (1, 1)$.*
  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1)$. Then it is isomorphic to the algebra $\mathbf{A}_{2,1} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*
  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1)$. Then it is isomorphic to $\mathbf{A}_{2,2}^{\alpha} : \begin{pmatrix} 0 & \alpha \\ 1 & 1 \end{pmatrix}$ for some $\alpha \in \mathbb{F}^*$.*
  - *If $\Delta(\mathbb{A}) = (\mathbf{E}_1, \mathbf{E}_1)$. Then it is isomorphic to $\mathbf{A}_{2,3}^{\alpha,\beta} : \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ for $(\alpha, \beta) \in \mathbb{F}^2$ and $\alpha\beta \neq 1$.*

  The only isomorphisms are between algebras in the same family, and they are permutations of the basis elements. Precisely, the only isomorphisms are $\mathbb{A}_{23}^{\alpha,\beta} \cong \mathbb{A}_{23}^{\beta,\alpha}$.
- *If $\delta(\mathbb{A}) = (2)$. Then it is either isomorphic to $\mathbf{E}_2$ or to $\mathbf{I}_2$.*

4. Classification of 3-dimensional complex non-degenerate evolution algebras

By a similar process, we can obtain the classification of the 3-dimensional non-degenerate evolution algebras. Fix a basis $\mathbf{B} = \{e_1, e_2, e_3\}$. In this section, the following remark will be used.

Remark 29. By Theorem 26 and Theorem 27, we have the following groups of automorphisms:

$$\mathrm{Aut}(\mathbf{E}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}; \qquad \mathrm{Aut}(\mathbf{I}_2) = \left\{ \begin{pmatrix} x & \mathbf{i} - \mathbf{i}x \\ \mathbf{i}x - \mathbf{i} & x \end{pmatrix} : x \in \mathbb{F} \setminus \left\{ \tfrac{1}{2} \right\} \right\}.$$

To classify the 3-dimensional non-degenerate evolution algebras, we have to study the following cases.

4.1. *Case* $\delta(\mathbb{A}) = (1,1,1)$

Suppose the standard extending evolution subspaces decomposition is given by $\mathbb{E}_1 = \langle e_1 \rangle$, $\mathbb{E}_2 = \langle e_2 \rangle$ and $\mathbb{E}_3 = \langle e_3 \rangle$.

4.1.1. *Subcase* $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1, \mathbf{O}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{O}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (1,1,1) \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{O}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \text{ where } x_1 x_2 x_3 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{O}_1)$ is given by:

$$\phi^{-1} \mathbf{M}_{\mathbf{B}}(\mathbb{A}) \phi^{(2)} = \begin{pmatrix} 0 & a_{12} x_2^2 x_1^{-1} & a_{13} x_3^2 x_1^{-1} \\ a_{21} x_1^2 x_2^{-1} & 0 & a_{23} x_3^2 x_2^{-1} \\ a_{31} x_1^2 x_3^{-1} & a_{32} x_2^2 x_3^{-1} & 0 \end{pmatrix}.$$

Let us denote by τ_{ij} the transposition that swaps elements e_i and e_j of the basis.

- If $a_{12} \neq 0$, $a_{23} \neq 0$ and $a_{31} \neq 0$, then we have $\begin{pmatrix} 0 & 1 & \alpha \\ \beta & 0 & 1 \\ 1 & \gamma & 0 \end{pmatrix}$, for any $\alpha, \beta, \gamma \in \mathbb{F}$.
- If $a_{12} = 0$, $a_{23} \neq 0$ and $a_{31} \neq 0$, then we have $\begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, for $\alpha, \beta \in \mathbb{F}$, $\beta \neq 0$. Two cases arise:
  - If $\alpha \neq 0$, choose the permutation τ_{12} of the basis to obtain an algebra of the first case.
  - If $\alpha = 0$, then we have a zero row, so it can not be isomorphic to the first case. Therefore, we have a second family $\begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, for $\beta \in \mathbb{F}^*$.
- If $a_{12} \neq 0$, $a_{23} = 0$ and $a_{31} \neq 0$, then we have $\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 0 & 0 \\ 1 & \beta & 0 \end{pmatrix}$, for $\alpha, \beta \in \mathbb{F}$, $\beta \neq 0$.
  - If $\alpha \neq 0$, again choose the permutation τ_{12} to obtain the first case.

  - If $\alpha = 0$, then choose τ_{12} to obtain the second family.
- If $a_{12} \neq 0$, $a_{23} \neq 0$ and $a_{31} = 0$, then we have $\begin{pmatrix} 0 & 1 & \beta \\ 1 & 0 & 1 \\ 0 & \alpha & 0 \end{pmatrix}$, for $\alpha, \beta \in \mathbb{F}$, $\beta \neq 0$.
  - If $\alpha \neq 0$, there is a suitable permutation of the basis to obtain an algebra of the first case.
  - If $\alpha = 0$, then choose τ_{13} to obtain the second family.
- If $a_{12} = 0$, $a_{23} = 0$ and $a_{31} \neq 0$, then we have $\begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, for $\alpha \in \mathbb{F}^*$.
- If $a_{12} = 0$, $a_{23} \neq 0$ and $a_{31} = 0$, then we have $\begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, for $\alpha \in \mathbb{F}^*$.
- If $a_{12} \neq 0$, $a_{23} = 0$ and $a_{31} = 0$, then we have $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}$, for $\alpha \in \mathbb{F}^*$.
- If $a_{12} = 0$, $a_{23} = 0$ and $a_{31} = 0$, then we have $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Note that applying τ_{12} to the last four cases give us the first case.

4.1.2. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (1,1,1) \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ where } x_1 x_2 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_1, \mathbf{O}_1, \mathbf{E}_1)$ is given by:

$$\phi^{-1} \mathbf{M}_{\mathbf{B}}(\mathbb{A}) \phi^{(2)} = \begin{pmatrix} 0 & a_{12} x_2^2 x_1^{-1} & a_{13} x_1^{-1} \\ a_{21} x_1^2 x_2^{-1} & 0 & a_{23} x_2^{-1} \\ a_{31} x_1^2 & a_{32} x_2^2 & 1 \end{pmatrix}.$$

At this point, the following cases arise:

- If $a_{31} \neq 0$ and $a_{32} \neq 0$, then we have $\begin{pmatrix} 0 & \alpha & \beta \\ \gamma & 0 & \epsilon \\ 1 & 1 & 1 \end{pmatrix}$ for $\alpha, \beta, \gamma, \epsilon \in \mathbb{F}$.
- If $a_{31} = 0$ and $a_{32} \neq 0$, then $a_{21} \neq 0$, and we have $\begin{pmatrix} 0 & \alpha & \beta \\ 1 & 0 & \gamma \\ 0 & 1 & 1 \end{pmatrix}$ for $\alpha, \beta, \gamma \in \mathbb{F}$.
- If $a_{31} \neq 0$ and $a_{32} = 0$, then $a_{12} \neq 0$, and choosing τ_{12}, we are in the previous case.
- If $a_{31} = 0$ and $a_{32} = 0$, then $a_{12} \neq 0$ and $a_{21} \neq 0$, and we have $\begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ 0 & 0 & 1 \end{pmatrix}$ for $\alpha, \beta \in \mathbb{F}$.

Recall that the parameters on the previous cases are subject to the condition $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$.

4.1.3. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (1,1,1) \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } x_1 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$ is given by:

$$\phi^{-1}\mathbf{M_B}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 0 & a_{12}x_1^{-1} & a_{13}x_1^{-1} \\ a_{21}x_1^2 & 1 & a_{23} \\ a_{31}x_1^2 & a_{32} & 1 \end{pmatrix}.$$

Note that here we have to distinguish the following two cases:

- If $a_{21} \neq 0$, then we have $\begin{pmatrix} 0 & \alpha & \beta \\ 1 & 1 & \gamma \\ \epsilon & \zeta & 1 \end{pmatrix}$ for $\alpha, \beta, \gamma, \epsilon, \zeta \in \mathbb{F}$, such that $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$.
- If $a_{21} = 0$, then $a_{31} \neq 0$ and using a suitable permutation we are in the previous case.

4.1.4. Subcase $\Delta(\mathbb{A}) = (\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (1,1,1) \right\}.$$

The group $\mathcal{G}_{\mathbf{D}}(\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$ is trivial and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$ sends it to itself. Then, we have one $\mathcal{G}_{\mathbf{D}}(\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$-orbit for every element in $\mathcal{E}(\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$.

4.2. Case $\delta(\mathbb{A}) = (2, 1)$

Suppose the standard extending evolution subspaces decomposition is given by $\mathbb{E}_1 = \langle e_1, e_2 \rangle$, $\mathbb{E}_2 = \langle e_3 \rangle$. Observe that since $\mathcal{G}_{\mathbf{P}}$ is trivial, then $\mathcal{G} = \mathcal{G}_{\mathbf{D}}$.

4.2.1. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_2, \mathbf{O}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_2, \mathbf{O}_1) = \left\{ \mathbb{A} : \mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{31} & 0 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{O}_2, \mathbf{O}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix},$$

where $x_{11}x_{22}x_{33} \neq x_{12}x_{21}x_{33}$ and $\phi_1^{\mathrm{T}}\phi_1 = \lambda \mathbf{1}_2$ for $\lambda \in \mathbb{F}$ and $\phi_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$.

The action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_2, \mathbf{O}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 0 & 0 & \frac{(a_{23}x_{12}-a_{13}x_{22})x_{33}^2}{x_{12}x_{21}-x_{11}x_{22}} \\ 0 & 0 & \frac{(a_{13}x_{21}-a_{23}x_{11})x_{33}^2}{x_{12}x_{21}-x_{11}x_{22}} \\ \frac{a_{31}(x_{11}^2+x_{21}^2)}{x_{33}} & \frac{a_{31}(x_{12}^2+x_{22}^2)}{x_{33}} & 0 \end{pmatrix},$$

where $x_{11}^2 + x_{21}^2 = x_{12}^2 + x_{22}^2 = \lambda$.

Here, we have two cases:

- If $a_{23}^2 + a_{13}^2 \neq 0$, we obtain the algebra corresponding to the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
- If $a_{23}^2 + a_{13}^2 = 0$, we obtain $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & 1 & 0 \end{pmatrix}$.

4.2.2. Subcase $\Delta(\mathbb{A}) = (\mathbf{O}_2, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{O}_2, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{31} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{O}_2, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_{11}x_{22} \neq x_{12}x_{21}$, $\phi_1^{\mathrm{T}}\phi_1 = \lambda \mathbf{1}_2$ for $\lambda \in \mathbb{F}$ and $\phi_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. The action of this group on an arbitrary element of $\mathcal{E}(\mathbf{O}_2, \mathbf{E}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 0 & 0 & \frac{a_{23}x_{12}-a_{13}x_{22}}{x_{12}x_{21}-x_{11}x_{22}} \\ 0 & 0 & \frac{a_{13}x_{21}-a_{23}x_{11}}{x_{12}x_{21}-x_{11}x_{22}} \\ a_{31}(x_{11}^2+x_{21}^2) & a_{31}(x_{12}^2+x_{22}^2) & 1 \end{pmatrix},$$

where $x_{11}^2 + x_{21}^2 = x_{12}^2 + x_{22}^2 = \lambda$.

Now, we have to distinguish two cases:

- If $a_{23}^2 + a_{13}^2 \neq 0$, we obtain the family $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \alpha & \alpha & 1 \end{pmatrix}$ for $\alpha \in \mathbb{F}^*$.
- If $a_{23}^2 + a_{13}^2 = 0$, we obtain $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{i} \\ 1 & 1 & 1 \end{pmatrix}$.

4.2.3. Subcase $\Delta(\mathbb{A}) = (\mathbf{E}_2, \mathbf{O}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{E}_2, \mathbf{O}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 1 & 1 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{31} & 0 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{E}_2, \mathbf{O}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}, \text{ where } x_1 \neq 0$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{E}_2, \mathbf{O}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 1 & 1 & a_{13}x_1^2 \\ 0 & 0 & \pm a_{23}x_1^2 \\ a_{31}x_1^{-1} & a_{31}x_1^{-1} & 0 \end{pmatrix}.$$

Here, we have two cases:

- If $a_{31} \neq 0$, we obtain the family $\mathbb{W}_1^{\alpha,\beta} : \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 0 & \beta \\ 1 & 1 & 0 \end{pmatrix}$ for $\alpha, \beta \in \mathbb{F}$, where $\mathbb{W}^{\alpha,\beta} \cong \mathbb{W}^{\alpha,-\beta}$.
- If $a_{31} = 0$, we obtain $\mathbb{W}_2^{\alpha} : \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ for $\alpha \in \mathbb{F}$, where $\mathbb{W}_2^{\alpha} \cong \mathbb{W}_2^{-\alpha}$.

4.2.4. Subcase $\Delta(\mathbb{A}) = (\mathbf{E}_2, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{E}_2, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 1 & 1 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{31} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{E}_2, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{E}_2, \mathbf{E}_1)$ is given by:

$$\phi^{-1}\mathbf{M}_{\mathbf{B}}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 1 & 1 & a_{13} \\ 0 & 0 & \pm a_{23} \\ a_{31} & a_{31} & 1 \end{pmatrix}.$$

Clearly, we have a single family $\mathbb{W}_3^{\alpha,\beta,\gamma} : \begin{pmatrix} 1 & 1 & \beta \\ 0 & 0 & \gamma \\ \alpha & \alpha & 1 \end{pmatrix}$ for $\alpha, \beta, \gamma \in \mathbb{F}$, where $\mathbb{W}_3^{\alpha,\beta,\gamma} \cong \mathbb{W}_3^{\alpha,\beta,-\gamma}$.

4.2.5. Subcase $\Delta(\mathbb{A}) = (\mathbf{I}_2, \mathbf{O}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{I}_2, \mathbf{O}_1) = \left\{ \mathbb{A} : \mathbf{M}_{\mathbf{B}}(\mathbb{A}) = \begin{pmatrix} 1 & 1 & a_{13} \\ \mathbf{i} & \mathbf{i} & a_{23} \\ a_{31} & a_{31} & 0 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{I}_2, \mathbf{O}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & \mathbf{i} - \mathbf{i}x_1 & 0 \\ \mathbf{i}x_1 - \mathbf{i} & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, \text{ where } 2x_1 \neq x_2$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{I}_2, \mathbf{O}_1)$ is given by:

$$\phi^{-1}\mathbf{M_B}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 1 & 1 & \frac{(\mathbf{i}a_{23}(x_1-1)+a_{13}x_1)x_2^2}{2x_1-1} \\ \mathbf{i} & \mathbf{i} & \frac{(\mathbf{i}a_{13}(1-x_1)+a_{23}x_1)x_2^2}{2x_1-1} \\ \frac{a_{31}(2x_1-1)}{x_2} & \frac{a_{31}(2x_1-1)}{x_2} & 0 \end{pmatrix}.$$

From here, we have the cases:

- If $a_{31} \neq 0$ and $a_{23}^2 + a_{13}^2 \neq 0$, we obtain the family $\begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & 0 \\ 1 & 1 & 0 \end{pmatrix}$ for $\alpha \in \mathbb{F}^*$.
- If $a_{31} \neq 0$ and $a_{23}^2 + a_{13}^2 = 0$, we have two algebras corresponding to $\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & \pm\mathbf{i} \\ 1 & 1 & 0 \end{pmatrix}$.
- If $a_{31} = 0$ and $a_{23}^2 + a_{13}^2 \neq 0$, we have $\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- If $a_{31} = 0$ and $a_{23}^2 + a_{13}^2 = 0$, we obtain the algebra $\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 0 & 0 & 0 \end{pmatrix}$.

4.2.6. Subcase $\Delta(\mathbb{A}) = (\mathbf{I}_2, \mathbf{E}_1)$

In this subcase, we are considering the algebras of the family:

$$\mathcal{E}(\mathbf{I}_2, \mathbf{E}_1) = \left\{ \mathbb{A} : \mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} 1 & 1 & a_{13} \\ \mathbf{i} & \mathbf{i} & a_{23} \\ a_{31} & a_{31} & 1 \end{pmatrix} \text{ and } \delta(\mathbb{A}) = (2,1) \right\}.$$

The group $\mathcal{G}(\mathbf{I}_2, \mathbf{E}_1)$ consists of maps ϕ such that:

$$\phi = \begin{pmatrix} x_1 & \mathbf{i} - \mathbf{i}x_1 & 0 \\ \mathbf{i}x_1 - \mathbf{i} & x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ where } 2x_1 \neq 1$$

and the action of this group on an arbitrary element of $\mathcal{E}(\mathbf{I}_2, \mathbf{E}_1)$ is given by:

$$\phi^{-1}\mathbf{M_B}(\mathbb{A})\phi^{(2)} = \begin{pmatrix} 1 & 1 & \frac{\mathbf{i}a_{23}(x_1-1)+a_{13}x_1}{2x_1-1} \\ \mathbf{i} & \mathbf{i} & \frac{\mathbf{i}a_{13}(1-x_1)+a_{23}x_1}{2x_1-1} \\ a_{31}(2x_1-1) & a_{31}(2x_1-1) & 1 \end{pmatrix}.$$

From here, we have the cases:

- If $a_{31} \neq 0$, we obtain the family $\begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & \beta \\ 1 & 1 & 1 \end{pmatrix}$ for $\alpha, \beta \in \mathbb{F}$, where $(\alpha, \beta) \neq (1, \mathbf{i})$.
- If $a_{31} = 0$ and $a_{23}^2 + a_{13}^2 \neq 0$, we have the algebras $\begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $\alpha \in \mathbb{F}^*$.

- If $a_{31} = 0$ and $a_{23}^2 + a_{13}^2 = 0$, we obtain the algebras $\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & \mathbf{i} \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & -\alpha\mathbf{i} \\ 0 & 0 & 1 \end{pmatrix}$ for $\alpha \in \mathbb{F}$.

4.3. Case $\delta(\mathbb{A}) = (3)$

By Theorem 17, $\mathbb{A}$ is either isomorphic to $\mathbf{E}_3$ or to $\mathbf{I}_3$.

We conclude this section with the following classification theorem.

Theorem 30. *Given a three-dimensional non-degenerate evolution algebra $\mathbb{A}$ over an algebraically closed field of characteristic zero $\mathbb{F}$, then it is isomorphic to only one of the following algebras:*

- *If $\delta(\mathbb{A}) = (1, 1, 1)$.*
  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1, \mathbf{O}_1)$. Then it is isomorphic to one of the following algebras:*

$$a)\ \mathbf{A}_{3,1}^{\alpha,\beta,\gamma} : \begin{pmatrix} 0 & 1 & \alpha \\ \beta & 0 & 1 \\ 1 & \gamma & 0 \end{pmatrix}, \qquad b)\ \mathbf{A}_{3,2}^{\alpha} : \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{O}_1, \mathbf{E}_1)$. Then it is isomorphic to one of the following algebras:*

$$a)\ \mathbf{A}_{3,3}^{\Lambda} : \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & 0 & \epsilon \\ 1 & 1 & 1 \end{pmatrix}, \qquad b)\ \mathbf{A}_{3,4}^{\alpha,\beta,\gamma} : \begin{pmatrix} 0 & \alpha & \beta \\ 1 & 0 & \gamma \\ 0 & 1 & 1 \end{pmatrix}, \qquad c)\ \mathbf{A}_{3,5}^{\alpha,\beta} : \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ 0 & 0 & 1 \end{pmatrix}.$$

  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_1, \mathbf{E}_1, \mathbf{E}_1)$. It is isomorphic to* $\mathbf{A}_{3,6}^{\Lambda} : \begin{pmatrix} 0 & \alpha & \beta \\ 1 & 1 & \gamma \\ \epsilon & \zeta & 1 \end{pmatrix}$.
  - *If $\Delta(\mathbb{A}) = (\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_1)$. It is isomorphic to* $\mathbf{A}_{3,7}^{\Lambda} : \begin{pmatrix} 1 & \alpha & \beta \\ \gamma & 1 & \epsilon \\ \zeta & \xi & 1 \end{pmatrix}$.

  The only isomorphisms are between algebras in the same family, and they are permutations of the basis elements.
- *If $\delta(\mathbb{A}) = (2, 1)$.*
  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_2, \mathbf{O}_1)$. Then it is isomorphic to one of the following algebras:*

$$a)\ \mathbf{B}_{3,1} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad b)\ \mathbf{B}_{3,2} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{i} \\ 1 & 1 & 0 \end{pmatrix}.$$

  - *If $\Delta(\mathbb{A}) = (\mathbf{O}_2, \mathbf{E}_1)$. Then it is isomorphic to one of the following algebras:*

$$a)\ \mathbf{B}_{3,3}^{\alpha} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \alpha & \alpha & 1 \end{pmatrix}, \qquad b)\ \mathbf{B}_{3,4} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{i} \\ 1 & 1 & 1 \end{pmatrix}.$$

  - *If $\Delta(\mathbb{A}) = (\mathbf{E}_2, \mathbf{O}_1)$. Then it is isomorphic to one of the following algebras:*

$$a)\ \mathbf{B}_{3,5}^{\alpha,\beta} : \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 0 & \beta \\ 1 & 1 & 0 \end{pmatrix}, \qquad b)\ \mathbf{B}_{3,6}^{\alpha} : \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isomorphisms in this section are $\mathbf{B}_{3,5}^{\alpha,\beta} \cong \mathbf{B}_{3,5}^{\alpha,-\beta}$ *and* $\mathbf{B}_{3,6}^{\alpha} \cong \mathbf{B}_{3,6}^{-\alpha}$*.*

- *If* $\Delta(\mathbb{A}) = (\mathbf{E}_2, \mathbf{E}_1)$*. It is isomorphic to* $\mathbf{B}_{3,7}^{\alpha,\beta,\gamma} : \begin{pmatrix} 1 & 1 & \beta \\ 0 & 0 & \gamma \\ \alpha & \alpha & 1 \end{pmatrix}$*. We have* $\mathbf{B}_{3,7}^{\alpha,\beta,\gamma} \cong \mathbf{B}_{3,7}^{\alpha,\beta,-\gamma}$*.*
- *If* $\Delta(\mathbb{A}) = (\mathbf{I}_2, \mathbf{O}_1)$*. Then it is isomorphic to one of the following algebras:*

a) $\mathbf{B}_{3,8}^{\alpha} : \begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & 0 \\ 1 & 1 & 0 \end{pmatrix},$ *b)* $\mathbf{B}_{3,9}^{\pm} : \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & \pm\mathbf{i} \\ 1 & 1 & 0 \end{pmatrix},$

c) $\mathbf{B}_{3,10} : \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 0 \end{pmatrix},$ *d)* $\mathbf{B}_{3,11} : \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 0 & 0 & 0 \end{pmatrix}.$

- *If* $\Delta(\mathbb{A}) = (\mathbf{I}_2, \mathbf{E}_1)$*. Then it is isomorphic to one of the following algebras:*

a) $\mathbf{B}_{3,12}^{\alpha,\beta} : \begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & \beta \\ 1 & 1 & 1 \end{pmatrix},$ *b)* $\mathbf{B}_{3,13}^{\alpha\neq 0} : \begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 1 \end{pmatrix},$

c) $\mathbf{B}_{3,14} : \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{i} & \mathbf{i} \\ 0 & 0 & 1 \end{pmatrix},$ *d)* $\mathbf{B}_{3,15}^{\alpha} : \begin{pmatrix} 1 & 1 & \alpha \\ \mathbf{i} & \mathbf{i} & -\alpha\mathbf{i} \\ 0 & 0 & 1 \end{pmatrix}$

- *If* $\delta(\mathbb{A}) = (3)$*. Then it is either isomorphic to* $\mathbf{E}_3$ *or to* $\mathbf{I}_3$*.*

The parameters in each family $\mathbf{A}_{3,*}^{*}$ *or* $\mathbf{B}_{3,*}^{*}$ *are taken freely in* $\mathbb{F}$*, but in such a way that the resulting algebra has the* δ*-index corresponding to that family.*

5. Derivations of non-degenerate evolution algebras

Given a non-degenerate evolution algebra $\mathbb{A}$ such that $\delta(\mathbb{A}) = (n_1, \ldots, n_r)$ and $\Delta(\mathbb{A}) := (\overline{\mathbb{A}_1}, \ldots, \overline{\mathbb{A}_r})$. Fix a basis $\mathbf{B} := \{e_1, \ldots, e_n\}$ as in Theorem 23, then for a derivation $\mathfrak{D}$ with matrix (d_{ij}) of $\mathbb{A}$ we have:

$$\mathfrak{D}(e_i e_j) = \mathfrak{D}(e_i)e_j + e_i\mathfrak{D}(e_j).$$

If $i \neq j$, then $d_{ij}e_i^2 + d_{ji}e_j^2 = 0$. Therefore, $d_{ij} = -d_{ji}$ if $e_i, e_j \in \mathbb{A}_k$ for some k or $d_{ij} = d_{ji} = 0$ otherwise. From here, the matrix of $\mathfrak{D}$ is a block diagonal matrix with blocks D_k for $1 \leq k \leq r$ of size $n_k \times n_k$, where D_k is a diagonal matrix plus an antisymmetric matrix. Now, if $i = j$, then $\mathfrak{D}(e_i^2) = 2e_i\mathfrak{D}(e_i)$ if and only if $\mathfrak{D}(e_i^2) = 2d_{ii}e_i^2$ (note that e_i^2 is an eigenvector of $\mathfrak{D}$ corresponding to the eigenvalue $2d_{ii}$). This is equivalent, for any $1 \leq i, j \leq n$, to the equations:

$$\sum_k d_{jk}w_{ki} = 2d_{ii}w_{ji},$$

where $\mathbf{M}_{\mathbf{B}}(\mathbb{A}) := (w_{ki})$. Also, note that if $e_i, e_j \in \mathbb{A}_k$, then $\mathfrak{D}(e_i^2) = \mathfrak{D}(e_j^2)$, $2d_{ii}e_i^2 = 2d_{jj}e_j^2$ and $d_{ii} = d_{jj}$. Let us assume in this section that the characteristic of the algebraically closed ground field $\mathbb{F}$ is not two.

Theorem 31. *The map corresponding to the matrix* D_k *is a derivation of* $\overline{\mathbb{A}_k}$ *for* $1 \leq k \leq r$*.*

Proof. It follows by observing that since $\mathfrak{D}$ is block diagonal, then $\mathfrak{D}(e_i^2) = 2e_i\mathfrak{D}(e_i)$ can be written

$$\begin{pmatrix} D_1\overline{\mathbb{A}_1} & D_1\mathbb{A}_{12} & \dots & D_1\mathbb{A}_{1r} \\ D_2\mathbb{A}_{21} & D_2\overline{\mathbb{A}_2} & \dots & D_2\mathbb{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ D_r\mathbb{A}_{r1} & D_r\mathbb{A}_{r2} & \dots & D_r\overline{\mathbb{A}_r} \end{pmatrix} = 2\begin{pmatrix} d_{11}\overline{\mathbb{A}_1} & d_{22}\mathbb{A}_{12} & \dots & d_{rr}\mathbb{A}_{1r} \\ d_{11}\mathbb{A}_{21} & d_{22}\overline{\mathbb{A}_2} & \dots & d_{rr}\mathbb{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ d_{11}\mathbb{A}_{r1} & d_{22}\mathbb{A}_{r2} & \dots & d_{rr}\overline{\mathbb{A}_r} \end{pmatrix},$$

where $\mathbf{M_B}(\mathbb{A}) = \begin{pmatrix} \overline{\mathbb{A}_1} & \dots & \mathbb{A}_{1r} \\ \vdots & \ddots & \vdots \\ \mathbb{A}_{r1} & \dots & \overline{\mathbb{A}_r} \end{pmatrix}$. Hence, we have $D_k\overline{\mathbb{A}_k} = 2d_{kk}\overline{\mathbb{A}_k}$, so D_k is a derivation of $\overline{\mathbb{A}_k}$. □

Observe that if $\mathbb{A}$ is the direct sum of the algebras $\overline{\mathbb{A}_k}$, then the converse of Theorem 31 is true. However, in general, it is not true, as we need the additional condition of e_i^2 being an eigenvector of the derivation corresponding to the eigenvalue $2d_{ii}$ for every $1 \le i \le \dim(\mathbb{A})$.

Now, by the previous result, we are interested in the derivations of $\mathbf{E}_n$ and $\mathbf{I}_n$ (for $\mathbf{O}_n$ the result is trivial).

Lemma 32. *The derivations $\mathfrak{D}$ of $\mathbf{E}_n$ are of the form $\mathfrak{D} = \begin{pmatrix} 0 & 0 \\ 0 & D_0 \end{pmatrix}$, where $D_0 = -D_0^{\mathrm{T}} = (d_{ij})_{i,j}^{n-1}$. Moreover,* $\dim \mathfrak{Der}(\mathbf{E}_n) = \frac{(n-1)(n-2)}{2}$.

Proof. Denote the matrix of a derivation $\mathfrak{D}$ as (d_{ij}) in the basis provided in Theorem 17. Since, for $i \neq j$, we have $\mathfrak{D}(e_ie_j) = \mathfrak{D}(e_i)e_j + \mathfrak{D}(e_j)e_i = 0$, then $d_{ij} = -d_{ji}$. Also, $\mathfrak{D}(e_i^2) = 2\mathfrak{D}(e_i)e_i$ implies that $\mathfrak{D}(e_1) = 2d_{ii}e_1$, so $d_{ii} = d_{jj}$. At last, the equation $\sum_k d_{k1}e_k = 2d_{11}e_1$ implies that $d_{k1} = 0$ for $1 \le k \le n$. □

Lemma 33. *The derivations $\mathfrak{D}$ of $\mathbf{I}_n$ are of the form:*

$$D = \begin{pmatrix} d & -d\mathbf{i} & -d_{23}\mathbf{i} & \dots & -d_{2(n-1)}\mathbf{i} & -d_{2n}\mathbf{i} \\ d\mathbf{i} & d & d_{23} & \dots & d_{2(n-1)} & d_{2n} \\ d_{23}\mathbf{i} & -d_{23} & d & \dots & d_{3(n-1)} & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{2(n-1)}\mathbf{i} & -d_{2(n-1)} & -d_{3(n-1)} & \dots & d & d_{(n-1)n} \\ d_{2n}\mathbf{i} & -d_{2n} & -d_{3n} & \dots & -d_{(n-1)n} & d \end{pmatrix}$$

Moreover, $\dim \mathfrak{Der}(\mathbf{I}_n) = \frac{(n-1)(n-2)}{2} + 1$.

Proof. Denote the matrix of a derivation $\mathfrak{D}$ as (d_{ij}) in the basis provided in Theorem 17. Again, for $i \neq j$, then $d_{ij} = -d_{ji}$. Since $\mathfrak{D}(e_i^2) = 2\mathfrak{D}(e_i)e_i$, we have $\mathfrak{D}(e_1 + \mathbf{i}e_2) = 2d_{ii}(e_1 + e_2)$. Hence,

$$2d_{jj}e_1 + 2d_{jj}\mathbf{i}e_2 = 2d_{ii}e_1 + 2d_{ii}\mathbf{i}e_2, \text{ so } d := d_{ii} = d_{jj}.$$

Moreover, we can write

$$\sum_k (d_{k1} + d_{k2}\mathbf{i})e_k = 2d_{ii}e_1 + 2d_{ii}\mathbf{i}e_2.$$

For $k = 1$, we have $d_{11} + d_{12}\mathbf{i} = 2d_{ii}$, thus, $d_{12} = -d\mathbf{i}$. For $k = 2$, we have $d_{21} + d_{22}\mathbf{i} = 2d_{ii}\mathbf{i}$, so $d_{21} = d\mathbf{i}$. Finally, for $k > 2$, we have $d_{k1} = -d_{k2}\mathbf{i}$. □

By the previous results, every derivation of $\mathbf{E_n}$ or $\mathbf{I_n}$ is singular. However, there are examples of non-degenerate evolution algebras with non-singular derivations, such as

$$\mathbb{A} : e_1^2 = e_3,\, e_2^2 = e_3,\, e_3^2 = e_1 + \mathbf{i}e_2.$$

Hence, we have the following corollary.

Corollary 34. *Given a non-degenerate evolution algebra with a non-singular derivation, then it has Δ-trace equal to $(\mathbf{O}_{n_1}, \dots, \mathbf{O}_{n_r})$, for some $n_1, \dots, n_r \in \mathbb{N}$.*

The well-known fact that every derivation of a perfect evolution algebra is zero can be generalized in the following way. This result was first proven in a recent paper [4], in order to enrich the evolution algebras literature we present a different, but interesting, approach to it.

Theorem 35. *Let $\mathbb{A}$ be an evolution algebra such that $\Delta(\mathbb{A}) = (\mathbf{O}_1, \dots, \mathbf{O}_1, \mathbf{E}_1, \dots, \mathbf{E}_1)$, where we have n_1 copies of $\mathbf{O}_1$ and n_2 copies of $\mathbf{E}_1$, over a field $\mathbb{F}$ with char$\mathbb{F} = p > 2$ and $p \nmid 2^s - 1$ for $s \leq n_1$. Then every derivation is zero.*

Proof. Suppose $\Delta(\mathbb{A}) = (\mathbf{O}_1, \dots, \mathbf{O}_1, \mathbf{E}_1, \dots, \mathbf{E}_1)$, where we have n_1 copies of $\mathbf{O}_1$ and n_2 copies of $\mathbf{E}_1$. Fix a basis $\mathbf{B} := \{e_1, \dots, e_n\}$ as in Theorem 23 and denote the structure matrix of $\mathbb{A}$ as (w_{ki}). Note that for every $1 \leq i \leq n$ there is some $k(i)$ such that $w_{k(i)i} \neq 0$. Choose any map $k : \{1, \dots, n\} \to \{1, \dots, n\}$ satisfying that $w_{k(i)i} \neq 0$ for every $1 \leq i \leq n$. It is clear that since $\mathbb{A}$ is non-degenerate, such a map exists. We assume that $n_1 \geq 1$, otherwise the result is proven.

Now, by Theorem 31, a derivation $\mathfrak{D}$ of $\mathbb{A}$ has a diagonal matrix with diagonal $(d_1, \dots, d_{n_1}, 0, \dots, 0)$ for some $d_i \in \mathbb{F}$. Then, the equation $\sum_k d_{jk} w_{ki} = 2 d_{ii} w_{ji}$ is equivalent to $d_j w_{ji} = 2 d_i w_{ji}$ and using the map k we can write $d_{k^s(i)} = 2^s d_i$, where k^s denotes applying the map k for s times. Now, for any $i \leq n_1$ three scenarios are possible:

(1) If $n_1 < k^s(i)$ for some s, then $d_{k^s(i)} = 0$ and $d_i = 0$.
(2) If $k^s(i) = i$ for some s, then we can assume $s \leq n_1$. So we have $d_i = 2^s d_i$ and, since $p \nmid 2^s - 1$, we have $d_i = 0$.
(3) If $k^s(i) \neq i$ and $k^s(i) \leq n_1$ for any s, then there exist some $s \neq 0$ and some $q \neq 0$ such that $k^s(i) = k^{s+q}(i)$. Again, we can assume that $s, q \leq n_1$. Then the relation $d_{k^s(i)} = 2^q d_{k^s(i)}$, implies that $d_{k^s(i)} = 0$ and $d_i = 0$.

Hence, we conclude that $d_1 = \ldots = d_{n_1} = 0$. $\square$

The previous result does not hold if the characteristic is a Mersenne prime.

Example 36. Suppose $p = 2^n - 1$ is a Mersenne prime. Consider the n-dimensional evolution algebra given by $e_i^2 = e_{i+1}$ for $i < n$ and $e_n^2 = e_1$. We show that the homomorphism $\mathfrak{D}$ defined by $\mathfrak{D}(e_i) = 2^{i-1} e_i$ for $1 \leq i \leq n$ is a derivation. If $i \neq j$, then $0 = \mathfrak{D}(e_i e_j) = \mathfrak{D}(e_i) e_j + e_i \mathfrak{D}(e_j) = 0$. Now, if $i < n$, then

$$\mathfrak{D}(e_i^2) = \mathfrak{D}(e_{i+1}) = 2^i e_{i+1} = 2(2^{i-1} e_i^2) = 2\mathfrak{D}(e_i) e_i.$$

If $i = n$, then

$$\mathfrak{D}(e_n^2) = \mathfrak{D}(e_1) = e_1 = e_n^2 = 2^n e_n^2 = 2(2^{n-1} e_n^2) = 2\mathfrak{D}(e_n) e_n.$$

So $\mathfrak{D}$ is a derivation.

6. The variety of evolution algebras with (one or zero)-dimensional square

Although it is known that the class of evolution algebras is not a variety, the study of some varieties contained in it is still an interesting topic. In this section, we study the class of complex evolution algebras with (one or zero)-dimensional square, and we prove that it is an irreducible variety.

Theorem 37. *Any complex n-dimensional commutative algebra $\mathbb{A}$ with $\dim \mathbb{A}^2 = 1$ is an evolution algebra.*

Proof. Let $\mathbb{A}$ be a complex commutative algebra such that $\dim \mathbb{A}^2 = 1$. Suppose $\mathbb{A}^2 = \langle e_1 \rangle$, then fix a basis $\mathbf{B} = \{e_1, \ldots, e_n\}$ and let $e_i e_j = m_{ij} e_1$ for any $1 \leq i, j \leq n$, where $m_{ij} \in \mathbb{C}$. Consider its multiplication structure matrices $M_1 = m_{ij}$, $M_k = (0)$ for $k \neq 1$. Then $\mathbb{A}$ is an evolution algebra if and only if $M_1, \ldots, M_n$ (see [3]) are simultaneously diagonalizable via congruence. But, we only have one matrix to diagonalize, and it is symmetric, so by Takagi theorem, there exists a unitary matrix V such that $V^T M_1 V = D$ for some real diagonal matrix with non-negative entries. Hence, $\mathbb{A}$ is an evolution algebra. □

As a consequence, we have the following result.

Corollary 38. *The class of complex n-dimensional evolution algebras with one-dimensional squares together with the trivial algebra is a variety, that we will call $\mathfrak{E}^{\mathfrak{n}}$.*

In the following result, we prove that this variety is irreducible. For that, we have to introduce some notions. For a further introduction to the topic of degeneration, see for example [20,21] and the references therein. Given an n-dimensional vector space $\mathbf{V}$, the set $\operatorname{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V}) \cong \mathbf{V}^* \otimes \mathbf{V}^* \otimes \mathbf{V}$ is a vector space of dimension n^3. This space has a structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1, \ldots, e_n$ of $\mathbf{V}$. Then any $\mu \in \operatorname{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ is determined by n^3 structure constants $c_{i,j}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{i,j}^k e_k$. A subset of $\operatorname{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables $c_{i,j}^k$ $(1 \leq i, j, k \leq n)$.

Let T be a set of polynomial identities. The algebra structures on $\mathbf{V}$ satisfying all the polynomial identities from T form a Zariski-closed subset of the variety $\operatorname{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\operatorname{GL}(\mathbf{V})$ acts on $\mathbb{L}(T)$ by conjugation:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbf{V}$, $\mu \in \mathbb{L}(T) \subset \operatorname{Hom}(\mathbf{V} \otimes \mathbf{V}, \mathbf{V})$ and $g \in \operatorname{GL}(\mathbf{V})$. Thus, $\mathbb{L}(T)$ decomposes into the $\operatorname{GL}(\mathbf{V})$-orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the $\operatorname{GL}(\mathbf{V})$-orbit of $\mu \in \mathbb{L}(T)$ and $\overline{O(\mu)}$ its Zariski closure. Let $\mathbf{A}$ and $\mathbf{B}$ be two n-dimensional algebras satisfying the identities from T and $\mu, \lambda \in \mathbb{L}(T)$ represent $\mathbf{A}$ and $\mathbf{B}$ respectively. We say that $\mathbf{A}$ *degenerates to* $\mathbf{B}$ and write $\mathbf{A} \to \mathbf{B}$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of degeneration does not depend on the choice of μ and λ. If $\mathbf{A} \not\cong \mathbf{B}$, then the assertion $\mathbf{A} \to \mathbf{B}$ is called a *proper degeneration.* We write $\mathbf{A} \not\to \mathbf{B}$ if $\lambda \notin \overline{O(\mu)}$. Let $\mathbf{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathbf{A}$ is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called *irreducible* if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component.* It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathbf{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

The last result characterizes the variety $\mathfrak{E}^{\mathfrak{n}}$.

Theorem 39. *The variety $\mathfrak{E}^{\mathfrak{n}}$ is irreducible and $\mathbf{E}_n$ is rigid.*

Proof. The case of $n = 1$ is trivial. Hence, we will consider $n > 1$. Thanks to [2], all algebras from $\mathfrak{E}^{\mathfrak{n}}$ have the following type:

(1) nilpotent algebras.
(2) non-nilpotent algebras of type $\mathbf{E}_k \oplus \mathbb{C}^{n-k}$ for $1 \leq k \leq n$.
(3) non-nilpotent algebras of type $\mathbf{I}_k \oplus \mathbb{C}^{n-k}$ for $2 \leq k \leq n$.

The non-degenerate algebras in this variety are precisely $\mathbf{E}_n$ and $\mathbf{I}_n$. In Lemmas 32 and 33, we have shown that the dimension of the algebra of derivations of the two evolution algebras with one-dimensional square and with trivial annihilator differ in one unit. Namely, $\dim \mathfrak{Der}(\mathbf{I}_n) = \dim \mathfrak{Der}(\mathbf{E}_n) + 1$. Recall that the dimension of the orbit of a n-dimensional algebra $\mathbb{A}$ is $n^2 - \dim \mathfrak{Der}(\mathbb{A})$. Hence, the dimensions of the orbits of $\mathbf{E}_n$ and $\mathbf{I}_n$ also differ in one unit. Moreover, we have $\mathbf{E}_n \to \mathbf{I}_n$, as we show below.

Consider the parametrized isomorphism given by

$$g(e_1)(t) = (1+t^2)e_1 + \mathbf{i}e_2, \quad g(e_2)(t) = \mathbf{i}(1-t^2)e_1 - e_2, \quad g(e_k)(t) = \sqrt{2}te_k, \quad \text{for} \quad 3 \leq k \leq n,$$

and its inverse, given by

$$g(e_1)(t) = \frac{1}{2t^2}e_1 + \frac{\mathbf{i}}{2t^2}e_2, \quad g(e_2)(t) = \frac{\mathbf{i}(1-t^2)}{2t^2}e_1 - \frac{1+t^2}{2t^2}e_2, \quad g(e_k)(t) = \frac{1}{\sqrt{2}t}e_k.$$

Then, the action of g gives us the family of commutative algebras $g(t) * \mathbf{E}_n$, given by

$$\begin{array}{lll} e_1^2 = (1+\frac{t^2}{2})e_1 + (1+\frac{t^2}{2})\mathbf{i}e_2, & e_1e_2 & = -\frac{\mathbf{i}t^2}{2}e_1 + \frac{t^2}{2}e_2, \\ e_2^2 = (1-\frac{t^2}{2})e_1 + (1-\frac{t^2}{2})\mathbf{i}e_2, & e_k^2 & = e_1 + \mathbf{i}e_2, \end{array}$$

for $3 \leq k \leq n$. From here, it is clear that $\lim\limits_{t\to 0} g(t) * \mathbf{E}_n = \mathbf{I}_n$. Moreover, it follows that this degeneration is primary by the dimension of the algebras of derivations.

Thanks to [21], the variety of complex n-dimensional nilpotent commutative algebras with dimension of the square not greater than one is irreducible with generic algebra $\mathbf{N}_n$ given by the following multiplication table $e_2^2 = \ldots = e_n^2 = e_1$.

Then, the rest of the necessary degenerations are (here g is inverse to the parametrized isomorphism):

$$\begin{array}{llllll} \mathbf{E}_n & \to \mathbf{N}_n & : g(e_1) = t^2e_1, & g(e_k) & = te_k, & \text{for } 2 \leq k \leq n. \\ \mathbf{E}_k \oplus \mathbb{C}^{n-k} & \to \mathbf{E}_{k-1} \oplus \mathbb{C}^{n-k+1} & : g(e_k) = te_k, & g(e_m) & = e_m, & \text{for } 1 \leq m \neq k \leq n. \\ \mathbf{I}_k \oplus \mathbb{C}^{n-k} & \to \mathbf{I}_{k-1} \oplus \mathbb{C}^{n-k+1} & : g(e_k) = te_k, & g(e_m) & = e_m, & \text{for } 1 \leq m \neq k \leq n. \end{array}$$

Summarizing, we have shown that every algebra in $\mathfrak{E}^{\mathfrak{n}}$ is in the closure of the orbit of $\mathbf{E}_n$. □

By Theorem 39, in order to describe the identities defining the variety $\mathfrak{E}^{\mathfrak{n}}$, it is enough to study the polynomial identities satisfied by $\mathbf{E}_n$. The following result slightly simplifies this problem.

Theorem 40. *Given a identity of degree n. Then if $\mathbf{E}_n$ satisfies the identity, then $\mathbf{E}_k$ satisfies the identity too for $k \in \mathbb{N}$.*

Proof. Suppose $\mathbf{E}_n$ satisfies an identity $p(x_1, \ldots, x_n) = 0$ of degree n. If $k \leq n$, then $\mathbf{E}_k$ satisfies the identity, because $\mathbf{E}_n \to \mathbf{E}_k \oplus \mathbb{C}^{n-k}$, so $\mathbf{E}_k \oplus \mathbb{C}^{n-k}$ satisfies it and then $\mathbf{E}_k \oplus \mathbb{C}^{n-k}/\mathbb{C}^{n-k} \cong \mathbf{E}_k$ satisfies it.

If $k > n$, then we have that $p(e_{i_1}, \ldots, e_{i_n}) = 0$ for $i_l \leq n$. Now, for $p(e_{j_1}, \ldots, e_{j_n})$ where $j_l \leq k$, we assign to every index j_l an index $i_l \leq n$ using the following rules: $j_l = 1$ if and only if $i_l = 1$ and $j_l = j_m$ if and only if $i_l = i_m$. Then we have $p(e_{j_1}, \ldots, e_{j_n}) = p(e_{i_1}, \ldots, e_{i_n}) = 0$. □

As a consequence, if we want to check if $\mathbf{E}_n$ satisfies an identity of degree $k \leq n$, we check it for $\mathbf{E}_k$.

Proposition 41. *The variety* $\mathfrak{E}^{\mathfrak{n}}$ *does not satisfy nontrivial identities of degree three, but it satisfies a nontrivial identity of degree four:* $((xy)z)t = ((xy)t)z$.

The present proposition can be obtained by a direct calculation and it gives the following question.

Open question. Characterize the variety $\mathfrak{E}^{\mathfrak{n}}$ with a set of identities.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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