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Research Paper

On the simple transposed Poisson algebras and Jordan superalgebras ☆

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ABSTRACT

We prove that a transposed Poisson algebra is simple if and only if its associated Lie bracket is simple. Consequently, any simple finite-dimensional transposed Poisson algebra over an algebraically closed field of characteristic zero is trivial. Similar results are obtained for transposed Poisson superalgebras. An example of a non-trivial simple finite-dimensional transposed Poisson algebra is constructed by studying the transposed Poisson structures on the modular Witt algebra. Furthermore, we show that the Kantor double of a transposed Poisson algebra is a Jordan superalgebra, that is, we prove that transposed Poisson algebras are Jordan brackets. Additionally, a simplicity criterion for the Kantor double of a transposed Poisson algebra is obtained.

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1. Introduction

In the last years, the study of Poisson algebras has led to various related Poisson type algebraic structures, including generic Poisson algebras, algebras of Jordan brackets, Lie-Yamaguti algebras, Gerstenhaber algebras, Novikov-Poisson algebras, among many others. In the paper [2], a dual class of the Poisson algebras was introduced, the transposed Poisson algebras, by changing the roles of the two multiplications in the Leibniz rule. Precisely, a transposed Poisson algebra is a vector space $\mathcal{P}$ over a field $\mathbb{F}$ of characteristic $p \neq 2$ endowed with two operations: an associative commutative multiplication $- \circ -$ and an associated Lie bracket $[\cdot, \cdot]$. Additionally, these two operations are required to satisfy the transposed Leibniz rule, that is, for any $x, y, z \in \mathcal{P}$ we have:

$$2x \circ [y, z] = [x \circ y, z] + [y, x \circ z]. \tag{1}$$

The authors show that transposed Poisson algebras share some common properties with Poisson algebras, including the closure undertaking tensor products and the Koszul self-duality as an operad. Since then, the interest in this class has been increasing. The variety of transposed Poisson algebras coincides with the variety of commutative Gelfand-Dorfman algebras, see [18]. The transposed Leibniz rule can be realized as the left multiplication of the associative commutative algebra is a $\frac{1}{2}$-derivation of the Lie bracket and this realization was fundamental on the classification of low-dimensional transposed Poisson algebras [3] or on the generalization of the notion to the n-ary case [4]. Recall that a $\frac{1}{2}$-derivation of a non-associative algebra $(\mathcal{A}, \cdot)$ is a linear map D in $\mathcal{A}$ such that $2D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$. Likewise, the notion of transposed Poisson superalgebra has been introduced in the usual way, with the $\mathbb{Z}_2$-graded version of the Leibniz rule being equivalent to the left multiplication of the associative commutative algebra being a $\frac{1}{2}$-superderivation of the Lie bracket.

In this paper, we focus our interest on simple transposed Poisson (super)algebras. Kac [9] used the classification of the simple Lie superalgebras and the TKK functor for Jordan superalgebras in order to classify all the simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero. Later, Kantor [10] introduced an invertible way to construct a Jordan superalgebra from a Poisson algebra (the Kantor double), this construction preserves the simplicity in both directions, so a classification of the simple finite-dimensional Poisson algebras over an algebraically closed field of characteristic zero was obtained. Also, one of the families of known infinite dimensional simple Jordan superalgebras is precisely the family of those superalgebras that are obtained by the Kantor double [5].

Recently, it was proven that if a Poisson algebra $(\mathcal{P}, \circ, [\cdot, \cdot])$ is simple, then the Lie algebra $[\mathcal{P}, \mathcal{P}]/([\mathcal{P}, \mathcal{P}] \cap Z)$, where Z denotes the center of $(\mathcal{P}, [\cdot, \cdot])$, is simple [1]. The "transposed" version of this result does not hold, namely, in a simple transposed Poisson algebra $\mathcal{P}$, the algebra $\mathcal{P} \circ \mathcal{P}/(\mathcal{P} \circ \mathcal{P} \cap Z)$, where Z is the center of $(\mathcal{P}, \circ)$, is not always simple. In fact, the Witt algebra together with the Laurent polynomials defines a simple transposed Poisson algebra, the algebra of Laurent polynomials is perfect and centerless, but not simple. However, a stronger result holds. It turns out that simple transposed Poisson (super)algebras are those that arise on simple Lie (super)algebras, if we omit the trivial case in which the Lie (super)bracket is zero. On the first part of this paper, we prove the following result for any field of characteristic $p \neq 2$.

Theorem. *A transposed Poisson (super)algebra is simple if and only if the associated Lie (super)bracket is simple.*

This result has various consequences over an algebraically closed field of characteristic zero. The classification of the simple finite-dimensional transposed Poisson (super)algebras follows thanks to the previous studies of the $\frac{1}{2}$-(super)derivations of the simple Lie (super)algebras [8,11]. Indeed, it turns out that there are no non-trivial simple finite-dimensional transposed Poisson (super)algebras. Moreover, this implies that any simple finite-dimensional weak Leibniz algebra (see [6]) is either an associative algebra or a Lie algebra. In addition, we study the case in which one of the multiplications is perfect, and we prove that the radical of the Lie algebra is an ideal of the transposed Poisson algebra.

The situation is different when we consider fields of positive characteristic. We show that there are non-trivial simple finite-dimensional transposed Poisson algebras by constructing all the transposed Poisson structures on the modular Witt algebra, which is the Lie algebra of derivations of the algebra of truncated polynomials in one variable $\mathbb{F}[x]/(x^p)$, where $\mathrm{char}(\mathbb{F}) = p$. Precisely, we prove that any transposed Poisson structure of this type has associative commutative part a mutation of an algebra isomorphic to $\mathbb{F}[x]/(x^p)$.

On the second part of this paper we are considering the Kantor double of a transposed Poisson algebra $\mathfrak{J}(\mathcal{P})$. Let us recall this construction briefly. Following Kantor's notation, given a vector space $\mathbb{V}$, we denote by $\mathbb{V}^s$ the vector space which is a copy of $\mathbb{V}$, but with odd parity. Now, suppose $(\mathcal{P}, \circ, [\cdot,\cdot])$ is an algebra equipped with two multiplications: the first one is associative and commutative and the second one is skew-symmetric. This class of algebras are called dot-bracket algebras [13]. Consider the algebra $\mathfrak{J}(\mathcal{P})$ with underlying vector space $\mathcal{P} \oplus \mathcal{P}^s$ and with the operation $*$ given for $a, b \in \mathcal{P}$ and corresponding $a^s, b^s \in \mathcal{P}^s$ by the following relations

$$a * b = a \circ b, \quad a^s * b = a * b^s = (a \circ b)^s, \quad a^s * b^s = [a, b].$$

Thus, $\mathfrak{J}(\mathcal{P}) = \mathcal{P} \oplus \mathcal{P}^s$ is a superalgebra with double the dimension of $\mathcal{P}$, which is called the Kantor double of $\mathcal{P}$. A similar construction, can be considered if $\mathcal{P}$ is a dot-bracket superalgebra [13]. Kantor proved that the double of a Poisson (super)algebra is a Jordan superalgebra [10], but there exist other non-Poisson (super)algebras whose Kantor doubles are Jordan superalgebras. Recall that a Jordan superalgebra is a superalgebra $(\mathcal{J} := \mathcal{J}_0 \oplus \mathcal{J}_1, \cdot)$ satisfying the supercommutativity, $x \cdot y = (-1)^{xy} y \cdot x$, and the Jordan superidentity

$$(-1)^{xz}[\![L_{x\cdot y}, L_z]\!] + (-1)^{yx}[\![L_{y\cdot z}, L_x]\!] + (-1)^{zy}[\![L_{z\cdot x}, L_y]\!] = 0, \tag{2}$$

for any homogeneous elements $x, y, z \in \mathcal{J}_0 \cup \mathcal{J}_1$ and where $L_x \in \mathrm{End}(\mathcal{J})$ denotes the linear operator of left multiplication by $x \in \mathcal{J}$ and the bracket of operators is $[\![L_x, L_y]\!] = L_xL_y - (-1)^{xy}L_yL_x$.

The algebras that produce a Jordan superalgebra through the Kantor double are usually called Jordan brackets [15]. In the case in which $(\mathcal{P}, \circ)$ is unital, Jordan brackets are characterized by a set of functional identities [13], involving the distinguished derivation $D(x) = [x, 1]$. Later, it was shown that in the unital case, Jordan brackets are in one to one correspondence with contact brackets [5], also known as generalized Poisson algebras. Unital transposed Poisson algebras are contact brackets and Jordan brackets, but for non-unital algebras they are not necessarily. In fact, one can prove that for unital transposed Poisson algebras [4], the associated Lie algebra is given by $[x, y] = D(x) \circ y - x \circ D(y)$, where $D(x) = [x, 1]$. Moreover, the Kantor double of the commutator of a Novikov-Poisson algebra is a Jordan superalgebra [19]. Recall that if we take the commutator of the Novikov multiplication of a Novikov-Poisson algebra, we obtain a transposed Poisson algebra [2]. Characterizing the subclass of transposed Poisson algebras that can be realized in this manner is an open problem. In any case, finding new classes of Jordan brackets is an interesting issue in Jordan algebras theory, and it turns out that transposed Poisson algebras are Jordan brackets. We will prove the next result.

Theorem. *The Kantor double $\mathfrak{J}(\mathcal{P})$ of a transposed Poisson algebra $\mathcal{P}$ is a Jordan superalgebra, i.e., transposed Poisson algebras are Jordan brackets. Moreover, the algebra $\mathfrak{J}(\mathcal{P})$ is simple if and only if $\mathcal{P}$ is simple and $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$.*

Moreover, we prove that simple transposed Poisson algebras with a perfect associative commutative multiplication are unital. Note that this is similar to the situation with the commutator of Novikov-Poisson algebras, see [19]. As a consequence, there are no simple Jordan superalgebras that can be constructed through the Kantor double from non-unital transposed Poisson algebras. In other words, they are all unital contact brackets.

Lastly, we close this manuscript by considering the Jordan superalgebra that arise from the Kantor double of the transposed Poisson structures constructed on the Witt algebra. This Jordan superalgebras are special and simple under certain conditions. We leave it as an open question whether there are exceptional Jordan superalgebras arising from a transposed Poisson algebras.

Any vector space, algebra or homomorphism in this paper will be considered over an arbitrary base field $\mathbb{F}$ of characteristic $p \neq 2$, if nothing is specified. We will denote the associative commutative multiplication $\circ$ by concatenation, and we will assume that it is not necessarily unital.

2. On the simple transposed Poisson algebras

This section is divided in three subsections. In the first one, we obtain some results on the existence of transposed Poisson structures on a given Lie algebra. In the second one, we prove that any simple transposed Poisson algebra has simple associated Lie bracket, and some important consequences that are derived from this fact. In the third part, we construct all the transposed Poisson structures on the modular Witt algebra, obtaining an example of a non-trivial finite-dimensional simple transposed Poisson algebra. Let us assume the Lie brackets of the transposed Poisson algebras in this section are not zero.

2.1. Note on the existence of transposed Poisson structures

As we have mentioned in the introduction, the existence of transposed Poisson structures on a given Lie algebra is related to the $\frac{1}{2}$-derivations of the algebra. The δ-derivations Der_δ of Lie algebras are well studied, see for example [8]. In Lie theory, the most important subspace of $\mathrm{Der}_{\frac{1}{2}}$ is the centroid.

Definition 1. The centroid of an algebra $(\mathcal{L}, [\cdot,\cdot])$ is the set $\Gamma(\mathcal{L})$ of linear maps $\phi : \mathcal{L} \to \mathcal{L}$ such that $\phi([x, y]) = [x, \phi(y)]$, for any $x, y \in \mathcal{L}$. An algebra is called central if for every $\phi \in \Gamma(\mathcal{L})$ we have that $\phi(x) = \alpha x$ for some $\alpha \in \mathbb{F}$ and any $x \in \mathcal{P}$.

If $\mathcal{L}$ is a Lie algebra, it is easy to see that any $\phi \in \Gamma(\mathcal{L})$ is a $\frac{1}{2}$-derivation. Now, let us introduce another definition.

Definition 2. A dot-Lie bracket is an algebra $(\mathcal{P}, \circ, [\cdot,\cdot])$ such that the algebra $\mathcal{P}_A := (\mathcal{P}, \circ)$ is associative and commutative and $\mathcal{P}_L := (\mathcal{P}, [\cdot,\cdot])$ is a Lie algebra.

It was shown by [2, Proposition 2.13] that a dot-Lie bracket $(\mathcal{P}, \circ, [\cdot,\cdot])$ is a Poisson algebra and a transposed Poisson algebra if and only if the identity $x[y, z] = [xy, z] = 0$ is satisfied for any $x, y, z \in \mathcal{P}$. However, it is important to mention this additional fact: the identity $x[y, z] = [xy, z]$ implies that $x[y, z] = [xy, z] = 0$. We have collected this fact and other equivalent statements in the following proposition.

Proposition 3. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a dot-Lie bracket. Then, the following statements are equivalent.*

(1) *The algebra $\mathcal{P}$ is a transposed Poisson algebra and a Poisson algebra.*
(2) *The identity $x[y, z] = [xy, z] = 0$ is satisfied for $x, y, z \in \mathcal{P}$.*
(3) *The identity $x[y, z] = [xy, z]$ is satisfied for $x, y, z \in \mathcal{P}$.*
(4) *$P_x \in \Gamma(\mathcal{P}_L)$ for any $x \in \mathcal{P}$, where $P_x : \mathcal{P} \to \mathcal{P}$ is the linear operator of left multiplication given by $P_x(y) = x \circ y$.*

Proof. We will prove that (3) implies (1), the other implications directly follow by the cited result in [2]. Note that we have the following relation $x[y,z] = -x[z,y] = -[xz,y] = [y,xz]$. So we can write $2x[y,z] = [xy,z]+[y,xz]$, implying that $\mathcal{P}$ is a transposed Poisson algebra. By [2, Theorem 2.5], the algebra $\mathcal{P}$ satisfies the identity $x[y,z]+y[z,x]+z[x,y] = 0$. Thus, we have the following relation

$$0 = x[y,z] + y[z,x] + z[x,y] = x[y,z] + [yz,x] + [x,zy] = x[y,z] = [xy,z].$$

Therefore, $\mathcal{P}$ is also a Poisson algebra. □

The next corollary follows directly by the previous proposition.

Corollary 4. *Let $\mathcal{L}$ be a Lie algebra such that $\Gamma(\mathcal{L}) = Der_{\frac{1}{2}}(\mathcal{L})$, then any transposed Poisson structure on $\mathcal{L}$ is also a Poisson algebra. On the contrary, if $\Gamma(\mathcal{L}) \cong \mathbb{F}$, then any non-trivial transposed Poisson structure on $\mathcal{L}$ is not a Poisson algebra.*

Observe that given a Lie algebra $\mathcal{L}$ such that $[\mathcal{L},\mathcal{L}] \neq \mathcal{L}$ and $C(\mathcal{L}) \neq 0$, where $C(\mathcal{L})$ denotes the center of $\mathcal{L}$, we can construct a non-trivial transposed Poisson structure. Indeed, suppose $\mathcal{L} = \mathcal{L}^2 \oplus \mathcal{M}$ as vector spaces. Then fix some $m \in \mathcal{M}$ and $c \in C(\mathcal{L})$. Now, the product $\circ$ given by $m \circ m = c$ defines a transposed Poisson structure on $\mathcal{L}$. Moreover, we can prove the following result about the existence of transposed Poisson structures on Lie algebras $\mathcal{L}$ such that $\Gamma(\mathcal{L}) = \mathrm{Der}_{\frac{1}{2}}(\mathcal{L})$.

Theorem 5. *Let $\mathcal{L}$ be a Lie algebra such that $\Gamma(\mathcal{L}) = Der_{\frac{1}{2}}(\mathcal{L})$. Then $\mathcal{L}$ admits a non-trivial transposed Poisson structure if and only if $C(\mathcal{L}) \neq 0$ and $\mathcal{L}$ is not perfect.*

Proof. Suppose $\mathcal{L}$ admits a non-trivial transposed Poisson structure $\mathcal{P} := (\mathcal{L}, \circ, [\cdot,\cdot])$. Then there are elements $x_0, y_0 \in \mathcal{P}$ such that $x_0y_0 \neq 0$. By Corollary 4, the algebra $\mathcal{P}$ is also a Poisson algebra, so we have $x[y,z] = [xy,z] = 0$ for any $x,y,z \in \mathcal{P}$. Now, observe that $[x_0y_0, z] = 0$, that is $x_0y_0 \in C(\mathcal{L})$. Also, if we suppose $\mathcal{L}$ is perfect, then $x[y,z] = 0$ would imply that $\mathcal{P}\mathcal{P} = 0$. □

Filippov proved an interesting fact that becomes relevant at this point, see [8, Theorem 6]. Recall that a non-zero Lie algebra $\mathcal{L}$ is called prime if for any two ideals $I, J \neq 0$, we have that $[I,J] \neq 0$. It is well known that prime Lie algebras have no non-zero solvable ideal; in particular, the algebra is not solvable itself and has trivial center. The mentioned result due to Filippov is the following.

Theorem 6. *Let $\mathcal{L}$ be a prime Lie algebra over a field of characteristic $p \neq 2, 3$ endowed with a non-degenerate symmetric invariant bilinear form. Then $\Gamma(\mathcal{L}) = Der_{\frac{1}{2}}(\mathcal{L})$.*

The following result, which is a consequence of Filippovs theorem and Theorem 5, generalizes the known fact that claims that a complex simple finite-dimensional Lie algebra do not admit a non-trivial transposed Poisson structure. Note that a similar fact can be obtained for Lie superalgebras using [20, Theorem 4.8].

Corollary 7. *Let $\mathcal{L}$ be a prime Lie algebras over a field of characteristic $p \neq 2, 3$ endowed with a non-degenerate symmetric invariant bilinear form. Then $\mathcal{L}$ does not admit a non-trivial transposed Poisson structure.*

2.2. On the simple transposed Poisson algebras

An ideal in a transposed Poisson algebra $\mathcal{P}$ is a proper subspace $\mathcal{I}$ such that $\mathcal{I}\mathcal{P} \subset \mathcal{I}$ and $[\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$. We say $\mathcal{P}$ is simple if it contains no ideals. Given a transposed Poisson algebra $\mathcal{P}$, if the Lie algebra $(\mathcal{P}, [\cdot,\cdot])$ is not perfect (that is, $\mathcal{P}^2_{[\cdot,\cdot]} := [\mathcal{P}, \mathcal{P}] \neq \mathcal{P}$), then by Leibniz identity (1) we have that $\mathcal{P}^2_{[\cdot,\cdot]}$ is an ideal of the associative commutative algebra $(\mathcal{P}, \circ)$, and so is an ideal of $\mathcal{P}$. Therefore, in any simple transposed Poisson algebra the Lie bracket must be perfect. Moreover, let us introduce the following notion.

Definition 8. A transposed quasi-ideal of a transposed Poisson algebra $(\mathcal{P}, \circ, [\cdot,\cdot])$ is a proper subspace $\mathcal{I}$ of $\mathcal{P}$ such that $[\mathcal{P}, \mathcal{I}] \subset \mathcal{I}$ and $[\mathcal{P}\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$.

This notion is the transposed version of the notion of a quasi-ideal of a Poisson algebra. Recall that a quasi-ideal is a proper subspace $\mathcal{I}$ of $\mathcal{P}$ such that $\mathcal{P}\mathcal{I} \subset \mathcal{I}$ and $[\mathcal{P}, \mathcal{I}]\mathcal{P} \subset \mathcal{I}$. Any simple Poisson algebra contains no quasi-ideals and the same is valid for transposed Poisson algebras and transposed quasi-ideals as we show in the next result.

Lemma 9. *A simple transposed Poisson algebra contains no transposed quasi-ideals.*

Proof. Suppose $\mathcal{I}$ is a transposed quasi-ideal of $\mathcal{P}$. Consider a maximal subspace $\mathcal{I}'$ such that $[\mathcal{P}, \mathcal{I}'] \subset \mathcal{I}$. We will show that $\mathcal{I}'$ is an ideal of $\mathcal{P}$. Observe that the Lie bracket is perfect, because $\mathcal{P}$ is simple, so $\mathcal{I}' \neq \mathcal{P}$. Also, we can assume $\mathcal{P}\mathcal{I} \subset \mathcal{I}'$, by the maximality of $\mathcal{I}'$ and since $\mathcal{I}$ is a transposed quasi-ideal. Now, since $x[y, z] + y[z, x] + z[x, y] = 0$ (see [2, Theorem 2.5]), we have that $\mathcal{I}'[y, z] \subset y[z, \mathcal{I}'] + z[\mathcal{I}', y] \subset \mathcal{I}'$. Finally, the Lie bracket is perfect so $\mathcal{I}'\mathcal{P} \subset \mathcal{I}'$, and also $[\mathcal{P}, \mathcal{I}'] \subset \mathcal{I} \subset \mathcal{I}'$. Therefore, $\mathcal{I}'$ is a proper ideal, which contradicts the simplicity of $\mathcal{P}$. □

Lemma 10. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a transposed Poisson algebra and suppose that the associated Lie bracket is perfect, i.e., $\mathcal{P}^2_{[\cdot,\cdot]} = \mathcal{P}$. Then any ideal in the Lie algebra $(\mathcal{P}, [\cdot,\cdot])$ is a transposed quasi-ideal.*

Proof. Suppose that $\mathcal{I}$ is an ideal of the associated Lie bracket of a transposed Poisson algebra $\mathcal{P}$. By Theorem 2.5 in [2], we have the identity $[hx, [y, z]] = -[hy, [z, x]] - [hz, [x, y]]$, so we can write

$$[\mathcal{P}\mathcal{I}, [\mathcal{P}, \mathcal{P}]] \subset [\mathcal{P}\mathcal{P}, [\mathcal{P}, \mathcal{I}]] + [\mathcal{P}\mathcal{P}, [\mathcal{I}, \mathcal{P}]] \subset \mathcal{I}.$$

Therefore $[\mathcal{P}\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$, that is, $\mathcal{I}$ is a transposed quasi-ideal. □

As a consequence of the previous two lemmas, we have the following result that shows that a transposed Poisson algebra is simple if and only if the Lie bracket is simple.

Theorem 11. *Any simple transposed Poisson algebra has simple Lie bracket.*

Proof. Suppose $\mathcal{P}$ is a simple transposed Poisson algebra, then the Lie bracket $(\mathcal{P}, [\cdot, \cdot])$ is perfect. Now, if we suppose that $(\mathcal{P}, [\cdot, \cdot])$ is not simple, then any ideal is a quasi-ideal by Lemma 10, but this contradicts Lemma 9, because $\mathcal{P}$ is simple. Therefore, the Lie bracket must be simple. □

It follows that any complex simple finite-dimensional transposed Poisson algebra is trivial.

Theorem 12. *Suppose that* $\mathbb{F}$ *is algebraically closed and* $char(\mathbb{F}) = 0$*, then any simple finite-dimensional transposed Poisson algebra is trivial.*

Proof. By Theorem 11, any simple transposed Poisson algebra is defined on a finite-dimensional simple Lie bracket. By Corollary 7, any transposed Poisson algebra has trivial associative commutative multiplication (take the Killing form). □

We can conclude the following about a finite-dimensional transposed Poisson algebra $\mathcal{P}$ over an algebraically closed field of characteristic zero, by looking at its Lie bracket. There are three non-trivial possibilities summarized below.

(1) If the Lie bracket is simple, then the associative commutative multiplication is trivial.
(2) If the Lie bracket is perfect and non-simple, then there exist a transposed quasi-ideal, and, therefore, an ideal. We will study this case in the next section and prove that the associative part must be nilpotent.
(3) If the Lie bracket is non-perfect and non-zero, then $\mathcal{P}^2_{[\cdot,\cdot]}$ is an ideal of $\mathcal{P}$.

Remark 13. Observe that there are simple infinite dimensional Lie algebras that admit a non-trivial transposed Poisson structure over the complex field, such as the Witt algebra (simple), with the algebra of Laurent polynomials (non-simple).

Another consequence of Theorem 11 is the next corollary.

Corollary 14. *A non-trivial transposed Poisson structure defined on a simple associative commutative algebra has simple associated Lie bracket.*

Note that an example of a non-trivial transposed Poisson algebra with simple associative commutative part can be constructed by defining the bracket $[x, y] = d(x)y - xd(y)$

on the field of formal series in one variable $\mathbb{C}[[x]]$, where d is a formal derivative. It is well known that this construction gives rise to a transposed Poisson algebra, see [2, Proposition 2.2]. By the previous corollary, this Lie bracket is simple, since it is clearly not zero. This example was found by A. Dzhumadil'daev.

Theorem 15. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a transposed Poisson algebra. The associated Lie algebra $(\mathcal{P}, [\cdot,\cdot])$ is a direct sum of simple ideals if and only if the transposed Poisson algebra $\mathcal{P}$ is a direct sum of simple ideals.*

Proof. Suppose the Lie algebra $(\mathcal{P}, [\cdot,\cdot])$ is a direct sum of the simple ideals $\mathcal{I}_r$ for r in some set of indices I, then it is perfect. Also, each of these ideals is a quasi-ideal, by Lemma 10. Now, observe that the maximal subspace $\mathcal{I}'_r$ such that $[\mathcal{P}, \mathcal{I}'_r] \subset \mathcal{I}_r$, from the proof of Lemma 9, is $\mathcal{I}'_r = \mathcal{I}_r$, because $[\mathcal{P}, \mathcal{I}_k] = \mathcal{I}_k$ for $k \in I$. Hence, by a similar argument, it follows that $\mathcal{I}_r\mathcal{P} \subset \mathcal{I}_r$, so every $\mathcal{I}_r$ is a ideal of the transposed Poisson algebra $\mathcal{P}$. Finally, since the ideals $\mathcal{I}_r$ are simple as Lie algebras, they are simple as transposed Poisson algebras. Conversely, it is clear that if the transposed Poisson algebra $\mathcal{P}$ is a direct sum of ideals, then the associated Lie algebra is also a direct sum of these ideals. Now, since they are simple as transposed Poisson algebras, their Lie bracket must be simple, by Theorem 11. □

In an algebraically closed field of characteristic zero, finite-dimensional semisimple (radical is zero) Lie algebras are precisely those that can be written as a direct sum of simple ideals. Also, semisimple Lie algebras have no non-trivial $\frac{1}{2}$-derivations [7]. Hence, they have no non-trivial transposed Poisson structures. By the previous theorem, we have a more general result than Theorem 12.

Theorem 16. *Suppose that $\mathbb{F}$ is algebraically closed and char$(\mathbb{F}) = 0$. Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a finite-dimensional transposed Poisson algebra. If the transposed Poisson algebra $\mathcal{P}$ is a direct sum of simple ideals, then it is trivial.*

The last consequence of Theorem 12 we are mentioning is related with the weak Leibniz algebras. A weak Leibniz algebra $\mathcal{L}$ is an algebra satisfying, for $x, y, z \in \mathcal{L}$, the identities

$$(xy)z - (yx)z = 2x(yz) - 2y(xz) \text{ and } x(yz) - x(zy) = 2(xy)z - 2(xz)y.$$

Given a transposed Poisson algebra $(\mathcal{P}, \circ, [\cdot,\cdot])$, we can consider a new multiplication $\cdot$ in $\mathcal{P}$, defined by $x \cdot y = x \circ y + [x, y]$ for any $x, y \in \mathcal{P}$. The new algebra $(\mathcal{P}, \cdot)$ is called the depolarization of $\mathcal{P}$. Conversely, given a weak Leibniz algebra $(\mathcal{L}, \cdot)$, we can consider the multiplications $\circ$ and $[\cdot,\cdot]$ in $\mathcal{L}$, defined by $x \circ y = \frac{1}{2}(x\cdot y + y\cdot x)$ and $[x, y] = \frac{1}{2}(x\cdot y - y\cdot x)$, for any $x, y \in \mathcal{L}$. This new algebra $(\mathcal{L}, \circ, [\cdot,\cdot])$ is called the polarization of $\mathcal{L}$. It was proven that any depolarized transposed Poisson algebra is a weak Leibniz algebra and that any polarized weak Leibniz algebra is a transposed Poisson algebra, see [6]. It is clear that simplicity is preserved by polarization and depolarization. Therefore, the following corollary is obtained.

Corollary 17. *Suppose that* $\mathbb{F}$ *is algebraically closed and char*$(\mathbb{F}) = 0$*, then any simple finite-dimensional weak Leibniz algebra is a commutative associative algebra or a Lie algebra.*

2.3. A non-trivial simple finite-dimensional transposed Poisson algebra

In this section we show that there are non-trivial simple finite-dimensional transposed Poisson algebras over fields of positive characteristic. Let us first discuss a non-example briefly. Consider the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic $p > 2$. It is well known that $\mathfrak{sl}_n(\mathbb{F})$ is simple if and only if $p \nmid n$. An straightforward calculation shows that the Killing form of $\mathfrak{sl}_n(\mathbb{F})$ is given by $k(x, y) = 2n \operatorname{tr}(xy)$ where $x, y \in \mathfrak{sl}_n(\mathbb{F})$ and xy denotes the composition, which is non-degenerate if and only if $p \nmid 2n$. For $p > 3$, it follows by Corollary 7 that any transposed Poisson structure on $\mathfrak{sl}_n(\mathbb{F})$ with $p \nmid n$ is trivial.

Now, we will provide an example of a non-trivial simple finite-dimensional transposed Poisson algebras in characteristic $p > 3$. The case in which $p = 3$ will be discussed later. Recall the definition of the modular Witt algebra. Let $\mathcal{A}_p = \mathbb{F}[x]/(x^p)$ be the truncated polynomial algebra in one variable, where (x^p) is the ideal generated by x^p and $\mathbb{F}$ has characteristic p. The modular Witt algebra $\mathcal{W}_p$ is the Lie algebra of derivations of $\mathcal{A}_p$. It is well known that $\mathcal{W}_p$ is simple for $p > 2$ and that it has a basis $e_{-1}, e_0, \ldots, e_{p-2}$ such that the multiplication is given by $[e_i, e_j] = (j - i)e_{i+j}$ for $-1 \leq i, j \leq p - 2$ and where we assume that $e_k = 0$ if $k < -1$ or $k > p - 2$. Moreover, note that $\mathfrak{sl}_2(\mathbb{F})$ coincides with the modular Witt algebra in characteristic $p = 3$, that is $\mathcal{W}_3$. We will construct the transposed Poisson structures on $\mathcal{W}_p$ for every $p > 3$. First, let us study its $\frac{1}{2}$-derivations.

Lemma 18. *The space of* $\frac{1}{2}$*-derivations of the algebra* $\mathcal{W}_p$ *for* $p > 3$ *is generated by the linear maps* $D_0, D_+ \in End(\mathcal{W}_p)$*, where* D_0 *is the identity and* $D_+(e_i) = e_{i+1}$*, for* $-1 \leq i \leq p - 2$*.*

Proof. Let D be a $\frac{1}{2}$-derivation of $\mathcal{W}_p$, then it is given by $D(e_i) = \sum_{-1 \leq k \leq p-2} \alpha_{k,i} e_k$ for $\alpha_{k,i} \in \mathbb{F}$. Observe that

$$\begin{aligned} 2i \sum_{-1 \leq k \leq p-2} \alpha_{k,i} e_k = 2iD(e_i) = 2D([e_0, e_i]) &= [D(e_0), e_i] + [e_0, D(e_i)] \\ &= \sum_{-1 \leq k \leq p-2} \alpha_{k,0}(i - k)e_{i+k} + \sum_{-1 \leq k \leq p-2} \alpha_{k,i} k e_k \\ &= \sum_{i-1 \leq k \leq i+p-2} \alpha_{k-i,0}(2i - k)e_k + \sum_{-1 \leq k \leq p-2} \alpha_{k,i} k e_k. \end{aligned} \tag{3}$$

Thus, we obtain the relation $(2i-k)\alpha_{k,i} = (2i-k)\alpha_{k-i,0}$ for $i-1 \le k \le i+p-2$ and $(2i-k)\alpha_{k,i} = 0$ for $-1 \le k \le i-2$. Moreover, we also have the following equation

$$\begin{aligned}2(i+1)\sum_{-1\le k\le p-2}\alpha_{k,i-1}e_k &= 2(i+1)D(e_{i-1}) = 2D([e_{-1},e_i])\\ &= [D(e_{-1}),e_i] + [e_{-1},D(e_i)]\\ &= \sum_{-1\le k\le p-2}\alpha_{k,-1}(i-k)e_{k+i} + \sum_{-1\le k\le p-2}\alpha_{k,i}(k+1)e_{k-1}\\ &= \sum_{i-1\le k\le i+p-2}\alpha_{k-i,-1}(2i-k)e_k + \sum_{-2\le k\le p-3}\alpha_{k+1,i}(k+2)e_k.\end{aligned} \tag{4}$$

From here, if we assume $i \ge 1$ we obtain the relation $2(i+1)\alpha_{k,i-1} = (2i-k)\alpha_{k-i,-1} + (k+2)\alpha_{k+1,i}$ for $i-1 \le k \le i+p-2$ and $2(i+1)\alpha_{k,i-1} = (k+2)\alpha_{k+1,i}$ for $k \le i-2$.

Now, by the relations obtained above we have the condition $\alpha_{k,i} = 0$ if $i > k$ and $\alpha_{k,i} = \alpha_{k-i-1,-1}$ otherwise. Consequently, setting $\alpha_i := \alpha_{i,-1}$ for $-1 \le i \le p-2$, we have $D(e_i) = \sum_{i\le j\le p+i-1}\alpha_{j-i-1}e_j$. At this point, a straightforward verification shows that we do not need any additional condition for D to be a $\frac{1}{2}$-derivation.

Finally, it is clear that D_0 and D_+ generates the space of $\frac{1}{2}$-derivations of the algebra $\mathcal{W}_p$. □

Let us construct the transposed Poisson structures on the modular Witt algebra $\mathcal{W}_p$. This kind of problem has been considered over the complex field in [3,7] for the 3-dimensional Lie algebras and for the infinite-dimensional Witt algebra (see Remark 42). Also, see [12, Section 7.3] and the references therein for similar studies. Now, recall the notion of a mutation of an algebra.

Definition 19. Let $(\mathcal{P}, \circ)$ be an associative commutative algebra and choose $q \in \mathcal{P}$. A mutation of the algebra $\mathcal{P}$ by the element q is a new algebra $(\mathcal{P}, \circ_q)$, where for any $x, y \in \mathcal{P}$, we have the product

$$x \cdot_q y = x \circ q \circ y.$$

If $\mathcal{P}$ is unital, a mutation of $\mathcal{P}$ is called trivial if $q = 1$.

Theorem 20. *Let $(\mathcal{W}_p, \circ, [\cdot,\cdot])$ be a transposed Poisson structure defined on the modular Witt algebra $\mathcal{W}_p$ with $p > 3$. Then the algebra $(\mathcal{W}_p, \circ)$ is a mutation of an algebra isomorphic to the algebra of truncated polynomials in one variable $\mathcal{A}_p$. In particular, this mutation can be trivial. Moreover, if it is non-trivial, it is not a Poisson algebra.*

Proof. For every e_i with $-1 \le i \le p-2$ there is an associated $\frac{1}{2}$-derivation φ_i, such that $\varphi_i(e_j) = e_i \circ e_j = e_j \circ e_i = \varphi_j(e_i)$. Denoting $\varphi_i(e_j) = \sum_{j \le k \le p-2} \alpha^i_{k-j-1} e_k$, we have the following equation

$$\varphi_i(e_j) = \sum_{j \le k \le p-2} \alpha^i_{k-j-1} e_k = \sum_{i \le k \le p-2} \alpha^j_{k-i-1} e_k = \varphi_j(e_i).$$

Hence, for $i = -1$, we have $\alpha^j_k = 0$ with $-1 \le k \le j-1$ and $\alpha^j_k = \alpha^{-1}_{k-j-1}$ with $j \le k \le p-2$. Observe that this condition is sufficient to guarantee the commutativity.

Therefore, we can write

$$\begin{aligned}\varphi_i(e_j) &= \sum_{j \le k \le p-2} \alpha^i_{k-j-1} e_k = \sum_{j \le k \le p-2} \alpha^{-1}_{k-i-j-2} e_k = \sum_{-i-2 \le k \le p-i-j-4} \alpha^{-1}_k e_{k+i+j+2} \\ &= \left(\sum_{-1 \le k \le p-i-j-4} \alpha^{-1}_k e_k \right) \cdot e_{i+j+1} = \left(\sum_{-1 \le k \le p-2} \alpha^{-1}_k e_k \right) \cdot e_{i+j+1},\end{aligned} \tag{5}$$

where the product $\cdot$ in $\mathcal{W}_p$ is given by $e_i \cdot e_j = e_{i+j+1}$, with the convention $e_k = 0$ if $k > p-2$. Note that the algebra $(\mathcal{W}_p, \cdot)$ is isomorphic to $\mathcal{A}_p$ setting $e_{-1} \to 1 \in \mathcal{A}_p$ and $e_0 \to x$. If we set $q = \sum_{-1 \le k \le p-2} \alpha^{-1}_k e_k$, we have that the initial product $e_i \circ e_j = q \cdot e_{i+j+1} = q \cdot e_i \cdot e_j$. Meaning that $(\mathcal{W}_p, \circ)$ is a mutation of an algebra isomorphic to $\mathcal{A}_p$. Since $\mathcal{A}_p$ is associative and $(\mathcal{W}_p, \circ)$ is a mutation of an algebra isomorphic to $\mathcal{A}_p$, the algebra $(\mathcal{W}_p, \circ)$ is also associative.

Finally, since the modular Witt algebra is perfect, the non-trivial transposed Poisson structures on $\mathcal{W}_p$ are not Poisson algebras, by virtue of Proposition 3. □

Now, let us discuss the case when $p = 3$. Observe that the algebra $\mathcal{W}_3$ is given by the products

$$[e_{-1}, e_0] = e_{-1}, \quad [e_{-1}, e_1] = 2e_0, \quad [e_0, e_1] = e_1.$$

Remark 21. The space of $\frac{1}{2}$-derivations of $\mathcal{W}_3$ is linearly spanned by the maps $D_{-2}, D_{-1}, D_0, D_1, D_2 \in \mathrm{End}(\mathcal{W}_3)$, where D_k is given by $D_k(e_i) = e_{i+k}$ for $-1 \le i \le 1$, with the usual convention. Note that $D^2_{-1} = D_{-2}$ and $D^2_1 = D_2$, but $D_{-1}D_1$ and $D_1 D_{-1}$ are not D_0, which makes the study harder. The proof of this fact is similar to that of Lemma 18, so we will omit it.

Theorem 22. *Let $(\mathcal{W}_3, \circ, [\cdot,\cdot])$ be a transposed Poisson structure. Then the algebra $(\mathcal{W}_3, \circ)$ is a mutation of an algebra isomorphic to the algebra of truncated polynomials in one variable $\mathcal{A}_3$. In particular, this mutation can be trivial. If it is non-trivial, it is not a Poisson algebra.*

Proof. Let us denote by φ_i the $\frac{1}{2}$-derivation corresponding to e_i for $-1 \leq i \leq 1$. By the previous remark, we can write $\varphi_i = \sum_{2\leq k\leq 2} \alpha^i_k D_k$ for some $\alpha^i_k \in \mathbb{F}$. Then, the commutativity $\varphi_i(e_j) = \varphi_j(e_i)$, gives us the following equation

$$\begin{aligned}\sum_{-1\leq k\leq 1} \alpha^i_{k-j} e_k = \sum_{-2\leq k\leq 2} \alpha^i_k D_k(e_j) = \varphi_i(e_j) = \varphi_j(e_i)\\ = \sum_{-2\leq k\leq 2} \alpha^j_k D_k(e_i) = \sum_{-1\leq k\leq 1} \alpha^j_{k-i} e_k.\end{aligned}$$

This equation implies the relation $\alpha^i_{k-j} = \alpha^j_{k-i}$. It follows that there are $\alpha_{-3}, \ldots, \alpha_3 \in \mathbb{F}$ such that $\varphi_i = \sum_{-2\leq k\leq 2} \alpha_{k-i} D_k$, by setting $\alpha_i := \alpha^0_i$ for $-2 \leq i \leq 2$, $\alpha_{-3} := \alpha^1_{-2}$ and $\alpha_3 := \alpha^{-1}_2$. This condition is sufficient to guarantee the commutativity.

Next, it can be proven that $(\mathcal{W}_3, \circ)$ is a mutation of one of the following algebras

$$\begin{aligned}&\mathcal{B}_1 : e_{\bar{i}} e_{\bar{j}} = e_{\overline{i+j}} \text{ where } \bar{i} \text{ denotes } i \bmod 3,\\ &\mathcal{B}_2 : e_i e_j = e_{i+j+1} \text{ with the convention } e_k = 0 \text{ for } k > 1,\\ &\mathcal{B}_3 : e_i e_j = e_{i+j-1} \text{ with the convention } e_k = 0 \text{ for } k < -1,\\ &\mathcal{B}_4 : e_{\bar{i}} e_{\bar{j}} = \beta_{ij} e_{\overline{i+j}} \text{ where } \beta_{ij} = 2 \text{ if } i = j = \pm 1 \text{ and } \beta_{ij} = 1 \text{ otherwise.}\end{aligned} \tag{6}$$

The four algebras defined above are isomorphic to $\mathcal{A}_3$, so the statement of the theorem follows. □

3. Transposed Poisson algebras with a perfect multiplication

Suppose $\mathbb{F}$ is an algebraically closed field of characteristic zero in this section. We have proven that there are no non-trivial simple finite-dimensional transposed Poisson algebras. In this section, we study how one of the multiplications being perfect affects the other one. Let us recall a useful result about generalized derivations of Lie algebras.

A generalized derivation of a Lie algebra $(\mathcal{L}, [\cdot, \cdot])$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ such that there exist two additional linear maps $D', D'' : \mathcal{L} \to \mathcal{L}$ such that

$$[D(x), y] + [x, D'(y)] = D''([x, y]).$$

Leger and Luks proved that the generalized derivations of a finite-dimensional Lie algebra preserve the radical of the algebra [14, Theorem 6.4]. Observe that $\frac{1}{2}$-derivations are generalized derivations, and recall that the left multiplication of the associative commutative operation in any transposed Poisson algebra constructed on $\mathcal{L}$ is a $\frac{1}{2}$-derivation of $\mathcal{L}$. Let us denote the descending derived series of an ideal as $\mathcal{I}^{(0)} = \mathcal{I}$ and $\mathcal{I}^{(k)} = [\mathcal{I}^{(k-1)}, \mathcal{I}^{(k-1)}]$ for $k \geq 1$. The next result follows.

Theorem 23. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a finite-dimensional transposed Poisson algebra. Denote by $\mathcal{R}$ the radical of the associated Lie algebra $(\mathcal{P}, [\cdot,\cdot])$, then $\mathcal{R}^{(i)}$ is a (not necessarily proper) ideal of the transposed Poisson algebra $\mathcal{P}$, for $i \geq 0$.*

Proof. The radical $\mathcal{R}$ of the associated Lie algebra is an ideal of $\mathcal{P}$, as we have shown above. Now, for the second part, we note that for an arbitrary ideal $\mathcal{I}$ of a transposed Poisson algebra $\mathcal{P}$ we have that

$$\mathcal{P}[\mathcal{I},\mathcal{I}] \subset [\mathcal{P}\mathcal{I},\mathcal{I}] + [\mathcal{I},\mathcal{P}\mathcal{I}] \subset [\mathcal{I},\mathcal{I}], \quad [\mathcal{P},[\mathcal{I},\mathcal{I}]] \subset [\mathcal{I},[\mathcal{P},\mathcal{I}]] + [\mathcal{I},[\mathcal{I},\mathcal{P}]] \subset [\mathcal{I},\mathcal{I}].$$

Hence, the subspace $[\mathcal{I},\mathcal{I}]$ is an ideal of $\mathcal{P}$. In particular, we conclude that $\mathcal{R}^{(i)}$ is an ideal of $\mathcal{P}$. □

Corollary 24. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a finite-dimensional transposed Poisson algebra. Then $\mathcal{P}\mathcal{P} \subset \mathcal{R}$ as subspaces.*

Proof. If $\mathcal{R} = \mathcal{P}$, this is clear. Suppose $\mathcal{R} \neq \mathcal{P}$, then the algebra $(\mathcal{P}/\mathcal{R}, [\cdot,\cdot])$ is semisimple, hence $(\mathcal{P}/\mathcal{R}, \circ, [\cdot,\cdot])$ is trivial and $(\mathcal{P}/\mathcal{R}, \circ)$ is the zero algebra. Therefore, it follows that $\mathcal{P}\mathcal{P} \subset \mathcal{R}$. □

Observe that the previous result does not hold in prime characteristic and a counterexample can be found in Theorem 22. The consequence is the next result about transposed Poisson algebras with perfect associative commutative part.

Corollary 25. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a finite-dimensional transposed Poisson algebra such that $\mathcal{P}\mathcal{P} = \mathcal{P}$, then the associated Lie algebra is solvable. In particular, if the associative commutative algebra is unital, then the associated Lie algebra is solvable.*

Note that the example in Remark 37 shows that $\mathcal{P}\mathcal{P}$ is not always an ideal of the Lie part. Also, the next remark shows that the solvability can not be replaced by nilpotency in Corollary 25.

Remark 26. The complex Lie algebra with a basis e_1, e_2, e_3 given by the products $[e_1, e_3] = e_1, [e_2, e_3] = e_2$ is non-nilpotent solvable, and together with the unital associative commutative algebra given by $e_1e_3 = e_1, e_2e_3 = e_2, e_3e_3 = e_3$, they form a transposed Poisson structure.

On the other hand, if the Lie part is perfect, we have the following result.

Corollary 27. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a finite-dimensional transposed Poisson algebra such that $[\mathcal{P},\mathcal{P}] = \mathcal{P}$, then the associative commutative algebra $(\mathcal{P}, \circ)$ is nilpotent.*

Proof. Let us show that $\mathcal{P}^{2^n+1} \subset \mathcal{R}^{(n)}$ for $n \geq 1$. We proceed by induction using the equation (13). For $n = 1$, we have that $\mathcal{P}\mathcal{P}[\mathcal{P},\mathcal{P}] \subset [\mathcal{P}\mathcal{P},\mathcal{P}\mathcal{P}] + [\mathcal{P}\mathcal{P},\mathcal{P}\mathcal{P}] \subset [\mathcal{R},\mathcal{R}] + [\mathcal{R},\mathcal{R}] = \mathcal{R}^{(1)}$, so $\mathcal{P}^3 \subset \mathcal{R}^{(1)}$. For $n > 1$, we have

$$\mathcal{P}^{2^{n-1}}\mathcal{P}^{2^{n-1}}[\mathcal{P},\mathcal{P}] \subset [\mathcal{P}\mathcal{P}^{2^{n-1}},\mathcal{P}\mathcal{P}^{2^{n-1}}] \subset [\mathcal{R}^{(n-1)},\mathcal{R}^{(n-1)}] \subset \mathcal{R}^{(n)}.$$

So $\mathcal{P}^{2^n+1} \subset \mathcal{R}^{(n)}$. Since $\mathcal{R}$ is solvable, there exist some k such that $\mathcal{P}^{2^k+1} \subset \mathcal{R}^{(k)} = 0$. Therefore, $(\mathcal{P},\circ)$ is nilpotent. □

4. On the simple transposed Poisson superalgebras

Let $\mathbb{F}$ be an arbitrary field of characteristic different from two. In this section, we show that the results in the second section are also valid for superalgebras. Although our arguments are the same, they deserve a special mention. A transposed Poisson superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ endowed with two multiplications: an associative supercommutative multiplication $-\circ-$ and a Lie superalgebra multiplication $[\cdot,\cdot]$. Recall that the Jacobi superidentity can be written as

$$\mathfrak{L}(x,y,z) := (-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x] + (-1)^{|z||y|}[[z,x],y] = 0. \tag{7}$$

Additionally, these two operations are required to satisfy the (transposed) Leibniz superidentity:

$$2x \circ [y,z] = [x \circ y, z] + (-1)^{|x||y|}[y, x \circ z], \tag{8}$$

for homogeneous elements $x, y, z \in \mathcal{P}_0 \cup \mathcal{P}_1$. As usual, $|x|$ denotes the parity of x. Although, we may write $(-1)^x := (-1)^{|x|}$.

The identities in [2, Theorem 2.5] can be generalized to the superalgebra case as follows. They can be constructed with the usual rules to construct superidentities from identities and their proof is analogous to the proof for identities in [2].

Proposition 28. *Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ be a transposed Poisson superalgebra. Then for $x, y, z, h, u, v \in \mathcal{P}_0 \cup \mathcal{P}_1$ we have:*

$$(-1)^{xz}x[y,z] + (-1)^{yx}y[z,x] + (-1)^{zy}z[x,y] = 0. \tag{9}$$

$$(-1)^{xz}[h[x,y],z] + (-1)^{yx}[h[y,z],x] + (-1)^{zy}[h[z,x],y] = 0. \tag{10}$$

$$(-1)^{xz}[hx,[y,z]] + (-1)^{yx}[hy,[z,x]] + (-1)^{zy}[hz,[x,y]] = 0. \tag{11}$$

$$(-1)^{xz}[h,x][y,z] + (-1)^{yx}[h,y][z,x] + (-1)^{zy}[h,z][x,y] = 0. \tag{12}$$

$$[xu,vy] + (-1)^{uv}[xv,uy] = 2(-1)^{ux+vx}uv[x,y]. \tag{13}$$

$$(-1)^{vx+yu}x[u,yv] + (-1)^{vu+vy}v[xy,u] + (-1)^{xu+xy}yu[v,x] = 0. \tag{14}$$

Recall that a $\mathbb{Z}_2$-graded ideal in a transposed Poisson superalgebra $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ is a $\mathbb{Z}_2$-graded vector space $\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$ such that $\mathcal{I}\mathcal{P} \subset \mathcal{I}$ and $[\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$. A superalgebra is simple if it contains no graded ideals. If the Lie superalgebra $(\mathcal{P}, [\cdot,\cdot])$ associated to a transposed Poisson superalgebra is not perfect, then by the Leibniz identity (8), $\mathcal{P}^2_{[\cdot,\cdot]}$ is a graded ideal of $\mathcal{P}$. Indeed, we have

$$\mathcal{P}\mathcal{P}^2_{[\cdot,\cdot]} = (\mathcal{P}_0 \oplus \mathcal{P}_1)[\mathcal{P}_0 \oplus \mathcal{P}_1, \mathcal{P}_0 \oplus \mathcal{P}_1] \subset \sum_{i,j,k \in \mathbb{Z}_2} [\mathcal{P}_i\mathcal{P}_j, \mathcal{P}_k] + \sum_{i,j,k \in \mathbb{Z}_2} [\mathcal{P}_j\mathcal{P}_i, \mathcal{P}_k] \subset \mathcal{P}^2_{[\cdot,\cdot]}.$$

Therefore, in any simple transposed Poisson superalgebra the Lie bracket must be perfect. Now, let us introduce the key notion of a graded transposed quasi-ideal.

Definition 29. A graded transposed quasi-ideal of a transposed Poisson superalgebra $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ is a $\mathbb{Z}_2$-graded proper subspace $\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$ of $\mathcal{P}$ such that $[\mathcal{P}, \mathcal{I}] \subset \mathcal{I}$ and $[\mathcal{P}\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$.

The following results are the equivalents of Lemma 9 and Lemma 10 for superalgebras. The proof is a repetition of the proof of the corresponding lemmas, but considering a graded ideal and using the equations in Proposition 28. The same applies for Theorem 32.

Lemma 30. *A simple transposed Poisson superalgebra contains no graded transposed quasi-ideals.*

Lemma 31. *Let $(\mathcal{P}, \circ, [\cdot,\cdot])$ be a transposed Poisson superalgebra and suppose that the associated Lie superalgebra is perfect, i.e., $\mathcal{P}^2_{[\cdot,\cdot]} = \mathcal{P}$. Then any graded ideal in the Lie superalgebra $(\mathcal{P}, [\cdot,\cdot])$ is a graded transposed quasi-ideal.*

Using the previous lemmas, the equivalent of Theorem 11 for superalgebras is obtained.

Theorem 32. *Any simple transposed Poisson superalgebra has simple (or trivial) Lie super bracket.*

By a similar argument as in Theorem 12, but using the fact that simple finite-dimensional Lie superalgebras have no non-trivial $\frac{1}{2}$-derivations (see [11,20] for details), we can conclude the classification theorem for simple finite-dimensional transposed Poisson algebras. See also [7, Corollary 12].

Theorem 33. *Suppose that $\mathbb{F}$ is algebraically closed and $char(\mathbb{F}) = 0$, then any simple finite-dimensional transposed Poisson superalgebra is trivial.*

5. Transposed Poisson algebras and Jordan superalgebras

This section contains three subsections. In the first part, we prove that transposed Poisson algebras are Jordan brackets. In the second part, we stablish a simplicity criterion for the Kantor double of a transposed Poisson algebra. In the third part, we study some properties of the Jordan superalgebras that arise from the transposed Poisson structures defined on the Witt algebra.

5.1. The Kantor double of a transposed Poisson algebra

On this section, let $\mathbb{F}$ be an arbitrary field of characteristic $p \neq 2$. Let us recall the Kantor double construction on the most general context. Let $(\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1, \circ, [\cdot,\cdot])$ be a superalgebra equipped with two multiplications: the first one is associative and supercommutative and the second one is super skew-symmetric. This class of algebras are called dot-bracket superalgebras [13]. We will assume that they are not necessarily unital, as in the rest of the paper. Denote by $\mathcal{P}^s$ a duplication of the space $\mathcal{P}$ with opposite parity of the elements, so if $x \in \mathcal{P}_i$, its corresponding copy $x^s \in \mathcal{P}^s$ has parity $i+1 \in \mathbb{Z}_2$. Then let $\mathfrak{J}(\mathcal{P})$ be the vector space $\mathcal{P} \oplus \mathcal{P}^s$ endowed with the multiplication $*$ given for homogeneous elements $x, y \in \mathcal{P}_0 \cup \mathcal{P}_1$ by

$$x * y = x \circ y, \quad x^s * y = (-1)^x x * y^s = (x \circ y)^s, \quad x^s * y^s = (-1)^x [x, y].$$

By this construction, the algebra $\mathfrak{J}(\mathcal{P})$ is a superalgebra with even space $\mathfrak{J}(\mathcal{P})_0 = \mathcal{P}_0 \oplus \mathcal{P}_1^s$ and odd space $\mathfrak{J}(\mathcal{P})_1 = \mathcal{P}_1 \oplus \mathcal{P}_0^s$. Also, $\mathfrak{J}(\mathcal{P}) = \mathfrak{J}(\mathcal{P})_0 \oplus \mathfrak{J}(\mathcal{P})_1$ has “double” the dimension of $\mathcal{P}$, hence its name: the Kantor double of $\mathcal{P}$.

We will prove that the Kantor double of a transposed Poisson (super)algebra is a Jordan superalgebra. We may refer to superalgebras simply by algebra if the context is clear. Let $(\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1, \circ, [\cdot,\cdot])$ be a transposed Poisson superalgebra and denote by $P_a, Q_a : \mathcal{P} \to \mathcal{P}$ the linear operators corresponding to the left multiplications given by $P_a(x) = ax$ and $Q_a(x) = [a, x]$, for an homogeneous element $a \in \mathcal{P}_0 \cup \mathcal{P}_1$. Then, we have the following useful relations between these linear operators.

Proposition 34. *Given $\mathcal{P}$ a transposed Poisson superalgebra and $x, y, z \in \mathcal{P}_0 \cup \mathcal{P}_1$, we have the relations:*

1. *The associativity of $\circ$ is equivalent to $[\![P_x, P_y]\!] = 0$, since $\circ$ is also supercommutative.*
2. *The Jacobi superidentity (7) is equivalent to $[\![Q_x, Q_y]\!] = Q_{[x,y]}$, due to the super skew-symmetry.*
3. *The Leibniz rule (8) can be written as $[\![P_x, Q_y]\!] = \frac{1}{2}(Q_{xy} - (-1)^{xy} Q_y P_x)$ or $Q_{xy} + (-1)^{xy} Q_y P_x = 2 P_x Q_y$.*
4. *Moreover, a consequence of this identity is*

$$(-1)^{xz} Q_{[x,y]z} + (-1)^{yx} Q_{[y,z]x} + (-1)^{zy} Q_{[z,x]y} = 0.$$

Note that the bracket of linear operators is graded, that is, for $A, B \in End(\mathcal{P})$, we have $[\![A, B]\!] := AB - (-1)^{AB} BA$.

Proof. The relations (1) and (2) are well known for associative commutative and Lie superalgebras. The first relation in (3) applied to an element $z \in \mathcal{P}$ gives us $x[y,z] - (-1)^{xy}[y, xz] = \frac{1}{2}\left([xy, z] - (-1)^{xy}[y, xz]\right)$, which is equivalent to the Leibniz rule. For the second relation in (3), when applied to $z \in \mathcal{P}$, we obtain $[xy, z] + (-1)^{xy}[y, xz] = 2x[y, z]$. The relation (4) is a consequence of the linearity of Q and the identity (9). □

Additionally, we have the next relations involving the left multiplication operators of a transposed Poisson superalgebra.

Proposition 35. *Given $\mathcal{P}$ a transposed Poisson superalgebra and $x, y, z \in \mathcal{P}$, we have the following relations:*

(1) $P_x Q_y - (-1)^{xy} P_y Q_x = -P_{[x,y]}$.
(2) $(-1)^{zy} Q_z P_{[x,y]} + (-1)^{xz} Q_x P_{[y,z]} + (-1)^{yx} Q_y P_{[z,x]} = 0$.
(3) $(-1)^{xz} Q_{[x,y]} P_z + (-1)^{yx} Q_{[y,z]} P_x + (-1)^{zy} Q_{[z,x]} P_y = 0$.
(4) $(-1)^{xz} P_{[x,y]} Q_z + (-1)^{yx} P_{[y,z]} Q_x + (-1)^{zy} P_{[z,x]} Q_y = 0$.
(5) $Q_{xy} P_z - (-1)^{zx+zy} Q_{zx} P_y = 2P_{x[y,z]}$ *and* $(-1)^{yx} Q_{yz} P_x + (-1)^{zy} Q_{zx} P_y = 2(-1)^{xz} P_{xy} Q_z$.
(6) $(-1)^{zy+zx} P_z Q_{xy} - (-1)^{yx} P_y Q_{xz} = P_{x[y,z]}$.

Proof. These relations are obtained, respectively, by rewriting the identities of the Proposition 28 using the maps P and Q. □

The following Theorem establishes a connection between the class of transposed Poisson superalgebras and the class of Jordan superalgebras through the Kantor double construction. Namely, transposed Poisson algebras are Jordan brackets.

Theorem 36. *If $\mathcal{P}$ is a transposed Poisson algebra, then the algebra $\mathfrak{J}(\mathcal{P})$ is a Jordan superalgebra.*

Proof. We verify the supercommutativity first. Given $x, y \in \mathcal{P}_0 \cup \mathcal{P}_1$ and corresponding $x^s, y^s \in \mathcal{P}_0^s \cup \mathcal{P}_1^s$, by the definition of the multiplication we have that $x * y = xy = (-1)^{xy} yx = (-1)^{xy} y * x$,

$$x^s * y = (xy)^s = (-1)^{xy}(yx)^s = (-1)^{(x+1)y} y * x^s,$$

$$x^s * y^s = (-1)^x [x, y] = -(-1)^{xy+x}[y, x] = (-1)^{xy+y+x+1} y^s * x^s = (-1)^{(x+1)(y+1)} y^s * x^s.$$

Since the parity of x^s is $|x| + 1$, we conclude that $\mathfrak{J}(\mathcal{P})$ is supercommutative.

Now, we check the Jordan superidentity (2). Observe that the left multiplication on the superalgebra $\mathfrak{J}(\mathcal{P})$ are the linear operators $L_a, L_{a^s} : \mathfrak{J}(\mathcal{P}) \to \mathfrak{J}(\mathcal{P})$ corresponding to the matrices

$$L_a \equiv \begin{pmatrix} P_a & 0 \\ 0 & (-1)^a P_a \end{pmatrix}, \quad L_{a^s} \equiv \begin{pmatrix} 0 & (-1)^a Q_a \\ P_a & 0 \end{pmatrix},$$

where $a \in \mathcal{P}_0 \cup \mathcal{P}_1$. A straightforward verification shows that $|L_a| = |a|$ and $|L_{a^s}| = |a|+1$. Thus, by the equations (1-3) in Proposition 34, we have the following relations between the multiplication operators in $\mathfrak{J}(\mathcal{P})$.

$$[\![L_x, L_y]\!] = \begin{pmatrix} [\![P_x, P_y]\!] & 0 \\ 0 & (-1)^{x+y}[\![P_x, P_y]\!] \end{pmatrix} = 0. \tag{15}$$

$$[\![L_x, L_{y^s}]\!] = \begin{pmatrix} 0 & \frac{1}{2}(-1)^y(Q_{xy} - (-1)^{xy}Q_y P_x) \\ 0 & 0 \end{pmatrix}. \tag{16}$$

$$[\![L_{x^s}, L_{y^s}]\!] = \begin{pmatrix} (-1)^x(Q_x P_y + (-1)^{xy}Q_y P_x) & 0 \\ 0 & (-1)^y(P_x Q_y + (-1)^{xy}P_y Q_x) \end{pmatrix}. \tag{17}$$

Proceed by considering the various cases that arise depending on the parity.

(1) For $x, y, z \in \mathcal{P}_0 \cup \mathcal{P}_1$. Then the Jordan superidentity (2) is verified, as a consequence of (15).

(2) For homogeneous elements $x, y \in \mathcal{P}$ and $z^s \in \mathcal{P}^s$, we have

$$\begin{aligned}&(-1)^{x(z+1)}[\![L_{x*y}, L_{z^s}]\!] + (-1)^{yx}[\![L_{y*z^s}, L_x]\!] + (-1)^{(z+1)y}[\![L_{z^s*x}, L_y]\!] \\ &= (-1)^{x(z+1)}[\![L_{xy}, L_{z^s}]\!] + (-1)^{yx}[\![L_{(-1)^y(yz)^s}, L_x]\!] + (-1)^{(z+1)y}[\![L_{(zx)^s}, L_y]\!].\end{aligned}$$

Applying (16), we can write the expression above as the sum of matrices

$$\begin{aligned}&(-1)^{xz+x}\begin{pmatrix} 0 & \frac{1}{2}(-1)^z(Q_{xyz} - (-1)^{xz+yz}Q_z P_{xy}) \\ 0 & 0 \end{pmatrix} \\ &-(-1)^{xz+y+x}\begin{pmatrix} 0 & \frac{1}{2}(-1)^{y+z}(Q_{xyz} - (-1)^{xy+xz}Q_{yz} P_x) \\ 0 & 0 \end{pmatrix} \\ &-(-1)^{yx}\begin{pmatrix} 0 & \frac{1}{2}(-1)^{x+z}(Q_{yzx} - (-1)^{yz+yx}Q_{zx} P_y) \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

The upper right term, up to $\frac{1}{2}(-1)^{x+z}$, is equal to

$$\begin{aligned}&(-1)^{xz}(Q_{xyz} - (-1)^{xz+yz}Q_z P_{xy}) - (-1)^{xz}(Q_{xyz} - (-1)^{xy+xz}Q_{yz} P_x) \\ &-(-1)^{yx}(Q_{yzx} - (-1)^{yz+yx}Q_{zx} P_y).\end{aligned}$$

After the simplification using the supercommutativity of the transposed Poisson superalgebra, we have

$$-(-1)^{xz}Q_{xyz} - (-1)^{yz}Q_zP_{xy} + (-1)^{xy}Q_{yz}P_x + (-1)^{yz}Q_{zx}P_y.$$

Using the second relation in (3) of Proposition 34 and the equation (5) in Proposition 35, this is zero.

(3) For homogeneous elements $x \in \mathcal{P}$ and $y^s, z^s \in \mathcal{P}^s$, we have

$$\begin{aligned}&(-1)^{x(z+1)}[\![L_{x*y^s}, L_{z^s}]\!] + (-1)^{(y+1)x}[\![L_{y^s*z^s}, L_x]\!] + (-1)^{(z+1)(y+1)}[\![L_{z^s*x}, L_{y^s}]\!]\\ &\quad= (-1)^{x(z+1)}[\![L_{(-1)^x(xy)^s}, L_{z^s}]\!] + (-1)^{(y+1)x}[\![L_{(-1)^y[y,z]}, L_x]\!]\\ &\qquad+ (-1)^{(z+1)(y+1)}[\![L_{(zx)^s}, L_{y^s}]\!].\end{aligned}$$

By the equations (15) and (17), the previous expression can be written as the following sum of matrices

$$\begin{aligned}&(-1)^{xz}\begin{pmatrix}(-1)^{x+y}(Q_{xy}P_z + (-1)^{xz+yz}Q_zP_{xy}) & 0\\ 0 & (-1)^z(P_{xy}Q_z + (-1)^{xz+yz}P_zQ_{xy})\end{pmatrix}\\ &\quad+ (-1)^{zy+z+y+1}\\ &\quad\times\begin{pmatrix}(-1)^{z+x}(Q_{zx}P_y + (-1)^{zy+xy}Q_yP_{zx}) & 0\\ 0 & (-1)^y(P_{zx}Q_y + (-1)^{zy+xy}P_yQ_{zx})\end{pmatrix}.\end{aligned}$$

The upper left term is, up to $(-1)^{x+y}$, equal to

$$\begin{aligned}&(-1)^{xz}(Q_{xy}P_z + (-1)^{xz+yz}Q_zP_{xy}) - (-1)^{zy}(Q_{zx}P_y + (-1)^{zy+xy}Q_yP_{zx})\\ &= (-1)^{xz}Q_{xy}P_z + (-1)^{yz}Q_zP_{xy} - (-1)^{zy}Q_{zx}P_y - (-1)^{xy}Q_yP_{zx}.\end{aligned}$$

By the relation (5) in Proposition 35, we have the Leibniz rule.

$$(-1)^{xz}2P_{x[y,z]} + (-1)^{yz}Q_zP_{xy} - (-1)^{xy}Q_yP_{zx} = 0.$$

The lower right term is, up to $(-1)^z$, equal to

$$\begin{aligned}&(-1)^{xz}(P_{xy}Q_z + (-1)^{xz+yz}P_zQ_{xy}) - (-1)^{zy}(P_{zx}Q_y + (-1)^{zy+xy}P_yQ_{zx})\\ &= (-1)^{xz}P_{xy}Q_z + (-1)^{yz}P_zQ_{xy} - (-1)^{zy}P_{zx}Q_y - (-1)^{xy}P_yQ_{zx}.\end{aligned}$$

From here, we can use equation (6) in Proposition 35 to obtain an expression which is zero, by (1) in Proposition 35.

$$\begin{aligned}&(-1)^{xz}P_{xy}Q_z - (-1)^{zy}P_{zx}Q_y + (-1)^{zx}P_{x[y,z]}\\ &\quad= (-1)^{xz}P_x(P_yQ_z - (-1)^{zy}P_zQ_y + P_{[y,z]}) = 0.\end{aligned}$$

(4) Lastly, for homogeneous elements $x^s, y^s, z^s \in \mathcal{P}^s$, the Jordan identity is

$$\begin{aligned}
&(-1)^{(x+1)(z+1)}[\![L_{x^s*y^s}, L_{z^s}]\!] + (-1)^{(y+1)(x+1)}[\![L_{y^s*z^s}, L_{x^s}]\!] \\
&\quad + (-1)^{(z+1)(y+1)}[\![L_{z^s*x^s}, L_{y^s}]\!] \\
&= (-1)^{(x+1)(z+1)}[\![L_{(-1)^x[x,y]}, L_{z^s}]\!] + (-1)^{(y+1)(x+1)}[\![L_{(-1)^y[y,z]}, L_{x^s}]\!] \\
&\quad + (-1)^{(z+1)(y+1)}[\![L_{(-1)^z[z,x]}, L_{y^s}]\!].
\end{aligned}$$

Applying the equation (16) and arguing as before, every term in the operator matrix is zero except for

$$\begin{aligned}
&(-1)^{xz}Q_{[x,y]z} - (-1)^{yz}Q_zP_{[x,y]} + (-1)^{yx}Q_{[y,z]x} - (-1)^{zx}Q_xP_{[y,z]} \\
&\quad + (-1)^{zy}Q_{[z,x]y} - (-1)^{xy}Q_yP_{[z,x]}.
\end{aligned}$$

Using the relation (4) in Proposition 34, we obtain the next expression, which is zero, by (2) in Proposition 35.

$$-(-1)^{yz}Q_zP_{[x,y]} - (-1)^{zx}Q_xP_{[y,z]} - (-1)^{xy}Q_yP_{[z,x]} = 0.$$

Due to the cyclic nature of the Jordan superidentity, the four cases considered here are enough to prove that $\mathfrak{J}(\mathcal{P})$ satisfies the Jordan superidentity (2), and therefore, it is a Jordan superalgebra. □

There is another functor introduced by Kantor in [10]. Kantor defines a second construction that produces a Lie superalgebra, given a Poisson algebra, by defining a new product $*_t$ on $\mathcal{P} \oplus \mathcal{P}^s$ in the following way. Given $a, b \in \mathcal{P}$ and corresponding $a^s, b^s \in \mathcal{P}^s$, we set

$$a *_t b = [a, b], \quad a^s *_t b = a *_t b^s = ([a, b])^s, \quad a^s *_t b^s = a \circ b.$$

Following the spirit of [2], we will call it the transposed Kantor double.

Remark 37. A straightforward verification of the Jacobi superidentity for the complex 3-dimensional transposed Poisson algebra $\mathcal{P}$ with a basis e_1, e_2, e_3 defined by the non-zero products:

$$\mathcal{P} : \begin{cases} e_3 \cdot e_3 = e_1, \\ [e_1, e_3] = e_1 + e_2, \end{cases}$$

shows that the transposed Kantor double is not a Lie superalgebra. Indeed, the Leibniz superidentity

$$\mathfrak{L}(e_3^s, e_3^s, e_3^s) = -3(e_1 + e_2)^s \neq 0.$$

We leave as an open problem the study of the class obtained by considering the transposed Kantor double of the variety of transposed Poisson algebras.

5.2. Simplicity criterion for the Kantor double of transposed Poisson algebras

Observe that if a transposed Poisson algebra $\mathcal{P}$ has an ideal $\mathcal{I}$, then its Kantor double has an ideal $\mathcal{I} \oplus \mathcal{I}^s$. Therefore, if $\mathfrak{J}(\mathcal{P})$ is simple, then $\mathcal{P}$ is simple. The converse is more complicate. Kantor proved that the double of a non-trivial Poisson (super)algebra $\mathcal{P}$ is a simple Jordan superalgebra if and only if $\mathcal{P}$ is simple [10]. Later, King and McCrimmon extended this result to unital non-trivial Jordan brackets [13]. For non-unital Jordan brackets and, in particular, for non-unital transposed Poisson algebras, this result does not hold, as we can see in the next remark.

Remark 38. Consider the family of simple transposed Poisson algebras $\mathcal{P}(\alpha, \beta)$ with $\alpha\beta = 0$ over a field of characteristic 3 that can be extracted from Theorem 22.

$$\mathcal{P}^{\alpha,\beta} : \begin{cases} e_1 \circ e_1 = \alpha e_2, \quad e_2 \circ e_2 = \beta e_1, \\ [e_1, e_2] = e_3, \quad [e_3, e_2] = -2e_2, \quad [e_3, e_1] = 2e_1. \end{cases}$$

The Kantor double of $\mathcal{P}^{\alpha,\beta}$ is the Jordan superalgebra defined on $\mathcal{P} \oplus \mathcal{P}^s$, with multiplication given by

$$\begin{array}{lll} e_1 * e_1 = \alpha e_2, & e_1 * e_1^s = \alpha e_2^s, & e_1^s * e_1 = \alpha e_2^s, \\ e_2 * e_2 = \beta e_1, & e_2 * e_2^s = \beta e_1^s, & e_2^s * e_2 = \beta e_1^s, \\ e_1^s * e_2^s = e_3, & e_3^s * e_2^s = -2e_2, & e_3^s * e_1^s = 2e_1. \end{array}$$

Note that $\mathfrak{J}(\mathcal{P})e_3 = e_3\mathfrak{J}(\mathcal{P}) = 0$. Hence, the superalgebra $\mathfrak{J}(\mathcal{P})$ is not simple. Also, note that $\mathcal{P}\mathcal{P} \neq \mathcal{P}$ and that the proper subspace $\mathcal{I} = \text{span}(\beta e_1, \alpha e_2)$ is a quasi-ideal of $\mathcal{P}$.

Moreover, suppose $\mathcal{P}$ is a simple transposed Poisson algebra such that $\mathcal{P}\mathcal{P} \neq \mathcal{P}$, then $\mathcal{P} \oplus (\mathcal{P}\mathcal{P})^s$ is an ideal of $\mathfrak{J}(\mathcal{P})$. Indeed, we have

$$(\mathcal{P} \oplus (\mathcal{P}\mathcal{P})^s) * (\mathcal{P} \oplus \mathcal{P}^s) \subset \mathcal{P}\mathcal{P} + (\mathcal{P}\mathcal{P})^s + ((\mathcal{P}\mathcal{P})\mathcal{P})^s + [\mathcal{P}\mathcal{P}, \mathcal{P}] \subset (\mathcal{P} \oplus (\mathcal{P}\mathcal{P})^s).$$

Hence, if $\mathfrak{J}(\mathcal{P})$ is simple, then $\mathcal{P}$ is simple and $\mathcal{P}\mathcal{P} = \mathcal{P}$. The converse of this fact also holds, as we will see. To prove it, we need the following lemma on simple transposed Poisson algebras with perfect associative part.

Lemma 39. *Any simple transposed Poisson (super)algebra* $(\mathcal{P}, \circ, [\cdot, \cdot])$ *such that* $\mathcal{P}\mathcal{P} = \mathcal{P}$ *contains no quasi-ideals.*

Proof. Suppose $\mathcal{I}$ is a quasi-ideal of $\mathcal{P}$. Choose a maximal subspace $\mathcal{I}'$ such that $\mathcal{I}'\mathcal{P} \subset \mathcal{I}$. Then $\mathcal{I} \subset \mathcal{I}'$ and $\mathcal{I}' \neq \mathcal{P}$, because $\mathcal{P}\mathcal{P} = \mathcal{P}$. Now, we have $\mathcal{I}'\mathcal{P} \subset \mathcal{I} \subset \mathcal{I}'$ and

$[\mathcal{I},\mathcal{P}] \subset \mathcal{I}'$, by the maximality of $\mathcal{I}'$. Moreover, using the transposed Leibniz rule, we have the following inclusion

$$[\mathcal{I}',\mathcal{P}\mathcal{P}] \subset \mathcal{P}[\mathcal{I}',\mathcal{P}] + [\mathcal{P}\mathcal{I}',\mathcal{P}].$$

So $[\mathcal{I}',\mathcal{P}] \subset \mathcal{P}[\mathcal{I}',\mathcal{P}] + [\mathcal{P}\mathcal{I}',\mathcal{P}]$. The first term satisfies $\mathcal{P}[\mathcal{I}',\mathcal{P}] \subset \mathcal{I}'$, since $\mathcal{P}\mathcal{P}[\mathcal{I}',\mathcal{P}] \subset [\mathcal{I}'\mathcal{P},\mathcal{P}\mathcal{P}] \subset \mathcal{I}'$, using equation (13) and $\mathcal{P}\mathcal{P} = \mathcal{P}$. The second term is $[\mathcal{P}\mathcal{I}',\mathcal{P}] \subset [\mathcal{I},\mathcal{P}] \subset \mathcal{I}'$. Therefore, the subspace $\mathcal{I}'$ is an ideal, which contradicts the simplicity. □

We can prove the simplicity criterion for the Kantor double of a transposed Poisson algebra. The converse was proved above.

Theorem 40. *Let $(\mathcal{P},\circ,[\cdot,\cdot])$ be a simple transposed Poisson (super)algebra such that $\mathcal{P}\mathcal{P} = \mathcal{P}$, then $\mathfrak{J}(\mathcal{P})$ is simple.*

Proof. Suppose $\mathcal{I}$ is an ideal with projections $\mathcal{I}_0$, $\mathcal{I}_1^s$ on $\mathcal{P}$ and $\mathcal{P}^s$, respectively. By the definition of the Kantor double, we have the following relations

$$\mathcal{P}\mathcal{I}_0 \subset \mathcal{I}_0 \cap \mathcal{I}_1, \quad \mathcal{P}\mathcal{I}_1 \subset \mathcal{I}_1, \quad [\mathcal{P},\mathcal{I}_1] \subset \mathcal{I}_0.$$

The subspace $\mathcal{J} := \mathcal{I}_0 \cap \mathcal{I}_1$ is a quasi-ideal of $\mathcal{P}$, since $\mathcal{J}\mathcal{P} \subset \mathcal{J}$ and $[\mathcal{J},\mathcal{P}]\mathcal{P} \subset \mathcal{I}_0\mathcal{P} \subset \mathcal{J}$. So either $\mathcal{J} = 0$ or $\mathcal{J} = \mathcal{P}$, by Lemma 39. If $\mathcal{J} = 0$, then $\mathcal{I}_0\mathcal{P} = 0$ and, using the equation (13) we have $\mathcal{P}[\mathcal{I}_0,\mathcal{P}] = 0$, since $\mathcal{P}\mathcal{P}[\mathcal{I}_0,\mathcal{P}] \subset [\mathcal{I}_0\mathcal{P},\mathcal{P}\mathcal{P}] + [\mathcal{I}_0\mathcal{P},\mathcal{P}\mathcal{P}] \subset 0$, so $\mathcal{I}_0$ is a quasi ideal, and then either $\mathcal{I}_0 = 0$ or $\mathcal{I}_0 = \mathcal{P}$. In the first case, we have that $\mathcal{I}_1$ is a non-zero ideal of $\mathcal{P}$, so $\mathcal{I}_1 = \mathcal{P}$. But then $\mathcal{I} * \mathcal{P} = \mathcal{I}_1^s * \mathcal{P}^s = [\mathcal{P},\mathcal{P}] \neq 0$, which is a contradiction. In the second case, $\mathcal{I}_0 = \mathcal{P}$ implies that $\mathcal{P}\mathcal{P} \subset \mathcal{J} = 0$, another contradiction. If $\mathcal{J} = \mathcal{P}$, then the arguments are analogous to [10, Theorem 3.4]. □

Now, we want to investigate whether it is possible to obtain a simple Jordan superalgebra from a non-unital transposed Poisson algebra. It turns out that it is not possible. They have to be unital (and therefore, contact brackets) to give rise to a simple Jordan superalgebra, as the following theorem shows.

Recall that an algebra is differentiably simple for a family of derivations $\mathfrak{D}$ if it is non-trivial and it contains no $\mathfrak{D}$-invariant ideals. We say an algebra is differentiably simple if this is true for some family of derivations. Posner proved that differentiably simple associative commutative algebras are unital [17].

Theorem 41. *Let $(\mathcal{P},\circ,[\cdot,\cdot])$ be a simple transposed Poisson algebra such that $\mathcal{P}\mathcal{P} = \mathcal{P}$, then $(\mathcal{P},\circ)$ is differentiably simple and, therefore, unital.*

Proof. Let us prove that $(\mathcal{P},\circ)$ is differentiably simple. Consider the set $\mathfrak{D}$ of linear endomorphisms of $\mathcal{P}$ given by

$$\mathfrak{D} = \{D_{xy} : x, y \in \mathcal{P}\},$$

where $D_{xy} : \mathcal{P} \to \mathcal{P}$ is given by $D_{xy}(a) = [ax, y] - a[x, y]$. We have

$$\begin{aligned} D_{xy}(a)b + aD_{xy}(b) &= b[ax, y] - ab[x, y] + a[bx, y] - ab[x, y] \\ &= \frac{1}{2}[abx, y] + \frac{1}{2}[ax, by] - ab[x, y] + \frac{1}{2}[abx, y] + \frac{1}{2}[bx, ay] - ab[x, y] \\ &= [abx, y] - ab[x, y] = D_{xy}(ab). \end{aligned}$$

Therefore, $\mathfrak{D}$ is a family of derivations of the algebra $(\mathcal{P}, \circ)$. Suppose there is some $\mathfrak{D}$-invariant ideal $\mathcal{I}$ of $(\mathcal{P}, \circ)$. Let us show that it is a quasi-ideal of $\mathcal{P}$. Indeed, using equation (13), we have

$$\mathcal{P}\mathcal{P}[\mathcal{I}, \mathcal{P}] \subset [\mathcal{I}\mathcal{P}, \mathcal{P}\mathcal{P}] \subset [\mathcal{I}\mathcal{P}, \mathcal{P}].$$

Since $\mathcal{I}$ is $\mathfrak{D}$-invariant, we have $[ax, y] - a[x, y] \in \mathcal{I}$, for any $x, y \in \mathcal{P}$ and $a \in \mathcal{I}$. That is, $[ax, y] \in \mathcal{I}$ and $[\mathcal{I}\mathcal{P}, \mathcal{P}] \subset \mathcal{I}$. So $\mathcal{P}\mathcal{P}[\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$. Since $\mathcal{P}\mathcal{P} = \mathcal{P}$, we can write $\mathcal{P}[\mathcal{I}, \mathcal{P}] \subset \mathcal{I}$, implying that $\mathcal{I}$ is a quasi-ideal. This contradicts Lemma 39, so $(\mathcal{P}, \circ)$ is differentiably simple. Hence, by [17, Theorem 5], it is unital. □

5.3. The Jordan superalgebra that arise from the Laurent-Witt transposed Poisson algebra

Let us fix $\mathbb{F}$ to be the complex field in this section. The Lie algebra of derivations of the algebra of Laurent Polynomials $\mathcal{L}$ is the Witt algebra $\mathcal{W}$. The Witt algebra is the complex vector space generated by $\{e_i : i \in \mathbb{Z}\}$ together with the multiplication $[e_i, e_j] = (j - i)e_{i+j}$.

Remark 42. In [7, Theorem 20], the authors proved that any transposed Poisson structure on the Witt algebra is a mutation of the algebra of Laurent polynomials. Precisely, if $(\mathcal{W}, \circ, [\cdot, \cdot])$ is a transposed Poisson algebra, then there exist some $q \in \mathcal{W}$ such that $e_i \circ e_j = q \cdot e_{i+j}$, where $(\mathcal{W}, \cdot) \cong \mathcal{L}$ is given by $e_i \cdot e_j = e_{i+j}$. Let us denote by $\mathcal{P}^q_{\mathcal{W}}$ the transposed Poisson algebra consisting of the Witt algebra with the mutation of the Laurent polynomials corresponding to the element q.

The algebras in the family $\mathcal{P}^q_{\mathcal{W}}$ are simple, none of them is a Poisson algebra and they are unital if and only if the element q in the mutation is an invertible Laurent polynomial. Let us construct the Jordan superalgebras that arises from these algebras.

Remark 43. Fixed some $q \in \mathcal{W}$, the Kantor double of $\mathcal{P}^q_{\mathcal{W}}$ is the vector space $\mathfrak{J}(\mathcal{P}^q_{\mathcal{W}}) = \mathcal{W} \oplus \mathcal{W}^s$ with the multiplication:

$$e_i * e_j = qe_{i+j}, \quad e_i^s * e_j = (qe_{i+j})^s, \quad e_i * e_j^s = (qe_{i+j})^s, \quad e_i^s * e_j^s = (j - i)e_{i+j}.$$

The associative commutative part of $\mathcal{P}_{\mathcal{W}}^{q}$ is perfect if and only if it is unital, by Theorem 41. Therefore, the Jordan superalgebra $\mathfrak{J}(\mathcal{P}_{\mathcal{W}}^{q})$ is simple if and only if q is invertible in $\mathcal{L}$, by Theorem 40.

Let us recall the definition of being special for a Jordan superalgebra. Let $(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \bullet)$ be an associative superalgebra. Consider a new product given for $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$ by $x \cdot y = \frac{1}{2}(x \bullet y + (-1)^{ab} y \bullet x)$, then $\mathcal{A}^{(+)} := (\mathcal{A}, \cdot)$ is a Jordan superalgebra. Now, a Jordan superalgebra $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1$ is called special if it can be embedded in a superalgebra $\mathcal{A}^{(+)}$, for some associative superalgebra $\mathcal{A}$. Otherwise, it is called exceptional.

Moreover, a transposed Poisson algebra $(\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1, \circ, [\cdot,\cdot])$ is called a transposed Poisson algebra of vector type if there is a derivation D of $(\mathcal{P}, \circ)$ such that $[x, y] = x \circ D(y) - (-1)^{xy} y \circ D(x)$ for any $x, y \in \mathcal{P}_0 \cup \mathcal{P}_1$. In particular, unital transposed Poisson algebras are of vector type, by choosing the derivation D such that $D(x) = [x, 1]$.

Remark 44. It was proven by McCrimmon that unital associative commutative algebras with a derivation $(\mathcal{A}, \circ, D)$, together with the bracket $[x, y] = x \circ D(y) - (-1)^{xy} y \circ D(x)$ produce a special Jordan superalgebra when considering their Kantor double [16, Theorem 4.4]. It follows that if q is an invertible Laurent polynomial, then $\mathfrak{J}(\mathcal{P}_{\mathcal{W}}^{q})$ is special.

Let us close the manuscript with a natural question.

Open question. Is there any exceptional Jordan superalgebra arising from the Kantor double of a transposed Poisson algebra?

Data availability

No data was used for the research described in the article.

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