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Attribute implications in multi-adjoint concept lattices with hedges [☆]

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ABSTRACT

The computation of (if-then) rules relating the most representative variables of a given dataset is a relevant goal in different areas of data analysis such as stock market prediction, disease diagnosis and census data analysis, among others. In Formal Concept Analysis (FCA), attribute implications follow the philosophy of the if-then rules establishing relationships between sets of attributes. This paper will focus on the theoretical development of attribute implications through the tools provided by the multi-adjoint concept lattice framework. We will show different notions to compute the validity of attribute implications, analyzing their properties and comparing the proposed alternatives. The use of linguistic hedges, and truth-stressing hedges in particular, is an interesting way of incorporating truth functions into attribute implications. Moreover, they have been considered in the computation of the concepts of a given context. This paper also introduces and studies the main properties of the multi-adjoint concept lattices framework with truth-stressing hedges.

1. Introduction

Formal Concept Analysis (FCA) is a useful mathematical tool in the process of extracting and representing knowledge from data sets, which are formally translated into a set of objects, a set of attributes and a relationship between them. FCA has widely been studied from a theoretic [5,10,21,31,42,43] and applied [1,3,9,24,29,30,36,45] point of view. One of the main features of FCA is the possibility of computing relations among the variables using the concept-forming operators, which form a Galois connection. This kind of if-then rule has a particular interpretation and has had a great impact in the science community and in the applications. For example, different logics have been applied [11,37,44] to use a set of attribute implications representing the original data set as part of the design of a decision support system.

Truth stressing linguistic hedges were mainly used to reduce the concept lattice of a context [6]. Moreover, they were successfully used in attribute implications [7]. Linguistic hedges try to model expressions like “very” or “extremely” in order to modify the meaning of natural language expressions [32–34,38]. They have been studied in different frameworks due to their relevance, indeed Zadeh asserted that linguistic hedges play an important role in human reasoning [47].

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Multi-adjoint concept lattices provide a fuzzy generalization of FCA [42], based on adjoint triples [16,23,41], which has obtained important advances in different areas, such as in attribute reduction [2,22] and size reduction [19,21], and in the comparison with related fields, such as rough set theory [8], fuzzy relation equations [40] and logic programming [12].

An initial study about attribute implications in multi-adjoint concept lattices was given in [20,39]. Considering fuzzy notions of support and confidence, a new kind of attribute implication was introduced in [39]. In [20], an initial approach of the definition of validity of attribute implications in multi-adjoint concept lattices was presented. In order to advance in this line, truth stressing hedges will be incorporated in the definition of the concept-forming operators of the multi-adjoint framework and different properties will be analyzed. Several of these properties will be used in the second part of the paper in which we are interested in applying the philosophy of the multi-adjoint paradigm to the computation of attribute implications, which will increase their flexibility and expressiveness. Specifically, we will provide different definitions to compute the validity of attribute implications in multi-adjoint concept lattices. The advantages and drawbacks of these definitions are widely analyzed, studying their properties and making a comparison with the proposed results in the residuated concept lattice framework [6,7]. The main contribution shows that the degree in which an attribute implication is valid in a multi-adjoint concept lattice is obtained computing only the degree of validity on the set of join-irreducible elements of the lattice.

The paper is organized as follows. Section 2 includes the basic notions and properties associated with adjoint triples and truth-stressing hedges, which are the operators involved in the computations. Section 3 recalls the basic definitions of the multi-adjoint concept lattice framework, and introduces its extension with the use of truth-stressing hedges, together with some technical results relating both environments and illustrative examples. Section 4 is devoted to the study of attribute implications in multi-adjoint concept lattices with hedges and their validity. Finally, the paper finishes with some conclusions and prospects for future work.

2. Adjoint triples and truth-stressing hedges

This section introduces certain basic notions and properties about the operators involved in the considered multi-adjoint formal concept analysis framework, for a better understanding of the theoretical development presented in this paper.

Adjoint triples [16,18,23] arise as an interesting generalization of a t-norm and its residuated implication [33], because they preserve their main properties and require only the minimal mathematical conditions for ensuring operability. Requiring few constraints increases remarkably the fields of application of these operators, for instance, they can be considered in non-commutative and/or non-associative frameworks. The formal definition of adjoint triple is given below.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called *adjoint property*.

The following properties are straightforwardly deduced by using the adjoint property.

Proposition 2. *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, we have that:*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*
3. *$z \nwarrow \perp_1 = \top_2$ and $z \swarrow \perp_2 = \top_1$, for all $z \in P_3$, when $(P_1, \leq_1, \perp_1, \top_1)$ and $(P_2, \leq_2, \perp_2, \top_2)$ are bounded posets.*
4. *When the infimum exists:*
    - (4.1) *$(\bigwedge_{z_i \in Z} z_i) \swarrow y = \bigwedge_{z_i \in Z} (z_i \swarrow y)$, for all $Z \subseteq P_3$ and $y \in P_2$.*
    - (4.2) *$(\bigwedge_{z_i \in Z} z_i) \nwarrow x = \bigwedge_{z_i \in Z} (z_i \nwarrow x)$, for all $Z \subseteq P_3$ and $x \in P_1$.*

A detailed study of adjoint triples and their useful properties was included in [14,16,18,23]. Specifically, the results exposed in the proposition below were presented in [16]. The first item states the conditions that the posets and the adjoint conjunctor must satisfy in order to ensure that the adjoint implications verify a property which can be seen as a generalization of the boundary conditions of the classical Boolean implications. The second item relates two boundary conditions in adjoint triples.

Proposition 3. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$.*

1. *If $P_1 \subseteq P_3$ and P_2 has a maximum $\top_2$ as a right identity element for $\&$, that is, the equality $x \,\&\, \top_2 = x$ is satisfied, for all $x \in P_1$, then we have:*

$$\top_2 = z \nwarrow x \quad \text{if and only if} \quad x \leq_3 z, \text{ for all } x \in P_1 \text{ and } z \in P_3.$$

2. *If $P_1 = P_3$ and P_2 has a maximum $\top_2$, then we obtain:*

$$z \swarrow \top_2 = z \quad \text{if and only if} \quad x \,\&\, \top_2 = x, \text{ for all } x \in P_1 \text{ and } z \in P_3.$$

The following proposition gives the sufficient and necessary conditions which guarantee the satisfiability of the well-known exchange principle property, by the residuated implications of an adjoint triple [25].

Proposition 4. *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to the poset* $(P, \leq)$*. Then, for all* $x, y, z \in P$*, the following equivalence holds:*

$$(z \swarrow y) \nwarrow x = (z \nwarrow x) \swarrow y \quad \textit{if and only if} \quad \& \text{ \textit{is associative.}}$$

After recalling the technical properties about adjoint triples which will be used later, we introduce some examples of these operators. The Gödel, product and Łukasiewicz t-norms together with their residuated implications are the most usual examples of adjoint triples defined on the unit interval $[0,1]$. Notice that, their implications are equal due to the commutativity of the t-norm. These adjoint triples are defined, for every $x, y, z \in [0,1]$, as follows:

$$x \,\&_{\mathrm{G}}\, y = \min\{x, y\} \qquad z \swarrow^{\mathrm{G}} y = \begin{cases} 1 & \text{si } y \leq z \\ z & \text{otherwise} \end{cases}$$

$$x \,\&_{\mathrm{P}}\, y = x \cdot y \qquad z \swarrow^{\mathrm{P}} y = \begin{cases} 1 & \text{si } y \leq z \\ z/y & \text{otherwise} \end{cases}$$

$$x \,\&_{\text{Ł}}\, y = \max\{0, x + y - 1\} \qquad z \swarrow^{\mathrm{L}} y = \min\{1, 1 - y + z\}$$

Next example shows an adjoint triple whose operators are defined on a partition of the unit interval $[0,1]$ and whose conjuctors could be neither commutative nor associative.

Example 5. Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces, for example, $[0,1]_4 = \{0, 0.25, 0.5, 0.75, 1\}$ divide the unit interval in four pieces. We can consider a discretization of a given left-continuous t-norm $T\colon [0,1] \times [0,1] \to [0,1]$ as the operator $\&_{\mathrm{DT}}\colon [0,1]_n \times [0,1]_m \to [0,1]_k$ with $n, m, k \in \mathbb{N}^+$, which is defined, for every $x \in [0,1]_n$ and $y \in [0,1]_m$ as follows:

$$x \,\&_{\mathrm{DT}}\, y = \frac{\lceil k \cdot T(x, y) \rceil}{k}$$

where $\lceil \_ \rceil$ is the ceiling function. For this operator, their corresponding residuated implications $\swarrow^{\mathrm{DT}}\colon [0,1]_k \times [0,1]_m \to [0,1]_n$ y $\nwarrow_{\mathrm{DT}}\colon [0,1]_k \times [0,1]_n \to [0,1]_m$ are defined, for all $x \in [0,1]_n$, $y \in [0,1]_m$ and $z \in [0,1]_k$, as:

$$z \swarrow^{\mathrm{DT}} y = \frac{\lfloor m \cdot (z \to y) \rfloor}{m} \qquad z \nwarrow_{\mathrm{DT}} x = \frac{\lfloor n \cdot (z \to x) \rfloor}{n}$$

where $\lfloor \_ \rfloor$ is the floor function and $\to$ is the residuated implication of the t-norm T.

In this case, we have that the tuple $(\&_{\mathrm{DT}}, \swarrow^{\mathrm{DT}}, \nwarrow_{\mathrm{DT}})$ is an adjoint triple and the conjuctor $\&_{\mathrm{DT}}$ could be neither commutative nor associative. Similar adjoint triples can be obtained from the Gödel, product and Łukasiewicz t-norms, which are denoted as $(\&_{\mathrm{DG}}, \swarrow^{\mathrm{DG}}, \nwarrow_{\mathrm{DG}})$, $(\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}})$, $(\&_{\mathrm{DL}}, \swarrow^{\mathrm{DL}}, \nwarrow_{\mathrm{DL}})$, respectively. □

Linguistic hedges have widely been studied in diverse papers related to the applied point of view of approximate reasoning [34], fuzzy logic [13,28,32,33], fuzzy relation equations [4] and formal concept analysis [6]. Specifically, this paper focuses on the following notion of truth-stressing hedge, which is a straightforward extension of the notion presented in [6] based on adjoint triples.

Definition 6. Let $(P, \leq, \top)$ be an upper bounded poset and $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(P, \leq)$. A *truth-stressing* $\swarrow$*-hedge* is a unary function $*\colon P \to P$ satisfying the following properties:

$$*(\top) = \top \qquad \text{(Boundary condition)}$$

$$*(x) \leq x \qquad \text{(Subdiagonal condition)}$$

$$*(z \swarrow y) \leq *(z) \swarrow *(y) \qquad (\swarrow\text{-regularity condition})$$

$$*(*(x)) = *(x) \qquad \text{(Idempotency condition)}$$

for each $x, y, z \in P$. Similarly, a *truth-stressing* $\nwarrow$*-hedge* is defined on $(P, \leq, \top)$.

Notice that, if we consider an adjoint triple with respect to three different upper bounded posets $(P_1, \leq_1, \top_1)$, $(P_2, \leq_2, \top_2)$ and $(P_3, \leq_3, \top_3)$ in the above definition, by the $\swarrow$-regularity condition, the three upper bounded posets must be related by inclusion. In order to simplify the definition of truth-stressing hedge, we have assumed that $P_1 = P_2 = P_3 = P$.

Next, we recall two boundary cases of hedges, that is, the identity mapping which is the greatest truth-stressing $\swarrow$-hedge and the least one which is called globalization [46]. This operator $*\colon P \to P$ is defined on an upper bounded poset $(P, \leq, \top)$, for each $x \in P$, as:

$$* (x) = \begin{cases} \top & \text{if} \quad x = \top \\ \bot & \text{otherwise} \end{cases} \tag{1}$$

The following result shows that the monotonicity of truth-stressing hedges is obtained if the adjoint conjunctor satisfies the boundary conditions with the top element of the poset.

Proposition 7. *Given an adjoint triple* $(\&, \swarrow, \nwarrow)$ *with respect to an upper bounded poset* $(P, \leq, \top)$*, we have that:*

1. *If the equality* $x \,\&\, \top = x$ *holds, for all* $x \in P$*, and* $* :\ P \to P$ *is a truth-stressing* $\nwarrow$*-hedge, then* $*$ *is order-preserving.*
2. *If the equality* $\top \,\&\, y = y$ *holds, for all* $y \in P$*, and* $* :\ P \to P$ *is a truth-stressing* $\swarrow$*-hedge, then* $*$ *is order-preserving.*

Proof. (1) Let x_1 and x_2 be two elements in P such that $x_1 \leq x_2$. By hypothesis, the boundary condition $x \,\&\, \top = x$ holds, for all $x \in P$, then applying Statement (1) in Proposition 3, we obtain that $x_2 \nwarrow x_1 = \top$. Now, taking into account Definition 6, the following chain of inequalities is obtained:

$$\top = *(\top) = *(x_2 \nwarrow x_1) \leq *(x_2) \nwarrow *(x_1)$$

Finally, the inequality $\top \leq *(x_2) \nwarrow *(x_1)$ is equivalent to the inequality $*(x_1) \leq *(x_2)$, by Statement (1) in Proposition 3, which provides the monotonicity of the $\nwarrow$-hedge.

The proof of Statement (2) follows similarly. ☐

3. Multi-adjoint concept lattices with truth-stressing hedges

The multi-adjoint concept lattice framework embeds several fuzzy extensions of concept lattices studied in the literature such as [5,10,31,35], among others. In order to give a more general (non-commutative and/or non-associative) and flexible setting, it becomes natural to consider different adjoint triples in multi-adjoint concept lattices. Next definition recalls the notions of multi-adjoint frame, context, concept-forming operators and multi-adjoint concept lattice [27,42].

Definition 8.

- The tuple $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ is a *multi-adjoint frame* where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(P, \leq)$, for all $i \in \{1, \dots, n\}$.
- The tuple (A, B, R, σ) is a *context* where A and B are non-empty sets, R is a P-fuzzy relation $R :\ A \times B \to P$ and $\sigma :\ A \times B \to \{1, \dots, n\}$ is a mapping which associates any element in $A \times B$ with a particular adjoint triple in the multi-adjoint frame.
- Let L_2^B and L_1^A be the sets of fuzzy subsets $g :\ B \to L_2$ and $f :\ A \to L_1$, respectively. The *concept-forming operators* $^{\uparrow} :\ L_2^B \to L_1^A$ and $^{\downarrow} :\ L_1^A \to L_2^B$ are defined as:

$$g^{\uparrow_\sigma}(a) = \inf\{R(a, b') \swarrow^{\sigma(a,b')} g(b') \mid b' \in B\} \tag{2}$$

$$f^{\downarrow^\sigma}(b) = \inf\{R(a', b) \nwarrow_{\sigma(a',b)} f(a') \mid a' \in A\} \tag{3}$$

  for all $g \in L_2^B$, $f \in L_1^A$, $a \in A$, $b \in B$.
- A *concept* is a pair $\langle g, f \rangle$ satisfying that $g^{\uparrow_\sigma} = f$, $f^{\downarrow^\sigma} = g$, for all $g \in L_2^B$ and $f \in L_1^A$. The fuzzy subsets g and f are usually known as *the extent and intent of the concept*, respectively.
- The *multi-adjoint concept lattice*, denoted as $(\mathcal{M}, \preceq)$ and associated with the multi-adjoint frame and the context, is the set:

$$\mathcal{M} = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A, g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

  together with the ordering $\preceq$, defined as $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ if and only if $g_1 \preceq_2 g_2$ (or equivalently, $f_2 \preceq_1 f_1$). The sets of extents and intents of the concepts of $(\mathcal{M}, \preceq)$ will be denoted by $\mathrm{Ext}(\mathcal{M})$ and $\mathrm{Int}(\mathcal{M})$, respectively.

From now on, we will write $g^{\uparrow}$ and $f^{\downarrow}$ instead of $g^{\uparrow_\sigma}$ and $f^{\downarrow^\sigma}$, respectively, in order to simplify the notation. It is convenient to mention that the pair of concepts-forming operators $(^{\uparrow}, ^{\downarrow})$ forms an antitone Galois connection, which provides $(\mathcal{M}, \preceq)$ with the algebraic structure of a complete lattice [42]. We recall the notion of antitone Galois connection since its properties will play a key role in the proofs.

Definition 9. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets. We say that the pair $(^{\uparrow}, ^{\downarrow})$ of mappings $^{\downarrow} :\ P_1 \to P_2$, $^{\uparrow} :\ P_2 \to P_1$ is an *antitone Galois connection* between P_1 and P_2 if the following properties are satisfied:

1. $^{\uparrow}$ and $^{\downarrow}$ are order-reversing.
2. $x \leq_1 x^{\downarrow\uparrow}$ e $y \leq_2 y^{\uparrow\downarrow}$
3. $x^{\downarrow} = x^{\downarrow\uparrow\downarrow}$ e $y^{\uparrow} = y^{\uparrow\downarrow\uparrow}$

for all $x \in P_1$ and $y \in P_2$.

The notion of join-irreducible element of a multi-adjoint concept lattice also plays an important role in this work. Before presenting a characterization of these elements, we need to consider a multi-adjoint frame $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ and a context (A, B, R, σ), as well as define the following specific family of fuzzy subsets of L_2^B [8].

Definition 10. For each $b \in B$, the fuzzy subsets of objects $\phi_{b,y} \in L_2^B$ defined, for all $y \in L_2$, as:

$$\phi_{b,y}(b') = \begin{cases} y & \text{if } b' = b \\ \bot_1 & \text{if } b' \neq b \end{cases}$$

are called *fuzzy-objects*. The set of all fuzzy-objects is denoted as $\Psi = \{\phi_{b,y} \mid b \in B, y \in L_2\}$.

Now, we present two easy properties related to the fuzzy-objects which will be used later.

Lemma 11. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P, \leq)$. For all $a \in A$, $b \in B$, $y \in L_2$, we have that:*

(1) $\phi^{\uparrow}_{b,y}(a) = R(a,b) \swarrow y$.

(2) $\phi^{\uparrow}_{b,\top_2}(a) \preceq_1 \phi^{\uparrow}_{b,y}(a)$.

Proof. (1) Given $a \in A$, $b \in B$ and $y \in L_2$, the next chain of equivalences is obtained:

$$\begin{aligned} \phi^{\uparrow}_{b,y}(a) &\overset{(i)}{=} \inf\{R(a,b') \swarrow \phi_{b,y}(b') \mid b' \in B\} \\ &\overset{(ii)}{=} \inf\{R(a,b) \swarrow y, \inf\{R(a,b') \swarrow \bot_2 \mid b' \in B \setminus \{b\}\}\} \\ &\overset{(iii)}{=} \inf\{R(a,b) \swarrow y, \top_1\} \\ &= R(a,b) \swarrow y \end{aligned}$$

where (i) holds by the definition of the operator $^{\uparrow}$, (ii) is satisfied by Definition 10 and (iii) is obtained since $z \swarrow \bot_2 = \top_1$, for all $z \in P$, by Statement (3) in Proposition 2.

(2) Given $b \in B$ and $y \in L_2$, since $y \preceq_2 \top_2$, by the definition of the fuzzy-objects we clearly obtain $\phi_{b,y} \preceq_2 \phi_{b,\top_2}$. Hence, by the monotonicity of the concept-forming operator $^{\uparrow}$, we have that: $\phi^{\uparrow}_{b,\top_2} \preceq_1 \phi^{\uparrow}_{b,y}$. $\square$

Notice that, every row of the table associated with the relation R can be expressed as a fuzzy subset of attributes, for each $b \in B$, that is, as a mapping $R_b : A \to P$ such that $R_b(a) = R(a,b)$, for all $a \in A$. In addition, for each $b \in B$, R_b is the intent of a concept in $(\mathcal{M}, \preceq)$, as the following theorem shows.

Theorem 12. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $x \,\&\, \top_1 = x$, for all $x \in L_1$. For each $b \in B$, we have that $R_b : A \to L_1$ is an intent of a concept in $(\mathcal{M}, \preceq)$, that is, $R_b \in Int(\mathcal{M})$.*

Proof. Given $a \in A$, $b \in B$, by Statement (2) in Proposition 3, we have $R_b(a) = R(a,b) \swarrow \top_1$. Moreover, by Lemma 11, we have that $R(a,b) \swarrow \top_1 = \phi^{\uparrow}_{b,\top_1}(a)$, which implies $R_b(a) = \phi^{\uparrow}_{b,\top_1}(a)$. Since the operators $^{\uparrow}$ and $^{\downarrow}$ defined by Equations (2) and (3) form an antitone Galois connection, by Statement (3) in Definition 9, we obtain that $\phi^{\uparrow}_{b,\top_1} = (\phi^{\uparrow}_{b,\top_1})^{\downarrow\uparrow}$ for all $b \in B$ and, therefore, we can conclude that $R_b \in \mathrm{Int}(\mathcal{M})$ for all $b \in B$. $\square$

The following theorem states that the join-irreducible elements of a concept lattice are uniquely generated by fuzzy-objects [8]. In addition, they cannot be expressed as supremum of elements that are less than them. This is a dual result to the one given in [17] for meet-irreducible elements of a concept lattice.

Theorem 13. *The set of join-irreducible elements of $(\mathcal{M}, \preceq)$, denoted as $J_F(\mathcal{M})$, is composed of the pairs $\langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y} \rangle$ in $\mathcal{M}$, with $b \in B$ and $y \in L_2$, such that:*

$$\phi^{\uparrow}_{b,y} \neq \bigwedge\{\phi^{\uparrow}_{b_i,y_i} \mid \phi_{b_i,y_i} \in \Psi, \phi^{\uparrow}_{b,y} \prec_1 \phi^{\uparrow}_{b_i,y_i}\}$$

and $\phi^{\uparrow}_{b,y} \neq f_{\top_1}$, where Ψ denotes the set of all of the fuzzy-objects, $\top_1$ is the maximum element in L_1 and $f_{\top_1} : A \to L_1$ is the fuzzy subset defined as $f_{\top_1}(a) = \top_1$, for all $a \in A$.

Table 1
Relation R of Example 1.

R	a_1	a_2	a_3
b_1	0.5	1	0.5
b_2	1	0.5	0
b_3	0	0.5	1

Table 2
Values required to compute $f^{\downarrow^*}$.

	$*_a(f)$	$R(a,b_1)\nwarrow_{\mathrm{DP}}*_a(f(a))$	$R(a,b_2)\nwarrow_{\mathrm{DP}}*_a(f(a))$	$R(a,b_3)\nwarrow_{\mathrm{DP}}*_a(f(a))$
a_1	1	0.5	1	0
a_2	0	1	1	1
a_3	0	1	1	1

In this paper, we are also interested in considering a multi-adjoint concept lattice framework enriched with truth-stressing hedges. In order to introduce formally this framework, given a multi-adjoint frame and a context, we need to define the following families of truth-stressing hedges on the sets of attributes and objects, that is:

$$*_A=\{*_a\colon L_1\to L_1 \mid *_a \text{ is a truth-stressing } \swarrow\text{-hedge, for all } a\in A\}$$

$$*_B=\{*_b\colon L_2\to L_2 \mid *_b \text{ is a truth-stressing } \nwarrow\text{-hedge, for all } b\in B\}$$

The following notion of concept-forming operators generalizes the one given in [6].

Definition 14. Let $(L_1,L_2,P,\preceq_1,\preceq_2,\leq,\&_1,\swarrow^1,\nwarrow_1,\dots,\&_n,\swarrow^n,\nwarrow_n)$ be a multi-adjoint frame, (A,B,R,σ) be a context, $*_A$ be a family of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $*_B$ be a family of arbitrary truth-stressing $\nwarrow$-hedges on L_2.

- The *concept-forming operators* ${}^{\uparrow_{*\sigma}}\colon L_2^B\to L_1^A$ and ${}^{\downarrow^{*\sigma}}\colon L_1^A\to L_2^B$ are defined as:

$$g^{\uparrow_{*\sigma}}(a)=\inf\{R(a,b')\swarrow^{\sigma(a,b')}*_{b'}(g(b'))\mid b'\in B\} \tag{4}$$

$$f^{\downarrow^{*\sigma}}(b)=\inf\{R(a',b)\nwarrow_{\sigma(a',b)}*_{a'}(f(a'))\mid a'\in A\} \tag{5}$$

  for all $g\in L_2^B$, $f\in L_1^A$, $a\in A$, $b\in B$.
- The *context* associated with the previous concept-forming operators is denoted as $(A_{*_A},B_{*_B},R,\sigma)$. A *concept* is a pair $\langle g,f\rangle$ satisfying that $g^{\uparrow_{*\sigma}}=f$, $f^{\downarrow^{*\sigma}}=g$, for all $g\in L_2^B$, $f\in L_1^A$.
- The *multi-adjoint concept lattice with truth-stressing hedges*, denoted as $(\mathcal{M}_*,\preceq)$, associated with the multi-adjoint frame and the context $(A_{*_A},B_{*_B},R,\sigma)$ is the set:

$$\mathcal{M}_*=\{\langle g,f\rangle\mid g\in L_2^B, f\in L_1^A, g^{\uparrow_{*\sigma}}=f, f^{\downarrow^{*\sigma}}=g\}$$

  where the ordering is defined by $\langle g_1,f_1\rangle\preceq\langle g_2,f_2\rangle$ if and only if $g_1\preceq_2 g_2$ (or equivalently, $f_2\preceq_1 f_1$). The sets of extents and intents of the concepts of $(\mathcal{M}_*,\preceq)$ will be denoted by $\mathrm{Ext}(\mathcal{M}_*)$ and $\mathrm{Int}(\mathcal{M}_*)$, respectively.

Once again, in order to simplify the notation, we will write $g^{\uparrow_*}$ and $f^{\downarrow^*}$ instead of $g^{\uparrow_{*\sigma}}$ and $f^{\downarrow^{*\sigma}}$, respectively. In this case, the pair of concepts-forming operators $({}^{\uparrow_*},{}^{\downarrow^*})$ does not form an antitone Galois connection in general, as we will show below.

Example 15. Consider the multi-adjoint frame $([0,1]_2,\leq,\&_{\mathrm{DP}},\swarrow^{\mathrm{DP}},\nwarrow_{\mathrm{DP}})$, the contexts (A,B,R,σ) and $(A_{*_A},B_{*_B},R,\sigma)$, where the set of attributes is $A=\{a_1,a_2,a_3\}$, the set of objects is $B=\{b_1,b_2,b_3\}$, the relation $R\colon A\times B\to[0,1]_2$ is given by Table 1, the triple $(\&_{\mathrm{DP}},\swarrow^{\mathrm{DP}},\nwarrow_{\mathrm{DP}})$ is defined from the discretization of the product t-norm on $[0,1]_2=\{0,0.5,1\}$, σ is constantly $(\&_{\mathrm{DP}},\swarrow^{\mathrm{DP}},\nwarrow_{\mathrm{DP}})$ and

$$*_A=\{*_a\colon[0,1]_2\to[0,1]_2\mid *_a \text{ is the globalization truth-stressing } \swarrow\text{-hedge, for all } a\in A\}$$

$$*_B=\{*_b\colon[0,1]_2\to[0,1]_2\mid *_b \text{ is the identity truth-stressing } \nwarrow\text{-hedge, for all } b\in B\}$$

Now, we will see that the pair $({}^{\uparrow_*},{}^{\downarrow^*})$ does not satisfy that $f\leq f^{\downarrow^*\uparrow_*}$ and $g\leq g^{\uparrow_*\downarrow^*}$, for all $f\in[0,1]_2^A$, $g\in[0,1]_2^B$, and therefore, it does not form an antitone Galois connection. For this purpose, we will consider the fuzzy subset of attributes $f=\{1/a_1,0.5/a_3\}$ and we will compute its closure, that is $f^{\downarrow^*\uparrow_*}$.

Applying the definition of the operator ${}^{\downarrow^*}$ and considering the data presented in Table 2, we have that:

$$f^{\downarrow^*}(b_1)=\inf\{R(a,b_1)\nwarrow_{\mathrm{DP}}*_a(f(a))\mid a\in A\}=0.5$$

$$f^{\downarrow^*}(b_2)=\inf\{R(a,b_2)\nwarrow_{\mathrm{DP}}*_a(f(a))\mid a\in A\}=1$$

Table 3
Values required to compute $f^{\downarrow^*\uparrow_*}$.

	$*_b\,(f^{\downarrow^*})$	$R(a_1,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))$	$R(a_2,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))$	$R(a_3,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))$
b_1	0.5	1	1	1
b_2	1	1	0.5	0
b_3	0	1	1	1

$$f^{\downarrow^*}(b_3)=\inf\{R(a,b_3)\nwarrow_{\mathrm{DP}}*_a\,(f(a))\mid a\in A\}=0$$

Then, we obtain that $f^{\downarrow^*}=\{0.5/b_1,1/b_2\}$. Applying the operator $^{\uparrow_*}$ to the set $f^{\downarrow^*}$ and using computed data presented in Table 3, we obtain that:

$$f^{\downarrow^*\uparrow_*}(a_1)=\inf\{R(a_1,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))\mid b\in B\}=1$$

$$f^{\downarrow^*\uparrow_*}(a_2)=\inf\{R(a_2,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))\mid b\in B\}=0.5$$

$$f^{\downarrow^*\uparrow_*}(a_3)=\inf\{R(a_3,b)\swarrow^{\mathrm{DP}}*_b\,(f^{\downarrow^*}(b))\mid b\in B\}=0$$

Hence, we have that $f^{\downarrow^*\uparrow_*}=\{1/a_1,0.5/a_2\}$. It is clear that $f\not\leq f^{\downarrow^*\uparrow_*}$, then Statement (2) in Definition 9 does not hold. As a consequence, we can conclude that the pair $(^{\uparrow_*},{}^{\downarrow^*})$ does not form an antitone Galois connection. □

Next, we will present basic properties of $^{\uparrow_*}$ and $^{\downarrow^*}$ which have a certain similarity with the well-known properties of antitone Galois connections. Before presenting the mentioned properties, we need to introduce the following fuzzy subset of attributes:

$$\bar{*}_A(f)\colon\ A\to L_1,\ \text{defined as}\ \bar{*}_A(f)(a)=*_a\,(f(a)),\ \text{for all}\ *_a\in *_A,\ a\in A\ \text{and}\ f\in L_1^A,$$

and the following fuzzy subset of objects:

$$\bar{*}_B(g)\colon\ B\to L_2,\ \text{defined as}\ \bar{*}_B(g)(b)=*_b\,(g(b)),\ \text{for all}\ *_b\in *_B,\ b\in B\ \text{and}\ g\in L_2^B.$$

Proposition 16. *Let $(L_1,L_2,P,\preceq_1,\preceq_2,\leq,\&_1,\swarrow^1,\nwarrow_1,\dots,\&_n,\swarrow^n,\nwarrow_n)$ be a multi-adjoint frame, (A,B,R,σ) and $(A_{*_A},B_{*_B},R,\sigma)$ be contexts where $*_A$ is a family of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $*_B$ is a family of arbitrary truth-stressing $\nwarrow$-hedges on L_2. The concept-forming operators $^{\uparrow_*}$ and $^{\downarrow^*}$ satisfy the following properties,*[1] *for all $g\in L_2^B$, $f\in L_1^A$, $a\in A$, $b\in B$:*

(1) $g^{\uparrow}\preceq_1 g^{\uparrow_*}$
$f^{\downarrow}\preceq_2 f^{\downarrow^*}$
(2) $\bar{*}_B(g)\preceq_2 g^{\uparrow_*\downarrow^*}$
$\bar{*}_A(f)\preceq_1 f^{\downarrow^*\uparrow_*}$
(3) $g^{\uparrow_*}=\bar{*}_B(g)^{\uparrow_*}$
$f^{\downarrow^*}=\bar{*}_A(f)^{\downarrow^*}$

Proof. Because of symmetry, we will only prove the first statement for each item. Considering the definition of the operators $^{\uparrow_*}$ and $^{\downarrow^*}$, it is clear to see that $\bar{*}_B(g)^{\uparrow}=g^{\uparrow_*}$ and $\bar{*}_A(f)^{\downarrow}=f^{\downarrow^*}$, for all $g\in L_2^B$, $f\in L_1^A$. This fact will be used in this proof.

(1) By subdiagonal condition in Definition 6, we have that $\bar{*}_B(g)\preceq_2 g$, for all $g\in L_2^B$. Applying Definition 9, as $^{\uparrow}$ is order-reversing, we obtain the inequality $g^{\uparrow}\preceq_1\bar{*}_B(g)^{\uparrow}$, that is, $g^{\uparrow}\preceq_1 g^{\uparrow_*}$.
(2) Since the operators $^{\uparrow}$ and $^{\downarrow}$ defined by Equations (2) and (3) form an antitone Galois connection, by Statement (2) in Definition 9, we have that $\bar{*}_B(g)\preceq_2\bar{*}_B(g)^{\uparrow\downarrow}$ for all $\bar{*}_B(g)\in L_2^B$. In addition, as $\bar{*}_B(g)^{\uparrow}=g^{\uparrow_*}$, we obtain that $\bar{*}_B(g)\preceq_2 g^{\uparrow_*\downarrow}$. Then, from the previous Statement (1), we deduce that $\bar{*}_B(g)\preceq_2 g^{\uparrow_*\downarrow^*}$.
(3) The identity that we want to prove follows from the following chain of equalities:

$$g^{\uparrow_*}\overset{(i)}{=}\bar{*}_B(g)^{\uparrow}\overset{(ii)}{=}\bar{*}_B(\bar{*}_B(g))^{\uparrow}\overset{(iii)}{=}\bar{*}_B(g)^{\uparrow_*}$$

where (i) and (iii) are satisfied due to the fact that $\bar{*}_B(g)^{\uparrow}=g^{\uparrow_*}$ for all $g\in L_2^B$ and (ii) because of the idempotency condition in Definition 6, that is, $\bar{*}_B(\bar{*}_B(g))=\bar{*}_B(g)$ for all $g\in L_2^B$. □

More properties can be obtained if we assume that the conjuctors of the different adjoint triples of the multi-adjoint frame satisfy the boundary conditions.

---

[1] Notice that, the mapping $^{\uparrow_*}$ in Statement (3) of Proposition 16 is applied to $\bar{*}_B(g)$, that is, $\bar{*}_B(g)^{\uparrow_*}=(\bar{*}_B(g))^{\uparrow_*}$.

Proposition 17. *Let $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ be a multi-adjoint frame, (A, B, R, σ) and $(A_{*_A}, B_{*_B}, R, \sigma)$ be contexts where $*_A$ is a family of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $*_B$ is a family of arbitrary truth-stressing $\nwarrow$-hedges on L_2. If $L_1 = L_2 = P$, $x \&_i \top_1 = x$ and $\top_1 \&_i y = y$, for all $x, y \in L_1$ and $i \in \{1, \dots, n\}$, then the concept-forming operators ${}^{\uparrow_*}$ and ${}^{\downarrow^*}$ satisfy the following properties:*

(1) *If $g_1 \preceq_1 g_2$ then $g_2^{\uparrow_*} \preceq_1 g_1^{\uparrow_*}$*
If $f_1 \preceq_1 f_2$ then $f_2^{\downarrow^*} \preceq_1 f_1^{\downarrow^*}$
(2) $\bar{*}_A(g^{\uparrow_*}) \preceq_1 g^{\uparrow_*\downarrow^*\uparrow_*} \preceq_1 g^{\uparrow_*}$
$\bar{*}_B(f^{\downarrow^*}) \preceq_1 f^{\downarrow^*\uparrow_*\downarrow^*} \preceq_1 f^{\downarrow^*}$
(3) $\bar{*}_A(g^{\uparrow_*}) = \bar{*}_A(g^{\uparrow_*\downarrow^*\uparrow_*})$
$\bar{*}_B(f^{\downarrow^*}) = \bar{*}_B(f^{\downarrow^*\uparrow_*\downarrow^*})$
(4) $g^{\uparrow_*\downarrow^*} = g^{\uparrow_*\downarrow^*\uparrow_*\downarrow^*}$
$f^{\downarrow^*\uparrow_*} = f^{\downarrow^*\uparrow_*\downarrow^*\uparrow_*}$

for all $g, g_1, g_2 \in L_1^B$ and $f, f_1, f_2 \in L_1^A$.

Proof. Once again, by symmetry, we will only prove the first statement for each item.

(1) Given $g_1, g_2 \in L_1^B$ such that $g_1 \preceq_1 g_2$, we obtain that $\bar{*}_B(g_1) \preceq_1 \bar{*}_B(g_2)$, since $\bar{*}_B$ is order-preserving by Proposition 7. As the pair $({}^\uparrow, {}^\downarrow)$ form an antitone Galois connection, ${}^\uparrow$ is order-reversing and we have that $\bar{*}_B(g_2)^\uparrow \preceq_1 \bar{*}_B(g_1)^\uparrow$. This last inequality is equivalent to $g_2^{\uparrow_*} \preceq_1 g_1^{\uparrow_*}$, because $\bar{*}_B(g)^\uparrow = g^{\uparrow_*}$, for all $g \in L_1^B$.
(2) Given $g \in L_1^B$, the first inequality $\bar{*}_A(g^{\uparrow_*}) \preceq_1 g^{\uparrow_*\downarrow^*\uparrow_*}$, follows directly applying Statement (2) of Proposition 16. Now, we will prove the inequality $g^{\uparrow_*\downarrow^*\uparrow_*} \preceq_1 g^{\uparrow_*}$ holds. By Statement (2) of Proposition 16, we have that $\bar{*}_B(g) \preceq_1 g^{\uparrow_*\downarrow^*}$. Applying Statement (1) of Proposition 17, we obtain that $g^{\uparrow_*\downarrow^*\uparrow_*} \preceq_1 \bar{*}_B(g)^{\uparrow_*}$. Finally, taking in account Statement (3) of Proposition 16, we obtain that $g^{\uparrow_*\downarrow^*\uparrow_*} \preceq_1 g^{\uparrow_*}$.
(3) Given $g \in L_1^B$, on the one hand, the inequality $g^{\uparrow_*\downarrow^*\uparrow_*} \preceq_1 g^{\uparrow_*}$ holds, by Statement (2). As $\bar{*}_A$ is order-preserving, by Proposition 7, we have that $\bar{*}_A(g^{\uparrow_*\downarrow^*\uparrow_*}) \preceq_1 \bar{*}_A(g^{\uparrow_*})$. On the other hand, the inequality $\bar{*}_A(g^{\uparrow_*}) \preceq_1 g^{\uparrow_*\downarrow^*\uparrow_*}$ holds, by Statement (2). Applying Proposition 7, $\bar{*}_A$ is order-preserving and we obtain the inequality $\bar{*}_A(\bar{*}_A(g^{\uparrow_*})) \preceq_1 \bar{*}_A(g^{\uparrow_*\downarrow^*\uparrow_*})$. By the idempotency condition of $\bar{*}_A$, we obtain $\bar{*}_A(g^{\uparrow_*}) \preceq_1 \bar{*}_A(g^{\uparrow_*\downarrow^*\uparrow_*})$.
Hence, the equality $\bar{*}_A(g^{\uparrow_*}) = \bar{*}_A(g^{\uparrow_*\downarrow^*\uparrow_*})$ holds, for all $g \in L_1^B$.
(4) Finally, the last equalities straightforwardly follow from the previous item and item (3) in Proposition 16. □

The following theorem allows us to establish an isomorphism between the concepts of $(\mathcal{M}, \preceq)$ and $(\mathcal{M}_*, \preceq)$ associated with the contexts (A, B, R, σ) and $(A_{*_A}, B_{*_B}, R, \sigma)$, respectively.

Theorem 18. *Let $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ be a multi-adjoint frame, (A, B, R, σ) and $(A_{*_A}, B_{*_B}, R, \sigma)$ be contexts where:*

$$*_A = \{*_a : L_1 \to L_1 \mid *_a \text{ is the identity truth-stressing} \swarrow \text{-hedge, for all } a \in A\}$$

$$*_B = \{*_b : L_2 \to L_2 \mid *_b \text{ is a truth-stressing} \nwarrow \text{-hedge, for all } b \in B\}$$

If $L_1 = L_2 = P$, $x \&_i \top_1 = x$ and $\top_1 \&_i y = y$, for all $x, y \in L_1$ and $i \in \{1, \dots, n\}$, then the concept lattice $(\mathcal{M}_*, \preceq)$ can be defined as follows:

$$\mathcal{M}_* = \{\langle g, f\rangle \in \mathcal{M} \mid f = h^{\uparrow_*} \text{ for some } h \in L_1^B \text{ with } h(b) \in \mathit{fix}(*_b) \text{ for all } b \in B\}$$

where fix$(*_b)$ is the set of all fix-points of $*_b$.

Proof. First of all, we will prove that if $\langle g, f\rangle \in \mathcal{M}_*$ then $\langle g, f\rangle \in \mathcal{M}$ verifying that $f = h^{\uparrow_*}$ for some $h \in L_1^B$ with $h(b) \in \mathrm{fix}(*_b)$, for all $b \in B$. Given $\langle g, f\rangle \in \mathcal{M}_*$, by the definition of concept, we have that the equalities $f^\downarrow = g$ and $g^{\uparrow_*} = f$ hold. In order to conclude that $\langle g, f\rangle \in \mathcal{M}$, we need to deduce that $g^\uparrow = f$ is also satisfied. As $({}^\uparrow, {}^\downarrow)$ forms an antitone Galois connection, the inequality $f \preceq_1 f^{\downarrow\uparrow}$ holds. In addition, the equality $f^{\downarrow\uparrow} = g^\uparrow$ is obtained due to $f^\downarrow = g$. From these facts, we deduce that $f \preceq_1 g^\uparrow$. On the other hand, the inequality $g^\uparrow \preceq_1 f$ is straightforwardly obtained taking into account Statement (1) in Proposition 16 and $\langle g, f\rangle \in \mathcal{M}_*$, that is, $g^\uparrow \preceq_1 g^{\uparrow_*}$ and $g^{\uparrow_*} = f$ hold. Hence, we can conclude that $f^\downarrow = g$ and $g^\uparrow = f$, or equivalently, $\langle g, f\rangle \in \mathcal{M}$. It remains to demonstrate that $f = h^{\uparrow_*}$ for some $h \in L_1^B$ with $h(b) \in \mathrm{fix}(*_b)$, for all $b \in B$. To reach this goal, we consider the mapping $h \in L_1^B$ defined as $h(b) = \bar{*}_B(g)(b)$, for all $b \in B$. Then, $h(b) \in \mathrm{fix}(*_b)$ for each $b \in B$, since $*_b\,(h(b)) =*_b\,(\bar{*}_B(g)(b)) =*_b\,(*_b\,(g(b))) =*_b\,(g(b)) = \bar{*}_B(g)(b) = h(b)$. Finally, we have that

$$h^{\uparrow_*} \overset{(i)}{=} \bar{*}_B(g)^{\uparrow_*} \overset{(ii)}{=} g^{\uparrow_*} \overset{(iii)}{=} f$$

where (i) is obtained because we have considered that $h = \bar{*}_B(g)$, (ii) is given by Statement (3) in Proposition 16 and (iii) is obtained because $\langle g, f\rangle \in \mathcal{M}_*$.

Now, we consider a concept $\langle g, f\rangle$ belonging to the concept lattice $(\mathcal{M}, \preceq)$ such that $f = h^{\uparrow_*}$ for some $h \in L_1^B$ and $h(b) \in \mathrm{fix}(*_b)$ for all $b \in B$. We have to prove that $\langle g, f\rangle \in \mathcal{M}_*$. By hypothesis, the equality $f^{\uparrow} = g$ holds, then we only need to prove that $g^{\uparrow_*} = f$ is satisfied. This fact follows from the next chain of equalities:

$$f \overset{(i)}{=} h^{\uparrow_*} \overset{(ii)}{=} h^{\uparrow_*\downarrow\uparrow_*} \overset{(iii)}{=} f^{\downarrow\uparrow_*} \overset{(iv)}{=} g^{\uparrow_*}$$

where (i) and (iii) are obtained because we have considered that $f = h^{\uparrow_*}$, (ii) holds by Statement (3) in Proposition 17 and since $*_A$ is a family of identity truth-stressing $\swarrow$-hedges, and (iv) is satisfied due to $\langle g, f\rangle \in \mathcal{M}$. ☐

As a consequence of the previous theorem, we can guarantee that the concept lattice $(\mathcal{M}_*, \preceq)$ associated with the context $(A_{*_A}, B_{*_B}, R, \sigma)$ is a sublattice of the concept lattice $(\mathcal{M}, \preceq)$ associated with the context (A, B, R, σ), when $*_A$ is a family of identity truth-stressing $\swarrow$-hedges and $*_B$ is a family of arbitrary truth-stressing $\nwarrow$-hedges. In general, the concept lattice $(\mathcal{M}_*, \preceq)$ is not a sublattice of the concept lattice $(\mathcal{M}, \preceq)$. This fact can be proven considering the same counterexample presented in [6, Example 3.21] for the residuated case.

From now on, we fix $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ a multi-adjoint frame and (A, B, R, σ), $(A_{*_A}, B_{*_B}, R, \sigma)$ two contexts where:

$$*_A = \{*_a \colon L_1 \to L_1 \mid *_a \text{ is the identity truth-stressing} \swarrow \text{-hedge, for all } a \in A\}$$

$$*_B = \{*_b \colon L_2 \to L_2 \mid *_b \text{ is a truth-stressing} \nwarrow \text{-hedge, for all } b \in B\}$$

In order to simplify the notation, we will write (A, B_*, R, σ) instead of $(A_{*_A}, B_{*_B}, R, \sigma)$.

After introducing formally the multi-adjoint concept lattice with truth-stressing hedges framework, we will present the following technical result which shows that every row of the table associated with the relation R is also the intent of a concept in $(\mathcal{M}_*, \preceq)$.

Theorem 19. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $x \,\&\, \top_1 = \top_1 \,\&\, x = x$, for all $x \in L_1$. For each $b \in B$, we have that $R_b \colon A \to L_1$ is an intent of a concept in $(\mathcal{M}_*, \preceq)$, that is, $R_b \in Int(\mathcal{M}_*)$.*

Proof. Given $b \in B$, by Statement (2) in Proposition 3, we have $R_b(a) = R(a, b) \swarrow \top_1$. By the boundary condition of Definition 6, we have that $R_b(a) = R(a, b) \swarrow *_b(\top_1)$. Taking into account the definition of the operator $^{\uparrow_*}$ and the equality $*_b(\bot_1) = \bot_1$, applying Lemma 11, we have that $R(a, b) \swarrow *_b(\top_1) = \phi^{\uparrow_*}_{b,\top_1}(a)$, which implies $R_b(a) = \phi^{\uparrow_*}_{b,\top_1}(a)$, for all $a \in A$. By Statement (3) of Proposition 17 we have that $\bar{*}_A(\phi^{\uparrow_*}_{b,\top_1}) = \bar{*}_A((\phi^{\uparrow_*}_{b,\top_1})^{\downarrow\uparrow_*})$ for all $b \in B$, which is equivalent to $\phi^{\uparrow_*}_{b,\top_1} = (\phi^{\uparrow_*}_{b,\top})^{\downarrow\uparrow_*}$, since $*_A$ is a family of identity truth-stressing $\swarrow$-hedges. Therefore, we can conclude that $R_b \in \mathrm{Int}(\mathcal{M}_*)$ for all $b \in B$. ☐

From the proof of the previous result we obtain the following result.

Corollary 20. *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ such that $x \,\&\, \top_1 = \top_1 \,\&\, x = x$, for all $x \in L_1$. We have that*

$$R_b = \phi^{\uparrow}_{b,\top_1} = \phi^{\uparrow_*}_{b,\top_1}$$

for all $b \in B$.

Proof. The proof straightforwardly follows from Theorem 19 and Lemma 11. ☐

It is convenient to mention identity truth-stressing hedges were also considered on the attributes in the fuzzy framework of residuated concept lattices [6,7]. Indeed, this was also fundamental for defining the notion of validity of an attribute implication in this framework which is based on the mappings R_b. Thus, the above result reinforces the choice of $*_A$ as a family of identity truth-stressing $\swarrow$-hedges on L_1.

This section finishes with an example in order to illustrate the previous notions and results associated with the multi-adjoint concept lattice with truth-stressing hedges framework.

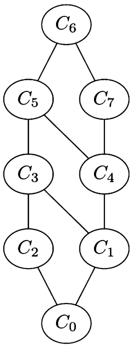
Example 21. Let $([0,1]_2, \leq, \&_{\mathrm{DG}}, \swarrow^{\mathrm{DG}}, \nwarrow_{\mathrm{DG}}, \&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}}, \&_{\mathrm{DL}}, \swarrow^{\mathrm{DL}}, \nwarrow_{\mathrm{DL}})$ be a multi-adjoint frame and (A, B, R, σ), (A, B_*, R, σ) be contexts where the set of attributes is $A = \{a_1, a_2, a_3, a_4, a_5\}$, the set of objects is $B = \{b_1, b_2, b_3\}$, $*_b \colon [0,1]_2 \to [0,1]_2$ is the globalization truth-stressing $\nwarrow$-hedge for all $b \in B$ and the relation $R \colon A \times B \to [0,1]_2$ and the mapping σ are given in Table 4. This particular selection of σ allows us to assign different adjoint triples to the objects and, therefore, to establish a preference between them. Namely, the object b_3 has more preference than the objects b_1 and b_2, hence the Łukasiewicz implication has been associated with it since it is greater than the product and Gödel implications. Moreover, the object the b_2 has more preference than the object b_1, for implementing that the product implication has been associated with b_2 and the Gödel implication with b_3, due to is greater than the Gödel one.

The multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ associated with (A, B, R, σ) is composed of 8 concepts, as it is shown in Fig. 1. Specifically, $C_0 = \langle \phi^{\uparrow\downarrow}_{b,0.0}, \phi^{\uparrow}_{b,0.0}\rangle$, for all $b \in B$, and $\langle \phi^{\uparrow\downarrow}_{b_1,0.5}, \phi^{\uparrow}_{b_1,0.5}\rangle$, $\langle \phi^{\uparrow\downarrow}_{b_2,0.5}, \phi^{\uparrow}_{b_2,0.5}\rangle$ also generate the concept C_0. The rest of fuzzy-objects are associated with the following concepts:

Table 4
Relation R and mapping σ of Example 21.

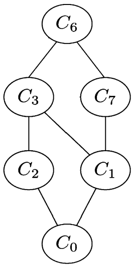
R	a_1	a_2	a_3	a_4	a_5
b_1	1	0.5	1	1	0.5
b_2	1	1	1	0.5	0.5
b_3	0	0	0.5	1	0

σ	a_1	a_2	a_3	a_4	a_5
b_1	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$
b_2	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$
b_3	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$



$$\begin{array}{lcl}
C_0 & = & \langle\{0.5/b_1, 0.5/b_2\}, \{1/a_1, 1/a_2, 1/a_3, 1/a_4, 1/a_5\}\rangle \\
C_1 & = & \langle\{1/b_1, 0.5/b_2\}, \{1/a_1, 0.5/a_2, 1/a_3, 1/a_4, 0.5/a_5\}\rangle \\
C_2 & = & \langle\{0.5/b_1, 1/b_2\}, \{1/a_1, 1/a_2, 1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
C_3 & = & \langle\{1/b_1, 1/b_2\}, \{1/a_1, 0.5/a_2, 1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
C_4 & = & \langle\{1/b_1, 0.5/b_2, 0.5/b_3\}, \{0.5/a_1, 0.5/a_2, 1/a_3, 1/a_4, 0.5/a_5\}\rangle \\
C_5 & = & \langle\{1/b_1, 1/b_2, 0.5/b_3\}, \{0.5/a_1, 0.5/a_2, 1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
C_6 & = & \langle\{1/b_1, 1/b_2, 1/b_3\}, \{0.5/a_3, 0.5/a_4\}\rangle \\
C_7 & = & \langle\{1/b_1, 0.5/b_2, 1/b_3\}, \{0.5/a_3, 1/a_4\}\rangle
\end{array}$$

Fig. 1. Hasse diagram of $(\mathcal{M}, \preceq)$ and its concepts.



$$\begin{array}{lcl}
C_0 & = & \langle\{0.5/b_1, 0.5/b_2\}, \{1/a_1, 1/a_2, 1/a_3, 1/a_4, 1/a_5\}\rangle \\
C_1 & = & \langle\{1/b_1, 0.5/b_2\}, \{1/a_1, 0.5/a_2, 1/a_3, 1/a_4, 0.5/a_5\}\rangle \\
C_2 & = & \langle\{0.5/b_1, 1/b_2\}, \{1/a_1, 1/a_2, 1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
C_3 & = & \langle\{1/b_1, 1/b_2\}, \{1/a_1, 0.5/a_2, 1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
C_6 & = & \langle\{1/b_1, 1/b_2, 1/b_3\}, \{0.5/a_3, 0.5/a_4\}\rangle \\
C_7 & = & \langle\{1/b_1, 0.5/b_2, 1/b_3\}, \{0.5/a_3, 1/a_4\}\rangle
\end{array}$$

Fig. 2. Hasse diagram of $(\mathcal{M}_*, \preceq)$ and its concepts.

$$\begin{array}{lcl}
C_1 & = & \langle \phi^{\uparrow\downarrow}_{b_1,1.0}, \phi^{\uparrow}_{b_1,1.0} \rangle \\
C_2 & = & \langle \phi^{\uparrow\downarrow}_{b_2,1.0}, \phi^{\uparrow}_{b_2,1.0} \rangle \\
C_4 & = & \langle \phi^{\uparrow\downarrow}_{b_3,0.5}, \phi^{\uparrow}_{b_3,0.5} \rangle \\
C_7 & = & \langle \phi^{\uparrow\downarrow}_{b_3,1.0}, \phi^{\uparrow}_{b_3,1.0} \rangle
\end{array}$$

The multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ associated with (A, B_*, R, σ) is composed of 6 concepts, as it is displayed in Fig. 2. Namely, $C_0 = \langle \phi^{\uparrow_*\downarrow}_{b,0.0}, \phi^{\uparrow_*}_{b,0.0} \rangle = \langle \phi^{\uparrow_*\downarrow}_{b,0.5}, \phi^{\uparrow_*}_{b,0.5} \rangle$, for all $b \in B$. The rest of fuzzy-objects are associated with the following concepts:

$$\begin{array}{lcl}
C_1 & = & \langle \phi^{\uparrow_*\downarrow}_{b_1,1.0}, \phi^{\uparrow_*}_{b_1,1.0} \rangle \\
C_2 & = & \langle \phi^{\uparrow_*\downarrow}_{b_2,1.0}, \phi^{\uparrow_*}_{b_2,1.0} \rangle \\
C_7 & = & \langle \phi^{\uparrow_*\downarrow}_{b_3,1.0}, \phi^{\uparrow_*}_{b_3,1.0} \rangle
\end{array}$$

Taking into account both Hasse diagrams, we obtain that the set of join-irreducible elements associated with these concept lattices are:

$$\mathrm{J_F}(\mathcal{M}) = \{C_1, C_2, C_4, C_7\}$$

$$\mathrm{J_F}(\mathcal{M}_*) = \{C_1, C_2, C_7\}$$

It is easy to check that the concepts belonging to $\mathrm{J_F}(\mathcal{M})$ and $\mathrm{J_F}(\mathcal{M}_*)$ satisfy the conditions required in Theorem 13. For example, if we consider $C_1 = \langle \phi^{\uparrow\downarrow}_{b_1,1.0}, \phi^{\uparrow}_{b_1,1.0} \rangle$ in $(\mathcal{M}, \preceq)$, we see from Fig. 1 that this element cannot be expressed as the infimum of intents

$\phi^{\uparrow}_{b,y}$ which are greater than $\phi^{\uparrow}_{b_1,1.0}$, for all $b \in B$ and $y \in [0,1]_2$. Moreover, it is clear that $\phi^{\uparrow}_{b_1,1.0} \neq f_{\top}$. Clearly, the concept C_0 cannot be a join-irreducible element since $C_0 = f_{\top}$.

According to Theorem 18, we have that each concept from $\mathcal{M}_*$ is also a concept of $\mathcal{M}$. However, the counterpart is not true, which is easily seen from the concepts C_4 and C_5. The fact that a concept is a join-irreducible element of the original concept lattice $(\mathcal{M}, \preceq)$ is not a sufficient condition to be a concept of the concept lattice with hedges $(\mathcal{M}_*, \preceq)$, as is the case of C_5.

Finally, it is clear that the result presented in Theorems 12 and 19 is satisfied, since every row of Table 4 associated with the relation R is an intent of a concept in $(\mathcal{M}, \preceq)$ and $(\mathcal{M}_*, \preceq)$. Specifically, we have that:

$$R_{b_1} = \mathrm{Int}(C_1)$$

$$R_{b_2} = \mathrm{Int}(C_2)$$

$$R_{b_3} = \mathrm{Int}(C_7) \quad \square$$

4. Validity of fuzzy attribute implications

As we mentioned at the beginning of the paper, our goal focuses on the survey of attribute implications making use of the advantages provided by the multi-adjoint framework. Namely, our contribution will allow us to complement the study on attribute implications in multi-adjoint concept lattices presented in [39]. To begin with, it is convenient to recall the philosophy underlying the notion of attribute implication.

Definition 22. An *attribute implication* over A is an expression $f_2 \Leftarrow f_1$, where $f_1, f_2 \in L_1^A$ are two fuzzy subsets of attributes.

Specifically, an attribute implication indicates that if an object satisfies all the attributes from the antecedent f_1 of the rule, it also satisfies the attributes of the consequent f_2. The meaning and interpretation of attribute implications are reinforced by their validity in the given context. Then, we need to provide a definition of validity of an attribute implication in the multi-adjoint frame, which be compatible with the whole context. For this purpose, we will first extend the usual fuzzy inclusion considering adjoint triples.

Definition 23. Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1), (L_2, \preceq_2), (L_1, \preceq_1)$ and two fuzzy subsets of attributes $f_1, f_2 \in L_1^A$, *the degree in which* f_1 *is included in* f_2 is defined as:

$$\mathrm{S}^1(f_1, f_2) = \inf\{f_2(a) \nwarrow f_1(a) \mid a \in A\} \tag{6}$$

Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1 \preceq_1), (L_2, \preceq_2), (L_2, \preceq_2)$ and two fuzzy subsets of objects $g_1, g_2 \in L_2^B$, *the degree in which* g_1 *is included in* g_2 is defined as:

$$\mathrm{S}^2(g_1, g_2) = \inf\{g_2(b) \swarrow g_1(b) \mid b \in B\} \tag{7}$$

An extra truth-stressing $\nwarrow$-hedge is considered in the definition of the degree in which a fuzzy implication is valid.

Definition 24. Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ and three fuzzy subsets of attributes $f_1, f_2, f_3 \in L_1^A$, the degree in which a fuzzy implication $f_2 \Leftarrow f_1$ is valid in f_3 is:

$$\|f_2 \Leftarrow f_1\|_{f_3} = \mathrm{S}^1(f_2, f_3) \swarrow *_2\, (\mathrm{S}^1(f_1, f_3))$$

where $*_2 \colon L_2 \to L_2$ is a truth-stressing $\nwarrow$-hedge.

This notion is extrapolated to a set of fuzzy subsets of attributes $\mathcal{F} \subseteq L_1^A$, defining the degree in which the fuzzy implication $f_2 \Leftarrow f_1$ is valid in $\mathcal{F}$ as:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{F}} = \bigwedge_{f \in \mathcal{F}} \|f_2 \Leftarrow f_1\|_f$$

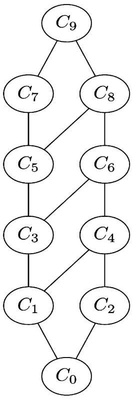
It is possible to extend the notion of validity of a fuzzy implication on a multi-adjoint concept lattice. Notice that in lattices satisfying the descendent chain condition, every element of the lattice can be written as the supremum of join-irreducible elements [26]. In the Boolean case, the rows of the relation contains the set of join-irreducible elements, and so considering these elements are enough for computing the validity of the whole context. However, in the fuzzy case, this is not true and the rows only provides part of the join-irreducible elements. Hence, it is natural to consider the whole set of intents in the definition of validity of an attribute implication, as we consider next.

Definition 25. Given two subsets of attributes $f_1, f_2 \in L_1^A$, the degree in which the fuzzy implication $f_2 \Leftarrow f_1$ is valid in the multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ is the truth degree:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*} = \|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathcal{M}_*)}$$

Table 5
Relation R of Example 26.

R	a_1	a_2
b_1	0.4	0.2
b_2	0	0.8



$$
\begin{aligned}
\mathrm{Int}(C_0) &= \{1/a_1, 1/a_2\}\\
\mathrm{Int}(C_1) &= \{1/a_1, 0.8/a_2\}\\
\mathrm{Int}(C_2) &= \{1/a_2\}\\
\mathrm{Int}(C_3) &= \{0.8/a_1, 0.4/a_2\}\\
\mathrm{Int}(C_4) &= \{0.8/a_2\}\\
\mathrm{Int}(C_5) &= \{0.53/a_1, 0.26/a_2\}\\
\mathrm{Int}(C_6) &= \{0.4/a_2\}\\
\mathrm{Int}(C_7) &= \{0.4/a_1, 0.2/a_2\}\\
\mathrm{Int}(C_8) &= \{0.26/a_2\}\\
\mathrm{Int}(C_9) &= \{0.2/a_2\}
\end{aligned}
$$

Fig. 3. Hasse diagram of $(\mathcal{M}, \preceq)$ and intents.

Table 6
Values associated with the computations of the degree of validity $\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}$.

	f_1	f_2	$\mathrm{Int}(C_0)$	$\mathrm{Int}(C_1)$	$\mathrm{Int}(C_2)$	$\mathrm{Int}(C_3)$	$\mathrm{Int}(C_4)$	$\mathrm{Int}(C_5)$	$\mathrm{Int}(C_6)$	$\mathrm{Int}(C_7)$	$\mathrm{Int}(C_8)$	$\mathrm{Int}(C_9)$
a_1	1	0.4	1	1	0	0.8	0	0.53	0	0.4	0	0
a_2	0.2	0.8	1	0.8	1	0.4	0.8	0.26	0.4	0.2	0.26	0.2
$\mathrm{S}^1(f_1, f)$			1	1	0	0.75	0	0.5	0	0.25	0	0
$\mathrm{S}^1(f_2, f)$			1	1	0	0.5	0	0.25	0	0.25	0	0
$\|f_2 \Leftarrow f_1\|_f$			1	1	1	0.66	1	0.5	1	1	1	1

The drawback of using the above definition is that we need to calculate the whole set of intents of the concept lattice, and this procedure usually involves a large number of calculations. Then it would be interesting to provide an alternative definition in order to reduce the computations.

In the fuzzy framework of residuated concept lattices [7], the straightforward extension of the Boolean case is given, which allows to calculate the degree of validity of an attribute implication by means of the degree of validity in the whole set of rows of the table associated with the relation of the given context. Although Theorem 19 provides sufficient conditions to ensure that each row of the table is an intent of a concept, the definition of validity on the row is not equivalent to the definition on the whole set of intents in the multi-adjoint framework (Definition 25), as we will show below.

Example 26. Consider the multi-adjoint framework $([0,1]_{100}, [0,1]_4, [0,1]_{100}, \preceq, \&_{\mathrm{DG}}, \swarrow^{\mathrm{DG}}, \nwarrow_{\mathrm{DG}}, \&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}}, \&_{\mathrm{DL}}, \swarrow^{\mathrm{DL}}, \nwarrow_{\mathrm{DL}})$, the contexts (A, B, R, σ) and (A, B_*, R, σ), where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $R\colon A \times B \to [0,1]_{100}$ is given by Table 5, $*_2\colon [0,1]_4 \to [0,1]_4$ is the identity truth-stressing $\nwarrow$-hedge, $*_b$ is the identity truth-stressing $\nwarrow$-hedge for all $b \in B$, and σ is constantly $(\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}})$.

The concept lattice related to this framework and the context (A, B_*, R, σ) is composed of 10 concepts whose intents are given in Fig. 3.

Now, we will consider $(\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}})$ as an extra adjoint triple with respect to $([0,1]_{100}, \preceq)$, the fuzzy subsets of attributes $f_1 = \{1/a_1, 0.2/a_2\}$, $f_2 = \{0.4/a_1, 0.8/a_2\}$ and we will compute the degree in which the fuzzy implication $f_2 \Leftarrow f_1$ is valid in the multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ related to the context (A, B_*, R, σ). For this purpose, we will use the data presented in Table 6, which have been previously computed. Applying Definition 25, we have that:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*} = \bigwedge_{f \in \mathrm{Int}(\mathcal{M}_*)} \|f_2 \Leftarrow f_1\|_f = \bigwedge_{f \in \mathrm{Int}(\mathcal{M}_*)} \{\mathrm{S}^1(f_2, f) \swarrow^{\mathrm{DP}} *_2 (\mathrm{S}^1(f_1, f))\} = 0.5$$

Now, we will compute the degree $\|f_2 \Leftarrow f_1\|_{\{R_b \mid b \in B\}}$, using the previous computations shown in Table 7. Applying Definition 24, we have that:

$$\|f_2 \Leftarrow f_1\|_{\{R_b \mid b \in B\}} = \bigwedge_{b \in B} (\mathrm{S}^1(f_2, R_b) \swarrow *_2 (\mathrm{S}^1(f_1, R_b))) = 1$$

Table 7
Values associated with the computations of the degree of inclusion $\|f_2 \Leftarrow f_1\|_{\{R_b \mid b\in B\}}$.

	f_1	f_2	R_{b_1}	R_{b_2}
a_1	1	0.4	0.4	0
a_2	0.2	0.8	0.2	0.8
$\mathrm{S}^1(f_1, R_b)$			0.25	0
$\mathrm{S}^1(f_2, R_b)$			0.25	0
$\|f_2 \Leftarrow f_1\|_{R_b}$			1	1

Therefore, we obtain that $\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*} \prec_1 \|f_2 \Leftarrow f_1\|_{\{R_b \mid b\in B\}}$ and we conclude that both degrees are not equivalent. □

Moreover, it is convenient to mention that in the residuated framework, the degree of validity of an implication on the set of rows of the relation table can be characterized by a certain degree of inclusion [7, Theorem 7]. This result can be extended to the multi-adjoint framework, as the following theorem shows.

Theorem 27. *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$*. Given two subsets of attributes* $f_1, f_2 \in L_1^A$ *and a truth-stressing* $\nwarrow$*-hedge* $*_2 : L_2 \to L_2$*, we have that:*

$$\|f_2 \Leftarrow f_1\|_{\{R_b \mid b\in B\}} = S^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow})$$

where $*_2 \circ f_1^{\downarrow}$ *denotes the composition of* $*_2$ *and* $f_1^{\downarrow}$*, that is, the fuzzy subset of objects* $*_2 \circ f_1^{\downarrow} : B \to L_2$ *such that* $(*_2 \circ f_1^{\downarrow})(b) = *_2\,(f_1^{\downarrow}(b))$*, for all* $b \in B$*.*

Proof. Taking into account Definition 24, we can express the degree $\|f_2 \Leftarrow f_1\|_{\{R_b\mid b\in B\}}$ as follows:

$$\|f_2 \Leftarrow f_1\|_{\{R_b\mid b\in B\}} = \bigwedge_{b\in B} (\mathrm{S}^1(f_2, R_b) \swarrow *_2\, (\mathrm{S}^1(f_1, R_b))) \tag{8}$$

Now, given $b \in B$, since $R_b(a) = R(a, b)$ for all $a \in A$, we have

$$\mathrm{S}^1(f_2, R_b) = \inf\{R(a, b) \nwarrow f_2(a) \mid a \in A\} = f_2^{\downarrow}(b)$$
$$\mathrm{S}^1(f_1, R_b) = \inf\{R(a, b) \nwarrow f_1(a) \mid a \in A\} = f_1^{\downarrow}(b)$$

Therefore, Equality (8) is equivalent to:

$$\|f_2 \Leftarrow f_1\|_{\{R_b\mid b\in B\}} = \bigwedge_{b\in B} \{f_2^{\downarrow}(b) \swarrow *_2\, (f_1^{\downarrow}(b))\}$$

Applying Definition 23, we obtain that:

$$\mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow}) = \inf\{f_2^{\downarrow}(b) \swarrow *_2\, (f_1^{\downarrow}(b)) \mid b \in B\}$$

Hence, we have that the equality $\|f_2 \Leftarrow f_1\|_{\{R_b \mid b\in B\}} = \mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow})$ holds. □

Thus, in general, the degree of inclusion $\mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow})$ is not equal to the degree of validity proposed in Definition 25. The following theorem proofs that the value $\mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow})$ is an upper bound of the degree of validity $\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}$ in the multi-adjoint framework.

Theorem 28. *Let* $(L_1, \preceq_1)$ *be a complete lattice satisfying the descendent chain condition (DCC),* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$ *such that* $x \,\&\, \top_1 = x$*, for all* $x \in L_1$*, and* $f_1, f_2 \in L_1^A$ *be two fuzzy subsets of attributes. Then:*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*} \preceq_1 S^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow})$$

Proof. Since L_1 satisfies the DCC, the concept $\langle \phi_{b,\top_1}^{\uparrow_*\downarrow}, \phi_{b,\top_1}^{\uparrow_*} \rangle$ can be written as the supremum of the join-irreducible concepts $\langle \phi_{b',y'}^{\uparrow_*\downarrow}, \phi_{b',y'}^{\uparrow_*} \rangle \in \mathrm{J_F}(\mathcal{M}_*)$ strictly less than it, that is,

$$\langle \phi_{b,\top_1}^{\uparrow_*\downarrow}, \phi_{b,\top_1}^{\uparrow_*} \rangle = \sup\{\langle \phi_{b',y'}^{\uparrow_*\downarrow}, \phi_{b',y'}^{\uparrow_*} \rangle \in \mathrm{J_F}(\mathcal{M}_*) \mid \langle \phi_{b',y'}^{\uparrow_*\downarrow}, \phi_{b',y'}^{\uparrow_*} \rangle \preceq \langle \phi_{b,\top_1}^{\uparrow_*\downarrow}, \phi_{b,\top_1}^{\uparrow_*} \rangle\}$$

which is equivalent by the ordering in the concept lattice recalled in Definition 8 to the following equality

$$\phi_{b,\top_1}^{\uparrow_*} = \inf\{\phi_{b',y'}^{\uparrow_*} \mid \langle \phi_{b',y'}^{\uparrow_*\downarrow}, \phi_{b',y'}^{\uparrow_*} \rangle \in \mathrm{J_F}(\mathcal{M}_*), \phi_{b,\top_1}^{\uparrow_*} \preceq_1 \phi_{b',y'}^{\uparrow_*}\}$$

Therefore, this is also a decomposition of R_b, due to $R_b = \phi^{\uparrow_*}_{b,\top_1}$, by Corollary 20. Now, the inequality that we want to prove arises from the following chain of equalities and inequalities:

$$\begin{aligned}
\mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow}) &\overset{(i)}{=} \bigwedge_{b\in B} (f_2^{\downarrow}(b) \swarrow^{*_2} (f_1^{\downarrow}(b)))\\
&\overset{(ii)}{=} \bigwedge_{b\in B} (\mathrm{S}^1(f_2, R_b) \swarrow^{*_2} (\mathrm{S}^1(f_1, R_b)))\\
&\overset{(iii)}{=} \bigwedge_{b\in B} \left(\mathrm{S}^1 \left(f_2, \bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \phi^{\uparrow_*}_{b',y'} \right) \swarrow^{*_2} \left(\mathrm{S}^1 \left(f_1, \bigwedge_{\phi^{\uparrow_*}_{b'',y''} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \phi^{\uparrow_*}_{b'',y''} \right)\right)\right)\\
&\overset{(iv)}{=} \bigwedge_{b\in B} \left(\left(\bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \mathrm{S}^1(f_2, \phi^{\uparrow_*}_{b',y'}) \right) \swarrow^{*_2} \left(\bigwedge_{\phi^{\uparrow_*}_{b'',y''} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \mathrm{S}^1(f_1, \phi^{\uparrow_*}_{b'',y''}) \right)\right)\\
&\overset{(v)}{=} \bigwedge_{b\in B} \left(\bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \left(\mathrm{S}^1(f_2, \phi^{\uparrow_*}_{b',y'}) \swarrow^{*_2} \left(\bigwedge_{\phi^{\uparrow_*}_{b'',y''} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \mathrm{S}^1(f_1, \phi^{\uparrow_*}_{b'',y''}) \right)\right)\right)\\
&\overset{(vi)}{{}_1\!\succeq} \bigwedge_{b\in B} \left(\bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \left(\mathrm{S}^1(f_2, \phi^{\uparrow_*}_{b',y'}) \swarrow^{*_2} (\mathrm{S}^1(f_1, \phi^{\uparrow_*}_{b',y'}))\right)\right)
\end{aligned}$$

where $\Phi_{\phi^{\uparrow}_{b,\top_1}} = \{\phi^{\uparrow_*}_{b',y'} \mid \langle \phi^{\uparrow_*\downarrow}_{b',y'}, \phi^{\uparrow_*}_{b',y'}\rangle \in \mathrm{J_F}(\mathcal{M}_*), \phi^{\uparrow}_{b,\top_1} \preceq_1 \phi^{\uparrow_*}_{b',y'}\}$, (i) holds by Definition 23, (ii) by the definition of the operator $^{\downarrow}$, (iii) by the decomposition proved at the beginning of this proof, (iv) is satisfied by Definition 23 and because that the operator $\nwarrow$ preserves the infimum in the consequent. This last property of the implication $\swarrow$ is used in (v). Finally, (vi) is obtained by the infimum property and the monotonicity of the operator $\swarrow$.

Now, taking into account Definition 24, we obtain the following chain of equalities:

$$\begin{aligned}
\bigwedge_{b\in B} \left(\bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \left(\mathrm{S}^1(f_2, \phi^{\uparrow_*}_{b',y'}) \swarrow^{*_2} (\mathrm{S}^1(f_1, \phi^{\uparrow_*}_{b',y'}))\right)\right) &= \bigwedge_{b\in B} \left(\bigwedge_{\phi^{\uparrow_*}_{b',y'} \in \Phi_{\phi^{\uparrow}_{b,\top_1}}} \| f_2 \Leftarrow f_1 \|_{\phi^{\uparrow_*}_{b',y'}} \right)\\
&= \bigwedge_{b\in B} \| f_2 \Leftarrow f_1 \|_{\Phi_{\phi^{\uparrow}_{b,\top_1}}}
\end{aligned}$$

Due to $\Phi_{\phi^{\uparrow}_{b,\top_1}} \subseteq \mathrm{Int}(\mathcal{M}_*)$ and the infimum property, we have that:

$$\| f_2 \Leftarrow f_1 \|_{\mathrm{Int}(\mathcal{M}_*)} \preceq_1 \bigwedge_{b\in B} \| f_2 \Leftarrow f_1 \|_{\Phi_{\phi^{\uparrow}_{b,\top_1}}}$$

for all $b \in B$, and we can conclude that:

$$\| f_2 \Leftarrow f_1 \|_{\mathcal{M}_*} = \| f_2 \Leftarrow f_1 \|_{\mathrm{Int}(\mathcal{M}_*)} \preceq_1 \bigwedge_{b\in B} \| f_2 \Leftarrow f_1 \|_{\Phi_{\phi^{\uparrow}_{b,\top_1}}} \preceq_1 \mathrm{S}^2(*_2 \circ f_1^{\downarrow}, f_2^{\downarrow}) \quad \square$$

Following our goal of providing an alternative definition for the computation of the degree of validity, the next result reveals that the degree in which a fuzzy implication is valid in a multi-adjoint concept lattice is obtained computing only the degree of validity on the set of join-irreducible elements of the lattice. As a consequence, we do not need to calculate the degree of validity on the whole set of intents of the concepts of the lattice, which imply an important reduction in the number of computations.

Theorem 29. *Let $(L_1, \preceq_1)$ be a complete lattice satisfying the descendent chain condition, $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $x \,\&\, \top_1 = x$. Given two fuzzy subsets of attributes $f_1, f_2 \in L_1^A$, we have that:*

$$\| f_2 \Leftarrow f_1 \|_{Int(\mathcal{M}_*)} = \| f_2 \Leftarrow f_1 \|_{Int(J_F(\mathcal{M}_*))}$$

where $Int(J_F(\mathcal{M}_*))$ denotes the set of intents of the join-irreducible elements of $(\mathcal{M}_*, \preceq)$.

Proof. First of all, we will prove the following inequality:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathcal{M}_*)} \preceq_1 \|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \tag{9}$$

By Definition 24, we can deduce that:

$$\begin{aligned}\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} &= \bigwedge_{\phi_{b,y}^{\uparrow_*} \in \mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \|f_2 \Leftarrow f_1\|_{\phi_{b,y}^{\uparrow_*}} \\ &= \bigwedge_{\phi_{b,y}^{\uparrow_*} \in \mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \left(\mathrm{S}^1(f_2, \phi_{b,y}^{\uparrow_*}) \swarrow^{*_2} (\mathrm{S}^1(f_1, \phi_{b,y}^{\uparrow_*}))\right)\end{aligned}$$

Taking into account that $\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*)) \subseteq \mathrm{Int}(\mathcal{M}_*)$ and the infimum property, we can conclude that:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathcal{M}_*)} \preceq_1 \|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))}$$

Now, we will demonstrate the other inequality, that is:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \preceq_1 \|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathcal{M}_*)} \tag{10}$$

By Definition 24, Equation (10) is equivalent to the next inequality:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \preceq_1 \|f_2 \Leftarrow f_1\|_f = \mathrm{S}^1(f_2, f) \swarrow^{*_2} (\mathrm{S}^1(f_1, f)) \tag{11}$$

for all $f \in \mathrm{Int}(\mathcal{M}_*)$. Now, similarly as in the proof of Theorem 28 to obtain a decomposition of R_b, we will give a decomposition of f. Since L_1 satisfies the DCC, then any concept of $\mathcal{M}_*$ can be expressed as supremum of the join-irreducible elements of the concept lattice greater than it, that is, $\langle g, f\rangle = \sup\{\langle \phi_{b,y}^{\uparrow_*\downarrow}, \phi_{b,y}^{\uparrow_*}\rangle \in \mathrm{J}_{\mathrm{F}}(\mathcal{M}_*) \mid \langle g, f\rangle \preceq \langle \phi_{b,y}^{\uparrow_*\downarrow}, \phi_{b,y}^{\uparrow_*}\rangle\}$. Moreover, by the ordering in the concept lattice $(\mathcal{M}_*, \preceq)$, the previous equality implies that any fuzzy subset of attributes f can be expressed as infimum of the intents $\phi_{b,y}^{\uparrow_*}$ of the join-irreducible elements of the concept lattice greater than it, and hence, Equation (11) can be rewritten as:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \preceq_1 \mathrm{S}^1\left(f_2, \bigwedge_{\phi_{b,y}^{\uparrow_*} \in \Phi_f} \phi_{b,y}^{\uparrow_*}\right) \swarrow^{*_2} \left(\mathrm{S}^1\left(f_1, \bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \phi_{b',y'}^{\uparrow_*}\right)\right) \tag{12}$$

where Φ_f denotes the intents of the family of join-irreducible elements which generates f, that is, $\Phi_f = \{\phi_{b,y}^{\uparrow_*} \mid \langle \phi_{b,y}^{\uparrow_*\downarrow}, \phi_{b,y}^{\uparrow_*}\rangle \in \mathrm{J}_{\mathrm{F}}(\mathcal{M}_*), f \preceq_1 \phi_{b,y}^{\uparrow_*}\}$.

Applying Definition 23 and taking into account that the operator $\nwarrow$ preserves the infimum in the consequent, we obtain that Equation (12) is equivalent to:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \preceq_1 \left(\bigwedge_{\phi_{b,y}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_2, \phi_{b,y}^{\uparrow_*})\right) \swarrow^{*_2} \left(\bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_1, \phi_{b',y'}^{\uparrow_*})\right) \tag{13}$$

As the operator $\swarrow$ preserves the infimum in the consequent, we obtain that Equation (13) is equivalent to:

$$\|f_2 \Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J}_{\mathrm{F}}(\mathcal{M}_*))} \preceq_1 \bigwedge_{\phi_{b,y}^{\uparrow_*} \in \Phi_f} \left(\mathrm{S}^1(f_2, \phi_{b,y}^{\uparrow_*}) \swarrow^{*_2} \left(\bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_1, \phi_{b',y'}^{\uparrow_*})\right)\right) \tag{14}$$

Finally, Equation (14) is proven from the following chain of inequalities:

$$\bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_1, \phi_{b',y'}^{\uparrow_*}) \overset{(i)}{\preceq_1} \mathrm{S}^1(f_1, \phi_{b,y}^{\uparrow_*})$$

$$*_2\left(\bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_1, \phi_{b',y'}^{\uparrow_*})\right) \overset{(ii)}{\preceq_1} *_2\left(\mathrm{S}^1(f_1, \phi_{b,y}^{\uparrow_*})\right)$$

$$\mathrm{S}^1(f_2, \phi_{b,y}^{\uparrow_*}) \swarrow^{*_2} (\mathrm{S}^1(f_1, \phi_{b,y}^{\uparrow_*})) \overset{(iii)}{\preceq_1} \mathrm{S}^1(f_2, \phi_{b,y}^{\uparrow_*}) \swarrow^{*_2} \left(\bigwedge_{\phi_{b',y'}^{\uparrow_*} \in \Phi_f} \mathrm{S}^1(f_1, \phi_{b',y'}^{\uparrow_*})\right)$$

Table 8
Relation R and mapping σ of Example 30.

R	a_1	a_2	a_3	a_4	a_5
b_1	1.0	0.5	0.5	0.5	1.0
b_2	1.0	0.0	0.5	0.0	0.5
b_3	0.0	1.0	0.0	0.0	0.0

σ	a_1	a_2	a_3	a_4	a_5
b_1	$\&_{\mathrm{DG}}$	$\&_{\mathrm{DG}}$	$\&_{\mathrm{DG}}$	$\&_{\mathrm{DG}}$	$\&_{\mathrm{DG}}$
b_2	$\&_{\mathrm{DP}}$	$\&_{\mathrm{DP}}$	$\&_{\mathrm{DP}}$	$\&_{\mathrm{DP}}$	$\&_{\mathrm{DP}}$
b_3	$\&_{\mathrm{DL}}$	$\&_{\mathrm{DL}}$	$\&_{\mathrm{DL}}$	$\&_{\mathrm{DL}}$	$\&_{\mathrm{DL}}$

$$\bigwedge_{\phi^{\uparrow *}_{b,y}\in\Phi_f}\left(\mathrm{S}^1(f_2,\phi^{\uparrow *}_{b,y})\swarrow *_2\,(\mathrm{S}^1(f_1,\phi^{\uparrow *}_{b,y}))\right)\overset{(iv)}{\preceq}_1\bigwedge_{\phi^{\uparrow *}_{b,y}\in\Phi_f}\left(\mathrm{S}^1(f_2,\phi^{\uparrow *}_{b,y})\swarrow *_2\left(\bigwedge_{\phi^{\uparrow *}_{b',y'}\in\Phi_f}\mathrm{S}^1(f_1,\phi^{\uparrow *}_{b',y'})\right)\right)$$

$$\|f_2\Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J_F}(\mathcal{M}_*))}\overset{(v)}{\preceq}_1\bigwedge_{\phi^{\uparrow *}_{b,y}\in\Phi_f}\left(\mathrm{S}^1(f_2,\phi^{\uparrow *}_{b,y})\swarrow *_2\left(\bigwedge_{\phi^{\uparrow *}_{b',y'}\in\Phi_f}\mathrm{S}^1(f_1,\phi^{\uparrow *}_{b',y'})\right)\right)$$

where (i) and (iv) holds by the infimum property, (ii) is obtained by Proposition 7, (iii) is satisfied because $\swarrow$ is order-reversing in the antecedent, and (v) holds by Definition 24 and the infimum property. □

Now, we will include an illustrative example that computes the validity of a particular fuzzy attribute implication and shows that the characterization presented by Theorem 29 allow us to reduce the number of computations considerably.

Example 30. Let $([0,1]_{10},\leq,\&_{\mathrm{DG}},\swarrow^{\mathrm{DG}},\nwarrow_{\mathrm{DG}},\&_{\mathrm{DP}},\swarrow^{\mathrm{DP}},\nwarrow_{\mathrm{DP}},\&_{\mathrm{DL}},\swarrow^{\mathrm{DL}},\nwarrow_{\mathrm{DL}})$ be a multi-adjoint frame and (A,B,R,σ), (A,B_*,R,σ) be contexts where $A=\{a_1,a_2,a_3,a_4,a_5\}$, $B=\{b_1,b_2,b_3\}$, $*_b$ is the identity truth-stressing $\nwarrow$-hedge for all $b\in B$ and the relation $R\colon\ A\times B\to[0,1]_{10}$ and the mapping σ are given in Table 8.

The concept lattice related to this framework and the context (A,B,R,σ) is composed of 61 concepts. From Theorem 13, we obtain that the set of join-irreducible elements associated with the concept lattice is:

$$\mathrm{J_F}(\mathcal{M}_*)=\{C_1,C_2,C_3,C_{12},C_{39},C_{41},C_{43},C_{52},C_{53},C_{54},C_{55},C_{56},C_{57},C_{58},C_{59},C_{60}\}$$

Then, if we use the result presented in Theorem 29, instead of the whole intents of the concept lattice to calculate the degree of validity, we can only consider the intents of the join-irreducible elements, which are $|\mathrm{J_F}(\mathcal{M}_*)|=16<61$. Hence, the number of calculations is considerably reduced.

Now, we will consider $(\&_{\mathrm{DP}},\swarrow^{\mathrm{DP}},\nwarrow_{\mathrm{DP}})$ as an extra adjoint triple with respect to $([0,1]_{10},\leq)$, $*_2$ as the identity truth-stressing, the fuzzy subsets of attributes $f_1=\{1/a_1,0.5/a_3,0.9/a_5\}$, $f_2=\{1/a_4\}$ and we will compute the degree in which the fuzzy implication $f_2\Leftarrow f_1$ is valid in the multi-adjoint concept lattice $(\mathcal{M}_*,\preceq)$ related to the context (A,B_*,R,σ), using only the intents of the join-irreducible elements of the concept lattice (Fig. 4). Applying Definition 24, we have that:

$$\begin{aligned}\|f_2\Leftarrow f_1\|_{\mathrm{Int}(\mathrm{J_F}(\mathcal{M}_*))}&=\bigwedge_{f\in\mathrm{Int}(\mathrm{J_F}(\mathcal{M}_*))}\|f_2\Leftarrow f_1\|_f\\&=\bigwedge_{f\in\mathrm{Int}(\mathrm{J_F}(\mathcal{M}_*))}\{\mathrm{S}^1(f_2,f)\swarrow^{\mathrm{DP}}*_2\,(\mathrm{S}^1(f_1,f))\}=1\quad\square\end{aligned}$$

We must also note that the computation of the join-irreducible elements is also required for achieving other goals, such as, in the computation of reducts in attribute reduction [2]. Therefore, the computation of the validity of an attribute implication does not need a complex preprocessing, but it takes advantage of the calculations already done.

5. Conclusions and future work

This paper has introduced the multi-adjoint concept lattice framework using truth-stressing hedges and presented a theoretical development of the main notions and results on attribute implications incorporating the advantages provided by the multi-adjoint concept lattice framework in the treatment of these tools.

Specifically, we have considered truth-stressing hedges in the definition of the concept-forming operators of the multi-adjoint concept lattice framework. In addition, we have shown that, in this case, these operators do not form a Galois connection. However, we have presented different properties that are verified by the concept-forming operators with hedges and that have a certain

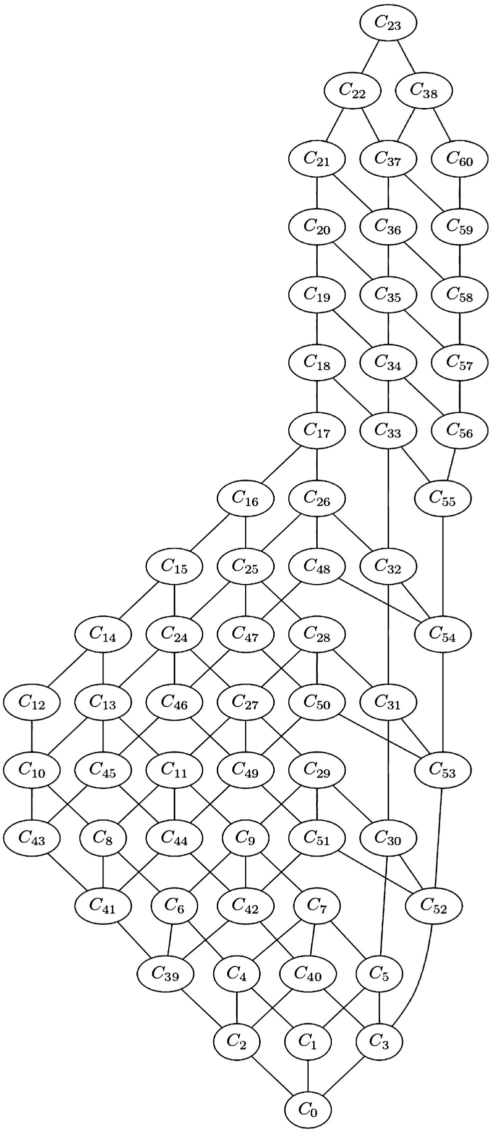


Fig. 4. Hasse diagram of $(\mathcal{M}_*, \preceq)$.

similarity with the well-known properties of Galois connections. Moreover, we have proven that, if we consider the identity truth-stressing hedge for each attribute of the given context, we can establish an isomorphism between the original concept lattice and the concept lattice enriched with truth-stressing hedges.

Furthermore, a new definition of validity of an attribute implication on a context has been introduced taking into account the whole set of intents, that is, the whole concept lattice. Moreover, we have shown that this definition of validity is not equivalent to the one given in the residuated framework, which relies on the set of rows of the relation in the given context. We have introduced a characterization of the introduced definition of validity in terms of the join-irreducible elements of the context. As a consequence, an important reduction arises in the number of operations, if we compare it with the needed computations from the whole set of intents of the concept lattice.

As a future work, we will study more properties of attribute implications in the multi-adjoint framework that can improve the computational efficiency and allow the extraction of more comprehensive information from a given context. The computation of bases will be a relevant challenge. Additionally, we are interested in conducting experimental studies on the generation of attribute implications. Initial studies about how attributes implications can be applied to a real crime data set, provided by the autonomous police of the Basque Country (Spain), have already been carried out [3,15]. Namely, different attribute implications have been used to analyze the criminal incidence in the city of Bilbao (Spain), when the local football club Athletic Club plays a football match at San Mamés Stadium. We will continue analyzing more implications associated with this context, as well as moving forward in the applied study of this research topic. These studies will involve to apply attribute implication techniques to databases, which will enable us to validate the utility and practical applicability of attribute implications in extracting valid conclusions from the considered databases.

CRediT authorship contribution statement

M. Eugenia Cornejo: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Writing – original draft, Writing – review & editing. **Jesús Medina:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Writing – original draft, Writing – review & editing. **Francisco José Ocaña:** Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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