



Stochastic Orders Applied to Insurance and Economics

Ordenaciones estocásticas aplicadas a los seguros y finanzas

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Ph.D. Dissertation

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A mi abuela Mari, la eterna mente inquieta.

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*“Cuando emprendas tu viaje a Ítaca
pide que el camino sea largo,
lleno de aventuras, lleno de experiencias.*

...

*Ten siempre a Ítaca en tu mente.
Llegar allí es tu destino.
Mas no apresures nunca el viaje.*

...

*Aunque la halles pobre, Ítaca no te ha engañado.
Así, sabio como te has vuelto, con tanta experiencia,
entenderás ya qué significan las Ítacas.”*

— *Konstantino Kavafis.*

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Abstract

In many real-life situations, the complexity of decision-making processes is often due to the stochastic nature of the random variables involved. Stochastic Ordering Theory, which addresses the problem of the comparison between two random variables (or vectors), arises as a robust tool for decision making under uncertainty.

In Financial Risk and Actuarial Theory, multiple risk factors interact among themselves. Hence, the intricacy of decision-making among different insurance and financial products increases manifold. In light of this fact, we must model risks considering the dependence structure of a random vector whose components are interrelated risks. Copula Theory serves as a fundamental instrument to model dependence among these components.

This Ph.D. dissertation aims to provide contributions in the scientific domain of Probability Theory, studying methodologies, based on Stochastic Orders and Copula Theory, which address a wide range of mathematical problems. In order to illustrate and highlight the applicability of the theoretical results, the present thesis falls within the context of Financial Risk and Actuarial Theory. In line with the illustrative purpose, several examples, either based on parametric families of distributions or dealing with real-world datasets, are provided throughout the dissertation.

The first part of the thesis studies the problem of consistency of co-risk measures with respect to stochastic orders in the following sense. Given two random vectors, $\mathbf{X}$ and $\mathbf{Y}$, we provide results of the form

$$\mathbf{X} \prec_P \mathbf{Y} \text{ implies } \rho[\mathbf{X}] \leq \rho[\mathbf{Y}],$$

where ρ is a co-risk measure and $\prec_P$ denotes the ordering that is induced by some distributional comparison principle P , based on univariate and multivariate stochastic orders. The considered measures of risk contagion are the conditional value at risk ($\text{CoVaR}^>$), the conditional expected shortfall (CoES) and the recently introduced marginal mean excess (MME).

Following with the study of co-risk measures, we tackle an important limitation, due to its scalar nature, of the Value at Risk (VaR). The addressed drawback is that the VaR does not take into account systemic risk. Hence, given a risk level $v \in (0, 1)$, we study the balancing point, in terms of the distress level, $u \in (0, 1)$, of a second risk, between the $\text{VaR}_v[Y]$ and its dependence-adjusted version, the $\text{CoVaR}_{v,u}^{\bar{}}[Y|X]$. This value, denoted by PELCov_v , can be roughly understood as an alert that the spillover effect has led to a higher risk than the one considered by the unconditional VaR. Sufficient conditions are provided for its existence, uniqueness, monotonicity and boundness. The latter property is specially relevant because it determines the range of values $x \in \mathbb{R}$ for which $Y \leq_{st} [Y|X = x]$ holds.

The second part of the thesis is devoted to studying distances between pairs of random variables. Sufficient conditions are provided with the aim of ordering either the excesses or the absolute differences between each pair of variables. Comparisons are performed in terms of several stochastic orderings. Our main contribution to this problem is to extend well-known results to a more general framework, since, in this work, variables are not necessarily independent or identically distributed.

Moreover, we provide a family of variability measures regarding the expected absolute difference. Let X, X_h be two random variables, where X_h is a distortion of X , linked by a fixed dependence structure C . Then, the functional $X \mapsto E_C|X - X_h|$ is proved to be a measure of variability in the sense of Bickel and Lehmann (1979). This functional is a subadditive measure when the components are negatively dependent and identically distributed. A general family of premium principles, given by $T[X] = E[X] + \lambda E_C|X - X_h|$, for $\lambda > 0$, is defined in terms of the variability functional. $T[X]$ reduces to the well-known standard deviation premium principle on a location-scale family.

Well-known location measures as the mean, the median, the quantiles or the expectiles can be defined as the argument which minimizes a variability measure. Therefore, considering the variability functional previously defined, and fixing the dependence structure between X and X_h , the minimization problem $\min_{\{h: F_h \in \mathcal{F}_X\}} E_C|X - X_h|$ is addressed in terms of the distortion function. As a first approach, known the copula C and the distribution function F of X , we restrict the minimization problem to be performed among the distortion functions given by a power function. That is, considering $F_h \in \mathcal{F}_X$, which denotes the family of proportional reversed hazard rate distribution functions associated to F . Under this framework, counterexamples for the existence

of the minimizer are considered and sufficient conditions for its existence are provided.

To finalize, we study Efron's monotonicity in terms of the copula. This dependence notion refers to the stochastic monotonicity of the marginals given the value of their sum. Given a bivariate vector (X, Y) with copula C , the study of the stochastic monotonicity of $[X|X + Y = s]$ in terms of s is reduced, under some particular frameworks, to study well-known positive dependence notions on the copula.

Resumen

En muchas situaciones en las que trabajamos con problemas reales, la complejidad de los procesos de toma de decisiones se debe a la naturaleza estocástica de las variables aleatorias implicadas. La Teoría de las Ordenaciones Estocásticas, que trata el problema de la comparación entre dos variables, vectores o matrices aleatorias, surge como una herramienta robusta para abordar problemas de toma de decisiones en condiciones de incertidumbre.

En problemas relacionados con la Teoría de Riesgos Financieros y Actuariales, múltiples factores de riesgo interactúan entre sí, incrementando de esta forma la complejidad de la toma de decisiones entre varios productos financieros o actuariales. Por lo tanto, cuando tenemos como objetivo modelar el riesgo, ha de tenerse en cuenta la estructura de dependencia entre las componentes implicadas. Este objetivo se aborda empleando la Teoría de Cópulas, la cual supone un instrumento fundamental para modelar la dependencia entre las componentes de un vector aleatorio.

Esta tesis doctoral se enmarca en el ámbito de la Teoría de la Probabilidad, y tiene como contribución principal el desarrollo de metodologías, basadas en ordenaciones estocásticas y Teoría de Cópulas, que abordan un amplio abanico de problemas matemáticos. Con el fin de ilustrar y destacar la aplicabilidad de los resultados teóricos, esta tesis doctoral se enmarca en el contexto de la Teoría de Riesgos Financieros y Actuariales. En consonancia con este propósito ilustrativo, a lo largo del manuscrito se incluyen varios ejemplos, algunos de ellos basados en familias paramétricas de distribuciones y otros considerando datos reales.

La primera parte de la tesis tiene, como objetivo fundamental, el estudio de consistencia de medidas de co-riesgo con respecto a diversas ordenaciones estocásticas. Dados dos vectores aleatorios, $\mathbf{X}$ e $\mathbf{Y}$, se incluyen resultados que abordan el problema desde el siguiente enfoque:

$$\mathbf{X} \prec_P \mathbf{Y} \text{ implica } \rho[\mathbf{X}] \leq \rho[\mathbf{Y}],$$

donde ρ es una medida de co-riesgo y $\prec_P$ denota la ordenación inducida por algún principio de comparación distribucional, P , basado en ordenaciones estocásticas univariantes y multivariantes. Las medidas de co-riesgo consideradas son el valor en riesgo condicional (*conditional value at risk*, $\text{CoVaR}^>$), el déficit condicional esperado (*conditional expected shortfall*, CoES) y el recientemente introducido exceso medio marginal (*marginal mean excess*, MME).

En esta línea, continuando con el estudio de medidas de co-riesgo, abordamos una limitación del Valor en Riesgo (VaR). Esta medida, debido a su naturaleza escalar, no tiene en cuenta el riesgo sistémico. Por lo tanto, dado un nivel de riesgo $v \in (0, 1)$, estudiamos el punto de equilibrio entre el $\text{VaR}_v[Y]$ y su versión ajustada a la dependencia, el $\text{CoVaR}_{v,u}[Y|X]$. Este punto de equilibrio viene dado en referencia al nivel de riesgo, $u \in (0, 1)$, de una segunda variable aleatoria y es denotado por PELCoV_v . A grandes rasgos, podemos entender el PELCoV_v como una alerta inducida cuando el contagio de otra variable ha llevado a un riesgo mayor que el considerado por el VaR incondicional. En este trabajo, aportamos condiciones suficientes para su existencia, unicidad, monotonía y acotación. Esta última propiedad toma especial relevancia, debido a que determina el conjunto de valores $x \in \mathbb{R}$ para los que se verifica $Y \leq_{st} [Y|X = x]$.

La segunda parte de la tesis está dedicada al estudio de las distancias entre pares de variables aleatorias. Se aportan resultados que refieren condiciones suficientes para la ordenación de los excesos y las diferencias absolutas entre cada par de variables. Las comparaciones se realizan en términos de varias ordenaciones estocásticas que formalizan comparaciones en localización o variabilidad. Nuestra principal contribución a este problema consiste en extender resultados conocidos a un marco general, ya que, en este trabajo, las variables consideradas no son necesariamente independientes ni están idénticamente distribuidas.

La diferencia absoluta esperada entre variables aleatorias determina una familia de medidas de variabilidad. Consideramos X, X_h dos variables aleatorias, donde X_h es una distorsión de X , con una estructura de dependencia fijada, C . Bajo estas condiciones, se demuestra que el funcional $X \mapsto E_C|X - X_h|$ es una medida de variabilidad en el sentido de Bickel and Lehmann (1979). Además, este funcional es una medida subaditiva cuando los componentes son negativamente dependientes e idénticamente distribuidos. En términos de esta medida de variabilidad, se define una familia general de principios de prima, dada por $T[X] = E[X] + \lambda E_C|X - X_h|$, para $\lambda > 0$. Comprobamos que $T[X]$

se reduce al conocido principio de prima de desviación estándar en una familia de escalas de localización.

En la literatura, encontramos que medidas de localización como la media, la mediana, los cuantiles o los expectiles, pueden definirse como el argumento que minimiza una medida de variabilidad. Por tanto, considerando el funcional de variabilidad previamente definido, y fijada la estructura de dependencia entre X y X_h , abordamos el problema de minimización $\min_{\{h: F_h \in \mathcal{F}_X\}} E_C |X - X_h|$ en términos de la función de distorsión. Como primera aproximación al problema, conocidas la cópula C y la función de distribución F de X , nos restringimos a considerar como funciones de distorsión las funciones potenciales. Es decir, consideramos $F_h \in \mathcal{F}_X$, donde $\mathcal{F}_X$ denota la familia de distribuciones con función de riesgo inverso proporcional a la de F . Bajo este marco, se presentan contraejemplos para la existencia del argumento minimizador y se proporcionan condiciones suficientes para la existencia del mismo.

Para finalizar, se estudia la monotonía de Efron en términos de la cópula. Esta noción de dependencia hace referencia a la monotonía estocástica de las marginales dado el valor de la suma. Dado un vector bivariante (X, Y) con cópula C , vemos como el estudio de la monotonía estocástica de $[X|X + Y = s]$ (en términos de s) se reduce, bajo algunos marcos particulares, a estudiar determinadas nociones de dependencia positiva sobre la cópula.

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*“A house is only as strong as its foundation.
Build a strong foundation and you can reach
even the most unthinkable heights”.*



Preliminaries

This introductory chapter includes the general background of the present Ph.D. thesis, together with a short summary of the concepts and notions that will be used in the subsequent chapters. Probability Theory, which provides the mathematical framework under which this Ph.D. thesis is developed, is a branch of Mathematics concerned with the analysis of random phenomena, with applications in many different fields where the variables of interest have an stochastic nature. Its application, and more specifically, the application of Stochastic Ordering Theory to Financial Risk and Actuarial Theory, provided the motivation that drove the development of this Ph.D. thesis. Relevant references of probabilistic methods in the actuarial and financial science are the works of Embrechts et al. (2013); Denuit et al. (2005); Boland (2007); Klugman et al. (2012) and Kaas et al. (2008).

Both in Financial Risk and Actuarial Theory, the growing complexity of insurance and financial products has made clear the need for developing innovative tools and different risk models to facilitate decision-making. In addition, the challenge does not only rely on developing the tools but also on the comparison of risk models, as it is part of the standard practice in actuarial and financial work. Traditionally, selecting the most suitable risk model among the numerous existing ones is solved by selecting a particular measure and considering the choice based on that indicator. An alternative form of risk comparison, which is based on robust comparisons between risk models, is through Stochastic Ordering Theory. As it will be discussed throughout the dissertation, decision making based on such theory leads to comparisons consistent with many of the most extended risk measures.

In addition, the complexity of insurance and financial products has increased since they are affected by multiple risk factors that interact among themselves. In light of this fact, it is required to model risk taking into account the interactions among different risk variables and to study the dependence structure of a random vector whose components are the different risks.

In order to address these problems and develop new methodologies related with Financial Risk and Actuarial Theory, the main tools used in the present Ph.D. dissertation are:

- *Probability distributions*, both univariate, in order to model variables of interest with stochastic nature, and multivariate, to model the simultaneous behaviour of the elements of a portfolio, or, in general, to model a random vector.
- *Theory of Stochastic Orders* with the aim of comparing variables and risk models.
- *Copula Theory* to model the dependence structure of random vectors.

In addition, many of the results included in the manuscript deal with risk measures (Chapters 2 and 3) and variability measures (Chapters 4 and 5). This chapter will be devoted to give a brief overview of the mentioned tools, risk and variability measures, as well as to introduce the concepts, definitions and properties that will be needed later on. Finally, the structure of the document will be explained. The main references on this preliminary chapter are the monographs by Bickel and Lehmann (1979); Denuit et al. (2005); Nelsen (2007); Shaked and Shanthikumar (2007) and Belzunce et al. (2015).

Throughout this memory, the following notations will be considered. Equality in law will be denoted as $=_{st}$. Random variables will be written by capital letters ($X, Y \dots$) and random vectors will be denoted by capital letters in bold ($\mathbf{X}, \mathbf{Y} \dots$). Given any random variable X (or any random vector $\mathbf{X}$) and any event A , $[X | A]$ ($[\mathbf{X} | A]$) denotes the conditional random variable X (random vector $\mathbf{X}$) given the event A . Note also that $(x)^+ = \max\{x, 0\}$ denotes the positive part of x , that is, $(x)^+ = x$ if $x \geq 0$ and $(x)^+ = 0$ if $x < 0$. In addition, unless it is specified, “increasing” means nondecreasing and “decreasing” means “nonincreasing”. If the monotonicity is specified to be strict, we will mean that it is non-constant in a non-zero measure set.

1.1 Probability Distributions

The basics of Probability Theory are briefly reviewed in this first section, with special emphasis on functions and tools that will serve to introduce later concepts and results.

1.1.1 Univariate Distribution Notions

1.1.1.1 Distribution, density, survival and quantile function.

Firstly, we recall the basic definitions and notions related to scalar random variables. If the reader requires an exhaustive review of basic concepts on probability, they may refer to Ross (1989); Johnson et al. (1995); Rohatgi and Ehsanes Saleh (2015) or Anderson et al. (2017).

Definition 1.1. *Given a random variable X defined on a probability space, the distribution and survival function, denoted respectively by F and $\bar{F}$, are defined, for all $x \in \mathbb{R}$, as*

$$F(x) = P[X \leq x], \quad \bar{F}(x) = 1 - F(x).$$

If the distribution function can be represented as

$$F(x) = \int_{-\infty}^x f(t) dt \quad \text{for all } x \in \mathbb{R}, \quad (1.1)$$

then f is known as the density function and $f(x) = F'(x)$ almost everywhere.

Throughout the manuscript, we will denote the support of a random variable X as $\mathcal{S}_X$. When the support of a random variable X is bounded, the left and right extremes will be denoted by l_X and u_X , respectively. If the density function verifying (1.1) exists, we will say that X is continuous and the support of the variable is a union of intervals. If, instead of intervals, the support of X is a discrete set on $\mathbb{R}$, then the variable is discrete and we consider the function defined next.

Definition 1.2. *Given a discrete random variable X , the mass probability function, denoted by p , can be defined as:*

$$p(x) = P[X = x], \quad \text{for all } x \in \mathcal{S}_X.$$

Related to the distribution function, it is defined the following function.

Definition 1.3. Given a random variable X , the quantile function, denoted by F^{-1} , can be defined as:

$$F^{-1}(u) = \inf\{x \in \mathbb{R} \mid F(x) \geq u\}, \quad \text{for all } u \in (0, 1).$$

The quantile function represents the lower value for which the distribution function exceeds the value u . Some of its properties are the following:

- $F^{-1}(U) =_{st} X$ for a random variable U uniformly distributed on the interval $(0, 1)$. Note that, when F is continuous and strictly increasing, also $U =_{st} F(X)$.
- For any real valued increasing function h , the quantile function of $h(X)$ is given by $h(F^{-1}(p))$.
- $m_X = F^{-1}(0.5)$ is the median of X .

Remark 1.4. Unless it is otherwise specified or unless that there is no doubt on the variable that we are referring to, we will use the subscript Z to denote that a function refers to the random variable Z . For instance, we will denote as F_Z^{-1} the quantile function of the random variable Z and this applies to all functions defined in the chapter.

In terms of the defined functions, let us recall the main measure that serves to sum up the scale of a random variable.

Definition 1.5. The mean of a random variable X is defined as

$$E[X] = \int_{-\infty}^{\infty} x \, dF(x).$$

If the mean is finite, it can be also expressed as

$$E[X] = \int_0^1 F^{-1}(u) du = \int_{-\infty}^0 F(x) dx + \int_0^{\infty} \bar{F}(x) dx.$$

1.1.1.2 Stop-loss, hazard rate and mean residual life functions.

In addition to the functions considered above, let us recall some other functions and notions that will be later needed on.

Definition 1.6. Given a random variable X with finite mean, the stop-loss function is defined as

$$E[(X - x)^+] = \int_x^{+\infty} \bar{F}(y) \, dy, \quad \text{for all } x \in \mathbb{R}.$$

This function has great interest in relation to reinsurance contracts in the context of actuarial science. If an insurance company has a random risk X , then, it is common that the company assumes the whole risk as far that it is less than a given value x (handling $\min(X, x)$), and it passes part of the risk, that is, over the amount x , to a reinsurance company. Then, the reinsurance company handles $(X - x)^+$ and its expected cost is given by the stop-loss function, $E[(X - x)^+]$. Note that the expected loss can be written as $E[(X - x)^+] = E[X - x | X > x] P[X > x]$. The variable $X_x = [X - x | X > x]$, which is nonnegative, is known as the *residual lifetime at time x* . The residual lifetime has been studied not only in the framework of actuarial science but also in several other fields, such as finance, reliability, biometry or survival analysis. Some suggested references on the residual lifetimes are Willmot and Lin (2001) and Bryson and Siddiqui (1969).

Considering the residual lifetime, if we are interested in studying the probability of failure of the variable X in a small increment of time $\Delta x > 0$ knowing that it has not failed before x , it can be written as:

$$P[X_x \leq \Delta x] = P[x < X \leq x + \Delta x | X > x]. \quad (1.2)$$

Therefore, assuming that X is an absolutely continuous random variable, we can compute how the ratio between (1.2) and Δx , behaves when the increment of time goes to zero:

$$\lim_{\Delta x \rightarrow 0} \frac{P[X_x \leq \Delta x]}{\Delta x} = \frac{f(x)}{\bar{F}(x)}.$$

This limit can be interpreted as the “probability” of instant failure at time x . The function that maps each time x with the given ratio is the hazard rate function, well known in the reliability context, and it is defined as follows.

Definition 1.7. *Given a continuous random variable X with distribution and density functions F and f , its hazard rate function, denoted by r , is defined as*

$$r(x) = \frac{f(x)}{\bar{F}(x)}, \quad \text{for all } x \in S_X.$$

Analogously, the reversed hazard rate function denoted by $\tilde{r}$, is defined as

$$\tilde{r}(x) = \frac{f(x)}{F(x)}, \quad \text{for all } x \in S_X.$$

Due to its definition, we can see that, given X a random life time, the higher the hazard rate is, the smaller values of X are expected. The reversed hazard rate function has also many properties similar to those of the hazard rate function (see Block et al. (1998) and Bartoszewicz (2009)). These functions

are analogous in terms of interpretation, since the reversed hazard rate can be treated as the instantaneous failure rate occurring immediately before the time point x (that is, it defines the conditional probability of failure in $(x - \Delta x, x]$ given that the failure had occurred in $[0, x]$). The reversed hazard rate function characterizes the proportional reversed hazard rate (PRHR) model, which has interesting applications in areas such as statistics, insurance risk, reliability, engineering, demography, physics or forensic science (see Di Crescenzo and Ricciardi (2001); Oliveira and Torrado (2015) or Psarrakos and Sordo (2019)). The interested reader may consult Gupta and Gupta (2007) for an extensive list of further applications.

For all value $x \in \mathcal{S}_X$, if we consider the additional expected lifetime of a random variable X which has trespassed the value x , it arises another well-known function, the mean residual life function, closely related to the hazard rate function.

Definition 1.8. *Given a random variable X with finite mean, its mean residual life function is defined as*

$$m(x) = E[X_x] = \frac{1}{\bar{F}(x)} \int_x^{+\infty} \bar{F}(t) dt, \quad \text{for all } x \in \mathcal{S}_X.$$

In risk theory, the mean residual life function can be interpreted as the expected costs for the reinsurance company in a stop-loss contract with fixed retention x , as we described above.

The property of having an increasing (decreasing) hazard rate, reverse hazard rate and mean residual life functions is well-known and widely studied. This properties are known as aging notions.

Definition 1.9. *Let X be a random variable with distribution function F and density function f . X is said to have the:*

- (i) *increasing (decreasing) failure rate property (denoted as IFR (DFR)) if its hazard rate function $r(x)$, is increasing (decreasing).*
- (ii) *increasing (decreasing) reversed failure rate property (IRFR (DRFR)) if its reversed hazard rate function $\tilde{r}(x)$, is increasing (decreasing).*
- (iii) *increasing (decreasing) mean residual life property (IMRL (DMRL)) if its mean residual life function $m(x)$, is increasing (decreasing).*

Note that Definition 1.9(ii) does not make sense for non-negative random variables, since, as remarked in Marshall and Olkin (2007), these random variables can not have distributions with increasing reversed hazard rate function.

We want to point out that an alternative way to define the hazard and reversed hazard rate functions is as the derivative of the functions $-\log \bar{F}(t)$ and $\log F(t)$. Related to the IFR notion and following this alternative definition, appear the notions of ILR (or log-concave density function) and IPLR.

Definition 1.10. *Let X be a random variable with distribution function F and density function f . X is said to have the:*

- (i) *increasing (decreasing) likelihood ratio (ILR (DLR)) property if the ratio $f(x+y)/f(x)$ decreases (increases) in x for all $y \geq 0$. Alternatively, X has the ILR (DLR) property if $\log f(x)$ is concave (convex).*
- (ii) *increasing (decreasing) proportional likelihood ratio (IPLR (DPLR)) property if $f(\lambda x)/f(x)$ is increasing (decreasing) in x for any positive constant $\lambda < 1$.*

Similarly to the logconcavity of the density function as a characterization of the ILR concept, an alternative condition was given in Oliveira and Torrado (2015) in order to define the IPLR property. The authors proved that a random variable X with density f is IPLR (DPLR) if and only if $x \cdot \eta(x)$ increases (decreases) in x , where $\eta(x) = -f'(x)/f(x)$.

Note that ILR implies IPLR and IFR, but the reverses do not hold (see Ramos and Sordo (2001) and Navarro et al. (2020)).

1.1.2 Multivariate Distribution Notions

In this section, a brief overview of basic notions and results for multivariate distributions is provided. Again, we will only focus on defining the concepts that will be needed later. For a detailed review of the topic, the reader may refer to Chatfield and Collins (2018) or Kotz et al. (2000).

Definition 1.11. *Given a n -dimensional random vector $\mathbf{X} = (X_1, \dots, X_n)$, we define its joint distribution function, denoted by $\mathbf{F}$, as*

$$\mathbf{F}(\mathbf{x}) = P[\mathbf{X} \leq \mathbf{x}] = P[X_1 \leq x_1, \dots, X_n \leq x_n],$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. If the joint distribution function can be represented as

$$\mathbf{F}(\mathbf{x}) = \int_{\mathbf{S}} \mathbf{f}(\mathbf{u}) d\mathbf{u}, \quad \text{where } \mathbf{S} = \prod_{i=1}^n (-\infty, x_i]. \quad (1.3)$$

then $\mathbf{f}$ is known as the joint density function.

Analogously to the univariate case, $\mathbf{X}$ is said to be absolutely continuous when the joint distribution function can be represented as in (1.3). In this Ph.D. thesis we will focus on the study of absolutely continuous random variables and vectors. Hence, throughout the dissertation, unless otherwise stated, we will consider vectors having an absolutely continuous joint distribution.

Definition 1.12. *Given a n -dimensional random vector $\mathbf{X} = (X_1, \dots, X_n)$, its joint survival (or tail) function, denoted by $\bar{\mathbf{F}}$, is defined as*

$$\bar{\mathbf{F}}(\mathbf{x}) = P[X_1 > x_1, \dots, X_n > x_n], \quad \text{for all } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

One can notice that, when $n \geq 2$, $\bar{\mathbf{F}}(\mathbf{x}) \neq 1 - \mathbf{F}(\mathbf{x})$, but we can obtain the expression of $\bar{\mathbf{F}}$ from the joint distribution function and the marginal distribution functions. For example, for $n = 2$, we have:

$$\bar{\mathbf{F}}(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + \mathbf{F}(x_1, x_2).$$

Although, we should bear in mind that it is not straightforward to generalize the definition of quantile function from the univariate to the multivariate case. Defining a suitable multivariate extension of the univariate quantile function has been subject of study for a long time, and numerous extensions have arisen. Here, we consider the extension known as standard construction (see Rosenblatt (1952) and Rüschendorf (1981)). In the dissertation, the use of the standard construction targets simulation purposes.

Definition 1.13. *Given a n -dimensional continuous random vector $\mathbf{X} = (X_1, \dots, X_n)$. The standard construction of $\mathbf{X}$, denoted as $\mathbf{Q}_{\mathbf{X}}$ is defined, for $\mathbf{p} = (p_1, p_2, \dots, p_n) \in [0, 1]^n$, by*

$$\mathbf{Q}_{\mathbf{X}}(\mathbf{p}) = (Q_{\mathbf{X},1}(p_1), Q_{\mathbf{X},2}(p_1, p_2), \dots, Q_{\mathbf{X},n}(p_1, \dots, p_n)),$$

where $Q_{\mathbf{X},i} : [0, 1]^i \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n$ are real functions recursively given by

$$\begin{aligned} Q_{\mathbf{X},1}(p_1) &= F_{X_1}^{-1}(p_1) \\ Q_{\mathbf{X},2}(p_1, p_2) &= F_{X_2|X_1=Q_{\mathbf{X},1}(p_1)}^{-1}(p_2) \\ &\vdots \\ Q_{\mathbf{X},n}(p_1, p_2, \dots, p_n) &= F_{X_n|\cap_{j=1}^{n-1}\{X_j=Q_{\mathbf{X},j}(p_1, p_2, \dots, p_j)\}}^{-1}(p_n). \end{aligned}$$

Since it aims to extend the quantile function to the multivariate case, the standard construction plays the role of the quantile. Given a n -dimensional continuous random vector $\mathbf{X} = (X_1, \dots, X_n)$ and n independent uniformly

distributed random variables $U_1, \dots, U_n \sim U(0, 1)$:

$$(X_1, \dots, X_n) =_{\text{st}} \mathbf{Q}_{\mathbf{x}}(U_1, \dots, U_n).$$

The marginal distribution function of each component, that we will denote as F_i , can be derived from the joint distribution function in the following manner:

$$F_i(x) = P[X_i \leq x] = \lim_{\substack{x_j \rightarrow +\infty \\ j \neq i}} \mathbf{F}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$.

1.2 Stochastic Orders

As the title reveals, the application of Stochastic Orders is the methodological cornerstone of the present Ph.D. thesis. In consequence, the material introduced in this section will be extensively used throughout the manuscript.

The theory of Stochastic Orders (also known as Stochastic Dominance) is an area of Probability Theory that addresses the problem of the comparison of two random variables, vectors, or matrices. This theory has become a topic of great interest due to its range of applications in diverse areas. Such areas include economics, insurance, actuarial sciences, statistics, biology, reliability, survival analysis, engineering, operations research, social welfare, queuing theory or stochastic processes.

As we have already mentioned, the simplest way of comparing two probability distributions would be to select a particular measure and to compare the measure associated to each distribution. For example, we can compare the means of two random variables to order their averages or their standard deviations to address an ordering of variability. However, comparisons based on single measures present many drawbacks, mostly due to the fact that a single measure is often not very informative. In addition, sometimes the measures do not represent properly the distribution function or it can even occur that these single measures (for example, the mean) do not exist. Therefore, when we seek to quantify the idea that a random variable is “bigger” or “more variable” than another, Stochastic Orders appear as an efficient tool which allows us to compare two distributions under different criteria, without losing relevant information. However, we must point out that stochastic orders are partial orders. Hence, given two random variables, they are not necessarily ordered.

The concept of Stochastic Orderings and its formalization appeared with Mann and Whitney (1947) and began to gain weight in the context of Probability Theory with the work of Hardy et al. (1952). In the last five decades, this tool has been used at an accelerated rate and the number of stochastic orderings has grown enormously, alongside with its applications. Some of the earliest works of stochastic orders in actuarial science, where it became a very important tool for comparing risks, include Borch (1961); Rothschild and Stiglitz (1978) and Goovaerts et al. (1982). In the context of finance and insurance, some other important references are Goovaerts et al. (1990); Kaas et al. (1994); Denuit et al. (2005); Levy and Robinson (2006); Rachev et al. (2008a); Rüschendorf (2013) and Arnold and Sarabia (2018). The supervisors of this Ph.D. dissertation have co-authored many papers in this line. Among those works, we can list Shaked and Suárez-Llorens (2003); Sordo et al. (2007); Sordo (2009b, 2008); Sordo and Suárez-Llorens (2011); López-Díaz et al. (2012); Sordo et al. (2015b,a, 2016b,a); Belzunce et al. (2017); Mulero et al. (2017); Castaño-Martínez et al. (2020b, 2021) and Balakrishnan et al. (2022).

Note that, in the literature on stochastic orders, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ (or two random variables X and Y) ordered stochastically by certain order $\leq_*$, one uses equivalently the notations $\mathbf{X} \leq_* \mathbf{Y}$ ($X \leq_* Y$) and $\mathbf{F} \leq_* \mathbf{G}$ ($F \leq_* G$), where $\mathbf{F}$ and $\mathbf{G}$ are the respective joint distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ (F and G their respective distribution functions).

The purpose of this section is to collect the definitions and basic properties of the main stochastic orders that will be needed throughout the next chapters. For an exhaustive revision of the numerous stochastic orders, overviews on the topic and their applications, the works of Müller and Stoyan (2002); Shaked and Shanthikumar (2007); Szekli (2012) or Belzunce et al. (2015) serve as excellent references. For stochastic comparisons of some well-known parametric continuous distributions in terms of their parameters, we refer the reader to Tables 2.1-2.5 in Belzunce et al. (2015).

1.2.1 Univariate Stochastic Orders

Firstly, we introduce the usual stochastic order. If we consider two random quantities X, Y , the idea that the order formalizes is, roughly speaking, that X is less likely than Y to take on large values.

Definition 1.14. *Given two random variables X and Y with distribution functions F and G and finite expectations μ_X and μ_Y , respectively, we say*

that X is smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if

$$\bar{F}(t) \leq \bar{G}(t) \text{ for all } t \in \mathcal{S}_X.$$

Remark 1.15. Note that alternative definitions or characterizations of the usual stochastic orders are considered in Shaked and Shanthikumar (2007). We say that $X \leq_{st} Y$ if any of the following conditions hold:

1. $F(t) \geq G(t)$ for all $t \in \mathbb{R}$.
2. $F^{-1}(u) \leq G^{-1}(u)$ for all $u \in [0, 1]$.
3. $E[\phi(X)] \leq E[\phi(Y)]$, for all increasing functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for which the expectations exist.
4. $\phi(X) \leq_{st} \phi(Y)$ for all increasing functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

The characterization in Remark 1.15.3 provides a framework that can be extended to formalize some other stochastic orders. That is, considering a class or family of functions, C , it is common to define a stochastic ordering $\prec_P$ in the following manner:

$$X \prec_P Y \text{ if and only if } E[\Phi(X)] \leq E[\Phi(Y)] \text{ for all functions } \Phi \in C. \quad (1.4)$$

Well-known examples of stochastic orders that follows the structure of definition (1.4) are the next.

Definition 1.16. Given two random variables X and Y with distribution functions F and G and finite expectations μ_X and μ_Y , respectively, we say that X is smaller than Y in the increasing convex (concave) order (denoted by $X \leq_{icx} Y$ ($X \leq_{icv} Y$)) if $E[\phi(X)] \leq E[\phi(Y)]$, for all increasing convex (concave) functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$. Alternatively, we can say that

1. $X \leq_{icx} Y$ if $\int_t^\infty \bar{F}_X(x)dx \leq \int_t^\infty \bar{F}_Y(x)dx$ for all $t \in \mathbb{R}$.
2. $X \leq_{icv} Y$ if $\int_{-\infty}^t F_X(x)dx \geq \int_{-\infty}^t F_Y(x)dx$ for all $t \in \mathbb{R}$.

Intuitively, $X \leq_{icx} Y$ formalizes X being both “smaller” and “less variable” than Y in some stochastic sense, and $X \leq_{icv} Y$ formalizes X being both “smaller” and “more variable” than Y in some stochastic sense. Analogous definitions considering decreasing convex (concave) functions and further characterizations of the increasing convex and concave orders are provided in Shaked and Shanthikumar (2007).

We already saw that the usual stochastic order is defined by means of pointwise comparisons of a function (in this case, the distribution functions).

Similarly, but now considering the hazard rate function, we can define the following order.

Definition 1.17. *Let X and Y be two random variables with absolutely continuous distributions, survival functions $\bar{F}$ and $\bar{G}$ and hazard rate functions r_X and r_Y , respectively. We say that X is smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $r_X(t) \geq r_Y(t)$, for all $t \in \mathbb{R}$. Alternatively, we can say that $X \leq_{hr} Y$ if $\bar{G}(t)/\bar{F}(t)$ increases in $t \in \mathbb{R}$.*

If, instead of considering the ratio between survival functions, we study the monotonicity of the ratio between density functions, we have the definition of the likelihood ratio order.

Definition 1.18. *Let X and Y be two random variables with absolutely continuous distributions, density functions f and g , respectively. We say that X is smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(x)/g(x)$ is decreasing in the union of the supports.*

It is well-known (see Shaked and Shanthikumar (2007)) that

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{icx(icv)} Y \Rightarrow E[X] \leq E[Y] \quad (1.5)$$

but the converses are false. It is also well-known (see Lemma 2.1 in Sordo and Ramos (2007)) that

$$X \leq_{icx} Y \text{ if and only if } ES_p[X] \leq ES_p[Y], \text{ for all } p \in (0, 1), \quad (1.6)$$

where $ES_p(X)$ is the expected shortfall that we will define in Section 1.5.1 below and that stands for $ES_p[X] = \frac{1}{1-p} \int_p^1 F^{-1}(v)dv$ for $p \in (0, 1)$.

As we will see in the subsequent chapters, variability plays a fundamental role when we define risk in the financial or actuarial context. Hence, it is important to formalize the notion of random dominance in terms of dispersion. In this sense, the most relevant order is the dispersive order, which compares variables in terms of the distance between any two quantiles. The formal definition is as follows.

Definition 1.19. *Given two random variables X and Y with distribution functions F and G , respectively, we say that X is smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if $F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q)$ for all $0 \leq q < p \leq 1$.*

Note that in Muñoz-Pérez (1990), the following characterization for the dispersive order was provided:

$$X \leq_{\text{disp}} Y \text{ if and only if } (X - F^{-1}(p))^+ \leq_{st} (Y - G^{-1}(p))^+ \text{ for all } p \in (0, 1).$$

One can see that the dispersive order is very restrictive and a very strong condition to ask. Therefore, we can also consider a weaker order of variability, which is the next.

Definition 1.20. *Given two random variables X and Y with distribution functions F and G , respectively, we say that X is smaller than Y in the excess wealth order (denoted by $X \leq_{ew} Y$) if $\int_{F^{-1}(p)}^{\infty} \bar{F}(x)dx \leq \int_{G^{-1}(p)}^{\infty} \bar{G}(x)dx$, $\forall p \in (0, 1)$, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ are the tail (or survival) functions of X and Y , respectively.*

For characterizations, properties and applications of the stochastic orders defined above, the interested reader may refer to Shaked and Shanthikumar (2007). Related to the excess wealth, dispersive order and variance (V), a chain of implications connecting the variability orders can be defined (see Shaked and Shanthikumar (2007)):

$$X \leq_{\text{disp}} Y \implies X \leq_{ew} Y \implies V[X] \leq V[Y]$$

but the converses are false.

1.2.2 Multivariate Stochastic Orders

There are many problems and applications where, instead of comparing random variables, we are interested in comparing two random vectors. Analogously as in the univariate case, multivariate stochastic orders compare vectors from different perspectives (addressing comparisons in location, variability, etc.) and under different frameworks (by comparing pointwise a given function of two vectors, by ordering expected values of transformations of the variables, etc.) Among the different multivariate stochastic orders that can be found in literature, in this section, we will only define the one that we will use in this dissertation. For further knowledge on this topic, one can consult the monograph by Shaked and Shanthikumar (2007).

The following multivariate stochastic order, introduced by Hu et al. (2003), generalizes the univariate hazard rate order.

Definition 1.21. *Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two random vectors with respective joint survival functions $\bar{F}$ and $\bar{G}$, defined, respectively,*

by $\bar{F}(\mathbf{x}) = P[\mathbf{X} > \mathbf{x}]$ and $\bar{G}(\mathbf{x}) = P[\mathbf{Y} > \mathbf{x}]$, for $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the weak multivariate hazard rate order (denoted by $\mathbf{X} \leq_{whr} \mathbf{Y}$) if

$$\frac{\bar{G}(\mathbf{x})}{\bar{F}(\mathbf{x})} \quad \text{is increasing in } \mathbf{x} \in \{\mathbf{x} : \bar{F}(\mathbf{x}) > 0\}.$$

1.3 Dependence Modelling

As we already mentioned above, despite the fact that insurance and finance have traditionally been built on the assumption of independence (or using correlation as the unique form of stochastic dependence) the complexity of products has led recently to highlight the importance of developing more efficient tools for modeling dependent risks. In this section, we will introduce the concept of copula, as well as some definitions and dependence notions needed throughout the dissertation.

1.3.1 Copula Theory

The notion of copulas, as well as the employment of the word copula in a mathematical or statistical sense, was introduced by Sklar (1959). The term is derived from the Latin *copulare*, which means “a link, tie, bond”. As it was described by Sklar in the theorem that nowadays bears his name (Theorem 1.23) it describes the “functions that *join together* one-dimensional distribution functions to form multivariate distribution functions”. That is, it describes the inner dependence structure of a random vector, independently of the marginal behavior of the components. In the initial years of its development, the concept of copula was also studied under the names of “uniform representation” (see Kimeldorf and Sampson (1975)) or “dependence function” (see Deheuvels (1978)). For further discussion on the evolution of copulas, on its early developments or on the great variety of copulas, the interested reader may consult Dall’Aglio (1991); Schweizer (1991) or Nelsen (2007).

Copulas are widely employed as a multivariate modeling tool in areas such as biomedical studies, engineering, actuarial science or finance. In particular, in the fields of Financial Risk and Actuarial Theory, the modeling of the dependence structure among different risk factors plays a fundamental role in the development of decision-making tools. Therefore, in the actuarial and financial context, complex models were born based on the concept of copula. Frees and Valdez (1998); Carriere (2000); Embrechts (2009); Embrechts et al. (2003); Cousin and Di Bernardino (2013) or Denuit et al. (2005) are part of

the numerous examples where dependent risk modeling, on the basis of copula modeling, is introduced in insurance and finance.

The joint application of Copula and Stochastic Ordering Theory has become of great relevance, and some important works on their connection are Scarsini (1998); Müller and Scarsini (2001) or Colangelo et al. (2006). In recent years, the supervisors of this Ph.D. dissertation have co-authored different works in this line. See, among others, Belzunce et al. (2012); Balakrishnan et al. (2012); Shaked et al. (2012); Sordo et al. (2015a); Belzunce et al. (2016) or Sordo (2016).

Formally, a *copula* $C : [0, 1]^n \rightarrow [0, 1]$ is a cumulative distribution function with uniform margins on $[0, 1]$, characterized by the following properties.

Proposition 1.22. *Given a function $C : [0, 1]^n \rightarrow [0, 1]$, we say that C is a copula if and only if it verifies the following conditions:*

- (i) $C(p_1, \dots, p_n)$ increases in each $p_i \in [0, 1]$.
- (ii) $C(1, \dots, 1, p_i, 1, \dots, 1) = p_i$ for all $i = 1, 2, \dots, n$.
- (iii) $C(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_n) = 0$ for all $i = 1, 2, \dots, n$.
- (iv) C is n -increasing, that is, for $a_i \leq b_i$ with $i = 1, 2, \dots, n$,

$$\sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 (-1)^{i_1+\dots+i_n} C(p_{1i_1}, \dots, p_{ni_n}) \geq 0$$

where $p_{j1} = a_j$ and $p_{j2} = b_j$, for all $j = 1, 2, \dots, n$.

Note that, when $n = 2$, C is 2-increasing if for $a, b, c, d \in [0, 1]$ with $a \leq b$ and $c \leq d$, it holds that

$$C(b, d) - C(a, d) - C(b, c) + C(a, c) \geq 0.$$

Some of the main advantages of the employment of copulas as multivariate modeling tools are stated in the well-known Sklar Theorem. Sklar (1959) proved that there always exists a copula associated to any multivariate distribution function and that also, any copula, evaluated in certain marginal distribution functions, leads to a multivariate distribution function with such given marginals.

Theorem 1.23 (Sklar (1959)). *Let C be a copula and F_i be n distribution functions. Then, the function*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \text{ for all } (x_1, \dots, x_n) \in \mathbb{R}^n,$$

is a multivariate distribution function with marginals $F_1, \dots, F_n$. Furthermore, if F is the joint distribution function of a random vector with continuous marginals $F_1, \dots, F_n$, then C is unique and is given by

$$C(p_1, \dots, p_n) = F(F_1^{-1}(p_1), \dots, F_n^{-1}(p_n)), \quad \text{for all } (p_1, \dots, p_n) \in [0, 1]^n.$$

Throughout the dissertation, we denote by $\bar{C}$ the joint survival function for n uniform random variables whose joint distribution function is the copula C , that is,

$$\bar{C}(p_1, \dots, p_n) = P[X_1 > F_1^{-1}(p_1), \dots, X_n > F_n^{-1}(p_n)].$$

We also use the notation $\bar{C}(\mathbf{p})$, where $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$.

Given $n \in \mathbb{N}$, the considered dimension, it can be defined:

$$M^n(\mathbf{u}) = \min(u_1, u_2, \dots, u_n) \quad \text{and}$$

$$W^n(\mathbf{u}) = \max(u_1 + u_2 + \dots + u_n - n + 1, 0),$$

for $u_1, \dots, u_n \in [0, 1]$. Note that the superscript on M^n , and W^n denotes dimension rather than exponentiation. It can be proved (see Nelsen (2007)) that M^n is an n -copulas for all $n \geq 2$, whereas the function W^n is a copula for $n = 2$ but it is not for any $n > 2$. The functions $M^n(\mathbf{u})$ and $W^n(\mathbf{u})$ serve as what is known as the Fréchet-Hoeffding bounds, which are stated in the following theorem.

Theorem 1.24. *If C is any n -copula, then for every $\mathbf{u}$ in $[0, 1]^n$,*

$$W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u}).$$

For $n = 2$, we can see that for all copula C and every (u, v) on $[0, 1]^2$,

$$W(u, v) \leq C(u, v) \leq M(u, v), \quad (1.7)$$

where both Fréchet-Hoeffding bounds are copulas $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$, called, respectively *comonotonic* and *countermonotonic* copulas. When $n > 2$, even if the lower bound W^n is not a copula, it remains to be an useful bound, because, as seen in Nelsen (2007), for any $n > 2$ and any $\mathbf{u} \in [0, 1]^n$, there exists an n -copula C such that $C(\mathbf{u}) = W^n(\mathbf{u})$.

Independence is a crucial concept in probability theory. It has the purpose of formalizing the intuitive notion of “not influencing each other” for random events. In terms of copulas, the independent behavior of the components can be represented by:

$$\Pi^n(\mathbf{u}) = u_1 u_2 \dots u_n.$$

Again, the superscript of $\Pi^n(\mathbf{u})$ denotes dimension and Π^n is an n -copula for all $n \geq 2$.

In addition to the mentioned copulas, many other examples can be found in Nelsen (2007). Some of the most known copulas are: Gumbel, Frank, Clayton and Ali-Mikhail-Haq families. These kind of copulas are special cases of Archimedean copulas. We can find the expressions of the most important parametric families of Archimedean copulas in Table 4.1 in Nelsen (2007).

1.3.2 Copula Orders

As we already mentioned, when considering two random vectors $\mathbf{X}$ and $\mathbf{Y}$ with $\mathbf{F}$ and $\mathbf{G}$ their respective joint distribution functions, one can use indistinctly $\mathbf{X} \leq_* \mathbf{Y}$ and $\mathbf{F} \leq_* \mathbf{G}$ where $\leq_*$ stands for an stochastic order. As the copulas are distribution functions of vectors with uniform marginals, we will use the latter notation, for consistency with the literature, when we compare copulas. In this section, we will introduce some orders that depend on the copulas and that will be needed throughout the dissertation.

Firstly, we recall the definition of concordance order (Definition 2.8.1 in Nelsen (2007)).

Definition 1.25. *Given two n -dimensional copulas C and C' , we say that C is smaller than C' in the concordance order (and write $C \leq_c C'$ or, alternatively, $C \prec C'$) if $C(\mathbf{p}) \leq C'(\mathbf{p})$ for all $\mathbf{p} \in (0, 1)^n$.*

Note that, when $n = 2$, we say that $C \prec C'$ if $C(u, v) \leq C'(u, v)$ for all $u, v \in (0, 1)$.

The name of this order is due to the fact that some measures of concordance, such as Kendall's tau and Spearman's rho, are increasing with respect to $\prec$. The notion of weak multivariate hazard rate order can be also rewritten in terms of copulas. According to Definition 1.21, we will write $C \leq_{whr} C'$ if

$$\frac{\bar{C}'(\mathbf{p})}{\bar{C}(\mathbf{p})} \text{ is increasing in } \mathbf{p} \in \{\mathbf{p} \in [0, 1]^n : \bar{C}(\mathbf{p}) > 0\}.$$

It is easy to see, see section 3 in Hu et al. (2003), that

$$C \leq_{whr} C' \text{ implies } C \leq_c C'.$$

Note that these orders of copulas are also partial orders because not every pair of copulas are comparable.

1.3.3 Dependence notions

By this point, we are already aware of the importance of analyzing the dependence structure of a vector. Therefore, it is important to formalize the types of dependence structure, since identifying some dependence properties is fundamental in order to know whether a particular model might be suitable for a given application. Here, we will focus on formalizing some notions of positive and negative dependence that will be used throughout the manuscript. Although, these only represent a small portion of the large set of dependence notions that can be considered. The interested reader may consult Joe (1997) for an overview of the topic of dependence, as well as for a summary of the main dependence notions and properties. In addition to the classical monograph by Joe, an exhaustive and illustrative review of the most important notions of bivariate dependence can be found in the recent paper by Navarro et al. (2020). Other relevant references of dependence notions are Barlow and Proschan (1975); Block et al. (1985); Kimeldorf and Sampson (1989); Longobardi and Pellerey (2019) and Li and Li (2021).

In this section, the notions will be introduced for the multivariate framework. Bivariate vectors represent a particular case of the multivariate framework, which is usually more illustrative than higher dimensions cases. Since most of the notions can be more easily understood considering the bivariate framework, and since some of the results included in this dissertation deal only with bivariate vectors, some of the dependence notions will be also defined for the particular case $n = 2$.

Positive dependence notions aim to formalize the intuitive concept of X and Y being likely to be large or to be small together and negative dependence notions mean that they behave the other way round. As already mentioned, this concept can be formalized in many different forms. The first notion of positive dependence captures the fact that large values of the marginal variables coincide more often than would be expected under independence. In the subsequent chapters this notion will be only used with bivariate vectors, thus we only define it in the bivariate framework (see page 187 in Nelsen (2007)).

Definition 1.26. *Let (X, Y) be a random vector. We say that the vector is Positively Quadrant Dependent (PQD) if for all $t, x \in \mathbb{R}$,*

$$P[X > x] \leq P[X > x \mid Y > t]. \quad (1.8)$$

Analogously, we say that the vector is *Negatively Quadrant Dependent (NQD)* if for all $t, x \in \mathbb{R}$,

$$P[X > x] \geq P[X > x \mid Y > t]. \quad (1.9)$$

Note that, if the vector has a copula C , (1.8) holds if, for all $u, v \in [0, 1]$, $uv \leq C(u, v)$. Similarly, (1.9) holds if $uv \geq C(u, v)$.

The second notion of positive dependence works as follows: considering a random vector (X, Y) , positive dependence holds when the probability of X trespassing a value is greater as Y trespasses greater values. This notion is known as *right-tail increasing (decreasing)* and, in the multivariate framework, the formal definition is the following (see Joe (1997)).

Definition 1.27. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with joint survival function $\bar{\mathbf{F}}$. We say that $X_i, i \in A^c$, is *right-tail increasing (decreasing)* or *RTI (RTD)* in $X_j, j \in A$, if

$$P[X_i > x_i, i \in A^c \mid X_j > x_j, j \in A]$$

increases (decreases) in $x_k, k \in A$, where A is a non-empty subset of $\{1, \dots, n\}$ and A^c is the complementary set of A .

If $n = 2$, the definition reduces to the following: we say that X is *right-tail increasing (decreasing)* in Y ($RTI(X|Y)$) if for all $x \in \mathcal{S}_X$ and $y_1, y_2 \in \mathcal{S}_Y$ with $y_1 < y_2$,

$$P[X > x \mid Y > y_1] \geq P[X > x \mid Y > y_2].$$

Alternatively, we can say that $RTI(X|Y)$ if $\{X|Y > t\}$ is *st-increasing* in $t \in \mathcal{S}_Y$.

In view of the *RTI* notion, an alternative concept that comes to mind is to formalize positive dependence in a similar manner but, instead of considering that the conditional value is greater when it *trespasses* greater values, consider that it is greater when it *takes* greater values. This intuition refers to the *positive regression dependence (PRD)*, concept introduced by Lehmann (1966), also called *stochastically increasing (SI)* which can be defined as follows.

Definition 1.28. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with joint survival function $\bar{\mathbf{F}}$.

- (i) $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ is said to be *stochastically increasing (decreasing)* in X_i (denoted $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \uparrow_{SI} X_i$ ($\downarrow_{SD}$)) if the conditional distribution $[(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \mid X_i = x_i]$ is *stochastically increasing (decreasing)* as x_i increases.

- (ii) $\mathbf{X}$ is said to be positively (negatively) dependent through the stochastic order (denotes as PDS (NDS)) if $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \uparrow_{SI} X_i (\downarrow_{SD})$ for $i = 1, \dots, n$.

We consider separately the definition for bivariate vectors, as it is simpler and more illustrative. An analogous condition is stated in terms of the copula.

Definition 1.29. Let $\mathbf{X} = (X, Y)$ be a random vector with copula C .

- (i) We say that Y is stochastically increasing (decreasing) in X ($Y \uparrow_{SI} X$ ($Y \downarrow_{SD} X$)), if $P[Y > y \mid X = x]$ is an increasing (decreasing) function of x for all y . Analogously, $Y \uparrow_{SI} X$ ($Y \downarrow_{SD} X$) if $\partial_1 C(u, v)$ is a decreasing (increasing) function of u for all v .
- (ii) We say that $\mathbf{X} = (X, Y)$ is positively dependent through the stochastic ordering (PDS) if $Y \uparrow_{SI} X$ and $X \uparrow_{SI} Y$. We say that is negatively dependent through the stochastic ordering (NDS) if $Y \downarrow_{SD} X$ and $X \downarrow_{SD} Y$.

Again, intuitively, if $\mathbf{X}$ is PDS (NDS), one component of the vector will tend to be large when the other component is large (small). It is easy to see that a random vector $\mathbf{X}$ with continuous marginals is PDS (respectively, NDS) if and only if its copula $C(u, v)$ is componentwise concave (respectively, convex). It is also well-known (see Cai and Wei (2012)) that a continuous random vector $\mathbf{X}$ with copula C has the property PDS (resp. NDS) if and only if its copula C is PDS (resp. NDS).

It must be pointed out that, similarly as with the RTI property, we can say that $Y \uparrow_{SI} X$ if $[Y \mid X = t]$ is *st*-increasing in $t \in \mathcal{S}_X$. Besides RTI and SI, the other notion that will be used to formalize the idea of positive dependence is the notion of total positivity.

Definition 1.30. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with joint survival function $\bar{\mathbf{F}}$. If $\mathbf{X}$ has a density function $\mathbf{f}$, $\mathbf{X}$ is said to be multivariate totally positive of order two (or MTP₂) if

$$\mathbf{f}(\mathbf{x} \vee \mathbf{y})\mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \geq \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

where $\vee$ represents the maximum and $\wedge$ the minimum.

In the bivariate framework, the definition of totally positive functions can be simplified.

Definition 1.31. A bivariate function $p : \mathbb{R}^2 \mapsto \mathbb{R}^+$ is said to be totally positive of order two (TP_2) if, for all $x_1 \leq x_2, y_1 \leq y_2$:

$$p(x_1, y_1)p(x_2, y_2) \geq p(x_1, y_2)p(x_2, y_1).$$

Given a random vector (X, Y) , we say that it is Positive Likelihood Ratio Dependent (PRLD) (sometimes also denoted as $PLR(X, Y)$) if its joint density function is (TP_2). This conditions holds equivalently, if $[Y|X = t]$ is LR-increasing in $t \in \mathcal{S}_X$ (see Foschi and Spizzichino (2013))

Note that, for such p , if at each point (x, y) , the second order partial derivative $\frac{\partial^2}{\partial x \partial y} \log(p(x, y))$ exists, then p is TP_2 if and only if $\frac{\partial^2}{\partial x \partial y} \log(p(x, y)) \geq 0$ (Karlin (1967)).

It is well-known (see, for instance, Navarro et al. (2020)) that $PRLD(X, Y)$ implies $Y \uparrow_{SI} X$, and that, if $Y \uparrow_{SI} X$, then $RTI(Y|X)$, but the converses are false.

1.4 Distortion functions

We introduce in this section a concept that will be extensively used throughout the dissertation. It is the concept of distortion function and one can intuit that it is so named because it distorts the probabilities. Formally, it is defined as follows.

Definition 1.32. A distortion function $h : [0, 1] \rightarrow [0, 1]$ is a nondecreasing function with $h(0) = 0$ and $h(1) = 1$. If $h(u)$ is a distortion function, then

$$\bar{h}(u) = 1 - h(1 - u), \quad u \in [0, 1],$$

is again a distortion function called the dual distortion function of h .

Different examples of distortion functions will be found throughout the subsequent chapters. Given the survival function $\bar{F}$ of a random variable X , if the distortion function h is continuous, the transformation $h(\bar{F}) = \bar{F}_h$ defines a new survival function associated with some random variable X_h . This random variable is considered as the distorted random variable induced from X by h . Note that the distribution function of X_h is given in terms of the distribution function of X and the dual distortion function of h , as $F_h = \bar{h}(F)$. Let us point out that, whenever it is convenient, we will define, as the distortion, the function g such that $F_g = g(F)$ and, as the dual distortion, the function $\bar{g}$ such that $\bar{F}_g = \bar{g}(\bar{F})$. If we are considering the latter case, it will be specified (see, for instance, the introduction of Chapter 5).

Note that some of the main stochastic orders are preserved under distortions. The following result shows this property (see Theorems 2.6 and 2.9 in Navarro et al. (2013) for the proof).

Proposition 1.33. *Let X and Y be two random variables with distribution functions F and G , respectively. Then,*

- (i) *If $X \leq_{st} Y$ then $X_h \leq_{st} Y_h$ for all distortion h .*
- (ii) *If $X \leq_{disp} Y$ then $X_h \leq_{disp} Y_h$ for all distortion h .*
- (iii) *If $X \leq_{icx} Y$ then $X_h \leq_{icx} Y_h$ for all concave distortion h .*
- (iv) *If $X \leq_{ew} Y$ then $X_h \leq_{ew} Y_h$ for all concave distortion h .*

Distortion functions are very popular in the financial and actuarial academic literature. In particular, the expected value of the distorted random variable X_h is widely used in this context. The distorted expectation is given by

$$E[X_h] = \int_0^\infty h(\bar{F}(x))dx - \int_{-\infty}^0 \bar{h}(F(x))dx. \quad (1.10)$$

Denneberg (1990) was the first one that introduced the distorted expectation given by (1.10) in order to build premium principles. Note that, when the distortion is concave, (1.10) satisfies many convenient properties in insurance (see Wang (1996) and Denuit et al. (2005)). For example, if the distortion function h is concave, $E[X] \leq E[X_h]$ and, therefore, X_h gives more weight to higher risk events than X .

In the actuarial framework, it is common to find references where the results are under the “distorted expectations hypothesis context”. A decision-maker is said to base his preferences on the “distorted expectations hypothesis” if they act in order to maximize the distorted expectation of their wealth. That is, if given a distortion function h , the decision-maker prefers X to Y when $E[X_h] \leq E[Y_h]$.

1.5 Risk Measures

In the economic and actuarial contexts, the concept of risk has been widely discussed and diverse intuitive explanations can be found, each under different perspectives and with various connotations. Roughly speaking, when we refer to risk in financial and actuarial terms, we are addressing the probability that an actual outcome or an investment’s actual gains will differ from the expected ones. That is, risk includes the possibility of losing some or all of an original

investment. We find an excellent exposition and discussion on the concept of risk in Chapter 1 in Levy and Robinson (2006). Note that, when we refer to risks, we are considering random variables. In our context, we will normally consider financial positions, which are described by the uncertain monetary outcome, that is, as a real-valued random variable X on some set of possible scenarios.

Risk measurement is fundamental to the insurance and financial industry. We can understand these measures as functionals used to evaluate (or quantify) risks. The formal definition of risk measure given by Denuit et al. (2005) is the following.

Definition 1.34. *A risk measure is a functional ρ mapping a risk X to a non-negative real number $\rho[X]$, possibly infinite, representing the extra cash which has to be added to X to make it acceptable.*

In finance, risk measures can be used to assess the risk of investments and to optimize the capital allocation. Since the earliest researchers (see, for example, Cramer (1930)), the work on such measures is mainly focused on establishing capital requirements, that is, to determine, for each financial position, the amount of risk capital to be held in reserve. In the actuarial science, risk measures are mainly employed to price individual contracts or establish premium principles (see, among others, Goovaerts et al. (1984); Deprez and Gerber (1985); Young (2004); Castaño-Martínez et al. (2019) or Castaño-Martínez et al. (2020a)). Recall that, given a risk X , a *premium principle* $T(X)$ is a functional that sets the premium to be charged for accepting the risk X . For a further discussion on premium principles, the interested reader may refer to the monograph by Young (2004).

From the most elementary to the most elaborate, numerous risk measures have been proposed along the years in insurance and finance, each based on a different axiomatic approach and addressing different characteristics of the risks. In this section, we provide a list of the main risk measures that will be considered throughout the manuscript. Firstly, we will introduce some of the main scalar risk measures, that is, those that quantify the risk of the random variable as if the variable were in isolation, without taking into account any possible interaction among random losses. Secondly, we will list some risk measures that take into account such interactions. These measures aim to consider the contagion risk (also referred to as systemic risk) caused by interconnections among financial institutions, markets or services. Further comments on the importance of modeling and considering such interactions

will be given in Chapters 2 and 3 in order to motivate the work. Throughout the present chapter, we will only introduce the risk measures that will be needed in later chapters of the present Ph.D. dissertation. For an exhaustive review of risk measures, the interested reader is referred to Denuit et al. (2005) or Kaas et al. (2008).

1.5.1 Scalar Risk Measures

The most popular measure of market risk, which represents the standard tool for risk management in financial and insurance institutions, is the Value-at-Risk (commonly abbreviated VaR).

Definition 1.35. *The Value-at-Risk (VaR) of X at level $v \in (0, 1)$ is defined by*

$$\text{VaR}_v[X] = F_X^{-1}(v) = \inf \{x : F_X(x) \geq v\}.$$

In Artzner et al. (1999), a set of requirements is given in order to determine a risk measure T as “coherent”. It is required to verify the following properties:

1. Translation invariance: $T(X + c) = T(X) + c$ for all $c \in \mathbb{R}$.
2. Positive homogeneity or scale-invariance: $T(cX) = cT(X)$ for all constant $c > 0$.
3. Subadditivity: $T(X + Y) \leq T(X) + T(Y)$ for all risks X and Y .
4. Monotonicity: if $X \leq Y$ almost surely, then $T(X) \leq T(Y)$.

The VaR is translatable, positive homogeneous, monotone and has some other desirable properties. Among these properties we can list that it has non-excessive loading (for all $v \in (0, 1)$, the VaR is smaller than the maximum risk loss), $\text{VaR}_v[h(X)] = h(\text{VaR}_v[X])$ for any real valued increasing function h (see Section 1.1.1.1) and it is comonotonic additive.

Although, despite its advantages, the VaR has some undesirable features. For instance, it is not a coherent risk measure in the sense of Artzner et al. (1999), due to the fact that it is not subadditive. Among many other disadvantages of the VaR, we could also mention that it does not provide information on the size of the tails of the risk distributions and that it does not provide robust conclusions (different probability levels could lead to different conclusions). In spite of the drawbacks, due to its conceptual simplicity and easiness in operational implementation, it has remained for a long time as the benchmark risk measure. For more details, properties of the VaR and for an overview on the

topic, one may consult the textbooks of Artzner et al. (1999); Jorion (2000); Kupiec (2002) or Denuit et al. (2005).

Another important scalar measure in financial risk management is the Expected Shortfall (ES) (see Wirth (1999)). This measure puts the focus on right-tail risk analysis, which is of great importance when comparing risks in actuarial and financial sciences (see, among others, Wang (1998)). Many financial risk tools, as the VaR, overlook extremely large losses that occur with very small probability, but these losses can be an important threat for the solvency of the institutions. In order to overcome such threat, the ES seeks to convey information regarding the exceedances of the risk beyond its VaR. We can see the ES as an “arithmetic average” of the VaRs of X , from the risk-level v onwards. It is formally defined as follows.

Definition 1.36. *The expected shortfall (ES) of X at level $v \in (0, 1)$ is defined by*

$$\text{ES}_v[X] = E[X|X > \text{VaR}_v[X]].$$

If we assume that X has a continuous distribution function F , the following identity holds:

$$\text{ES}_v[X] = \frac{1}{1-v} \int_v^1 \text{VaR}_t[X] dt, \quad p \in (0, 1). \quad (1.11)$$

The expected shortfall has replaced the VaR as the standard risk measure for market risk in the recent Fundamental Review of the Trading Book (FRTB), as referred by Basel Committee on Banking Supervision (BCBS (2019)). It presents many interesting properties, the main one is that it is a coherent risk measure in the sense of Artzner et al. (1999). In actuarial science, the integral appearing in (1.11) is referred as Tail value-at-risk ($\text{TVaR}_v[X]$). Since X is a continuous random variable, $\text{TVaR}_v[X] = \text{ES}_v[X]$.

In literature, there are many applications of Stochastic Ordering Theory to the management of risk measures. Among them, for example, we can list Bäuerle and Müller (2006); Rachev et al. (2008b); Laniado et al. (2012); Di Bernardino et al. (2015) or Castaño-Martínez et al. (2019).

1.5.2 Systemic Risk and Co-Risk Measures

In the last decades, especially since the 2007-2009 financial crisis, there has been an increasing interest in modeling the contagion risk (also referred to as systemic risk) caused by interconnections among financial institutions, markets or services. In portfolio risk analysis, the risk of contagion appears

when the collapse of one or more components of the portfolio can eventually cause the collapse of other components, putting the entire portfolio at risk. From the point of view of risk measurement, the risk of contagion puts the focus on conditional rather than unconditional risk distributions. This makes necessary to adapt the scalar risk measures typically used in the financial industry, such as the Value-at-risk and the expected shortfall, to incorporate the effect of interaction and the dynamic feedback relations that take place within our highly interconnected financial-economic network.

Therefore, under the stated perspective of putting the focus on conditional risks, we will firstly introduce the multi-dimensional extensions to the already defined univariate risk measures VaR and ES. Note that we can define two versions of the dependence-adjusted version of the VaR, called CoVaR (see Girardi and Ergün (2013) and Adrian and Brunnermeier (2016)). The superscripts $>$ and $=$ denote the nature of the conditional event.

Definition 1.37. For $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$ we set

$$\text{CoVaR}^>_{\mathbf{p}} [X_1 | X_2, \dots, X_n] = \text{VaR}_{p_1} \left[X_1 | \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right], \quad (1.12)$$

and

$$\text{CoVaR}^=_{\mathbf{p}} [X_1 | X_2, \dots, X_n] = \text{VaR}_{p_1} \left[X_1 | \bigcap_{j=2}^n \{X_j = \text{VaR}_{p_j}(X_j)\} \right]. \quad (1.13)$$

Sometimes, to avoid ambiguity, we shall denote the $\text{CoVaR}^>$ and $\text{CoVaR}^=$ considered in Definition 1.37, respectively, by $\text{CoVaR}^>_{p_1, \dots, p_n} [X_1 | X_2, \dots, X_n]$ and $\text{CoVaR}^=_{p_1, \dots, p_n} [X_1 | X_2, \dots, X_n]$. In words, the $\text{CoVaR}^>$ in (1.12) is the VaR at level p_1 of the conditional distribution of X_1 given the joint event $[X_2 > \text{VaR}_{p_2}[X_2], \dots, X_n > \text{VaR}_{p_n}[X_n]]$. And the $\text{CoVaR}^=$ in (1.13) is, analogously, the VaR at level p_1 of the conditional distribution of X_1 given the joint event $[X_2 = \text{VaR}_{p_2}[X_2], \dots, X_n = \text{VaR}_{p_n}[X_n]]$.

When $n = 2$, considering a random vector (X, Y) and the risk values $u, v \in (0, 1)$, Definition 1.37 reduces to:

$$\text{CoVaR}^>_{u,v} [X | Y] = \text{VaR}_u [X | Y > \text{VaR}_v(Y)] = F_{X|Y > F_Y^{-1}(v)}^{-1}(u), \quad (1.14)$$

and

$$\text{CoVaR}^=_{u,v} [X | Y] = \text{VaR}_u [X | Y = \text{VaR}_v(Y)] = F_{X|Y = F_Y^{-1}(v)}^{-1}(u) \quad (1.15)$$

The expected shortfall is also generalized to incorporate dependence (see Mainik and Schaanning (2014)).

Definition 1.38. For $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$ we set

$$\begin{aligned} \text{CoES}^*_{\mathbf{p}} [X_1 | X_2, \dots, X_n] &= \\ &= \frac{1}{1 - p_1} \int_{p_1}^1 \text{CoVaR}^*_{t, p_2, \dots, p_n} [X_1 | X_2, \dots, X_n] dt \end{aligned} \quad (1.16)$$

where $*$ stands for $>$ or $=$.

In the present dissertation we will only consider $\text{CoES}^>$. Therefore, in order to simplify the notation, we will avoid the superscript and, from now on, we will denote by CoES to the $\text{CoES}^>$.

Other dependence-adjusted extensions of VaR and ES can be found in Adrian and Brunnermeier (2016). Some other measures have been recently introduced to manage and quantify the impact of risk contagion. We consider in this work the marginal mean excess (MME) measure introduced by Das and Fasen-Hartmann (2018a).

Definition 1.39. Assume $E |X_1| < \infty$. For $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$ we set

$$\text{MME}_{\mathbf{p}^*} [X_1 | X_2, \dots, X_n] = E \left[(X_1 - A_{\mathbf{X}, \mathbf{p}^*})_+ \left| \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right. \right]$$

with $A_{\mathbf{X}, \mathbf{p}^*} = a_1 \text{VaR}_{p_2}(X_2) + \dots + a_{n-1} \text{VaR}_{p_n}(X_n)$, where $a_1, \dots, a_{n-1}$ are given positive constants, such that $\sum_{i=1}^{n-1} a_i = 1$.

For $n = 2$ and $p \in (0, 1)$, the MME given in Definition 1.39 reduces to

$$\text{MME}_p [X_1 | X_2] = E [(X_1 - \text{VaR}_p(X_2))_+ | X_2 > \text{VaR}_p(X_2)], \quad p \in (0, 1),$$

as defined originally by Das and Fasen-Hartmann (2018a). In this case, we interpret the MME as the expected excess of the risk X_1 over the VaR of X_2 at level p , given that X_2 is greater than its VaR. The extension given in Definition 1.39 to the case $n > 2$, which is taken from Hashorva (2019), represents the expected excess of the risk X_1 over a linear combination of the values at risks $\text{VaR}_{p_2}(X_2), \dots, \text{VaR}_{p_n}(X_n)$ under a stress scenario for the other risks, given by

$$\{X_2 > \text{VaR}_{p_2}(X_2), \dots, X_n > \text{VaR}_{p_n}(X_n)\}.$$

1.6 Variability Measures

In any introductory book on Probability, there are many different well-known measures that give information about the variability of a random variable X . Among them, we can list the *variance* of a random variable

X , defined as $V[X] = E[(X - E[X])^2]$; the *standard deviation*, $SD[X] = \sqrt{E[(X - E[X])^2]}$ or the *median absolute deviation*, defined as $E(|X - m_X|)$. But, formally, what do we mean by variability measures?

A *measure of variability* ν is a map from the set of random variables to $\mathbb{R}^+$, such that $\nu(X)$ quantifies the variability of a random variable X . Next, we list a set of properties that a measure of variability should reasonably satisfy (see Furman et al. (2017) and references therein):

- (P0) *Law invariance*: if X and Y have the same distribution ($X =_{st} Y$), then $\nu(X) = \nu(Y)$.
- (P1) *Translation invariance*: $\nu(X + k) = \nu(X)$ for all X and all constant $k \in \mathbb{R}$.
- (P2) *Positive homogeneity*: $\nu(0) = 0$ and $\nu(\lambda X) = \lambda \nu(X)$ for all X and all $\lambda > 0$.
- (P3) *Non-negativity*: $\nu(X) \geq 0$ for all X , with $\nu(X) = 0$ if X is degenerated at $c \in \mathbb{R}$.

Bickel and Lehmann Bickel and Lehmann (1979) also require $\nu(X)$ to be consistent with the dispersive order. A functional ν satisfying properties (P0) to (P3) is said to be a measure of variability or spread in the sense of Bickel and Lehmann if it satisfies in addition (see Bickel and Lehmann (1979)):

- (P4) *Consistency with dispersive order*: if $X \leq_{disp} Y$, then $\nu(X) \leq \nu(Y)$.

A measure of variability in the sense of Bickel and Lehmann considers the variability or spread of a random variable throughout its distribution. Sometimes, however, there is an interest in measuring only the variability of X along the right tail of its distribution (in risk theory, for example, some popular measures focus on the variability of a risk X beyond the value at risk). When this is the case, the requirement on ν to be consistent with the dispersive order is too strong. A natural weaker requirement was introduced in Sordo (2009a) and it weakens (P4) by asking the measure to be consistent with the excess wealth order, which is defined as follows. This allows us to consider the following property.

- (P5) *Consistency with excess wealth order*: if $X \leq_{ew} Y$, then $\nu(X) \leq \nu(Y)$.

Measures of variability have received great attention in the actuarial and financial literature (see Rockafellar et al. (2006); Sordo and Suárez-Llorens (2011); Furman et al. (2017); Psarrakos and Sordo (2019); Hu and Chen (2020), among others). In actuarial science, for example, a variability measure

sometimes is combined with a location measure to build a premium principle (see Young (2004)). For particular applications in this context, one may wish ν to satisfy the following properties:

(P6) *Comonotonic additivity*: if X and Y are comonotonic random variables, then $\nu(X + Y) = \nu(X) + \nu(Y)$.

(P7) *Subadditivity*: $\nu(X + Y) \leq \nu(X) + \nu(Y)$ for all X and Y .

Furman et al. (Furman et al. (2017)) say that ν is a coherent measure of variability if it satisfies (P0)–(P3) and (P7).

1.7 Ph.D. Thesis Structure

As it has been already stated, this dissertation has the primary objective of studying methodologies from a mathematical point of view, more specifically, based on Stochastic Orders, in the fields of Financial Risk and Actuarial Theory. It is remarkable the range of potential mathematical problems that arise under this framework, as well as the diversity on how to use the theoretical results depending on the nature of each application. In light of this variety, the Ph.D. dissertation is divided into separate chapters, each of them defining their own framework, state of art and objectives, which helps to contextualize separately each part of the work.

The core of the dissertation is divided into five chapters (Chapters 2 to 6), which can be grouped into three main parts. In broad terms we can breakdown the content attending to the measure considered in the work of each of the chapters. In Chapters 2 and 3, we mainly deal with co-risk measures. Chapters 4 and 5 contain an analysis of theoretical aspects that lead to the study of variability measures. Chapter 6 has a different approach to the previous chapters and it is devoted to study a particular case of dependence monotonicity into the copula framework. Examples, either based on parametric families of distributions or dealing with real-world data-sets, are provided to illustrate the results included in each chapter. Finally, in addition to this core content, a concluding chapter is included.

After this introduction on the background and main concepts that set the framework of this thesis, we give a brief description of the contents of the mentioned chapters of this Ph.D. dissertation.

1. Chapter 2 presents sufficient conditions, based on stochastic orders, to compare the risk of contagion of two portfolios represented by n -dimensional random vectors ($n \geq 2$) with positive dependent components. The comparisons are based on co-risk and risk contributions measures, which address the risk of one asset of the portfolio, given that the others are under a stress scenario. In this chapter, the co-risk considered measures are the conditional value at risk ($\text{CoVaR}^>$), the conditional expected shortfall (CoES) and the marginal mean excess (MME).
2. Chapter 3 addresses one of the main drawbacks of the VaR , the lack of taking into account systemic risk. With this aim, we consider a random vector (X, Y) and risk levels $v \in [0, 1]$ and $u \in [0, 1]$. In this chapter, under the approach of using risk measures to determine risk capital, we are interested in comparing the $\text{VaR}_v[Y]$ and the $\text{CoVaR}_{v,u}^=[Y|X]$ to balance the convenience of replacing the first by the latter. In particular, assuming that the vector exhibits some kind of positive dependence, we study the existence of a bound $u^* \in (0, 1)$ such that for a risk level $u > u^*$, the $\text{CoVaR}_{v,u}^=[Y|X]$ lies above $\text{VaR}_v[Y]$ for all $v \in (0, 1)$.
3. Chapter 4 is devoted to two main objectives, illustrated with particular applications in actuarial and financial risk management. The first purpose is to show that, given a random vector (X, Y) with $X =_{st} Y$ and not necessarily independent components, then $E|X - Y|$ is a measure of its variability in the sense of Bickel and Lehmann (1979), which is a subadditive measure if the components are negatively dependent. Moreover, the aim of this chapter is to provide sufficient conditions for comparing several distances between pairs of random variables (with possibly different distribution functions) in terms of various stochastic orderings.
4. In light of Chapter 4, in Chapter 5, we develop sufficient conditions for the existence of a minimizer for the given variability measure. That is, in this chapter we consider the random vector (X, Y) with copula C , where X is a continuous random variable with distribution function F , $\mathcal{F}_X$ is the family of proportional reversed hazard rate distribution functions associated to F and $G \in \mathcal{F}_X$ is the distribution function of Y . The chapter is devoted to the development of mathematical results to obtain sufficient conditions for the existence of $G \in \mathcal{F}_X$ that minimizes the functional $E_C|X - Y|$.

5. Chapter 6 is devoted to give sufficient conditions, in terms of properties of the marginals and the copula, that extend the results in Pellerey and Navarro (2022). These results deal with the study of Efron's marginal monotonicity (EMM) in terms of the likelihood ratio order. Given a multivariate random vector, EMM refers to the stochastic monotonicity of one of the variables given the value of the sum of all the components.
6. Chapter 7 discusses the conclusions and contributions achieved in this Ph.D. thesis. Moreover, the future research lines in which this dissertation might be extended are presented. Finally, we provide a list of publications in scientific journals obtained from this research work, together with published conferences proceedings that arose from this Ph.D. thesis.

*“The beauty of mathematics only shows
itself to more patient followers”.*

Maryam Mirzakhani

2

Stochastic orders and multivariate measures of risk contagion

Co-risk measures and risk contribution measures are used in portfolio risk analysis to assess and quantify the risk of contagion, given that one or more assets in the portfolio are in distress. In this chapter, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ that represent two portfolios of n assets ($n \geq 2$) and exhibit some kind of positive dependence, we give sufficient conditions based on stochastic orders to compare the risk of contagion of the portfolios. The measures of risk contagion that we consider are the conditional value at risk ($\text{CoVaR}^>$), the conditional expected shortfall (CoES) and the recently introduced marginal mean excess (MME).

2.1 Introduction

In this chapter, we focus our attention on the study and comparison of risk measures which take into account the contagion risk caused by interconnections among the variables. We recall that, as it was introduced in Section 1.5.2, co-risk measures capture the potential for the spreading of financial distress across institutions by gauging how the tail behavior of some components affect the others.

Throughout the chapter, we will let $\mathbf{X} = (X_1, \dots, X_n)$ be a portfolio of risks (or random vector) with joint distribution and survival functions $\mathbf{F}$ and $\bar{\mathbf{F}}$, respectively. We will consider that $\mathbf{X}$ has respective marginal distribution functions $F_1, F_2, \dots, F_n$ and marginal survival functions $\bar{F}_1, \bar{F}_2, \dots, \bar{F}_n$, which we will assume to be continuous with finite expectations. We recall that the joint distribution function $\mathbf{F}$ can be expressed as

$$\mathbf{F}(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)),$$

where C is the copula of $\mathbf{X}$. By $\bar{C}$ we represent the joint survival function for uniform random variables whose joint distribution function is the copula, that is,

$$\bar{C}(p_1, \dots, p_n) = P[X_1 > F_1^{-1}(p_1), \dots, X_n > F_n^{-1}(p_n)]$$

Note that we also use the notation $C(\mathbf{p})$ and $\bar{C}(\mathbf{p})$, where $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$.

Under this framework, among the defined co-risk measures listed in Section 1.5.2, we next recall the ones considered in the present chapter.

The $\text{CoVaR}^>$ (Definition 1.37), is defined, for $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$, as:

$$\text{CoVaR}^>_{\mathbf{p}}[X_1 | X_2, \dots, X_n] = \text{VaR}_{p_1} \left[X_1 | \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right]. \quad (2.1)$$

From this expression, it is clear that, given $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$, (2.1) is increasing in p_1 . Fixed p_1 , however, the monotonicity of (2.1) with respect to $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$ reveals the structure of dependence between X_1 and $(X_2, \dots, X_n)$ known as right-tail increasing (decreasing) (RTI (RTD)) and defined in Section 1.3.3 (Definition 1.27).

Alongside the co-risk measures, and also on target to quantify the systemic risk in risk management, risk contribution measures are also considered. The risk contribution measure

$$\Delta \text{CoVaR}^>_{\mathbf{p}}[X_1 | X_2, \dots, X_n] = \text{CoVaR}^>_{\mathbf{p}}[X_1 | X_2, \dots, X_n] - \text{VaR}_{p_1}[X_1]$$

quantifies how a joint stress situation for the components $X_2, \dots, X_n$ affects X_1 by comparing $\text{CoVaR}^>_{\mathbf{p}}[X_1 | X_2, \dots, X_n]$, which represents the risk of X_1 when is affected by the joint behavior of $(X_2, \dots, X_n)$, with $\text{VaR}_{p_1}[X_1]$, which is the risk of X_1 when considered in isolation.

The CoES (Definition 1.38) is given, for $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$, by:

$$\begin{aligned} & \text{CoES}_{\mathbf{p}}[X_1 | X_2, \dots, X_n] \\ &= \frac{1}{1 - p_1} \int_{p_1}^1 \text{CoVaR}^>_{t, p_2, \dots, p_n}[X_1 | X_2, \dots, X_n] dt \end{aligned} \quad (2.2)$$

Clearly, fixed $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$, (2.2) is increasing with respect to p_1 and fixed p_1 , (2.2) inherits the monotonicity of $\text{CoVaR}^>$. Therefore, fixed p_1 , if X_1 is RTI (RTD) in $(X_2, \dots, X_n)$, then (2.2) increases (decreases) with respect to $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$.

Similarly as above, risk contributions can be measured in terms of CoES by using

$$\Delta \text{CoES}_{\mathbf{p}}[X_1 | X_2, \dots, X_n] = \text{CoES}_{\mathbf{p}}[X_1 | X_2, \dots, X_n] - \text{ES}_{p_1}[X_1].$$

Finally, we recall the most recently defined risk measure here considered, which is the marginal mean excess (MME) (Definition 1.39), defined, for $\mathbf{p}^* = (p_2, \dots, p_n) \in (0, 1)^{n-1}$, by

$$\text{MME}_{\mathbf{p}^*}[X_1 | X_2, \dots, X_n] = E \left[(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \mid \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right]$$

with $A_{\mathbf{X}, \mathbf{p}^*} = a_1 \text{VaR}_{p_2}(X_2) + \dots + a_{n-1} \text{VaR}_{p_n}(X_n)$, where $a_1, \dots, a_{n-1} \in \mathbb{R}^+$ and $\sum_{i=1}^{n-1} a_i = 1$.

The contribution of $(X_2, \dots, X_n)$ to the expected excess of X_1 over the linear combination $A_{\mathbf{X}, \mathbf{p}^*}$ can be quantified by:

$$\begin{aligned} \Delta \text{MME}_{\mathbf{p}^*}[X_1 | X_2, \dots, X_n] \\ = E \left[(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \mid \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right] - E \left[(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \right]. \end{aligned}$$

The study of $\text{CoVaR}^>$, CoES and MME has attracted growing attention among researchers, specially in the copula framework. To mention only some of the most recent works: Bernardi et al. (2017) provide a directory of $\text{CoVaR}^>$ values for different families of copulas, Jaworski (2017) deals with $\text{CoVaR}^>$ for copulas with tail dependence, Karimalis and Nomikos (2018) use dynamic copula models to estimate the $\text{CoVaR}^>$ and the CoES, Choi and Shin (2019) propose a new method for estimation of $\text{CoVaR}^>$, and Bernardi et al. (2018) and Bernardi and Catania (2019) provide applications in different contexts. Regarding the marginal mean excess, Das and Fasen-Hartmann (2018a,b) study its asymptotic behavior under hidden regular variation assumptions and Hashorva (2019) provides approximations for Gaussian random vectors. Some generalizations of MME can be found in Ling (2019).

An important reason for measuring the risk of contagion is to make comparisons and express preferences among different portfolios. However, given two portfolios $\mathbf{X}$ and $\mathbf{Y}$, there are several issues that complicate the process of decision-making based on the direct comparisons of $\text{CoVaR}^>$, CoES and MME. One issue is that, in general, these measures cannot be calculated explicitly since they depend on quantiles which sometimes do not have closed forms. Another issue is the frequent lack of consensus about the vector $\mathbf{p}$ of probabilities that defines the stress scenario. One possibility to overcome these

issues is to find conditions (that do not require the explicit expression of the underlying measures) to ensure that

$$\Phi_{\mathbf{p}}[X_1|X_2, \dots, X_n] \leq \Phi_{\mathbf{p}}[Y_1|Y_2, \dots, Y_n], \quad \text{for all } \mathbf{p} \in \Omega, \quad (2.3)$$

where Φ stands for $\text{CoVaR}^>$, CoES and MME and $\Omega = (0, 1)^n$ (in the case of $\text{CoVaR}^>$ and CoES) or $\Omega = (0, 1)^{n-1}$ (in the case of MME). In this chapter, we give sufficient conditions on the vectors $\mathbf{X}$ and $\mathbf{Y}$, in terms of well-known stochastic orders and dependence notions, under which (2.3) holds. Regarding $\text{CoVaR}^>$ and CoES, our results complement and extend those obtained by Sordo et al. (2018), who derived sufficient conditions for (2.3) in the bivariate case. The results connecting MME with stochastic orders are novel and offer an alternative to the approach of Hashorva (2019) and Das and Fasen-Hartmann (2018a,b), who derived approximations of MME under different asymptotic conditions. Other recent works that provide sufficient conditions based on stochastic orders for the comparisons of random vectors under different co-risk measures include Fang and Li (2018); Navarro and Sordo (2018); Longobardi and Pellerey (2019) or Dhaene et al. (2022).

In the present chapter, several notions of positive dependence will be considered. Recall that these notions mathematically describe the idea that large (respectively, small) values of one of the variables go together with large (respectively, small) values of the other. These notions have been formalized in Section 1.3.3 and in order to tackle the issues described in this chapter, we will consider vectors that are RTI (Definition 1.27), SI (Definition 1.28) or MTP2 (Definition 1.30). It is well-known that a continuous random vector $\mathbf{X}$ has the property MTP2 (resp. SI, RTI) if and only if its copula is MTP2 (resp. SI, RTI) (see Theorem 3.10.19 in Müller and Stoyan (2002) and the discussion on this issue in Cai and Wei (2012)).

The rest of the chapter is organized as follows. In Section 2.2, we give sufficient conditions for comparisons of $\text{CoVaR}^>$, CoES and their associated risk contribution measures. The results in this section extend to the multivariate setting those obtained by Sordo et al. (2018) for the bivariate case. Given the random vectors $\mathbf{X}$ and $\mathbf{Y}$, under the hypothesis that at least one of the vectors satisfies a particular property of positive dependence, the conditions are either given in terms of the order $\leq_{whr}$ of the vectors or in terms of the order $\leq_{whr}$ of their copulas and the order $\leq_{icx}$ or $\leq_{st}$ of their components. In Section 2.3, we give sufficient conditions for comparisons of the form (2.3), where Φ is MME or its associated risk contribution measure (2.3). In this case, the conditions are given in terms of the order $\leq_{whr}$ of their copulas and

a particular variability order of their components. Sections 2.2 and 2.3 contain several examples based on multivariate parametric families that are relevant in actuarial science. Section 2.4 is devoted to propose a new contagion index based on the MME. In Section 2.5, we provide an empirical illustration in the context of Spanish banking sector. Section 2.6 contains conclusions.

2.2 Sufficient conditions for the comparisons of CoVaRs and CoESs

In the remainder of the chapter, we will make repeatedly use of the following assumption, which, to improve the readability of the chapter, will be referred as Assumption 2.1.

Assumption 2.1. $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ are two random vectors with absolutely continuous joint distribution functions F and G , marginal distribution functions $F_1, F_2, \dots, F_n$ and $G_1, G_2, \dots, G_n$ and copulas C and C' respectively.

Given two random vectors, our first result shows the relationships between the orderings of their respective $\text{CoVaR}^>$ s, CoESs and MMEs.

Theorem 2.2. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 and let $\mathbf{p} = (p_1, p_2, \dots, p_n) \in [0, 1]^n$ and $\mathbf{p}^* = (p_2, p_3, \dots, p_n) \in [0, 1]^{n-1}$.

(i) If

$$\text{CoVaR}^>_{\mathbf{p}}[X_1|X_2, \dots, X_n] \leq \text{CoVaR}^>_{\mathbf{p}}[Y_1|Y_2, \dots, Y_n]$$

then

$$\text{CoES}_{\mathbf{p}}[X_1|X_2, \dots, X_n] \leq \text{CoES}_{\mathbf{p}}[Y_1|Y_2, \dots, Y_n]. \quad (2.4)$$

(ii) If (2.4) holds and $X_i \geq_{st} Y_i$ for all $i = 2, \dots, n$ (in particular, if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals) then

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n]. \quad (2.5)$$

Proof. Part (i) is obvious from Definition 1.38. In order to prove (ii), note that $\text{CoES}_{\mathbf{p}}[X_1|X_2, \dots, X_n]$ is the expected shortfall of the conditional random variable $\left[X_1 | \bigcap_{j=2}^n \{X_j > F_j^{-1}(p_j)\}\right]$. Therefore, it follows from (1.6) that (2.4) is the same as

$$\left[X_1 | \bigcap_{i=2}^n \{X_i > F_i^{-1}(p_i)\}\right] \leq_{icx} \left[Y_1 | \bigcap_{i=2}^n \{Y_i > G_i^{-1}(p_i)\}\right]. \quad (2.6)$$

Now observe that the assumption $X_i \geq_{st} Y_i$ for all $i = 2, \dots, n$ implies $A_{\mathbf{X}, \mathbf{p}^*} \geq A_{\mathbf{Y}, \mathbf{p}^*}$ (see Definition 1.39). Therefore, using that $Z \leq_{icx} Z + k$ for any random variable Z and for all $k > 0$, we have

$$\left[Y_1 \mid \bigcap_{i=2}^n \{Y_i > G_i^{-1}(p_i)\} \right] \leq_{icx} \left[Y_1 + A \mid \bigcap_{i=2}^n \{Y_i > G_i^{-1}(p_i)\} \right], \quad (2.7)$$

where $A = A_{\mathbf{X}, \mathbf{p}^*} - A_{\mathbf{Y}, \mathbf{p}^*}$. From (2.6), (2.7) and the transitivity of the increasing convex order we get

$$\left[X_1 \mid \bigcap_{i=2}^n \{X_i > F_i^{-1}(p_i)\} \right] \leq_{icx} \left[Y_1 + A \mid \bigcap_{i=2}^n \{Y_i > G_i^{-1}(p_i)\} \right] \quad (2.8)$$

with, again, $A = A_{\mathbf{X}, \mathbf{p}^*} - A_{\mathbf{Y}, \mathbf{p}^*}$. Since $Z_1 \leq_{icx} Z_2$ implies $E[\phi(Z_1)] \leq E[\phi(Z_2)]$ for all increasing and convex function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ (see Chapter 4 in Shaked and Shanthikumar (2007)) (2.5) follows from (2.8) by taking $\phi(x) = (x - A_{\mathbf{X}, \mathbf{p}^*})^+$.

Given two bivariate random vectors with the same marginals, Mainik and Schaanning (2014) prove that the concordance order is a sufficient condition for the ordering of the respective $\text{CoVaR}^>$ s. The following theorem is an extension of this result to the case of multivariate random vectors with possibly different marginals.

Theorem 2.3. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1. If $\mathbf{X} \leq_{whr} \mathbf{Y}$ and X_1 is RTI in X_j (or Y_1 is RTI in Y_j) $j = 2, \dots, n$, then, for all $\mathbf{p} \in (0, 1)^n$,*

$$\text{CoVaR}^>_{\mathbf{p}} [X_1 | X_2, \dots, X_n] \leq \text{CoVaR}^>_{\mathbf{p}} [Y_1 | Y_2, \dots, Y_n]. \quad (2.9)$$

Proof. Let $\mathbf{X} \leq_{whr} \mathbf{Y}$ be and assume that X_1 is RTI in $X_j, j = 2, \dots, n$. On the one hand, it follows from Theorem 2.5 of Hu et al. (2003) and from the relationship (1.5) between the hazard rate order and the usual stochastic order, that

$$\left[X_1 \mid \bigcap_{j=2}^n \{X_j > x_j\} \right] \leq_{st} \left[Y_1 \mid \bigcap_{j=2}^n \{Y_j > x_j\} \right], \quad x_j \in \mathbb{R}, \quad j = 2, \dots, n,$$

or, equivalently, with $0 \leq p_j \leq 1$, for $j = 2, \dots, n$,

$$\left[X_1 \mid \bigcap_{j=2}^n \{X_j > G_j^{-1}(p_j)\} \right] \leq_{st} \left[Y_1 \mid \bigcap_{j=2}^n \{Y_j > G_j^{-1}(p_j)\} \right]. \quad (2.10)$$

On the other hand, $\mathbf{X} \leq_{whr} \mathbf{Y}$ implies $X_i \leq_{st} Y_i$ for $i = 1, \dots, n$ (Hu et al. (2003)) or, equivalently, $F_j^{-1}(p_j) \leq G_j^{-1}(p_j)$ for $0 \leq p_j \leq 1$. Using that X_1 is

RTI in X_j , $j = 2, \dots, n$, it follows

$$\left[X_1 \mid \bigcap_{j=2}^n \{X_j > F_j^{-1}(p_j)\} \right] \leq_{st} \left[X_1 \mid \bigcap_{j=2}^n \{X_j > G_j^{-1}(p_j)\} \right], 0 \leq p_j \leq 1. \quad (2.11)$$

Combining (2.10) and (2.11) we get

$$\left[X_1 \mid \bigcap_{j=2}^n \{X_j > F_j^{-1}(p_j)\} \right] \leq_{st} \left[Y_1 \mid \bigcap_{j=2}^n \{Y_j > G_j^{-1}(p_j)\} \right], 0 \leq p_j \leq 1,$$

which is equivalent to (2.9) for all $\mathbf{p} \in (0, 1)^n$. The proof is similar when Y_1 is RTI in Y_j , $j = 2, \dots, n$.

Regarding marginal risk contributions, we have the following result, which extends Theorem 15 in Sordo et al. (2018).

Theorem 2.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $C \leq_{whr} C'$, $(X_2, \dots, X_n) \uparrow_{SI} X_1$, (or $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$) and $X_1 \leq_{disp} Y_1$ then*

$$\Delta \text{CoVaR}_{\mathbf{p}}^>[X_1 \mid X_2, \dots, X_n] \leq \Delta \text{CoVaR}_{\mathbf{p}}^>[Y_1 \mid Y_2, \dots, Y_n]$$

holds for all $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$.

Proof. The proof is similar to the proof of Theorem 15 in Sordo et al. (2018) with the difference that we must use the distortion function

$$h_C(u) = \frac{\bar{C}(1 - u, p_2, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)}, u \in [0, 1]. \quad (2.12)$$

Looking again at the proof of Theorem 2.3, we easily see that, under equal marginals, the RTI condition is no longer necessary since $F_i^{-1}(p_i) = G_i^{-1}(p_i)$ for $i = 1, \dots, n$. Moreover, in this case, (2.10) is equivalent to

$$\frac{\bar{C}(p_1, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)} \leq \frac{\bar{C}'(p_1, \dots, p_n)}{\bar{C}'(0, p_2, \dots, p_n)}, p_i \in [0, 1],$$

which is implied by the stronger condition $C \leq_{whr} C'$. This observation together with Theorem 2.2 proves the following result.

Corollary 2.5. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $X_i =_{st} Y_i$ for all $i = 1, \dots, n$. If $\mathbf{X} \leq_{whr} \mathbf{Y}$ or $C \leq_{whr} C'$, then for all $\mathbf{p} \in (0, 1)^n$*

$$\text{CoVaR}_{\mathbf{p}}^>[X_1 \mid X_2, \dots, X_n] \leq \text{CoVaR}_{\mathbf{p}}^>[Y_1 \mid Y_2, \dots, Y_n]. \quad (2.13)$$

and

$$\text{CoES}_{\mathbf{p}}[X_1 \mid X_2, \dots, X_n] \leq \text{CoES}_{\mathbf{p}}[Y_1 \mid Y_2, \dots, Y_n]. \quad (2.14)$$

In addition, for all $\mathbf{p}^* \in (0, 1)^{n-1}$,

$$\text{MME}_{\mathbf{p}^*} [X_1 | X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*} [Y_1 | Y_2, \dots, Y_n]. \quad (2.15)$$

Remark 2.6. In the case of bivariate random vectors, the condition $C \leq_{whr} C'$ in Corollary 2.5 can be weakened to the concordance order of the corresponding copulas since, under the assumption of equally distributed components, $C(p_1, p_2) \leq C'(p_1, p_2)$ is equivalent to

$$\text{CoVaR}_{\mathbf{p}}^> [X_1 | X_2] \leq \text{CoVaR}_{\mathbf{p}}^> [Y_1 | Y_2], \quad \mathbf{p} = (p_1, p_2).$$

We illustrate the above results by providing several examples based on parametric families of distributions.

Example 2.7 (Multivariate Pareto Distribution). A random vector $\mathbf{X} = (X_1, \dots, X_n)$ follows a multivariate Pareto distribution $\mathbb{P}_n(\alpha, a)$, with $\alpha > 0, a > 0$, if its joint survival function is given by

$$\bar{F}(x_1, \dots, x_n) = \left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^{-a}, \quad x_i \geq \alpha > 0, \quad a > 0.$$

It can be checked that $\mathbf{X}$ has identically distributed Pareto marginals (that is, $\bar{F}_1(x_i) = (\alpha/x_i)^a, i = 1, \dots, n$) linked by a Clayton-Oakes survival copula (see Arias-Nicolás et al. (2009)). Since, given $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the conditional tail function

$$\begin{aligned} \bar{F}_{X_1 | \cap_{j=2}^n \{X_j > x_j\}}(x_1) &= \left(\frac{\sum_{i=2}^n \frac{x_i}{\alpha} - (n-1)}{\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)} \right)^a \\ &= \left(1 - \frac{\frac{x_1}{\alpha}}{\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)} \right)^a \end{aligned}$$

is increasing in x_i for $i = 2, 3, \dots, n$, X_1 is RTI in $\{X_j, j = 2, \dots, n\}$. Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be another random vector that follows a multivariate Pareto distribution $\mathbb{P}_n(\gamma, b)$ with $\gamma > 0, b > 0$. Now it will be shown that if $\alpha \leq \gamma$ and $a \geq b$, then $\mathbf{X} \leq_{whr} \mathbf{Y}$. In order to prove it, we consider a third random vector $\mathbf{Z} = (Z_1, \dots, Z_n) \sim \mathbb{P}_n(\alpha, b)$, with joint survival function $\bar{H}$. Then,

$$\frac{\bar{H}(x_1, \dots, x_n)}{\bar{F}(x_1, \dots, x_n)} = \frac{\left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^{-b}}{\left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^{-a}} = \left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^{a-b}$$

which is increasing in x_i , for $i = 2, 3, \dots, n$, that is, $\mathbf{X} \leq_{whr} \mathbf{Z}$. Similarly,

$$\frac{\bar{G}(x_1, \dots, x_n)}{\bar{H}(x_1, \dots, x_n)} = \frac{\left(\sum_{i=1}^n \frac{x_i}{\gamma} - (n-1) \right)^{-b}}{\left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^{-b}} = \left(\frac{\sum_{i=1}^n \frac{x_i}{\gamma} - (n-1)}{\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)} \right)^{-b}. \quad (2.16)$$

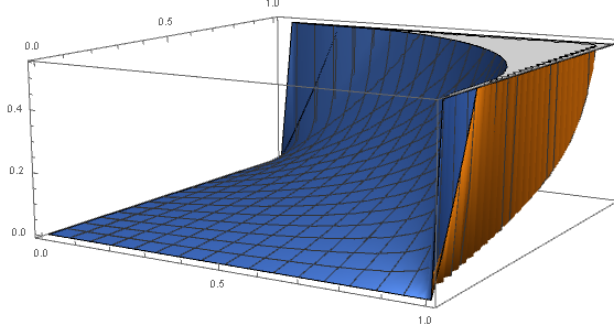


Figure 2.1. $\text{CoVaR}_{p_1, p_2}^>[X_1|X_2]$ as function of p_1 and p_2 . The orange surface corresponds to the $\text{CoVaR}^>$ of $(X_1, X_2) \sim \mathbb{P}_2(1, 6)$. The blue one corresponds to the $\text{CoVaR}^>$ of $(Y_1, Y_2) \sim \mathbb{P}_2(3, 4)$. Here $\mathbf{X} \leq_{whr} \mathbf{Y}$.

To study the monotony of (2.16) we write, for $i = 2, \dots, n$,

$$\frac{d}{dx_i} \left(\frac{\bar{G}(x_1, \dots, x_n)}{\bar{H}(x_1, \dots, x_n)} \right) = -b \cdot f(x_1, \dots, x_n)^{-b-1} \cdot \frac{df(x_1, \dots, x_n)}{dx_i} \quad (2.17)$$

where

$$f(x_1, \dots, x_n) = \left(\frac{\sum_{i=1}^n \frac{x_i}{\gamma} - (n-1)}{\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)} \right).$$

It can be checked that

$$\frac{df(x_1, \dots, x_n)}{dx_i} = \frac{(\alpha - \gamma)(n-1)}{\alpha \gamma \left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1) \right)^2}$$

which is negative if and only if $\alpha \leq \gamma$. From this fact and (2.17) it follows that (2.16) is increasing, that is, $\mathbf{Z} \leq_{whr} \mathbf{Y}$. By the transitivity of the order $\leq_{whr}$, we have that if $\alpha \leq \gamma$ and $b \leq a$, then $\mathbf{X} \leq_{whr} \mathbf{Y}$. It follows from Theorem 2.3 that (2.13) holds for all $\mathbf{p} \in (0, 1)^n$. This example is illustrated in the bivariate case in Figure 2.1.

Example 2.8 (Multivariate Gumbel Exponential Distributions). Given $\lambda = \{\lambda_I : I \subseteq \{1, \dots, n\}, \lambda_I \geq 0, I \neq \emptyset\}$, a random vector $\mathbf{X}_\lambda = (X_1, \dots, X_n)$ follows a multivariate Gumbel Exponential distribution (see Kotz et al. (2000)) if its joint survival function is given by

$$\bar{F}_\lambda(x_1, x_2, \dots, x_n) = \exp \left\{ - \sum_I \lambda_I \prod_{i \in I} x_i \right\}, \quad x_i \geq 0, \quad i = 1, \dots, n.$$

Let $\mathbf{X}_\lambda = (X_1, \dots, X_n)$ and $\mathbf{Y}_{\lambda^*} = (Y_1, \dots, Y_n)$ be two multivariate Gumbel Exponential vectors such that $X_i =_{st} Y_i$ for all $i = 1, \dots, n$ (that is, $\lambda_i = \lambda_i^*$). Under this assumption, Khaledi and Kochar (2005) show that $\lambda \geq \lambda^*$ implies

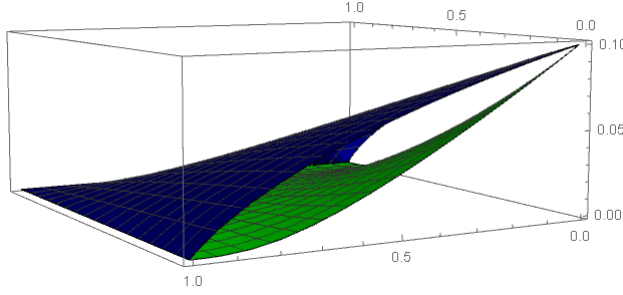


Figure 2.2. $\text{MME}_{(p_2, p_3)}[X_1|X_2, X_3]$ as function of p_2 and p_3 . The green surface corresponds to the MME of (X_1, X_2, X_3) a multivariate Gumbel Exponential Distribution with parameters $\lambda = \{\lambda_I = 10, I \subseteq \{1, \dots, n\}, I \neq \emptyset\}$. The blue one corresponds to the $\text{MME}_{(p_2, p_3)}[Y_1|Y_2, Y_3]$, where $\mathbf{Y} = (Y_1, Y_2, Y_3)$ is a multivariate Gumbel Exponential Distribution with parameters $\lambda^* = \{\lambda_i^* = 10 \text{ for } i = 1, 2, 3, \text{ and } \lambda_{12}^* = \lambda_{13}^* = \lambda_{23}^* = \lambda_{123}^* = 100\}$.

$\mathbf{X} \leq_{whr} \mathbf{Y}$. It follows from Corollary 2.5 that (2.13) and (2.14) hold for all $\mathbf{p} \in (0, 1)^n$ and (2.15) holds for all $\mathbf{p}^* \in (0, 1)^{n-1}$. This example is illustrated in Figure 2.2.

Intuitively, a random vector $\mathbf{X} = (X_1, \dots, X_n)$ with a positive dependence structure among the components exhibits more risk of contagion when compared to a random vector with the same marginals but independent components. The following example formalizes this intuition.

Example 2.9 (MTP₂ Distributions). *Hu et al. (2003) proved that if $\mathbf{X} = (X_1, \dots, X_n)$ is MTP₂ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ is a random vector with independent components such that $X_i =_{st} Y_i$ for all $i = 1, \dots, n$, then $\mathbf{Y} \leq_{whr} \mathbf{X}$. It follows from Corollary 2.5 that (2.13) and (2.14) hold for all $\mathbf{p} \in (0, 1)^n$ and (2.15) holds for all $\mathbf{p}^* \in (0, 1)^{n-1}$.*

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1. For $n = 2$, Sordo et al. (2018) showed that under SI dependence, the conditions $X_1 \leq_{icx} Y_1$ and $C \leq_c C'$ imply $\text{CoES}_{p_1, p_2}[X_1|X_2] \leq \text{CoES}_{p_1, p_2}[Y_1|Y_2]$ for $0 \leq p_i \leq 1, i = 1, 2$. The next result is an extension (under the stronger assumption $C \leq_{whr} C'$) to the general multivariate case. In order to prove it, we recall the notion of distortion function. A distortion function h is a non-decreasing function from $[0, 1]$ to $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$. Given a random variable Z with tail function $\bar{F}_Z$ and a continuous distortion

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function h , the transformation $h(\bar{F}_Z(x)) = h \circ \bar{F}_Z(x)$ defines the tail function associated to the distorted random variable variable, Z_h .

Theorem 2.10. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1. If $X_1 \leq_{icx} Y_1, C \leq_{whr} C'$ and $(X_2, \dots, X_n) \uparrow_{SI} X_1$, (or $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$) then, for all $\mathbf{p} \in (0, 1)^n$,*

$$\text{CoES}_{\mathbf{p}}[X_1|X_2, \dots, X_n] \leq \text{CoES}_{\mathbf{p}}[Y_1|Y_2, \dots, Y_n]. \quad (2.18)$$

Proof. Suppose $(X_2, \dots, X_n) \uparrow_{SI} X_1$ (the proof of the case $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$ is similar). It follows from theorems 2 and 4 in Sordo et al. (2015b) that

$$X_{1,h_C} = \left[X_1 \mid \bigcap_{i=2}^n \{X_i > F_i^{-1}(p_i)\} \right], \quad 0 \leq p_i \leq 1,$$

is a distorted random variable induced from X_1 by the concave distortion function

$$h_C(u) = \frac{\bar{C}(1-u, p_2, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)}, \quad u \in [0, 1]. \quad (2.19)$$

Similarly, given a random vector $\hat{\mathbf{Y}} = (\hat{Y}_1, \dots, \hat{Y}_n)$ with copula C and such that $\hat{Y}_i =_{st} Y_i, i = 1, \dots, n$, we have that

$$\hat{Y}_{1,h_C} = \left[\hat{Y}_1 \mid \bigcap_{i=2}^n \{\hat{Y}_i > G_i^{-1}(p_i)\} \right], \quad 0 \leq p_i \leq 1,$$

is a distorted random variable induced from $\hat{Y}_1$ (or by Y_1 , because $\hat{Y}_1$ and Y_1 have the same distribution function) by the distortion function (2.19). Therefore, since $X_1 \leq_{icx} \hat{Y}_1$ and h_C is concave, it follows from Theorem 13 in Sordo et al. (2015b) that

$$X_{1,h_C} \leq_{icx} \hat{Y}_{1,h_C}. \quad (2.20)$$

Now consider the conditional random variable

$$Y_{1,h_{C'}} = \left[Y_1 \mid \bigcap_{i=2}^n \{Y_i > G_i^{-1}(p_i)\} \right], \quad 0 \leq p_i \leq 1,$$

which is a distorted random variable induced from Y_1 by the (non-necessarily concave) distortion function

$$h_{C'}(u) = \frac{\bar{C}'(1-u, p_2, \dots, p_n)}{\bar{C}'(0, p_2, \dots, p_n)}, \quad u \in [0, 1].$$

Since $C \leq_{whr} C'$, we have $h_C(u) \leq h_{C'}(u)$ for all $u \in (0, 1)$. In particular, $h_C(\bar{F}_1(x)) \leq h_{C'}(\bar{F}_1(x))$ for all x , which means that

$$\hat{Y}_{1,h_C} \leq_{st} Y_{1,h_{C'}}. \quad (2.21)$$

Since the stochastic order is stronger than the increasing convex order, it follows from (2.20) and (2.21) that $X_{1,h_C} \leq_{icx} Y_{1,h_{C'}}$, which is equivalent to say that (2.18) holds for all $\mathbf{p} \in (0, 1)^n$.

Remark 2.11. Observe, using the same argument as in Example 2.9, that if $\hat{C}$ is any MTP_2 copula and $\hat{C}^I$ is the independence copula, then $\hat{C} \leq_{whr} \hat{C}^I$. Examples of MTP_2 copulas can be found in Müller and Scarsini (2005). For instance, if $\hat{C}_\psi$ is a survival Archimedean copula with a generator function ψ such that $(-1)^n \psi^{(n)}$ is log-convex, then $\hat{C}_\psi$ is MTP_2 .

As mentioned before, for $n = 2$, the condition $C \leq_{whr} C'$ in Theorem 2.10 can be weakened to the condition $C \leq_c C'$. This is shown in Theorem 12 of Sordo et al. (2018). Now, making use of Theorem 2.2(ii), this result can also be used to compare the respective MMEs. This is illustrated in the following example.

Example 2.12 (Gumbel copulas). Let (X_1, X_2) and (Y_1, Y_2) have Gumbel copulas with dependence parameters $\theta < \theta'$, which implies $C \leq C'$. Let $X_1 \sim G(\alpha_{X1}, \beta_{X1})$, $X_2 \sim G(\alpha_{X2}, \beta_{X2})$, $Y_1 \sim G(\alpha_{Y1}, \beta_{Y1})$ and $Y_2 \sim G(\alpha_{Y2}, \beta_{Y2})$, with $\alpha_{X1} > \alpha_{Y1}$, $\alpha_{X1}\beta_{X1} \leq \alpha_{Y1}\beta_{Y1}$ (which implies $X_1 \leq_{icx} Y_1$), and $\alpha_{X2} \geq \alpha_{Y2}$, $\beta_{X2} \geq \beta_{Y2}$ (which implies $X_2 \geq_{st} Y_2$). It follows from Theorem 12 of Sordo et al. (2018) and Theorem 2.2(ii) that $MME_p[X_1|X_2] \leq MME_p[Y_1|Y_2]$ for all $0 < p < 1$ (see Figure 2.3).

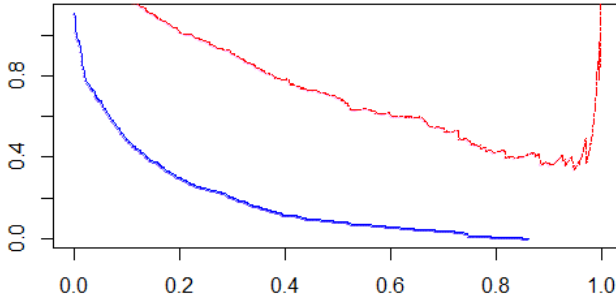


Figure 2.3. $MME_p[X_1|X_2]$ as a function of p . The blue line corresponds to a Gumbel copula C with dependence parameter $\rho = 1.05$ and marginals $X_1 \sim G(1.1, 1)$ and $X_2 \sim G(2, 2)$. The red one corresponds to a Gumbel copula C' with dependence parameter $\rho = 1.1$ and marginals $Y_1 \sim G(1, 1.5)$ and $Y_2 \sim G(2, 1)$. Here $X_1 \leq_{icx} Y_1$, $X_2 \geq_{st} Y_2$ and $C \leq C'$.

Theorem 2.10 admits a variant for the case of stochastically decreasing dependence between X_1 and $(X_2, \dots, X_n)$. Recall that, given a random vector

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$(X_1, \dots, X_n)$, we say that $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ is stochastically decreasing in X_i (denoted $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \downarrow_{SD} X_i$) if the conditional distribution $\{(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) | X_i = x_i\}$ is stochastically decreasing as x_i increases. We have the following result (observe that the order $\leq_{icx}$ is replaced by the order $\leq_{icv}$ and the CoES is replaced by a kind of dual conditional measure).

Theorem 2.13. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1. If $X_1 \leq_{icv} Y_1, C \leq_{whr} C'$ and $(X_2, \dots, X_n) \downarrow_{SD} X_1$, (or $(Y_2, \dots, Y_n) \downarrow_{SD} Y_1$) then*

$$\begin{aligned} & \int_0^{p_1} \text{VaR}_{p_1} \left[X_1 | \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right] dt \\ & \leq \int_0^{p_1} \text{VaR}_{p_1} \left[Y_1 | \bigcap_{j=2}^n \{Y_j > \text{VaR}_{p_j}(Y_j)\} \right] dt \end{aligned} \quad (2.22)$$

for all $(p_1, \dots, p_n) \in (0, 1)^n$.

Proof. The proof is similar to the proof of Theorem 2.10, with the following differences. Under the assumption $(X_2, \dots, X_n) \downarrow_{SD} X_1$, the distortion function $h_C(u)$ in (2.19) is convex. Recall that $\hat{\mathbf{Y}} = (\hat{Y}_1, \dots, \hat{Y}_n)$ is a random vector with copula C , such that $\hat{Y}_i =_{st} Y_i, i = 1, \dots, n$. Then, since the increasing concave order is preserved by convex distortion functions, the assumption $X_1 \leq_{icv} \hat{Y}_1$ implies $X_{1,h_C} \leq_{icv} \hat{Y}_{1,h_C}$. This, together with (2.21) implies $X_{1,h_C} \leq_{icv} Y_{1,h_{C'}}$, which is equivalent to (2.22).

To interpret the result, note that the risk measure in the left hand of inequality (2.22) represents (up to the scale factor $1/p_1$), the expected value of the conditional random variable

$$\left[X_1 | \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\}, X_1 \leq \text{VaR}_{p_1} \left[X_1 | \bigcap_{j=2}^n \{X_j > \text{VaR}_{p_j}(X_j)\} \right] \right]$$

which intuitively describes the behavior of small values of X_1 given that $X_2, \dots, X_n$ jointly take on large values (taking into account interactions between X_1 and $(X_2, \dots, X_n)$).

The following result compares marginal risk contributions based on CoES and extends Theorem 23 in Sordo et al. (2018).

Theorem 2.14. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $C \leq_{whr} C'$, $(X_2, \dots, X_n) \uparrow_{SI} X_1$, (or $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$) and*

$X_1 \leq_{ew} Y_1$. Then,

$$\Delta \text{CoES}_{\mathbf{p}}[X_1 \mid X_2, \dots, X_n] \leq \Delta \text{CoES}_{\mathbf{p}}[Y_1 \mid Y_2, \dots, Y_n]$$

holds for all $\mathbf{p} = (p_1, \dots, p_n) \in (0, 1)^n$.

Proof. The proof is similar to the proof of Theorem 23 in Sordo et al. (2018) by using the convex distortion function

$$A_{\mathbf{p}}(t) = \max \left\{ 1 - \frac{\bar{C}(1 - u, p_2, \dots, p_n)}{(1 - p_1)\bar{C}(0, p_2, \dots, p_n)}, 0 \right\}.$$

2.3 Sufficient conditions for the comparisons of MMEs

In this section, we provide sufficient conditions for the comparisons of MMEs in the general multivariate case. As in Theorem 2.10, the copulas C and C' are required to verify $C \leq_{whr} C'$. However, while in Theorem 2.10 the first components are required to be ordered in the increasing convex order ($X_1 \leq_{icx} Y_1$), in the following results they are required to be ordered according to how much they exceed a threshold given by a linear combination of the VaRs of the other $n - 1$ components. We recall from Definition 1.39 that, given a random vector X satisfying Assumption 2.1 and $\mathbf{p}^* = (p_2, p_3, \dots, p_n) \in [0, 1]^{n-1}$, this threshold is given by

$$A_{\mathbf{X}, \mathbf{p}^*} = a_1 \text{VaR}_{p_2}(X_2) + \dots + a_{n-1} \text{VaR}_{p_n}(X_n),$$

where $a_1, \dots, a_{n-1}$ are given positive constants, such that $\sum_{i=1}^{n-1} a_i = 1$. We start by providing an integral representation for the marginal mean excess.

Lemma 2.15. *Let $\mathbf{X}$ be a random vector satisfying Assumption 2.1 and let $(U_1, \dots, U_n)$ be the corresponding vector-copula, where $U_i = F_i(X_i)$, $i = 1, \dots, n$. Given $\mathbf{p}^* = (p_2, p_3, \dots, p_n) \in [0, 1]^{n-1}$, we have*

$$\text{MME}_{\mathbf{p}^*}[X_1 \mid X_2, \dots, X_n] = \int_{F_1(A_{\mathbf{X}, \mathbf{p}^*})}^1 (F_1^{-1}(u) - A_{\mathbf{X}, \mathbf{p}^*}) dF_{U_1 \mid \cap_{j=2}^n \{U_j > p_j\}}(u). \quad (2.23)$$

Proof. Let $Y = \{(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \mid \cap_{j=2}^n \{X_j > F_j^{-1}(p_j)\}\}$ be and denote by F_Y the distribution function of Y . Clearly, $F_Y(x) = 0$ for $x < 0$. For $x > 0$, we have

$$F_Y(x) = \frac{\bar{C}(0, p_2, \dots, p_n) - \bar{C}(F_1(A_{\mathbf{X}, \mathbf{p}^*} + x), p_2, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)}.$$

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Since $E[Y] = \text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n]$ and

$$dF_Y(x) = -\frac{\partial_1 \bar{C}(F_1(A_{\mathbf{p}^*} + x), p_2, \dots, p_n) \cdot f_1(A_{\mathbf{p}^*} + x)}{\bar{C}(0, p_2, \dots, p_n)},$$

we see that $\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n]$ can be written as

$$\int_0^\infty x \left(\frac{-\partial_1 \bar{C}(F_1(A_{\mathbf{p}^*} + x), p_2, \dots, p_n) \cdot f_1(A_{\mathbf{p}^*} + x)}{\bar{C}(0, p_2, \dots, p_n)} \right) dx.$$

The change of variable $u = F_1(x + A_{\mathbf{p}^*})$ yields

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] = \int_{F_1(A_{\mathbf{p}^*})}^1 (F^{-1}(u) - A_{\mathbf{p}^*}) \frac{-\partial_1 \bar{C}(u, p_2, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)} du. \quad (2.24)$$

By taking into account that

$$F_{U_1 \cap \bigcap_{j=2}^n \{U_j > p_j\}}(u) = \frac{\bar{C}(0, p_2, \dots, p_n) - \bar{C}(u, p_2, \dots, p_n)}{\bar{C}(0, p_2, \dots, p_n)},$$

we see that (2.24) is the same as (2.23).

Now we can prove the announced results.

Theorem 2.16. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 and let $\mathbf{p}^* = (p_2, \dots, p_n) \in [0, 1]^{n-1}$. If $C \leq_{whr} C'$ and $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ then*

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n].$$

Proof. Let $\mathbf{Z} = (Z_1, \dots, Z_n)$ be a random vector with the same copula C as $\mathbf{X}$ such that $Z_i =_{st} Y_i$ holds for $i = 1, \dots, n$. Since the excesses verify $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ and $A_{\mathbf{Y}, \mathbf{p}^*} = A_{\mathbf{Z}, \mathbf{p}^*}$, then, for all $u \in (0, 1)$, $F_{(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+}^{-1}(u) \leq F_{(Z_1 - A_{\mathbf{Z}, \mathbf{p}^*})^+}^{-1}(u)$. It follows, by using (2.23), that $\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Z_1|Z_2, \dots, Z_n]$. Since $\mathbf{Z}$ and $\mathbf{Y}$ have the same marginals and $C \leq_{whr} C'$, it follows from Corollary 2.5 that $\text{MME}_{\mathbf{p}^*}[Z_1|Z_2, \dots, Z_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n]$. The result follows by transitivity.

A particularly interesting corollary is the following statement concerning the dispersive order.

Corollary 2.17. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $X_i =_{st} X_1$ and $Y_i =_{st} Y_1$ for $i = 2, \dots, n$. If $C \leq_{whr} C'$ and $X_1 \leq_{disp} Y_1$ then, for all $\mathbf{p}^* = (p, \dots, p) \in (0, 1)^{n-1}$,*

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n]. \quad (2.25)$$

Proof. If $\mathbf{X}$ has identically distributed components and $\mathbf{p}^* = (p, \dots, p)$, then $A_{\mathbf{X}, \mathbf{p}^*} = F_1^{-1}(p)$, where F_1 is the common distribution function of $X_1, \dots, X_n$. If, in addition, $\mathbf{Y}$ has identically distributed components, then $A_{\mathbf{Y}, \mathbf{p}^*} = G_1^{-1}(p)$, where G_1 is the common distribution function of $Y_1, \dots, Y_n$. Then, the condition $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ can be rewritten as $(X_1 - F_1^{-1}(p))^+ \leq_{st} (Y_1 - G_1^{-1}(p))^+$. Taking into account that $X_1 \leq_{disp} Y_1$ if and only if $(X_1 - F_1^{-1}(p))^+ \leq_{st} (Y_1 - G_1^{-1}(p))^+$ for all $p \in (0, 1)$ (Muñoz-Pérez (1990)), the result follows by applying Theorem 2.16.

This result can be interpreted as follows: given a portfolio of identically distributed risks, the greater the dispersion of the individual risks (in the sense of the order $\leq_{disp}$) and the stronger the dependence structure of the vector (in the sense of the order $\leq_{whr}$), the greater the risk of contagion (as measured by MME).

Example 2.18. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 with multivariate Pareto distributions $\mathbb{P}_{\mathbf{X}, n}(\alpha, a)$ and $\mathbb{P}_{\mathbf{Y}, n}(\beta, a)$ respectively, with $0 < \alpha \leq \beta$ and $a > 0$. Then $\mathbf{X}$ has identically distributed Pareto marginals $\mathbb{P}(\alpha, a)$ and $\mathbf{Y}$ has identically distributed Pareto marginals $\mathbb{P}(\alpha, b)$. Moreover, $C = C'$ and $X_i \leq_{disp} Y_i$ for $i = 1, \dots, n$ (see Arias-Nicolás et al. (2009)). It follows from Corollary 2.17 that (2.25) holds for all $\mathbf{p}^* = (p, \dots, p) \in (0, 1)^{n-1}$. This example is illustrated in Figure 2.4.

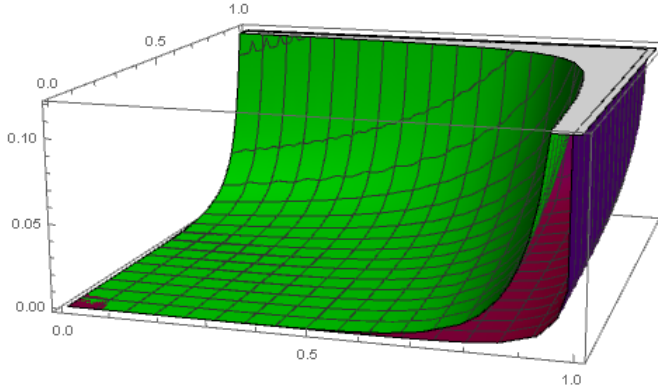


Figure 2.4. $\text{MME}_{(p_2, p_3)}[X_1 | X_2, X_3]$ as function of p_2 and p_3 . The purple surface corresponds to the MME of $(X_1, X_2, X_3) \sim \mathbb{P}(1, 1, 1, 5)$. The green one corresponds to the MME of $(Y_1, Y_2, Y_3) \sim \mathbb{P}(10, 10, 10, 5)$. Here $X_i \leq_{disp} Y_i$ for $i = 1, 2, 3$.

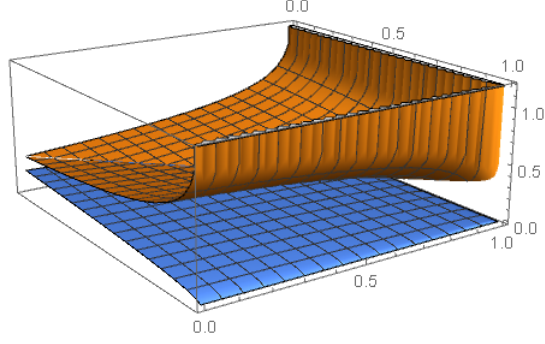


Figure 2.5. $\text{MME}_{(p_2, p_3)} [X_1 | X_2, X_3]$ as function of p_2 and p_3 . The blue surface corresponds to the Marginal mean Excess of $\mathbf{X} = (X_1, X_2, X_3)$ with the independence copula and marginals $X_1, X_2, X_3 \sim \text{Exp}(5)$. The orange one corresponds to the Marginal mean Excess of $\mathbf{Y} = (Y_1, Y_2, Y_3)$ a vector with the comonotonic copula and marginals $Y_1, Y_2, Y_3 \sim \text{Exp}(3)$. Here $X_i \leq_{\text{disp}} Y_i$ for $i=1, 2, 3$ and $\Pi^n \leq_{\text{whr}} M^n$.

Example 2.19. A particular example of Example 2.9, together with Corollary 2.17 is to consider the comonotonic copula. We assume $\mathbf{X}$ and $\mathbf{Y}$ be two n -random vectors satisfying Assumption 2.1 with copulas M^n and Π^n respectively (described in Section 1.3.1). From Example 2.9 it is clear that $\Pi^n \leq_{\text{whr}} M^n$ and $(X_2, \dots, X_n) \uparrow_{SI} X_1$. Then, considering $X_1 \leq_{\text{disp}} Y_1$, we get:

$$\text{MME}_{\mathbf{p}^*} [X_1 | X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*} [Y_1 | Y_2, \dots, Y_n].$$

This is illustrated in Figure 2.5.

Example 2.20. Let us consider two multivariate normal distribution copulas with ρ such that all off-diagonal elements of the covariance matrix in the Gaussian copula are non-negative. As seen by Joe (1997), the vector is PDS. Therefore, for example, considering $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ multivariate normal vectors with respective correlation coefficients $\rho_1 < \rho_2$ (which implies the concordance order) and with $X_i \sim N(\mu_X, \sigma_X)$ and $Y_i \sim N(\mu_Y, \sigma_Y)$ for $i=1, 2$. If $\sigma_X \leq \sigma_Y$ we have that $X_1 \leq_{ew} Y_1$. Hence,

$$\text{MME}_p [X_1 | X_2] \leq \text{MME}_p [Y_1 | Y_2], \text{ for all } 0 < p < 1.$$

This is illustrated in Figure 2.6

Given two random vectors with the same marginals, comparisons of marginal risk contributions of the form (2.3) are equivalent to comparisons of MME. The following result follows directly from Corollary 2.17.

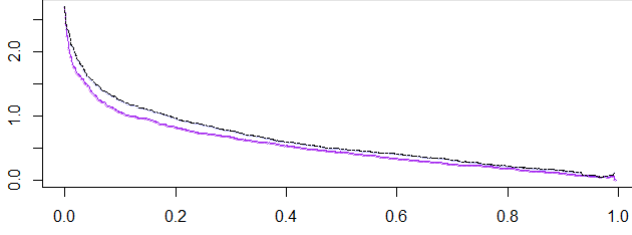


Figure 2.6. $\text{MME}_p[X_1|X_2]$ as function of p . The purple line corresponds to a multivariate normal copula C with dependence parameter $\rho = 0.4$ and marginals $X_1 \sim N(0, 0.7)$ and $X_2 \sim N(0, 0.7)$. The black one corresponds to a multivariate normal vector with the same copula C and marginals $Y_1 \sim N(0, 0.9)$ and $Y_2 \sim N(0, 0.9)$. Here $X_1 \leq_{ew} Y_1$.

Theorem 2.21. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $X_i =_{st} Y_i$ for all $i = 1, \dots, n$. If $\mathbf{X} \leq_{whr} \mathbf{Y}$ or $C \leq_{whr} C'$, then

$$\Delta \text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \Delta \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n]$$

holds for all $\mathbf{p}^* \in (0, 1)^{n-1}$.

By using the same argument as in Corollary 2.5, for $n = 2$ the condition $C \leq_{whr} C'$ can be weakened to the concordance order of the corresponding copulas.

The following result says that if $(X_2, \dots, X_n) \uparrow_{SI} X_1$, then the condition $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ in Theorem 2.16 can be weakened by replacing $\leq_{st}$ by $\leq_{icx}$.

Theorem 2.22. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 and let $\mathbf{p}^* = (p_2, \dots, p_n) \in [0, 1]^{n-1}$. If $(X_2, \dots, X_n) \uparrow_{SI} X_1$ (or $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$), $C \leq_{whr} C'$ and $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{icx} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ then

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n].$$

Proof. As in the proof of Theorem 2.16, let $\mathbf{Z} = (Z_1, \dots, Z_n)$ be a random vector with the same copula C as $\mathbf{X}$ such that $Z_i =_{st} Y_i, i = 1, \dots, n$. Since $(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+ \leq_{icx} (Y_1 - A_{\mathbf{Y}, \mathbf{p}^*})^+$ and $A_{\mathbf{Y}, \mathbf{p}^*} = A_{\mathbf{Z}, \mathbf{p}^*}$, it follows from Theorem 2.1 in Sordo and Ramos (2007) that

$$\int_0^1 F_{(X_1 - A_{\mathbf{X}, \mathbf{p}^*})^+}^{-1}(u) dg(u) \leq \int_0^1 F_{(Z_1 - A_{\mathbf{Z}, \mathbf{p}^*})^+}^{-1}(u) dg(u) \quad (2.26)$$

for all increasing and convex function g . Now observe that (2.23) can be rewritten as

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] = \int_0^1 F_{(X_1 - A_{\mathbf{p}^*})^+}^{-1}(u) dF_{U_1|\cap_{j=2}^n \{U_j > p_j\}}(u), \quad (2.27)$$

where $(U_1, \dots, U_n)$ is the corresponding vector-copula of $\mathbf{X}$. From the assumption $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \uparrow_{SI} X_i$ and Theorem 4(i) of Sordo et al. (2015b) it follows that $F_{U_1|\cap_{j=2}^n \{U_j > p_j\}}(u)$ is an increasing and convex function of u . Then, by taking $g(u) = F_{U_1|\cap_{j=2}^n \{U_j > p_j\}}(u)$ in (2.26) and using (2.27), we have $\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Z_1|Z_2, \dots, Z_n]$. Reasoning again as in the proof of Theorem 2.16, since $\mathbf{Z}$ and $\mathbf{Y}$ have the same marginals and $C \leq_{whr} C'$, it follows from Corollary 2.5 that $\text{MME}_{\mathbf{p}^*}[Z_1|Z_2, \dots, Z_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n]$. The result follows by transitivity. The proof is similar if we assume $(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n) \uparrow_{SI} Y_i$.

A corollary similar to Corollary 2.17 can be stated in terms of the excess wealth order.

Corollary 2.23. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying Assumption 2.1 such that $X_i =_{st} X_1$ and $Y_i =_{st} Y_1$ for $i = 2, \dots, n$. If $(X_2, \dots, X_n) \uparrow_{SI} X_1$ (or $(Y_2, \dots, Y_n) \uparrow_{SI} Y_1$), $C \leq_{whr} C'$ and $X_1 \leq_{ew} Y_1$ then, for all $\mathbf{p}^* = (p, \dots, p) \in (0, 1)^{n-1}$,*

$$\text{MME}_{\mathbf{p}^*}[X_1|X_2, \dots, X_n] \leq \text{MME}_{\mathbf{p}^*}[Y_1|Y_2, \dots, Y_n].$$

Proof. The proof follows from Theorem 2.22 in the same manner as the proof of Corollary 2.17 follows from Theorem 2.16. The only difference is that now, instead of using the characterization of the dispersive order, we use $X_1 \leq_{ew} Y_1$ if and only if $(X_1 - F_1^{-1}(p))^+ \leq_{icx} (Y_1 - G_1^{-1}(p))^+$ for all $p \in (0, 1)$ (Belzunce (1999)).

The interpretation of this result is similar to that of Corollary 2.17 with the difference that when $(X_2, \dots, X_n)$ is positive dependent on X_1 in the sense SI (or when $(Y_2, \dots, Y_n)$ is positive dependent on Y_1), we can weaken the condition $X_1 \leq_{disp} Y_1$ to the condition $X_1 \leq_{ew} Y_1$.

Remark 2.24. *It follows from the comment below Corollary 2.5 that, in the case of bivariate random vectors, the condition $C \leq_{whr} C'$ in corollaries 2.17 and 2.23 can be weakened to the condition $C \leq_c C'$.*

2.4 A contagion index based on the MME

The Marginal mean excess is a measure of contagion, therefore, if we consider fixed marginal distributions, comparisons of the MME reflect an ordering among random vectors based on the contagion between their dependence structures. However, a particular value of the MME is not very informative if it is not compared with a reference level. For example, if the value of the MME is 0.5, does it reflect great contagion? Or does it reflect that there is no contagion? In fact, the same value, depending on the marginal distributions that were fixed, can be endowed with different meanings. Hence, with the purpose of obtaining a normalized value, which is informative on the grade of contagion, this section is devoted to an initial proposal of a contagion index. The building of different contagion indexes or methodologies based on the copula, in order to analyze contagion, is not new (see, for instance, Durante and Jaworski (2010)). However, it is innovative to construct the index based on the MME.

From Theorem 2.16, we know that, if we consider two vectors equally distributed with copulas C and C' verifying $C \leq_{whr} C'$, the MMEs of the vectors are ordered in the same direction as the copulas. In fact, the result reduces to the comparison in the concordance order when we consider bivariate orders (see Theorem 12 in Sordo et al. (2018) and Lemma 2.2 from Section 2.2). In the bivariate case and considering fixed marginals, as the MME is consistent with the concordance order, the maximum and minimum values will be reached in the Fréchet-Hoeffding bounds (see (1.7)), which are, respectively, the comonotonic ($M(u, v) = \min(u, v)$) and countermonotonic ($W(u, v) = \max(u + v - 1, 0)$) copulas. In fact, if we consider PQD copulas (see Definition 1.26), the maximum and minimum are given, respectively, by the comonotonic and independence ($\Pi(u, v) = uv$) copulas. Considering vectors $\mathbf{X} = (X_1, X_2)$ with a PQD copula C and such that X_1 and X_2 have the same marginal distribution F , we propose the following index for a given risk level $p \in (0, 1)$:

$$I_{(C,F)}(p) = \frac{\text{MME}_p[X_1|X_2] - \text{MME}_p^I[X_1|X_2]}{\text{MME}_p^M[X_1|X_2] - \text{MME}_p^I[X_1|X_2]} \quad (2.28)$$

where MME stands for the MME of the vector $\mathbf{X}$ and MME^M and MME^I stand for vectors with equal marginals as $\mathbf{X}$ and with, respectively, the comonotonic and independence copulas. The subscripts C, F on $I_{(C,F)}(p)$ point out the fact that the index depends on those arguments.

We recall that, for bivariate vectors with equal marginals, and taking into account the fact that $\text{VaR}_p(X_1) = \text{VaR}_p(X_2)$ (so, we refer to it as $\text{VaR}_p(X)$),

the MME reduces to

$$\text{MME}_p[X_1|X_2] = E[(X_1 - \text{VaR}_p(X))^+ | X_2 > \text{VaR}_p(X)], \quad p \in (0, 1).$$

Note that the excess is conditioned to the event $\{X_2 > \text{VaR}_p(X)\}$. An interesting property of the proposed contagion index is the fact that the contagion index can be given by an alternative expression, where the excess is conditioned to an event on the form $\{X_2 < \text{VaR}_p(X)\}$. In that form, the expressions transfer the study of contagion in the upper tails to the lower tail.

Theorem 2.25. *Let $\mathbf{X} = (X_1, X_2)$ be a random vector verifying Assumption 2.1, with the same marginal distribution function F , where $\text{VaR}_p(X)$ stands for the VaR of both X_1 and X_2 . The contagious index at level $p \in (0, 1)$ satisfies:*

$$I_{(C,F)}(p) = 1 - \frac{E[(X_1 - \text{VaR}_p(X))^+ | X_2 < \text{VaR}_p(X)]}{E[(X_1 - \text{VaR}_p(X))^+]}. \quad (2.29)$$

Similarly, we define the contagion index at $t \in \mathbb{R}$ by:

$$I_{(C,F)}(p) = 1 - \frac{E[(X_1 - t)^+ | X_2 < t]}{E[(X_1 - t)^+]}. \quad (2.30)$$

Proof. From now on we will use $F^{-1}(p)$ instead of $\text{VaR}_p(X)$ (see Definition 1.35). Let us recall that if we consider a vector (X_1, X_2) with marginal distribution functions F_1, F_2 and $U_i = F_i(X_i)$ for $i = 1, 2$, we can write:

$$\begin{aligned} \text{MME}_p^I[X_1|X_2] &= \int_{F_2^{-1}(p)}^{\infty} \bar{F}_1(x) dx, \\ \text{MME}_p^M[X_1|X_2] &= \frac{1}{1-p} \int_{\max(F_1(F_2^{-1}(p)), p)}^1 (F_1^{-1}(u) - F_2^{-1}(p)) du, \\ \text{MME}_p[X_1|X_2] &= \int_{F_1(F_2^{-1}(p))}^1 (F_1^{-1}(u) - F_2^{-1}(p)) f_{U_1|U_2 \geq p}(u) du \\ &= \int_{F_2^{-1}(p)}^{\infty} \bar{F}_{X_1|X_2 \geq F_2^{-1}(p)}(x) dx. \end{aligned}$$

Taking into account that both the components X_1 and X_2 have the same marginal distribution functions, it reduces to:

$$\begin{aligned} \text{MME}_p^I[X_1|X_2] &= \int_{F^{-1}(p)}^{\infty} \bar{F}(x) dx, \\ \text{MME}_p^M[X_1|X_2] &= \frac{1}{1-p} \int_p^1 (F^{-1}(t) - F^{-1}(p)) dt, \\ \text{MME}_p[X_1|X_2] &= \int_{F^{-1}(p)}^{\infty} \bar{F}_{X_1|X_2 \geq F^{-1}(p)}(x) dx. \end{aligned} \quad (2.31)$$

For simplicity of notation, let us just call D to the denominator and N to the numerator of (2.28). The expression of D reduces to

$$\begin{aligned}
 D &= \text{MME}_p^M [X_1|X_2] - \text{MME}_p^I [X_1|X_2] \\
 &= \frac{1}{1-p} \int_p^1 (F^{-1}(t) - F^{-1}(p)) dt - \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= \frac{1}{1-p} \int_{F^{-1}(p)}^\infty \bar{F}(x) dx - \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= \frac{p}{1-p} \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= \frac{p}{1-p} E[(X - F^{-1}(p))^+].
 \end{aligned}$$

And, in the numerator, we get

$$\begin{aligned}
 N &= \text{MME}_p [X_1|X_2] - \text{MME}_p^I [X_1|X_2] \\
 &= \int_{F^{-1}(p)}^\infty \bar{F}_{X_1|X_2 \geq F^{-1}(p)}(x) dx - \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= \int_{F^{-1}(p)}^\infty \left(\frac{\bar{C}(F(x), p) - \bar{F}(x)(1-p)}{1-p} \right) dx.
 \end{aligned}$$

It can be simplified into

$$\begin{aligned}
 N &= \int_{F^{-1}(p)}^\infty \left(\frac{1 - F(x) - p + C(F(x), p) - \bar{F}(x) + \bar{F}(x)p}{1-p} \right) dx \\
 &= \frac{1}{1-p} \int_{F^{-1}(p)}^\infty (-p + C(F(x), p)) dx + \frac{p}{1-p} \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= - \int_{F^{-1}(p)}^\infty \frac{p - C(F(x), p)}{1-p} dx + \frac{p}{1-p} \int_{F^{-1}(p)}^\infty \bar{F}(x) dx \\
 &= \frac{-p}{1-p} \int_{F^{-1}(p)}^\infty 1 - F_{X_1|X_2 < F^{-1}(p)}(x) dx + \frac{p}{1-p} E[(X_1 - F^{-1}(p))^+] \\
 &= \frac{-p}{1-p} E[(X_1 - F^{-1}(p))^+ | X_2 < F^{-1}(p)] + \frac{p}{1-p} E[(X_1 - F^{-1}(p))^+].
 \end{aligned}$$

So that, as we wanted to see, (2.29) holds. Considering $t = F^{-1}(p)$, is direct to see that (2.30) holds too.

Fixed the marginal distributions, in Remark 2.6 we saw that, for bivariate vectors, the ordering of the Marginal Mean Excesses is consistent with respect to the concordance order. Hence, the order of the index is also consistent with respect to the concordance order. It is direct to see that, for all $p \in (0, 1)$, for comotonic vectors, we get that $I_{(C,F)}(p) = 1$ and for independent vectors, $I_{(C,F)}(p) = 0$. Therefore, among PQD copulas, when the index value is close to zero, there is low contagion and a high level of contagion can be addressed when the index takes values close to one.

In addition, we can study which would be the lower value that, for any bivariate copula, the index can take. As already stated, such bound is reached when we consider the countermonotonic copula. In order to obtain such lower bound, we must distinguish different cases. In the integral that appears in (2.31), $x \in (F^{-1}(p), \infty)$. Hence, depending if $F^{-1}(p) > F^{-1}(1-p)$ or not, the integral would be split or not. $F^{-1}(p) > F^{-1}(1-p)$ if and only if $p > 1-p$, which occurs if $p > \frac{1}{2}$. Therefore, we distinguish the value of the upper bound in terms of p .

Case 2.26 (Lower bound on $I_{(C,F)}(p)$ when $p < \frac{1}{2}$). *Taking into account that:*

$$\int_{F^{-1}(p)}^{\infty} \left(1 - \frac{\max\{F(x) + p - 1, 0\}}{p} \right) dx = \int_{F^{-1}(p)}^{F^{-1}(1-p)} dx + \int_{F^{-1}(1-p)}^{\infty} \frac{\bar{F}(x)}{p} dx,$$

one can easily check that

$$\begin{aligned} E[(X_1 - F^{-1}(p))^+ | X_2 < F^{-1}(p)] &= \int_{F^{-1}(p)}^{\infty} (1 - F_{X_1|X_2 < F^{-1}(p)}(x)) dx \\ &= \int_{F^{-1}(p)}^{\infty} 1 - \frac{C(F(x), p)}{p} dx \\ &= \int_{F^{-1}(p)}^{\infty} \frac{p - \max\{F(x) + p - 1, 0\}}{p} dx \\ &= \Delta_p + \frac{1}{p} E[(X_1 - F^{-1}(p))^+], \quad (2.32) \end{aligned}$$

where $\Delta_p = F^{-1}(1-p) - F^{-1}(p)$. Therefore, when $p < \frac{1}{2}$, we have that, for any copula C and marginal F ,

$$I_{(C,F)}(p) \geq \frac{(F^{-1}(p) - F^{-1}(1-p))}{E[(X - F^{-1}(p))^+]} - \frac{1-p}{p}. \quad (2.33)$$

And the lower bound (2.33) is reached when considering the countermonotonic copula.

Case 2.27 (Lower bound on $I_{(C,F)}(p)$ when $p \geq \frac{1}{2}$). *Analogously as the previous case, we can now consider $p \geq \frac{1}{2}$ and, proceeding as in (2.32), it can be checked that:*

$$E[(X_1 - F^{-1}(p))^+ | X_2 < F^{-1}(p)] = \frac{1}{p} E[(X_1 - F^{-1}(p))^+]$$

Hence, when $p \geq \frac{1}{2}$, for any copula C and marginal F ,

$$I_{(C,F)}(p) \geq -\frac{1-p}{p}. \quad (2.34)$$

With the lower bound (2.34) being reached when C is the countermonotonic copula.

2.4.1 Properties of the index

An exhaustive study of the properties of the index remain for future research. Although, two important properties will be here stated. They are related with the fact that, among all PQD copulas, the ones whose value $I_{(C,F)}(p)$ reaches the bounds 0 or 1, are determined in the half-plane $U_1 > U_2$.

Lemma 2.28. *Let $\mathbf{X} = (X_1, X_2)$ be a random vector verifying Assumption 2.1 with PQD copula C , joint distribution and survival functions $\mathbf{F}$ and $\bar{\mathbf{F}}$ and marginal distribution F for X_1 and X_2 . Then, considering the index defined in (2.4):*

- (i) *If, for all $p \in (0, 1)$, $I_{(C,F)}(p) = 0$, then $C(u, v) = uv$ for all $u, v \in [0, 1]$ such that $u \geq v$.*
- (ii) *If, for all $p \in (0, 1)$, $I_{(C,F)}(p) = 1$, then $C(u, v) = \min\{u, v\}$ for all $u, v \in [0, 1]$ such that $u \geq v$.*

Proof. Given $x > 0$, it is direct to check that:

$$\begin{aligned} \bar{F}_{\{(X_1-t)^+ | X_2 < t\}}(x) &= P[(X_1 - t)^+ > x | X_2 < t] = \\ \frac{P[X_1 > x + t, X_2 < t]}{P[X_2 < t]} &= 1 - \frac{\mathbf{F}(x + t, t)}{F(t)}. \end{aligned}$$

For a given value $p \in (0, 1)$, taking $t = F^{-1}(p)$, note that $I_{(C,F)}(p) = 0$ if and only if $E[(X_1 - t)^+ | X_2 < t] = E[(X_1 - t)^+]$. It is direct to see that this happens if and only if

$$\int_0^\infty \left(F(x + t) - \frac{\mathbf{F}(x + t, t)}{F(t)} \right) dx = 0.$$

Since the copula is PQD , the argument of the integral has a constant sign. Therefore, as the is null, so should be the argument. Replacing $u = F(t + x)$ and $v = F(t)$, the assertion 2.28(i) follows. Analogously, $I_{(C,F)}(p) = 1$ if and only if $E[(X_1 - t)^+ | X_2 < t] = 0$, which occurs if and only if

$$\int_0^\infty \left(1 - \frac{\mathbf{F}(x + t, t)}{F(t)} \right) dx = 0.$$

Since it is clear that, for all $x, t \in \mathbb{R}$, $\mathbf{F}(x + t, t) \leq F(t)$. Therefore, the integral is zero if, for all $x, t \in \mathbb{R}$, we have $\mathbf{F}(x + t, t) = F(t)$. Replacing again $u = F(t + x)$ and $v = F(t)$, the assertion 2.28(ii) follows.

2.5 A real numerical example

As an empirical illustration, we apply the methodology based on stochastic orders to compare the risk of contagion in the context of Spanish banking sector. We consider three of the largest banks operating in Spain, namely, the Santander bank, the BBVA and Bankinter. At the end of March, 2019, the Santander bank is the largest one in terms of assets, deposits, loans, number of branches and employers. It is also one of the seven banks in the euro area labelled by the Financial Stability Board as global systemically important banks. This means that distress of this bank could have a significant impact on economic activity and financial system. The BBVA is the second largest bank in Spain in terms of assets and Bankinter is the sixth one. Our aim is to compare how larger positions of Santander affect the behavior of BBVA and Bankinter.

We follow the common approach of testing for contagion based on the analysis of interaction among asset log returns, which are weekly logarithm price differentials. If we denote as p_t the price of an asset at week t , the log return at week t is defined by $r_t = \log(p_t/p_{t-1})$. Our model considers two vectors of log returns, $\mathbf{X} = (X_{BV}, X_S)$, where X_{BV} and X_S represent the log returns of BBVA and Santander, respectively and $\mathbf{Y} = (Y_{BK}, Y_S)$ where X_{BK} and Y_S represent the log returns of Bankinter and Santander, with the assumption $X_S = Y_S$. We have used samples of size $n = 209$ for each financial institution, measuring the share value from June 2015 until June 2019. Data were gathered from the public website <http://es.finance.yahoo.com> and they are also available from authors upon request. In order to eliminate the time dependent effect, data are related to the weekly close of trading.

Figure 2.7 plots the sample marginal mean excess of BBVA on Santander (blue curve) and the sample marginal mean excess of Bankinter on Santander (green curve) as a function of the p -level. The empirical measures generated from the samples suggest that $\text{MME}_p[X_{BK}|X_S] \leq \text{MME}_p[X_{BV}|X_S]$ for $p \geq p_0$, where p_0 is close to 0.3, whereas for p -levels lower than p_0 the curves cross each other slightly, experiencing very small differences that could be due to sampling variability. Consequently, we use statistical inference to check the hypothesis

$$\text{MME}_p[X_{BK}|X_S] \leq \text{MME}_p[X_{BV}|X_S], \text{ for all } 0 < p < 1. \quad (2.35)$$

We first perform some tests to study the marginal distributions of X_S , X_{BV} and Y_{BK} . For the classical runs test for randomness, the p -values are 0.6276,

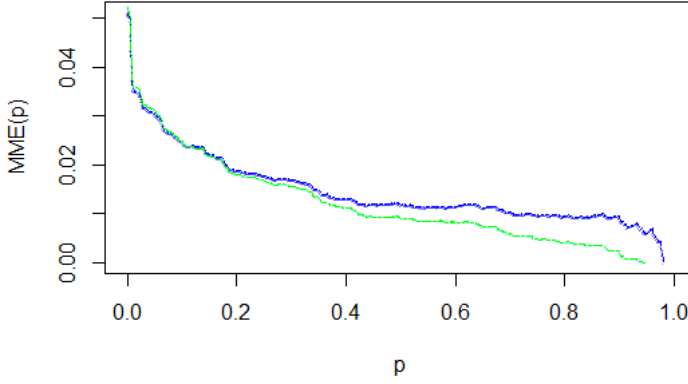


Figure 2.7. *Marginal mean excess in function of the p -level. The blue curve corresponds to $\mathbf{X} = (X_{BV}, X_S)$ and the green curve to $\mathbf{Y} = (Y_{BK}, Y_S)$.*

0.6272 and 0.945, respectively. For the Kolmogorov-Smirnov test for normality, the p -values are 0.7431, 0.9804 and 0.6482, respectively. We conclude that there is not significant evidence to reject the hypothesis that the three log return distributions are random, symmetric and normal.

Next, we compare means and standard deviations of X_{BV} and Y_{BK} . An unilateral F-test for paired data performed for testing the hypothesis of equality of variances against $\sigma_{BK} < \sigma_{BV}$ gives a p -value of 0.000014, showing significant evidence that $\sigma_{BK} < \sigma_{BV}$. The p -value of the t-test for testing $\mu_{BV} = \mu_{BK}$ against $\mu_{BV} \neq \mu_{BK}$ is 0.8593, so we can not reject the equality of means. From the assumptions of normality, $\mu_{BV} = \mu_{BK}$ and $\sigma_{BK} < \sigma_{BV}$, it follows $Y_{BK} \leq_{icx} X_{BV}$ (see Table 2.2 in Belzunce et al. (2015)).

Finally, we adjust the copulas C and C' of the vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively, by using the goodness of fit test based on the Rosenblatt transformation (see Genest et al. (2009)). This test compares the empirical copula with a parametric estimation of a given copula. The elliptical family of copulas is one of the most extensively applied models in finance (see, for example, Owen and Rabinovitch (1983)) and the normal distribution is the archetype of this family. By considering a bivariate normal copula we obtain p -values 0.6079 and 0.9196, respectively, so there is not statistical evidence to reject the null hypothesis. For a bivariate random vector $\mathbf{X}$ the bivariate normal

copula parameter is the Pearson correlation coefficient $\rho_{\mathbf{X}}$. We perform the Williams's Test (unilateral) (Steiger (1980) and Williams (1959)) for testing the hypothesis of equality of these coefficients against $\rho_{\mathbf{X}} < \rho_{\mathbf{Y}}$ when the vectors share one component ($X_S = Y_S$). A p -value of 0.010 was obtained, so there is significant evidence that $\rho_{\mathbf{X}} < \rho_{\mathbf{Y}}$. This implies that $C \leq_c C'$ (see Example 3.8.6 in Müller and Stoyan (2002)). Reasoning as in Example 2.12, it follows from Theorem 12 of Sordo et al. (2018) and Theorem 2.2(ii) that (2.35) is supported by statistical significance, which indicates that higher positions of Santander affect more to BBVA than to Bankinter.

2.6 Conclusions

Conditional risk measures (or co-risk measures) and risk contributions measures are used in portfolio risk analysis to quantify the risk of contagion given that one or more assets in the portfolio are in distress. Although these measures can be used directly to compare the systemic risk among different portfolios, there are several issues that hinder the process. One issue is that, sometimes, these measures cannot be calculated explicitly. Another is that different stress scenarios may produce different conclusions, something that happens, for example, when $\text{CoVaR}_{\mathbf{p}}^> [X_1|X_2]$ and $\text{CoVaR}_{\mathbf{p}}^> [Y_1|Y_2]$ produce different orderings for different choices of $\mathbf{p} \in [0, 1]^2$.

To overcome these issues, given the vectors $\mathbf{X}$ and $\mathbf{Y}$, we have provided sufficient conditions, in terms of well-known stochastic orders and dependence notions, under which the multivariate $\text{CoVaR}_{\mathbf{p}}^>$ s, CoESs and MMEs are ordered whatever the probability level used to define the stress scenario. While some results concerning $\text{CoVaR}^>$ and CoES have extended to a multivariate setting the bivariate results obtained by Sordo et al. (2018), all results concerning MMEs are new even for the bivariate case. The results have been illustrated with examples based on parametric families of multivariate distributions and with real data.

“To be is to be a value of a variable”.

Willard Van Orman Quine

3

Comparisons of VaR and CoVaR in terms of the value of the conditional variable.

Given a random vector $\mathbf{X} = (X, Y)$ and the risk levels $v \in [0, 1]$ and $u \in [0, 1]$, in order to study the risk of the component Y , two well-known measures of risk are $\text{VaR}_v(Y)$ and $\text{CoVaR}_{v,u}^-(Y|X)$, that stands for $\text{VaR}_v(Y|X = \text{VaR}_u(X))$. We compare the risk measures $\text{VaR}_v(Y)$ and $\text{CoVaR}_{v,u}^-(Y|X)$ in terms of the risk-level of the conditional variable, u . When $Y \uparrow_{SI} X$, sufficient conditions are provided, in terms of the copula, under which there exists $u^* \in (0, 1)$ such that, for all $u \geq u^*$, $\text{CoVaR}_{v,u}^-(Y|X)$ lies above $\text{VaR}_v(Y)$ for all $v \in (0, 1)$. The analytical expression of such u^* is provided for several bivariate copula families.

3.1 Introduction

The study of risk measures to determine capital-at-risk has generated considerable attention in the economic and actuarial literature. We recall that capital-at-risk refers to the amount of cash a company should hold in reserve to cover unforeseen losses, a capital threshold that will not be likely trespassed. We assume in this chapter that capital-at-risk calculation follows a value-at-risk approach. Since the Basel Committee on Banking Supervision adopted it in the nineties of the previous century, the value-at-risk (VaR) has become the predominant measure of risk and has been employed in regulatory frameworks of banking (Basel III/IV) and insurance (Solvency II).

We recall that, given a random risk Y with distribution function F_Y the value-at-risk (VaR) of Y at level $v \in (0, 1)$, given by $\text{VaR}_v[Y]$, (see Definition 1.35) is just the inverse (or quantile function) of F_Y at v . It represents “the worst expected loss over a given horizon under normal market conditions at a

Chapter 3. Comparisons of VaR-CoVaR in terms of the conditional variable.

given level of confidence” and, as it was addressed in Section 1.5.1, its popularity is mainly due to its simplicity and ease of computation. However, as it has been widely recognized, the VaR possesses some limitations. In a multivariate risk scenario, one of its main disadvantages comes from not considering the system as a whole or the interactions with other risks. One possibility to take care of these interactions is to calculate a conditional VaR that captures contagion effects and tail dependence among risks. Since the first version in the paper by Adrian and Brunnermeier (2016), an active area of research has been devoted to the quantification, management and comparison of contagion risk measures (see, for example, Mainik and Schaanning (2014), Sordo et al. (2015b), Acharya et al. (2017), Sordo et al. (2018), Das and Fasen-Hartmann (2018a), Ortega-Jiménez et al. (2021) and Dhaene et al. (2022)). In particular, Adrian and Brunnermeier (2016) considered the dependence-adjusted version of the VaR called $\text{CoVaR}^=$ considered in Definition 1.37. We recall that, formally, the co-value-at-risk ($\text{CoVaR}^=$) of Y at level $v \in (0, 1)$, given that X is at level $u \in (0, 1)$, denoted by $\text{CoVaR}_{v,u}^=[Y|X]$, is defined by the VaR of the conditional variable $[Y|X = \text{VaR}_u[X]]$ at level risk v as follows

$$\text{CoVaR}_{v,u}^=[Y|X] = \text{VaR}_v[Y|X = \text{VaR}_u[X]] = F_{Y|X=\text{VaR}_u[X]}^{-1}(v).$$

Although risk-capital calculation is based on the unconditional VaR, it would be prudent to combine VaR and $\text{CoVaR}^=$ approaches to determine provisions. This strategy puts the focus on the risk level of the conditional risk such that VaR and $\text{CoVaR}^=$ coincide, an issue to which we will refer to as “the balance of the replacement of VaR by $\text{CoVaR}^=$ ”. To fix ideas, suppose that the risk-capital calculation is based on the VaR at level v of Y , for some $v \in (0, 1)$. If, given a risk X and $u \in [0, 1]$, $\text{CoVaR}_{v,u}^=[Y|X] < \text{VaR}_v[Y]$ holds, replacing VaR by CoVaR does not make sense as CoVaR contemplates a more optimistic scenario. However, if $\text{CoVaR}_{v,u}^=[Y|X] > \text{VaR}_v[Y]$ holds, then $\text{VaR}_v[Y]$ underestimates the spillover effect and it is prudent to replace it with $\text{CoVaR}_{v,u}^=[Y|X]$ as a more conservative provision to determine the capital at risk. The point in question here is the existence of a risk-level threshold $u_v \in (0, 1)$, as a function of v , such that $\text{CoVaR}_{v,u_v}^=[Y|X] = \text{VaR}_v[Y]$. This threshold determines, in practice, the balance of the replacement. One of the primary purposes of this chapter is to analyze such a point and its properties.

From now on, for the sake of simplicity, any random vector (X, Y) is assumed to have absolutely continuous strictly increasing marginal distributions $F_X(\cdot)$ and $F_Y(\cdot)$ and strictly increasing conditional distribution $F_{Y|X=x}(\cdot)$, for all x , on their respective supports. These assumptions will be referred to as regularity conditions. If K is the joint distribution of the random vector

(X, Y) , by the Sklar's theorem (see Theorem 1.23), we can write $K(x, y) = C(F_X(x), F_Y(y))$, where C is the copula, that is, the joint distribution function of the vector-copula (U, V) given by $U = F_X(X)$ and $V = F_Y(Y)$. Under the regularity conditions, the copula is unique and differentiable.

In finance and insurance, decision makers are often interested in positive dependence structures among risks. We mean by positive dependence that X and Y are likely to be large or to be small together (see Section 1.3.3). In particular, in this chapter, we focus on the positive regression dependence (PRD) concept introduced by Lehmann (1966), also called stochastically increasing (SI) (see Definition 1.28). We recall that, considering a random vector (X, Y) with copula C , say that Y is SI in X , in short $Y \uparrow_{SI} X$, if

$$[Y \mid X = x_1] \leq_{st} [Y \mid X = x_2], \text{ for all } x_1 \leq x_2. \quad (3.1)$$

In some sense, if $Y \uparrow_{SI} X$ we expect that Y is likely to take on large values when the conditional random variable X increases. It is clear from (1.14) that (3.1) holds if and only if the survival probability $P[Y > y \mid X = x]$ is an increasing function of x , for all y . As highlighted in Definition 1.29, it is well-known that the SI concept is characterized by the copula. From Nelsen (2007), $Y \uparrow_{SI} X$ is equivalent to check whether the partial derivative $\partial_1 C(u, v) = P[Y \leq F_Y^{-1}(v) \mid X = F_X^{-1}(u)]$ is a decreasing function of u , for all v . Therefore, many authors refer to a vector (X, Y) having a SI copula. As seen in Definition 1.29, if $Y \uparrow_{SI} X$ and $X \uparrow_{SI} Y$, we say that the vector (X, Y) is positively dependent through stochastic ordering (PDS).

The following two remarks will be useful in the sequel.

Remark 3.1. *Note that the partial derivative $\partial_1 C(u, v)$ can be interpreted as the conditional distribution of the vector-copula (U, V) , given by $\partial_1 C(u, v) = P[V \leq v \mid U = u]$ (see Nelsen (2007)).*

Remark 3.2. *The $\text{CoVaR}_{v,u}^{\leftarrow}[Y \mid X]$ is increasing in u , for all v , if and only if the copula C is SI (see Fernández-Ponce et al. (2011)). Throughout the chapter, we will only consider the case where $Y \uparrow_{SI} X$ in a strict sense, i.e., when the conditional distribution $P[Y > y \mid X = x]$ strictly increases in x , for all y . We will denote this property by $Y \uparrow_{SSI} X$ (strict SI property). Note that we consciously avoid the problem of having an independent structure which, in some sense, can be considered as an extreme positive dependence structure.*

We are ready to define the Probability Equivalent Level of CoVar-VaR at risk level v (PELCoV_v). Since we are interested in the case where X and Y are

positively dependent, we restrict the definition to random vectors satisfying the SSI property.

Definition 3.3. Let (X, Y) be a random vector having a copula C and verifying the regularity conditions such that $Y \uparrow_{SSI} X$ and let assume that $\partial_1 C(u, v)$ is continuous u , for all v . Given $v \in (0, 1)$, the Probability Equivalent Level of CoVar-VaR at risk level v ($PELCoV_v$) is defined as the unique risk level $u_v \in (0, 1)$ for X such that $\text{CoVaR}_{v, u_v}^{\equiv}[Y|X] = \text{Var}_v[Y]$.

Remark 3.4. The continuity assumption in Definition 3.3 guarantees the existence of $PELCoV_v$, whereas the SSI structure ensures uniqueness. We address both existence and uniqueness in Corollary 3.9 below. It is remarkable that the $PELCoV_v$ depends only on the copula, as seen in Lemma 3.6 below.

Remark 3.5. The notation $PELCoV_v$ is clearly inspired in Li and Wang (2022). Those authors study the balancing point for the equivalence between the Value-at-risk of Y and the Expected Shortfall of Y (ES). The ES is a well-known risk measure that seeks to convey information regarding the exceedances of the risk beyond its VaR and, in consequence, is more conservative than the VaR in the context of economic capital computation. The expected shortfall presents the advantage of being a coherent risk measure in the sense of Artzner et al. (1999) and has recently replaced the VaR as the standard risk measure for market risk in the recent Fundamental Review of the Trading Book (FRTB), as referred by Basel Committee on Banking Supervision (BCBS (2019)). Although, it presents some undesirable features since it is harder to interpret with respect to the risk-level and the difficultness in the computation. The interested reader is referred to Fiori and Rosazza Gianin (2022) and Barczy et al. (2022) for further details on the topic.

In order to clarify the exposition, we present in Figure 3.1 an illustration of the $u_v = PELCoV_v$ for a bivariate normal vector (X, Y) with $\mu_X = 10000$, $\sigma_X = 1000$, $\mu_Y = 40000$, $\sigma_Y = 2500$ and $\rho = 0.9$ (the positive correlation coefficient guarantees that $Y \uparrow_{SSI} X$). Details on the $PELCoV_v$ computation for a bivariate normal vector can be found in Section 3.4.2.1. Note that u_v only depends on the copula (that is, on ρ). The graphic plots u_{v_i} , $i = 1, 2, 3$, for $v_1 = 0.4, v_2 = 0.7$ and $v_3 = 0.95$, where u_{v_i} is the abscissa of the intersection point between the conditional quantile curve $u \mapsto \text{CoVaR}_{v, u}^{\equiv}[Y|X]$ and the line $y = \text{Var}_v[Y]$. Given $v \in [0, 1]$, we see from the graphic that

$$u_v = \min_{u \in [0, 1]} \{ \text{CoVaR}_{v, u'}^{\equiv}[Y|X] \geq \text{Var}_v[Y], \text{ for all } u' > u \},$$

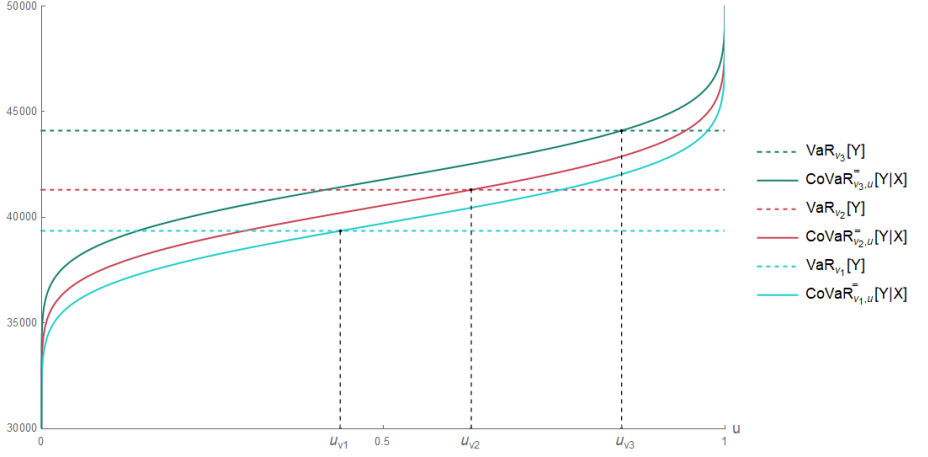


Figure 3.1. $PELCoV_v$ values for $v_1 = 0.4, v_2 = 0.7$ and $v_3 = 0.95$ for a normal vector (X, Y) with $\mu_X = 10000$, $\sigma_X = 1000$, $\mu_Y = 40000$, $\sigma_Y = 2500$ and $\rho = 0.9$.

representing the lowest risk level u of X from which $CoVaR_{v,u}^-[Y|X]$ will be a more conservative reserve position than $VaR_v[Y]$. For the bivariate normal distribution shown in Figure 3.1, the $PELCoV_v$ is an increasing function of v and tends to 1 as v goes to 1. However, this is not always the case: the $PELCoV_v$ is not necessarily increasing in v , as we show in Section 3.4. The existence of an upper bound for the $PELCoV_v$ deserves further discussion. If there exists a $u^* < 1$ such $u_v \leq u^*$ for all v , then

$$\begin{aligned} CoVaR_{v,u}^-[Y|X] &\geq CoVaR_{v,u^*}^-[Y|X] \\ &\geq CoVaR_{v,u_v}^-[Y|X] \\ &= VaR_v[Y], \end{aligned}$$

for all $v \in (0, 1)$ and for any risk-level u in the interval $[u^*, 1)$, where the inequalities follow from the fact that $CoVaR_{v,u}^-[Y|X]$ is increasing in u , as noted in Remark 3.2, and the equality is by definition of u_v . Using Remark 1.15, this can be rewritten as

$$Y \leq_{st} [Y|X = F_X^{-1}(u)], \text{ for all } u \in [u^*, 1). \quad (3.2)$$

Note that comparisons of the form (3.2) are discussed in Navarro and Sordo (2018). We interpret (3.2) as follows. Given $v \in (0, 1)$, if X reaches its quantile u^* , $VaR_v[Y]$ underestimates the risk spillover effect and it is more prudent to determine reserves in terms of $CoVaR_{v,u}^-[Y|X]$, for all $u > u^*$. This suggests

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to consider u^* as an alert for possible undervaluation of the risk. We provide in Theorem 3.18 sufficient conditions for the existence of such u^* .

The rest of the chapter is organized as follows. In Section 3.2, we give sufficient conditions for the existence of the PELCoV_v and show some of its properties. In Section 3.3, we study the monotonicity of PELCoV_v as a function of v as well as the existence of the upper bound u^* . In Section 3.4, we study the PELCoV_v for several parametric copula models. Section 3.5 is devoted to study the PELCoV_v and its boundness for vectors of consecutive record values. Section 3.6 contains an empirical parametric illustration dealing with two companies from IBEX35: International Consolidated Airlines Group S.A. and Meliá Hotels International S.A. Finally, conclusions and further work are addresses in Section 3.7.

3.2 On the definition of the PELCoV_v .

Note that, in Definition 3.3, the PELCoV_v is defined under some conditions on the dependence structure. In this section, we will see how these conditions guarantee the existence and uniqueness of the PELCoV_v and, therefore, it is well defined. Firstly, we will show that under the regularity conditions, the underlying comparisons between the CoVaR and VaR just depend on the copula.

Lemma 3.6. *Let (X, Y) be a random vector under the regularity conditions having a copula C . Given two risk-levels $v, u \in (0, 1)$, the $\text{CoVaR}_{v,u}^{\bar{=}}[Y | X]$ is greater, smaller or equal than the $\text{VaR}_v[Y]$ if, and only if the partial derivative $\partial_1 C(u, v)$ is smaller, greater or equal than v , respectively. Therefore, the PELCoV_v only depends on the copula.*

Proof. The proof is straightforward from the regularity conditions and using Remark 3.1.

Lemma 3.7. *Let (X, Y) be a random vector under the regularity conditions having a copula C . Given a particular risk-level $v \in (0, 1)$, the $\text{CoVaR}_{v,u}^{\bar{=}}[Y | X]$ is continuous in $u \in (0, 1)$, for all $v \in (0, 1)$ if, and only if the partial derivative $\partial_1 C(u, v)$ is continuous in $u \in (0, 1)$, for all $v \in (0, 1)$.*

Proof. We only prove the “if” part. The other part is analogous. Given an $\epsilon > 0$ and $u_0 \in (0, 1)$, we obtain that the inequality

$$|\text{CoVaR}_{v,u}^{\bar{=}}[Y | X] - \text{CoVaR}_{v,u_0}^{\bar{=}}[Y | X]| < \epsilon,$$

is equivalent to

$$\partial_1 C(u, F_Y(y_1)) < v < \partial_1 C(u, F_Y(y_2)), \quad (3.3)$$

where $y_1 = \text{CoVaR}_{v, u_o}^=[Y | X] - \epsilon$ and $y_2 = \text{CoVaR}_{v, u_o}^=[Y | X] + \epsilon$. Under the regularity conditions and using the hypothesis assumption,

$$\lim_{u \rightarrow u_o} \partial_1 C(u, F_Y(y_1)) = \partial_1 C(u_o, F_Y(y_1)) < v, \quad (3.4)$$

where the last inequality follows from the fact that the partial derivative $\partial_1 C(u, v)$ is strict increasing in v . Analogously, we obtain that

$$\lim_{u \rightarrow u_o} \partial_1 C(u, F_Y(y_2)) = \partial_1 C(u_o, F_Y(y_2)) > v. \quad (3.5)$$

From (3.4) and (3.5), we easily obtain (3.3) in an interval $|u - u_o| < \delta$.

The Proposition 3.8 plays an important role to ensure the existence of the PELCoV_v which is stated in Corollary 3.9.

Proposition 3.8. *Let (X, Y) be a random vector under the regularity conditions having a copula C . Given a risk-level $v \in (0, 1)$, there exist $u_v^+, u_v^- \in (0, 1)$ such that $\text{CoVaR}_{v, u_v^-}^=[Y|X] \leq \text{VaR}_v[Y] \leq \text{CoVaR}_{v, u_v^+}^=[Y|X]$.*

Proof. By reduction to absurd, given a fix level $v \in (0, 1)$ we will assume that the values $u_v^+, u_v^- \in (0, 1)$ do not exist. Therefore, the conditional quantile curve $u \mapsto \text{CoVaR}_{v, u}^=[Y|X]$ lies strictly either above or below the $\text{VaR}_v[Y]$. We will assume that the curve lies strictly above where if it lies strictly below the proof follows by using an equivalent argument. From Lemma 3.6, the inequality $\text{CoVaR}_{v, u}^=[Y|X] > \text{VaR}_v[Y]$, for all $u \in (0, 1)$, is equivalent to check that $v > \partial_1 C(u, v)$, for all $u \in (0, 1)$. Therefore, we obtain that

$$v = \int_0^1 v du > \int_0^1 \partial_1 C(u, v) du = C(1, v) = v,$$

which clearly leads to a contradiction and concludes the proof.

Proposition 3.8 is interesting in itself and also states that it does not exist any random vector (X, Y) under the regularity conditions where the conditional quantile curve $u \mapsto \text{CoVaR}_{v, u}^=[Y|X]$ lies strictly either above or below the $\text{VaR}_v[Y]$, for all u . By using Remark 1.15, it means that it is not possible to state the ordering $[Y|X = \text{VaR}_u[X]] \geq_{st} Y$ for all $u \in (0, 1)$, analogously for $[Y|X = \text{VaR}_u[X]] \leq_{st} Y$. This fact reinforces the interest of finding a bound for the PELCoV_v as mentioned in the introduction section.

Corollary 3.9. *Let (X, Y) be a random vector under the regularity conditions having a copula C where $\partial_1 C(u, v)$ is continuous in $u \in (0, 1)$, for all $v \in$*

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$(0, 1)$. Given a particular risk-level $v \in (0, 1)$ there exists at least a level u_v such that $\text{CoVaR}_{v, u_v}^-[Y|X] = \text{VaR}_v[Y]$. Furthermore, if $Y \uparrow_{SI} X$ then the $u_v = \text{PELCoV}_v$ is unique.

Proof. Using Proposition 3.8, jointly with Lemma 3.7, the result follows by applying the well-known Darboux property to the continuous function given by $u \mapsto \text{CoVaR}_{v, u}^-[Y|X]$ in the closed interval between the values u_v^- and u_v^+ . The uniqueness is a direct consequence of Remark 3.2.

We will present in Example 3.10 a situation where the continuity of $\partial_1 C(u, v)$ fails. Given a risk-level $v \in (0, 1)$, we will see that equation $\partial_1 C(u, v) = v$ does not hold for any level u .

Example 3.10. Let C be copula having density c as follows

$$c(u, v) = \begin{cases} 2 & \text{if } (u - 1/2)(v - 1/2) > 0, \\ 0 & \text{if } (u - 1/2)(v - 1/2) \leq 0. \end{cases}$$

A straightforward computation shows that

$$\partial_1 C(u, v) = \begin{cases} 2v & \text{if } 0 \leq u, v \leq 1/2, \\ 1 & \text{if } 0 \leq u \leq 1/2, 1/2 < v \leq 1, \\ 0 & \text{if } 1/2 < u \leq 1, 0 \leq v \leq 1/2, \\ 2v - 1 & \text{if } 1/2 < u, v \leq 1. \end{cases}$$

By construction, C is absolutely continuous and satisfies the regularity conditions. However, $\partial_1 C(u, v)$ is not continuous in $u \in (0, 1)$, for all $v \in (0, 1)$. In particular, it is always discontinuous at $u = 0.5$. It is apparent that $\partial_1 C(u, v) \neq v$ for any $u, v \in (0, 1)$. Hence the PELCoV_v never exists, for all $v \in (0, 1)$.

After this introduction on the PELCoV , we can address the potential usefulness of the concept. If $Y \uparrow_{SI} X$, then the Positive Quadrant Dependence (PQD) notion holds (see Nelsen (2007)). It implies that, given any risk-levels $u, v \in (0, 1)$,

$$P[Y > \text{VaR}_v[Y] \mid X > \text{VaR}_u[X]] \geq 1 - v.$$

Hence, X trespassing any quantile implies an increase in the probability of Y trespassing its capital-risk (determined by the VaR). A natural question arises

at this point: how can the PELCoV_v be a helpful tool? The answer follows from the fact that the PELCoV_v is the first value for which the following stochastic inequality holds:

$$\left[F_{Y|X}^{-1}(v) \mid X > VaR_u[X] \right] \geq_{st} VaR_v[Y], \text{ for all } u \geq u_v.$$

Note that in the previous inequality, the term of the right is a degenerated random variable. In other words, this inequality represents that if the “alert” variable trespasses its $VaR_u[X]$, we are certain that the unconditioned capital risk of Y is greater than the non-conditional capital risk. This stochastic inequality represents a notably more pessimistic scenario. The PELCoV_v is the first value of X that leads to such an alarmist framework.

Let us now consider how we can measure the extent of the risk implied in this scenario. Roughly speaking, the average added risk associated to the event $\{X > VaR_{u_v}[X]\}$ can be given by:

$$E \left[F_{Y|X}^{-1}(v) \mid X > VaR_{u_v}[X] \right] = \frac{1}{1 - u_v} \int_{u_v}^1 CoVaR_{v,u}^{\bar{=}}[Y|X] du, \quad (3.6)$$

which is obviously greater than $VaR_v[Y]$. The VaR is only reached under the extreme scenario of independence, where the “alarm” becomes irrelevant. If X and Y are comonotonic, (3.6) equals the expected shortfall, $ES_v[Y]$. In general, we can think on (3.6) as a modified $ES_v[Y]$ since we can rewrite the latter as

$$ES_v[Y] = E \left[F_{Y|Y}^{-1}(v) \mid Y > VaR_v[Y] \right].$$

Replacing the event $\{Y > VaR_v[Y]\}$ by $\{X > VaR_{u_v}[X]\}$, we obtain (3.6). Both measures convey information regarding the average value due to the exceedances of the risk beyond the VaR and both are greater than $VaR_v[Y]$.

Three main factors intuitively affect the problem of determining which positive dependence structure leads to a potentially helpful “alert”-value PELCoV_v. The first is to find an interpretative covariate X with a dependence structure with Y which can be easily determined. The second factor is the range of the “alert”-value, PELCoV_v. A greater PELCoV_v implies a lower probability (1-PELCoV_v) of occurrence. The third factor is the extent of the added risk due to trespassing the alarm value. It must be pointed out that the copula determines both the PELCoV_v and the monotonicity of $CoVaR_{v,u}^{\bar{=}}[Y|X]$ when u trespasses the PELCoV_v. However, greater positive dependence does not imply greater PELCoV_vs. One can check that the ordering of the PELCoV_vs is not consistent with the concordance nor the Copula Hazard Rate Ordering defined in Unnikrishnan Nair and Vineshkumar (2021).

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To sum up, the PELCoV_v serves us well if it is not too large and if the associated risk load is large enough to rectify a possible alarmist scenario.

Considering again the bivariate normal vector with $\rho = 0.4$, given $v = 0.95$, one can check that $\text{PELCoV}_{0.95} = 0.6344$ (see Section 3.4.2.1). We can also check that:

$$\begin{aligned} E \left[F_{Y|X}^{-1}(v) \mid X > \text{VaR}_{uv}[X] \right] &= \mu_Y + (\rho ES_{uv}[Z] + \sqrt{1 - \rho^2} \text{VaR}_v[Z]) \sigma_Y, \\ &= \mu_Y + k_{\rho,v} \sigma_Y \geq \mu_Y + \text{VaR}_v[Z] \sigma_Y, \end{aligned}$$

where $Z \sim N(0,1)$ and $k_{\rho,v}$ measures the added risk. One can compute $k_{0.4,0.95} = 1.9189$, which is significantly higher than $\text{VaR}_{0.95}[Z] = 1.6448$, which represents the added risk for the $\text{VaR}_{0.95}[Y] = \mu_Y + \text{VaR}_{0.95}[Z] \sigma_Y$. Therefore it implies an increase on the risk.

3.3 Boundness of the PELCoV_v.

To identify whether (3.2) holds, we must study sufficient conditions for the existence of an upper bound of the PELCoV_vs. To tackle this problem, a first approach consists of analyzing under which dependence structures the map $v \mapsto \text{PELCoV}_v$ is increasing. It will imply that the upper bound is given by $\lim_{v \rightarrow 1} \text{PELCoV}_v$.

Under the regularity conditions, the PELCoV_v is well defined as the unique value $u_v \in (0,1)$ such that $\partial_1 C(u_v, v) = v$. From this expression, and assuming $\partial_1 C(u, v)$ differentiable in both of its arguments, by the implicit function theorem, the $u_v = \text{PELCoV}_v$ is differentiable in terms of v . In addition, its derivative is given by:

$$\frac{du_v}{dv} = \frac{1 - c(u_v, v)}{\partial_{11} C(u_v, v)}. \quad (3.7)$$

If we consider a random vector (X, Y) where $Y \uparrow_{SSI} X$, $\partial_{11} C(u, v) < 0$ for all $u, v \in (0,1)^2$. Hence:

$$\frac{du_v}{dv} > 0 \iff c(u_v, v) > 1. \quad (3.8)$$

However, the study of the monotonicity of the PELCoV_v in terms of the condition given by (3.8) may not be the most advisable option since it requires knowing the analytical expression of the PELCoV_v. A sufficient condition can be given in terms of the dispersive order. We recall that X_1 is said to be smaller than X_2 in the dispersive order, denoted by $X_1 \leq_{disp} X_2$, if the distances between quantiles satisfies

$$F_{X_1}^{-1}(u_1) - F_{X_1}^{-1}(u_2) \leq F_{X_2}^{-1}(u_1) - F_{X_2}^{-1}(u_2) \text{ for all } 0 \leq u_1 < u_2 \leq 1.$$

Proposition 3.11. *Let (X, Y) be a random vector under the regularity conditions having a copula C such that $\partial_1 C(u, v)$ is continuous in both arguments and $Y \uparrow_{SSI} X$. If $[Y|X = F^{-1}(u)] \leq_{disp} Y$ for all $u \in (0, 1)$, then the PELCoV_v increases with respect to v .*

Proof. Let us assume that $[Y|X = F^{-1}(u)] \leq_{disp} Y$ for all $u \in (0, 1)$. By considering $v_1 < v_2$, we obtain that for all $u \in (0, 1)$

$$F_{Y|X=F^{-1}(u)}^{-1}(v_2) - F_{Y|X=F^{-1}(u)}^{-1}(v_1) \leq F_Y^{-1}(v_2) - F_Y^{-1}(v_1).$$

That is,

$$F_{Y|X=F^{-1}(u)}^{-1}(v_2) - F_Y^{-1}(v_2) \leq F_{Y|X=F^{-1}(u)}^{-1}(v_1) - F_Y^{-1}(v_1). \quad (3.9)$$

Inequality (3.9) holds for all $u \in (0, 1)$. Therefore, considering $u = \text{PELCoV}_{v_2}$, we have that for $v_1 < v_2$, $0 \leq F_{Y|X=F^{-1}(u_{v_2})}^{-1}(v_1) - F_Y^{-1}(v_1)$. Since $Y \uparrow_{SI} X$, then $u \mapsto F_{Y|X=F^{-1}(u)}^{-1}$ increases and $\text{PELCoV}_{v_1} \leq \text{PELCoV}_{v_2}$.

Example 3.12 (Bivariate Normal Distribution). *The fact that the PELCoV_v of a bivariate normal distribution having $\rho > 0$ is increasing in v is a direct consequence of Proposition 3.11. In that case, it is well-known that Y and $[Y|X = F^{-1}(u)]$ are normally distributed with variances σ_Y^2 and $\sigma_Y^2(1 - \rho^2)$, respectively. It is also well-known that two normal distributions satisfy the dispersive order if the variances are ordered in the same sense, see Table 2.2 in Belzunce et al. (2015).*

Remark 3.13. *It is well known that $\{Y|X = F^{-1}(u)\} \leq_{disp} Y$ for all $u \in (0, 1)$ if and only if $f_{\{Y|X=F^{-1}(u)\}}(F_{Y|X=F^{-1}(u)}^{-1}(v)) > f_Y(F_Y^{-1}(v))$ for all $u, v \in (0, 1)$. Considering the PELCoV_v, $u = u_v$, $\{Y|X = F^{-1}(u)\} \leq_{disp} Y$ implies that $f_{\{Y|X=F^{-1}(u_v)\}}(F_Y^{-1}(v)) > f_Y(F_Y^{-1}(v))$. Since $c(u_v, v) = \frac{f_{\{Y|X=F^{-1}(u_v)\}}(F_Y^{-1}(v))}{f_Y(F_Y^{-1}(v))}$, condition (3.8) also follows.*

We would like to emphasize that the ordering $[Y|X = F^{-1}(u)] \leq_{disp} Y$, for all $u \in (0, 1)$, corresponds to the conditional dispersive order defined in Belzunce et al. (2008). Here we compare two bivariate random vectors with the same marginal distributions F_X and F_Y , but having different copulas, C and the independence copula, respectively. For further details about multivariate dispersion orderings and the notion of copula we refer the reader to Belzunce et al. (2008), Fernández-Ponce and Suárez-Llorens (2003) and Arias-Nicolás et al. (2005). Next we show that this conditional dispersive ordering is also a necessary condition in some particular models.

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Proposition 3.14. *Let (X, Y) be a random vector under the regularity conditions such that $Y = \psi(X) + W$, where X and W are independent random variables and ψ is a strictly increasing function. Then the PELCoV_v increases with respect to v if and only if $W \leq_{\text{disp}} \psi(X) + W$. Moreover, this last inequality is always true when W has a logconcave density function.*

Proof. First, $Y \uparrow_{SSI} X$ trivially holds. From the location independent property of $\leq_{\text{disp}}$ it is apparent that the ordering $[Y|X = F^{-1}(u)] \leq_{\text{disp}} Y$, for all $u \in (0, 1)$, is equivalent to $W \leq_{\text{disp}} \psi(X) + W$. Therefore, the “if” part is straightforward just using Proposition 3.11. Now, we will prove the “only if” part. From the fact that $F_{Y|X=F^{-1}(u)}^{-1}(v) = F_Y^{-1}(v)$ holds if and only if $F_{\psi(X)+W}^{-1}(v) = F_W^{-1}(v) + \psi(F_X^{-1}(u))$ and using that ψ is strictly increasing we easily obtain the expression of $u_v = \text{PELCoV}_v$ given by

$$u_v = F_X \left(\psi^{-1} \left(F_{\psi(X)+W}^{-1}(v) - F_W^{-1}(v) \right) \right). \quad (3.10)$$

Since both ψ^{-1} and F_X are strictly increasing functions, if u_v increases with respect to v implies that $F_{\psi(X)+W}^{-1}(v) - F_W^{-1}(v)$ also increases in v , which is trivially equivalent to $W \leq_{\text{disp}} \psi(X) + W$. The rest of the proof is a direct consequence of Theorem 3.B.7 in Shaked and Shanthikumar (2007).

One could have the intuition that if $Y \uparrow_{SSI} X$ holds, the PELCoV_v always increases when v increases. Now we will provide a counter-example, based on the previous result, where this intuition is false.

Example 3.15. *Let us consider $(X, X + W)$ such that $X \sim \text{Exp}(1)$ and $W \sim \text{Weibull}(0.5, 1)$, where it is well-known and easy to check that W has not a logconcave density. Then*

$$\begin{aligned} F_{W+X}(x) &= \int_0^x F_W(x-y)f_X(y)dy = \int_0^x \left(1 - e^{-\sqrt{x-y}}\right) \cdot e^{-y} dy = \\ &= 1 - e^{-\sqrt{x}} - e^{-x} \cdot DF\left(\frac{1}{2}\right) + e^{-\sqrt{x}} DF\left(\frac{1}{2} - \sqrt{x}\right), \end{aligned}$$

where $DF(x)$ is the Dawson’s integral $DF(x) = e^{-x^2} \int_0^x e^{-y^2} dy$. We can obtain numerically the inverse of $F_{X+W}(x)$ and plot the PELCoV_v , whose expression is given by (3.10). We graphically see in Figure 3.2 that PELCoV_v is not always increasing in v and the maximum is reached at $v^* = 0.74816$ where $u_{v^*} = 0.764213$.

Bearing in mind that the PELCoV_v s do not have an upper bound if and only if $\sup_{v \in (0,1)} \text{PELCoV}_v = 1$, we wonder if such supremum can be reached in the open interval, $v \in (0,1)$. The answer is negative and follows from Proposition 3.8 as a corollary.

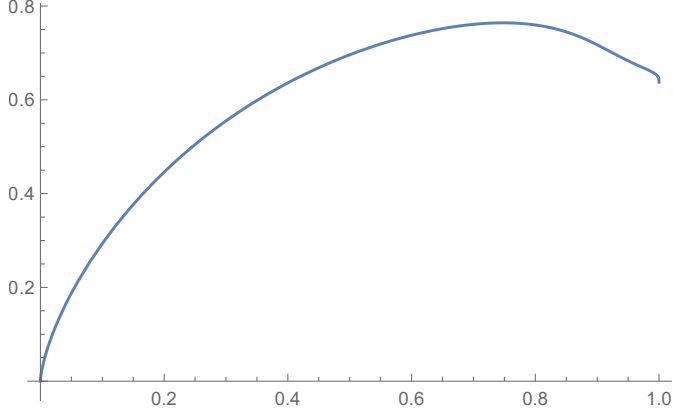


Figure 3.2. Function $v \mapsto \text{PELCoV}_v$ for the vector $(X, X+W)$ where $X \sim \text{Exp}(1)$ and $W \sim \text{Weibull}(0.5, 1)$.

Corollary 3.16. Let $\mathbf{X} = (X, Y)$ be a random vector, where $Y \uparrow_{SSI} X$, with copula C . Then, there is no $v \in (0, 1)$ such that $v = \partial_1 C(1, v)$.

Proof. Assume that there exists $v^* \in (0, 1)$ such that $v^* = \partial_1 C(1, v^*)$. Then, as $Y \uparrow_{SSI} X$, $\partial_1 C(u, v)$ decreases in u and, for all $u \in (0, 1)$, $v^* = \partial_1 C(1, v^*) < \partial_1 C(u, v^*)$. It contradicts Proposition 3.8.

As a direct consequence of this result, the value 1 can never be the PELCoV_v for $v \in (0, 1)$. As already mentioned, under the regularity conditions, the function $v \mapsto \text{PELCoV}_v$ is continuous. Therefore, Corollary 3.16 reduces the study on the existence of an upper bound to check how the PELCoV_v behaves in the extremes of the interval $v \in (0, 1)$. The corollary follows from the fact that $\sup_{v \in (0, 1)} \text{PELCoV}_v = 1$ can only hold if either $\lim_{v \rightarrow 0} \text{PELCoV}_v = 1$ or $\lim_{v \rightarrow 1} \text{PELCoV}_v = 1$.

Corollary 3.17. Let $\mathbf{X} = (X, Y)$ be a random vector, where $Y \uparrow_{SSI} X$, with continuously differentiable copula C . Then, the following assertions are equivalent:

1. There exists $u^* < 1$ such that $\text{PELCoV}_v \leq u^*$ for all $v \in (0, 1)$.
2. $\lim_{v \rightarrow 0} \text{PELCoV}_v < 1$ and $\lim_{v \rightarrow 1} \text{PELCoV}_v < 1$.

We can now formulate the main result of this section.

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Theorem 3.18. *Let $\mathbf{X} = (X, Y)$ be a random vector with copula C such that its density c is continuous, $Y \uparrow_{SI} X$. Let us assume that the following conditions hold:*

1. *There exists $0 < m < 1$ such that $c(u, v)$ increases in v for all $m \leq u < 1$ and for all $0 < v \leq u$.*
2. *The limit $\lim_{(u,v) \rightarrow (1,1)} c(u, v)$ exists and $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = l > 1$.*

Then there exists $u^* < 1$ such that $PELCoV_v \leq u^*$ for all $v \in (0, 1)$.

Proof. As $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = l > 1$, for all $l - 1 > 0$ there exists $\delta > 0$ such that, for all $u, v > 1 - \delta$, $c(u, v) > 1$. Therefore, it is clear that, for all $u, v > 1 - \delta$, $\frac{\int_v^1 c(u, z) dz}{1-v} > 1$, which is analogous as $\frac{\int_0^v c(u, z) dz}{v} < 1$, that is,

$$\frac{\partial_1 C(u, v)}{v} < 1 \text{ for all } u, v > 1 - \delta. \quad (3.11)$$

Therefore, it is clear that, for $v > 1 - \delta$, $u_v < 1 - \delta$.

Let us see that, for $u \geq m$ and $v \leq u$, as $c(u, v)$ increases in v , so does the mean value $\frac{\int_0^v c(u, s) ds}{v} = \frac{\partial_1 C(u, v)}{v}$.

In order to see that there exists a bound u^* , we would need to see, by (3.17), that $\lim_{v \rightarrow 0} u_v \neq 1$ and $\lim_{v \rightarrow 1} u_v \neq 1$. So, let us proceed by reduction to absurd and let us assume that $\lim_{v \rightarrow 0} u_v = 1$ or $\lim_{v \rightarrow 1} u_v = 1$.

Let us assume that $\lim_{v \rightarrow 1} u_v = 1$. Then, for $\delta > 0$, there exists $\lambda > 0$ such that, for all $v > 1 - \lambda$, $u_v > 1 - \delta$. Considering any $v > \max(1 - \lambda, 1 - \delta)$, it contradicts (3.11), so $\lim_{v \rightarrow 1} u_v \neq 1$.

Let us now assume that $\lim_{v \rightarrow 0} u_v = 1$. Then, for $\epsilon = \min(1 - m, \delta)$, there exists $\alpha > 0$ such that for all $v < \alpha$, $u_v > 1 - \epsilon/2$. So, for all $u < 1 - \epsilon/2$ and all $v < \alpha$, we have that $\frac{\partial_1 C(u, v)}{v} > 1$. Now, considering $u \in (1 - \epsilon, 1 - \epsilon/2)$ ($u \geq m$), and $v_1 < \alpha$, $v_2 > u$ (with $v_1 < v_2$) we have:

$$\frac{\partial_1 C(u, v_1)}{v_1} \leq \frac{\partial_1 C(u, v_2)}{v_2},$$

which is impossible since

$$1 < \frac{\partial_1 C(u, v_1)}{v_1} \text{ and } \frac{\partial_1 C(u, v_2)}{v_2} > 1.$$

Therefore, $\lim_{v \rightarrow 0} u_v \neq 1$ and we can conclude that there exists $u^* < 1$ such that $u_v \leq u^*$ for all $v \in (0, 1)$.

3.4 Parametric examples of PELCOV_vs.

In this section, several parametric examples are presented in order to illustrate Theorem 3.18, both for the case of unbounded and bounded sets of PELCOV_vs. Details on studying the analytical expression, monotonicity and bounds of the PELCOV_vs can be found in Appendix A.

Since the PELCOV_v, under the regularity conditions, does not depend on the marginals, in this section we will assume, for all parametric examples, the following framework. We consider a random vector (X, Y) satisfying the regularity conditions with the copula given in each of the subsections. Note that the values of the parameters are considered in order to have $Y \uparrow_{SSI} X$.

3.4.1 Bounded PELCOV_vs.

3.4.1.1 Farlie–Gumbel–Morgenstern family of copulas

Let us consider the Farlie–Gumbel–Morgenstern (FGM) copula, given by $C(u, v) = uv(1 + \theta(1 - u)(1 - v))$, for $\theta \in (0, 1]$, $u, v \in [0, 1]$. Since

$$\partial_1 C(u, v) = v + \theta v(2u - 1)(1 - v),$$

we have $Y \uparrow_{SSI} X$. One can check that the density of this copula is given by $c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$. It is easy to see that, if we consider $\theta > 0$, taking $m = 1/2$ and being $\partial_2 c(u, v) = 2(2u - 1)\theta$, $c(u, v)$ increases in v for all $u \geq m$. In addition, $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = 1 + \theta > 0$. Conditions of Theorem 3.18 are satisfied and, therefore, we can conclude that there exists an upper bound $u^* < 1$ for the PELCOV_vs.

In this case, we can easily found the upper bound because the PELCOV_v is constant in v . As seen in Example 2 in Navarro and Sordo (2018), from the expression of $\partial_1 C(u, v)$ we can observe that $\partial_1 C(u, v) = v$ if and only if $u = 1/2$. Hence, $u_v = 1/2$ for all $v \in (0, 1)$. Now, if $\theta > 0$, the vector is SI and, therefore, $u^* = 1/2$ and

$$Y \leq_{st} (Y \mid X = x) (\geq_{st}) \text{ for all } x \geq F^{-1}(u^*)(<).$$

It is also relevant to note that, for the FGM family of copulas, since the PELCOV_v is constant, for any given risk value $v \in (\frac{1}{2}, 1)$, the median always represents the “alert”-level on the variable X to see that the VaR of Y underestimates systemic risk.

3.4.1.2 Frank family of copulas.

Let us consider the Frank copula (copula 5 in Table 4.1 in Nelsen (2007)) given by $C(u, v) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-u\theta} - 1)(e^{-v\theta} - 1)}{e^{-\theta} - 1} \right)$ for $\theta > 0$. Computing $\partial_1 C(u, v)$, we can see that $Y \uparrow_{SSI} X$. The density is given by:

$$c(u, v) = \frac{e^{(1+u+v)\theta} (e^\theta - 1) \theta}{(e^{(u+v)\theta} - e^\theta (e^{u\theta} + e^{v\theta} - 1))^2}.$$

Note that $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = \frac{e^\theta \theta}{e^\theta - 1}$, which is greater than 1 for $\theta > 0$. In addition,

$$\partial_2 c(u, v) = \frac{e^{(1+u+v)\theta} (e^\theta - 1) (e^{(u+v)\theta} + e^\theta (e^{u\theta} - e^{v\theta} - 1)) \theta^2}{(e^{(u+v)\theta} - e^\theta (e^{u\theta} + e^{v\theta} - 1))^3},$$

so, considering $m = 1/2$, $u \geq 1/2, v \leq u$, it is direct to see that $\partial_2 c(u, v) > 0$, that is, conditions of Theorem 3.18 hold and, therefore, an upper bound exists for the PELCoV_vs.

Once we have illustrated the conditions of the Theorem, for this particular family of copulas, we can find the analytical form of the PELCoV_v and the bound. Details on the computation can be found in Appendix A. The PELCoV_v and its upper bound are given by

$$u_v = \frac{1}{\theta} \ln \left(\frac{e^\theta (1 - e^{\theta v}) (1 - v)}{v (e^{\theta v} - e^\theta)} \right) \text{ and } u^* = \frac{1}{\theta} \ln \left(\frac{e^\theta - 1}{\theta} \right).$$

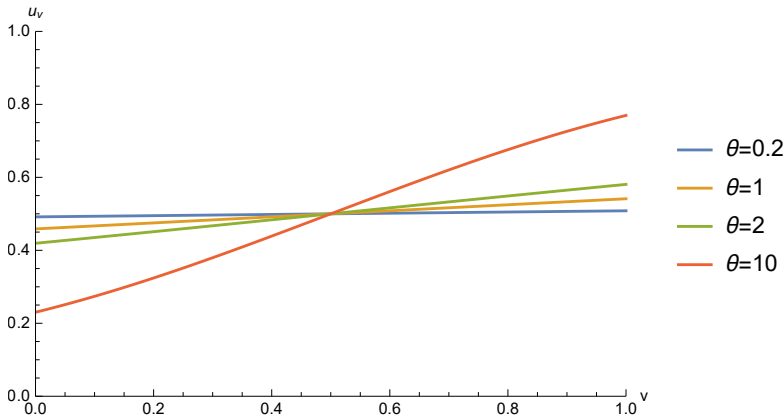


Figure 3.3. Function $v \mapsto \text{PELCoV}_v$ for the Frank copula with different values of the parameter.

In Figure 3.3, the PELCOV_v is plotted as a function of v for different values on the dependence parameter of the Frank copula.

Note that, from the expression of the PELCOV_v, and from Figure 3.3, we can see that $v \geq \text{PELCOV}_v$ if and only if $v \geq \frac{1}{2}$. Therefore, for any given risk value $v \in (\frac{1}{2}, 1)$, the ‘alert’-level on the variable X to replace VaR by CoVaR is always smaller than the considered risk-level.

3.4.1.3 Clayton family of copulas

Let consider the copula Clayton, which appears as copula 1 in Table 4.1 in Nelsen (2007), and it is given by $C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ for $\theta > 0$, parameters for which $Y \uparrow_{SSI} X$. The density copula is given by

$$c(u, v) = u^{-1-\theta} v^{-1-\theta} \left(u^{-\theta} + v^{-\theta} - 1 \right)^{-2-\frac{1}{\theta}} (1 + \theta).$$

Taking the partial derivative, we get that $\partial_2 c(u, v)$ is equal to

$$\frac{u^{-1+\theta} v^{-2+\theta} (-1 + u^{-\theta} + v^{-\theta})^{-1/\theta} (1 + \theta) (u^\theta \theta + (u^\theta - 1) v^\theta (1 + \theta))}{(v^\theta + u^\theta (1 - v^\theta))^3}.$$

Thus, $c(u, v)$ increases in v if $u^\theta \theta + (u^\theta - 1) v^\theta (1 + \theta) > 0$. This happens if $u > \left(\frac{v^\theta (1 + \theta)}{\theta + v^\theta (1 + \theta)} \right)^{\frac{1}{\theta}} = \left(1 - \frac{\theta}{\theta + v^\theta (1 + \theta)} \right)^{\frac{1}{\theta}}$. This bound clearly increases in v and

$$\lim_{v \rightarrow 1} \left(\frac{v^\theta (1 + \theta)}{\theta + v^\theta (1 + \theta)} \right)^{\frac{1}{\theta}} = \left(\frac{1 + \theta}{1 + 2\theta} \right)^{1/\theta}.$$

Therefore, for all $u \geq m$ and $v \leq u$, considering $m = \left(\frac{1 + \theta}{1 + 2\theta} \right)^{1/\theta}$, $c(u, v)$ increases in v . In addition, $\lim_{(u, v) \rightarrow (1, 1)} c(u, v) = 1 + \theta$, which is greater than 1 for $\theta > 0$. Hence, conditions of theorem 3.18 hold and the PELCOV_s are bounded.

For this example, we can also obtain the expression of the PELCOV_v and its upper bound u^* analytically, since the PELCOV_v increases in terms of v . As seen in (3.26), we get

$$u_v = \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1} \right)^{1/\theta} \quad \text{and} \quad u^* = \lim_{v \rightarrow 1} u_v = (1 + \theta)^{-\frac{1}{\theta}}.$$

In Figure 3.4, we can observe the function $v \mapsto \text{PELCOV}_v$ considering different parameters.

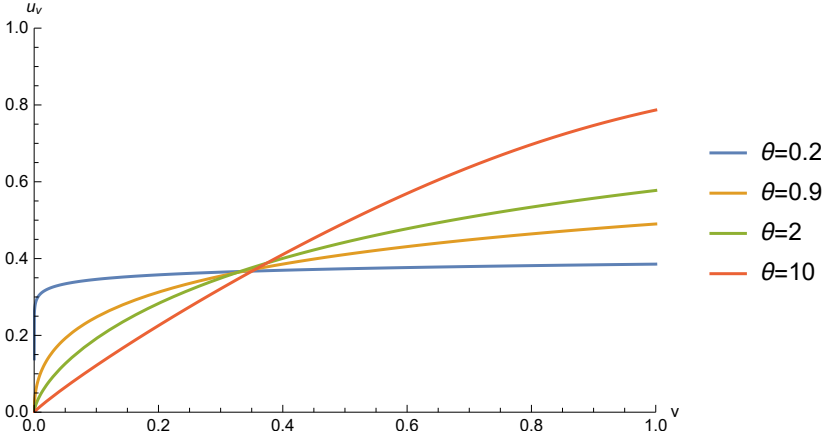


Figure 3.4. Function $v \mapsto \text{PELCoV}_v$ for the Clayton copula with different values of the parameter.

3.4.1.4 Ali-Mikhail-Haq family of copulas

Let us now consider the copula 3 in Table 4.1 in Nelsen (2007), given by $C(u, v) = \frac{uv}{(1-\theta(1-u)(1-v))}$ for $\theta \in (0, 1]$, values for which the vector is PDS. Note that the case for $\theta = 1$ is considered in Example 1 in Navarro and Sordo (2018). The density of the copula is given by

$$c(u, v) = \frac{1 + \theta(u + v + uv - 2 + (1 - u)(1 - v)\theta)}{(1 - (1 - u)(1 - v)\theta)^3},$$

and its partial derivative verifies that

$$\partial_2 c(u, v) = \frac{2\theta(2u - 1 + (u - 1)(-2 + v + u(2 + v))\theta - (1 - u)^2(1 - v)\theta^2)}{(1 - (1 - u)(1 - v)\theta)^4}.$$

It can be checked that, if $u \geq \frac{2}{3}$, for all $\theta \in (0, 1)$, $\partial_2 c(u, v)$ is positive and, therefore, $c(u, v)$ increases in v . Also, we can see that $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = 1 + \theta > 1$, for $\theta > 0$. Therefore, conditions of Theorem 3.18 hold, and, therefore, there exists an upper bound for the PELCoV_v s. We can also obtain the analytical expression of the PELCoV_v and the upper bound (see Appendix A), which are given by:

$$u_v = \frac{\theta(1 - v) - 1 + \sqrt{1 - \theta(1 - v)}}{\theta(1 - v)} \text{ and } u^* = \lim_{v \rightarrow 1} u_v = \frac{1}{2}$$

The PELCoV_v of the AMH copula is considered in Figure 3.5 for different values of the parameter.

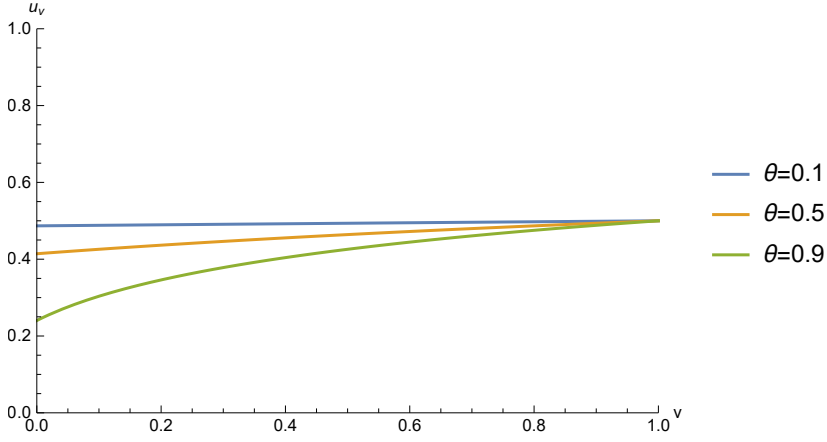


Figure 3.5. Function $v \mapsto PELCoV_v$ for the Ali-Mikhail-Haq copula with different values of the parameter.

3.4.1.5 Family of copulas 13 in Table 4.1 in Nelsen (2007).

Let us now consider the copula 13 in Table 4.1 in Nelsen (2007), given by $C(u, v) = \exp\left(1 - [(1 - \ln u)^\theta + (1 - \ln v)^\theta - 1]^{1/\theta}\right)$ for $\theta > 1$ and that, for such values of the parameter, $Y \uparrow_{SSI} X$. The density of the copula is given by

$$c(u, v) = \frac{e^{1-d(u,v)^{1/\theta}} (1 - d(u,v)^{1/\theta}) (d(u,v))^{1/\theta-2} (1 - \ln u)^{\theta-1} (1 - \ln v)^{\theta-1}}{uv},$$

where $d(u, v) = (1 - \ln u)^\theta + (1 - \ln v)^\theta - 1$.

We can compute the limit and obtain that $\lim_{(u,v) \rightarrow (1,1)} c(u, v) = \theta > 1$. Graphically, we can also see that, if $u \geq \frac{1}{2}$, $v \leq v$ and considering parameter $\theta \in (1, 1000)$, then $\partial_2 c(u, v) \geq 0$ holds. Therefore, by Theorem 3.18, there exists an upper bound for the $PELCOV_v$ s.

The bound can not be obtained analytically, but we numerically obtain the $PELCoV_v$ and plot $(v, \theta) \mapsto PELCoV_v$, for $(v, \theta) \in (0, 1) \times (1, 1000)$, and it can be observed that $v \mapsto PELCoV_v$ increases in v for all $\theta \in (1, 1000)$. Therefore, if we numerically compute the bound as a limit, $u^* = \lim_{v \rightarrow 1} PELCoV_v$, we can found, for each $\theta \in (1, 1000)$, the numerical approximation of the bound, which can be observed in Figure 3.6.

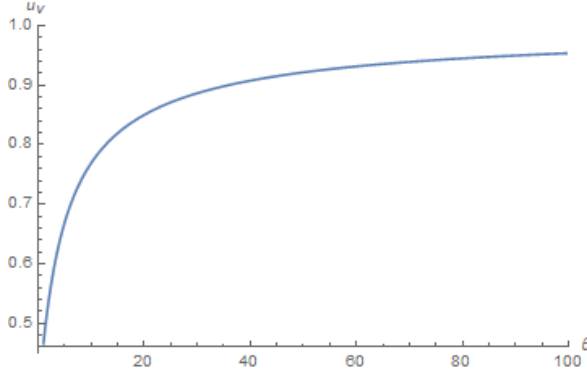


Figure 3.6. Function $\theta \mapsto u^* = \lim_{v \rightarrow 1} PELCoV_v$ for the copula 13 in Table 4.1 in Nelsen (2007).

3.4.1.6 Family of copulas 17 in Table 4.1 in Nelsen (2007).

Let us now consider the copula 17 in Table 4.1 in Nelsen (2007), given by $C(u, v) = \left(1 + \frac{[(1+u)^{-\theta}-1][(1+v)^{-\theta}-1]}{2^{-\theta}-1}\right)^{-1/\theta} - 1$ for $\theta > -1$ ($\theta \neq 0$). Note that $Y \uparrow_{SSI} X$ and that the density of the copula is given by $c(u, v) = a(u, v)b(u, v)$, where

$$a(u, v) = \frac{2^\theta \left(\frac{(1+u)^{-\theta}(1+v)^{-\theta} (2^\theta (-1+(1+u)^\theta) + (2^\theta - (1+u)^\theta)(1+v)^\theta)}{-1+2^\theta} \right)^{-1/\theta}}{(1+u)(1+v) (2^\theta (-1+(1+u)^\theta) + (2^\theta - (1+u)^\theta)(1+v)^\theta)^2},$$

$$b(u, v) = \left(2^\theta (-1+(1+u)^\theta) (-1+(1+v)^\theta) + (-1+2^\theta)(1+u)^\theta(1+v)^\theta \right).$$

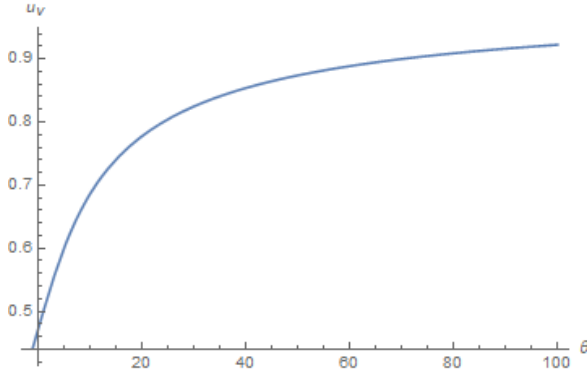


Figure 3.7. Function $\theta \mapsto u^* = \lim_{v \rightarrow 1} PELCoV_v$ for the copula 17 in Table 4.1 in Nelsen (2007).

We can compute $\lim_{(u,v) \rightarrow (1,1)} c(u,v) = \frac{1}{2} \left(1 + \theta + \frac{\theta}{2\theta-1} \right)$, which is greater than 1 for $\theta > -1$ ($\theta \neq 0$). Analogously as the previous example, we can numerically see that, for $u \geq \frac{1}{2}$, $u \leq v$ and considering $\theta \in (-1, 1000) \setminus \{0\}$, $\partial_2 c(u,v) \geq 0$. By Theorem 3.18, there exists an upper bound for the PELCOV_v s for this copula. Also analogously as the previous example, it can be computed that the PELCoV_v increases in v for all $\theta \in (-1, 1000) \setminus \{0\}$. Obtaining $u^* = \lim_{v \rightarrow 1} \text{PELCoV}_v$, the numerical approximation of the bound can be obtained and plotted (see Figure 3.7) for $\theta \in (-1, 100) \setminus \{0\}$.

3.4.1.7 Family of copulas 19 in Table 4.1 in Nelsen (2007).

As a last example, let us now consider the copula 19 in Table 4.1 in Nelsen (2007), given by $C(u,v) = \frac{\theta}{\ln(e^{\theta/u} + e^{\theta/v} - e^{\theta})}$ for $\theta > 0$. For such values of the parameter θ , if $\mathbf{X} = (X, Y)$ follows the given copula, $Y \uparrow_{SSI} X$. The density is given by:

$$c(u,v) = \frac{e^{(\frac{1}{u} + \frac{1}{v})\theta} \theta^3 \left(2 + \log \left[e^{\frac{\theta}{u}} + e^{\frac{\theta}{v}} - e^{\theta} \right] \right)}{\left(e^{\frac{\theta}{u}} + e^{\frac{\theta}{v}} - e^{\theta} \right)^2 u^2 v^2 \log \left[e^{\frac{\theta}{u}} + e^{\frac{\theta}{v}} - e^{\theta} \right]^3},$$

We can obtain the limit $\lim_{(u,v) \rightarrow (1,1)} c(u,v) = \theta + 2 > 1$, for $\theta > 0$. Again, we can plot, for $u \geq \frac{1}{2}$, $u \leq v$ and considering $\theta \in (0, 1000)$, $\partial_2 c(u,v) \geq 0$. Therefore, the existence of an upper bound for the PELCoV_v s is ensured. This example is analogous as the two previous ones, where we can numerically obtain such bounds (see Figure 3.8), due to the fact that PELCoV_v is increasing.

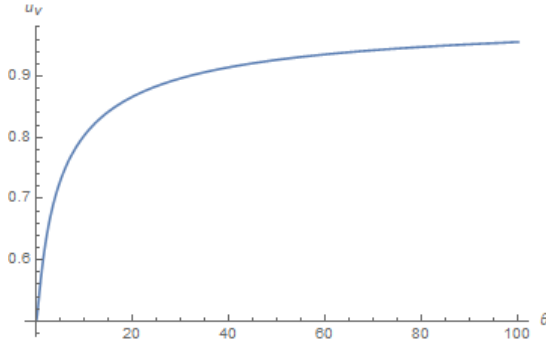


Figure 3.8. Function $\theta \mapsto u^* = \lim_{v \rightarrow 1} \text{PELCoV}_v$ for the copula 19 in Table 4.1 in Nelsen (2007).

This section is summarized in Table 3.1.

Copulas with bounded PELCoV	PELCoV	Bound
$(X, \psi(X) + W)$ with ψ strictly increasing and X, W independent	$u_v = F_X \left(\psi^{-1} \left(F_{\psi(X)+W}^{-1}(v) - F_W^{-1}(v) \right) \right)$	If W has a logconcave density function
FGM copula: for $\theta \in (0, 1]$, $C(u, v) = uv(1 + \theta(1 - u)(1 - v))$	$u_v = 1/2$	$u^* = 1/2$
Frank copula: for $\theta > 0$, $C(u, v) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-u\theta} - 1)(e^{-v\theta} - 1)}{e^{-\theta} - 1} \right)$	$u_v = \frac{1}{\theta} \ln \left(\frac{e^{\theta(1-e^{\theta v})(1-v)}}{v(e^{\theta v} - e^{\theta})} \right)$	$u^* = \frac{1}{\theta} \ln \left(\frac{e^{\theta} - 1}{\theta} \right)$
Clayton copula: for $\theta > 0$, $C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$	$u_v = \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1} \right)^{1/\theta}$	$u^* = (1 + \theta)^{-\frac{1}{\theta}}$
AMH copula: for $\theta \in (0, 1]$, $C(u, v) = \frac{uv}{(1 - \theta(1-u)(1-v))}$	$u_v = \frac{\theta(1-v) - 1 + \sqrt{1 - \theta(1-v)}}{\theta(1-v)}$	$u^* = 1/2$
Copula 13 in Table 4.1 in Nelsen (2007): for $\theta > 0$, $C(u, v) = \exp \left(1 - [(1 - \ln u)^{\theta} + (1 - \ln v)^{\theta} - 1]^{1/\theta} \right)$	Numerically computed	Corollary 3.17 holds. Numerically computed
Copula 17 in Table 4.1 in Nelsen (2007): for $\theta > 0$, $C(u, v) = \left(1 + \frac{[(1+u)^{-\theta} - 1][(1+v)^{-\theta} - 1]}{2^{-\theta} - 1} \right)^{-1/\theta} - 1$	Numerically computed	Corollary 3.17 holds. Numerically computed
Copula 19 in Table 4.1 in Nelsen (2007): for $\theta > 0$, $C(u, v) = \frac{\theta}{\ln(e^{\theta/u} + e^{\theta/v} - e^{\theta})}$	Numerically computed	Corollary 3.17 holds. Numerically computed

Table 3.1. Values of the PELCoVs and their bounds for vectors (X, Y) such that $Y \uparrow_{SSI} X$.

3.4.2 Unbounded cut points

We can also find many different examples where the PELCOV_s do not have an upper bound.

3.4.2.1 Gaussian copula

Let the vector have a bivariate normal distribution, that is, let $\mathbf{X} = (X, Y) \sim \mathbf{N}(\mu, \Sigma)$ with $\mu = (\mu_1, \mu_2)$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. We consider $\rho > 0$, values for which the vector is PDS. It was already seen that the PELCOV_v increases in terms of v (Example 3.12). In order to obtain the analytical expression, let us consider $Z \sim N(0, 1)$, and:

$$F_Y^{-1}(v) = \mu_Y + F_Z^{-1}(v)\sigma_Y$$

$$F_{Y|X=x}^{-1}(v) = x \left(\rho \frac{\sigma_Y}{\sigma_X} \right) + \left(\mu_Y - \mu_X \rho \frac{\sigma_Y}{\sigma_X} + F_Z^{-1}(v)\sigma_Y \sqrt{1 - \rho^2} \right)$$

Therefore, by solving $F_{Y|X=x_v}^{-1}(v) = F_Y^{-1}(v)$, we get that:

$$x_v = \mu_X + \sigma_X \frac{1 - \sqrt{1 - \rho^2}}{\rho} F_Z^{-1}(v).$$

Taking into account that $u_v = F_X(x_v)$, we have the expression of the PELCOV_v given by:

$$u_v = F_Z \left(\frac{1 - \sqrt{1 - \rho^2}}{\rho} F_Z^{-1}(v) \right). \quad (3.12)$$

In Figure 3.9, we see the PELCOV_v in terms of the risk level v , when we consider a vector with the Gaussian copula, for different values of the correlation coefficient ρ .

We can observe that for all $v \in (0, 1)$, $\text{PELCOV}_v < v$. That is, a lower risk-value of X alerts us about the risk value v . Since $v \mapsto \text{PELCOV}_v$ is a continuous function such that $\lim_{v \rightarrow 0} \text{PELCOV}_v = 0$ and $\lim_{v \rightarrow 1} \text{PELCOV}_v = 1$, it does not exist any $x \in \mathbb{R}$ such that $Y|X = x \geq_{st} Y$ and neither any $x \in \mathbb{R}$ such that $Y|X = x \leq_{st} Y$.

3.4.2.2 Gumbel Hougaard family of copulas.

Let us now consider the copula 4 in Table 4.1 in Nelsen (2007), given by $C(u, v) = \exp \left(- [(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta} \right)$ with $\theta > 1$, which is PDS. The density is given by

$$c(u, v) = \frac{e^{-((-\ln[u])^\theta + (-\ln[v])^\theta)^{1/\theta}} (\ln[v] \ln[u])^{\theta-1} h(u, v)}{uv}$$

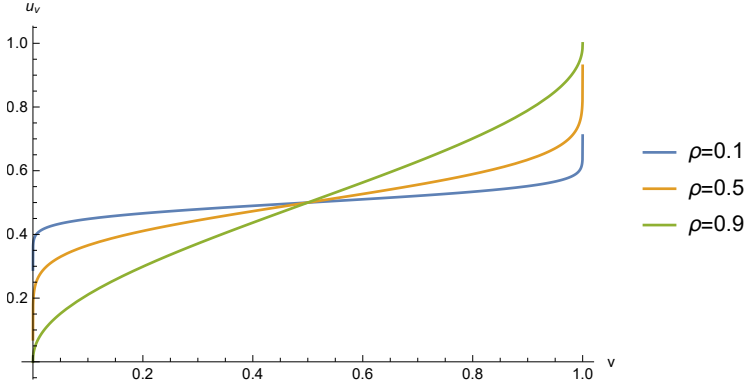


Figure 3.9. Function $v \mapsto PELCoV_v$ for the Gaussian copula with different values of the parameter.

where

$$h(u, v) = \left(\theta - 1 + \left((-\ln[u])^\theta + (-\ln[v])^\theta \right)^{\frac{1}{\theta}} \right) \left((-\ln[u])^\theta + (-\ln[v])^\theta \right)^{-2 + \frac{1}{\theta}}.$$

For all $u \in (0, 1)$ (respectively, $v \in (0, 1)$), $\lim_{v \rightarrow 1} c(u, v) = 0$ (respectively, $\lim_{u \rightarrow 1} c(u, v) = 0$). Although, $\lim_{u \rightarrow 1} c(u, u) = \infty$. Therefore, the limit $\lim_{(u,v) \rightarrow (1,1)} c(u, v)$ does not exist, and conditions of Theorem 3.18 do not hold.

In addition, we can plot numerically the values of the $PELCoV_v$ in terms of v for different values of $\theta > 1$ and we can see, in Figure 3.10, that there does not exist a bound.

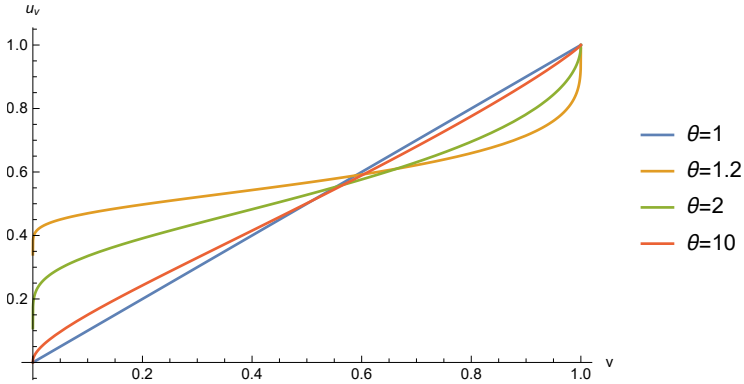


Figure 3.10. Function $v \mapsto PELCoV_v$ for the Gumbel Hougaard copula with different values of the parameter.

3.4.2.3 Family of copulas 2 in Table 4.1 in Nelsen (2007).

Let us now consider the copula 2 in Table 4.1 in Nelsen (2007), given by $C(u, v) = 1 - [(1-u)^\theta + (1-v)^\theta]^{1/\theta}$ with $\theta > 1$. The density is given by

$$c(u, v) = (\theta - 1)(1-u)^{\theta-1}(1-v)^{\theta-1} \left((1-u)^\theta + (1-v)^\theta \right)^{-2+\frac{1}{\theta}}$$

Here, we can see that, again, for all $u \in (0, 1)$ (respectively, $v \in (0, 1)$), $\lim_{v \rightarrow 1} c(u, v) = 0$ (respectively, $\lim_{u \rightarrow 1} c(u, v) = 0$), but $\lim_{u \rightarrow 1} c(u, u) = \infty$. Therefore, the limit $\lim_{(u,v) \rightarrow (1,1)} c(u, v)$ does not exist, and conditions of Theorem 3.18 do not hold. Here, we can obtain the expression of u_v , considering that:

$$\partial_1 C(u, v) = (1-u)^{\theta-1} \left((1-u)^\theta + (1-v)^\theta \right)^{-1+\frac{1}{\theta}}.$$

Solving $\partial_1 C(u_v, v) = v$, we can get that the PELCOV_v is given by:

$$u_v = 1 - (1-v) \left(v^{\frac{-\theta}{\theta-1}} - 1 \right)^{-\frac{1}{\theta}} \quad (3.13)$$

We can plot the expression (3.13) for different values of θ . Figure 3.11 shows that the set of PELCOV_v does not have an upper bound for this copula.

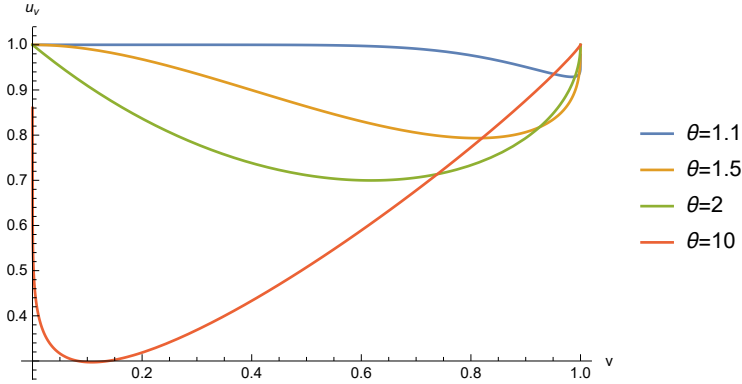


Figure 3.11. Function $v \mapsto \text{PELCOV}_v$ for the copula 2 in Table 4.1 in Nelsen (2007) with different values of the parameter.

3.4.2.4 Family of copulas 6 in Table 4.1 in Nelsen (2007).

Let us now consider the copula 6 in Table 4.1 in Nelsen (2007), given by $C(u, v) = 1 - [(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta]^{1/\theta}$ with $\theta > 1$. The partial derivative $\partial_1 C(u, v)$ is given by

$$\frac{\left((1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta \right)^{\frac{1}{\theta}-1} \left((1-u)^{-1+\theta}\theta - (1-u)^{\theta-1}(1-v)^\theta\theta \right)}{\theta}$$

Chapter 3. Comparisons of VaR-CoVaR in terms of the conditional variable.

Solving $\partial_1 C(u_v, v) = v$, we numerically plot the $PELCoV_v$ for different values of θ and, as it can be observed in Figure 3.12, it does not have an upper bound.

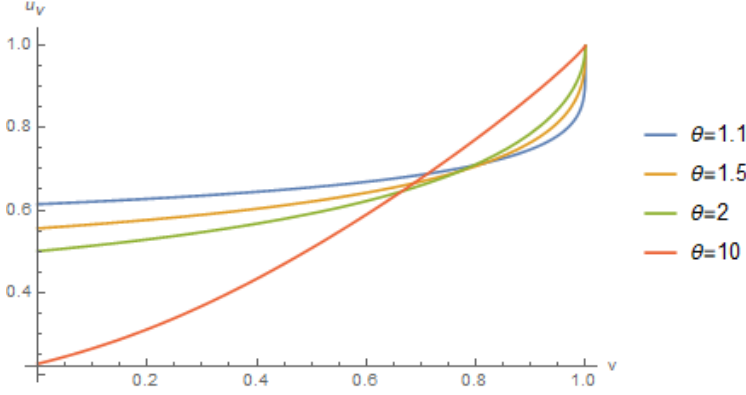


Figure 3.12. Function $v \mapsto PELCoV_v$ for the copula 6 in Table 4.1 in Nelsen (2007) with different values of the parameter.

3.4.2.5 Family of copulas 12 in Table 4.1 in Nelsen (2007).

The copula 12 in Table 4.1 in Nelsen (2007) is given, for $\theta > 1$, by $C(u, v) = \left(1 + \left[(u^{-1} - 1)^\theta + (v^{-1} - 1)^\theta\right]^{1/\theta}\right)^{-1}$. The partial derivative is given by

$$\partial_1 C(u, v) = \frac{\left(\frac{1}{u} - 1\right)^{\theta-1} \left(\left(\frac{1}{u} - 1\right)^\theta + \left(\frac{1}{v} - 1\right)^\theta\right)^{\frac{1}{\theta}-1}}{u^2 \left(1 + \left(\left(\frac{1}{u} - 1\right)^\theta + \left(\frac{1}{v} - 1\right)^\theta\right)^{\frac{1}{\theta}}\right)^2}.$$

Analogously as in the previous examples, we can solve numerically $\partial_1 C(u_v, v) = v$, and plot the $PELCoV_v$ for different values of θ . In Figure 3.13 we can again see that there does not exist an upper bound.

3.4.2.6 Family of copulas 14 in Table 4.1 in Nelsen (2007).

In Table 4.1 in Nelsen (2007), the copula 14 is given, for $\theta > 1$, by $C(u, v) = \left(1 + \left[\left(u^{-1/\theta} - 1\right)^\theta + \left(v^{-1/\theta} - 1\right)^\theta\right]^{1/\theta}\right)^{-\theta}$. The partial derivative is:

$$\partial_1 C(u, v) = \frac{u^{-\frac{1}{\theta}} \left(u^{-1/\theta} - 1\right)^{-1+\theta} \left(\left(u^{-1/\theta} - 1\right)^\theta + \left(v^{-1/\theta} - 1\right)^\theta\right)^{\frac{1}{\theta}-1}}{\left(1 + \left(\left(u^{-1/\theta} - 1\right)^\theta + \left(v^{-1/\theta} - 1\right)^\theta\right)^{\frac{1}{\theta}}\right)^{\theta+1}}.$$

3.4. Parametric examples of $PELCOV_v$ s.

This example is analogous as the previous ones. Computing numerically the solution of $\partial_1 C(u_v, v) = v$, we can plot the $PELCOV_v$ for different values of θ , observing again, in Figure 3.14, that such values are not bounded.

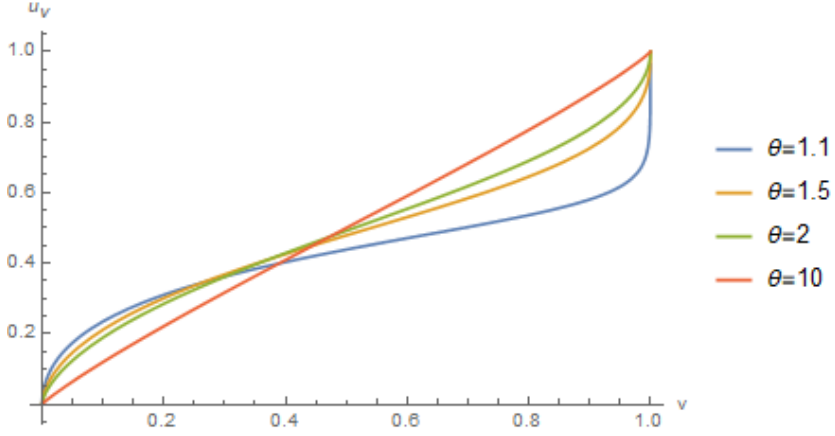


Figure 3.13. Function $v \mapsto PELCOV_v$ for the copula 12 in Table 4.1 in Nelsen (2007) with different values of the parameter.

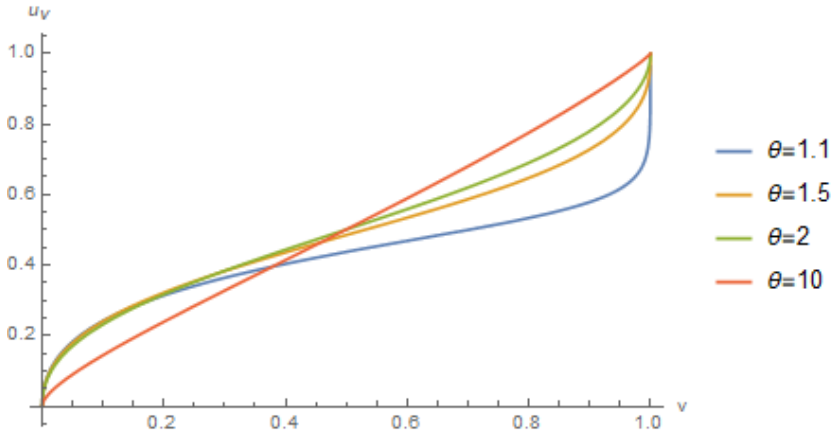


Figure 3.14. Function $v \mapsto PELCOV_v$ for the copula 14 in Table 4.1 in Nelsen (2007) with different values of the parameter.

This section is summarized in Table 3.2.

Chapter 3. Comparisons of VaR-CoVaR in terms of the conditional variable.

Copulas with unbounded PELCoV	PELCoV
Gaussian copula: for $\rho \in (0, 1]$ $(X, Y) \sim \mathbf{N}(\mu, \Sigma)$	$u_v = F_Z\left(\frac{1-\sqrt{1-\rho^2}}{\rho} F_Z^{-1}(v)\right)$
Gumbel Hougaard copula: for $\theta > 1$ $C(u, v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right)$	Numerically computed
Copula 2 in Table 4.1 in Nelsen (2007): for $\theta > 1$, $C(u, v) = 1 - \left[(1-u)^\theta + (1-v)^\theta\right]^{1/\theta}$	$u_v = 1 - (1-v) \left(v^{\frac{-\theta}{\theta-1}} - 1\right)^{-\frac{1}{\theta}}$
Copula 6 in Table 4.1 in Nelsen (2007): for $\theta > 1$, $C(u, v) = 1 - \left[(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\right]^{1/\theta}$	Numerically computed
Copula 12 in Table 4.1 in Nelsen (2007): for $\theta > 1$, $C(u, v) = \left(1 + \left[(u^{-1} - 1)^\theta + (v^{-1} - 1)^\theta\right]^{1/\theta}\right)^{-1}$	Numerically computed
Copula 14 in Table 4.1 in Nelsen (2007): for $\theta > 1$, $C(u, v) = \left(1 + \left[(u^{-1/\theta} - 1)^\theta + (v^{-1/\theta} - 1)^\theta\right]^{1/\theta}\right)^{-\theta}$	Numerically computed

Table 3.2. Values of PELCoVs with no upper bound for vectors (X, Y) such that $Y \uparrow_{SSI} X$.

3.5 An applied example with record values

In Navarro (2022), a general method was proposed to predict record values from preceding records. The method was based on conditional quantile curves, analogous as the ones given by $u \mapsto \text{CoVaR}_{v,u}^-(Y|X)$, and the predictions were based of the conditional median. Predictions obtained from such curve are based on quantile regression, which is known to be a robust methodology (see Koenker and Bassett (1978)). In this example, employing the theoretical findings on the PELCOV_v for a vector of two consecutive record values, we will establish thresholds on preceding records that lead to an increase in predictions on future record values.

With this purpose, let us consider $(X_n)_{n \in \mathbb{N}}$ an infinite sequence of independent and identically distributed (iid) random variables that will be assumed to have an absolutely continuous cumulative distribution function F , with $\bar{F}$ and f its respective survival and density functions. We will also assume that the support of such variables is $S = [l_X, u_X]$, where l_X can take the value $-\infty$ and u_X can take the value ∞ . We will consider as an upper record value the

observation X_i that trespasses all the previous observed values $X_1, \dots, X_{i-1}$. We can define the *upper record times* where the records occur in the following manner:

$$T(1) = 1 \text{ and } T(n+1) = \min \{j : X_j > X_{T(n)}\} \text{ for } n = 1, 2, \dots$$

Given the times, the sequence of *upper records* $(R_n)_{n \in \mathbb{N}}$ is defined as $R_n = X_{T(n)}$, for $n = 1, 2, 3, \dots$. Note that $R_1 = X_1$. We denote as $F_n, \bar{F}_n$ the distribution and survival function of the n -th record value R_n . The survival function of a record is a distortion of the original survival function (see, for example Nevzorov (2001)) and it is given, for $n = 1, 2, \dots$, by:

$$\bar{F}_n(t) = \bar{F}(t) \sum_{k=0}^{n-1} \frac{(-\log(\bar{F}(t)))^k}{k!} = \bar{q}_n(\bar{F}(t)) \quad (3.14)$$

$$\text{with } \bar{q}_n(u) = u \sum_{k=0}^{n-1} \frac{(-\log(u))^k}{k!}.$$

It can be easily checked that $\bar{q}_n$ is a continuous distortion function (i.e. it is increasing, $\bar{q}_n(0) = 0$ and $\bar{q}_n(1) = 1$). The dual distortion function q_n such that $F_n(u) = q_n(F(u))$ is given by:

$$q_n(u) = 1 - (1 - u) \sum_{k=0}^{n-1} \frac{(-\log(1 - u))^k}{k!}. \quad (3.15)$$

We can easily see that, for a given value $u \in (0, 1)$, $q_{n+1}(u) < q_n(u)$. In addition, as $(q_n)_{n \in \mathbb{N}}$ are distortion functions, it is also direct to see that $q_{n+1}^{-1}(u) > q_n^{-1}(u)$. Furthermore, q_n is derivable and one can compute that:

$$q'_n(u) = \frac{(-\log(1 - u))^{n-1}}{(n-1)!}. \quad (3.16)$$

Let us now consider, for $n = 1, 2, \dots$, a vector $\mathbf{X}_n = (R_n, R_{n+1})$. Let us denote as $F_{m|n}(x_m|x_n)$ and $\bar{F}_{m|n}(x_m|x_n)$ the distribution and survival function evaluated in x_m of the random variable $\{R_m | R_n = x_n\}$. The quantile function of such variable in u will be denoted as $F_{m|n}^{-1}(u|x_n)$. In Navarro (2022) we can see that:

$$\bar{F}_{n+1|n}(x_{n+1}|x_n) = \frac{\bar{F}(x_{n+1})}{\bar{F}(x_n)}. \quad (3.17)$$

It is easy to see that (3.17) increases in x_n and, therefore, $R_{n+1} \uparrow_{SSI} R_n$. This is already well-known for the MTP2 property of the vector.

3.5.1 PELCoV_v values for a vector of consecutive upper records

Let us consider $\mathbf{X}_n = (R_n, R_{n+1})$ the vector of two consecutive upper records, with $n = 1, 2, \dots$. Given $v \in (0, 1)$, in order to study the PELCOV_v of the vector $\mathbf{X}_n$, that will be denoted as $u_{v,n}$, we know that it is the value $u_{v,n} \in (0, 1)$ that satisfies:

$$F_{n+1}^{-1}(v) = F_{n+1|n}^{-1}(v|F_n^{-1}(u_{v,n})). \quad (3.18)$$

One can check that equation (3.18) holds when $F_{n+1|n}(F_{n+1}^{-1}(v)|F_n^{-1}(u_{v,n})) = v$. Using (3.17), we can get that $u_{v,n}$, verifies:

$$v = \frac{F(F_{n+1}^{-1}(v)) - F(F_n^{-1}(u_{v,n}))}{1 - F(F_n^{-1}(u_{v,n}))}. \quad (3.19)$$

As $F_n(u) = q_n(F(u))$, $q_n(u) = F_n(F^{-1}(u))$. Hence, $q_n^{-1}(u) = F(F_n^{-1}(u))$. From (3.19) we get that the $\text{PELCoV}_{v,n}$ of the vector $\mathbf{X}_n = (R_n, R_{n+1})$, for $n = 1, 2, \dots$ is given by:

$$u_{v,n} = q_n \left(\frac{q_{n+1}^{-1}(v) - v}{1 - v} \right). \quad (3.20)$$

Since $\{q_n\}_{n \in \mathbb{N}}$ are strictly increasing continuous functions, by expression (3.20), also $u_{v,n}$ is a continuous function of v . Therefore, by Corollary 3.17, in order to check if the PELCoV_v values are bounded or not, and in consequence, to see if $\{R_{n+1} | R_n = x\}$ and R_{n+1} can be ordered stochastically, it would be enough to study the behavior of the PELCoV_v in the extremes of the interval $(0, 1)$.

Proposition 3.19. *Let $\mathbf{X}_n = (R_n, R_{n+1})$ be a vector of two consecutive upper records. Then, the PELCoV_v value of such vector satisfies that*

$$\lim_{v \rightarrow 0} \text{PELCoV}_{v,n} = 0 \quad \text{and} \quad \lim_{v \rightarrow 1} \text{PELCoV}_{v,n} = 1. \quad (3.21)$$

Proof. By the expression (3.20), since $(q_n)_{n \in \mathbb{N}}$ are distortion functions, it is direct to check that $\lim_{v \rightarrow 0} \text{PELCoV}_{v,n} = 0$. For the second limit, since q_n is continuous, using l'Hôpital rule and expression (3.16), we have that:

$$\begin{aligned} \lim_{v \rightarrow 1} \text{PELCoV}_{v,n} &= q_n \left(\lim_{v \rightarrow 1} \frac{q_{n+1}^{-1}(v) - v}{1 - v} \right) = \\ q_n \left(\lim_{v \rightarrow 1} 1 - (q_{n+1}^{-1})'(v) \right) &= q_n \left(\lim_{v \rightarrow 1} 1 - \frac{1}{q'_{n+1}(q_{n+1}^{-1}(v))} \right) = \\ q_n(1) &= 1. \end{aligned}$$

In the last step, it was taken into account that $\lim_{v \rightarrow 1} q'_{n+1}(u) = \infty$.

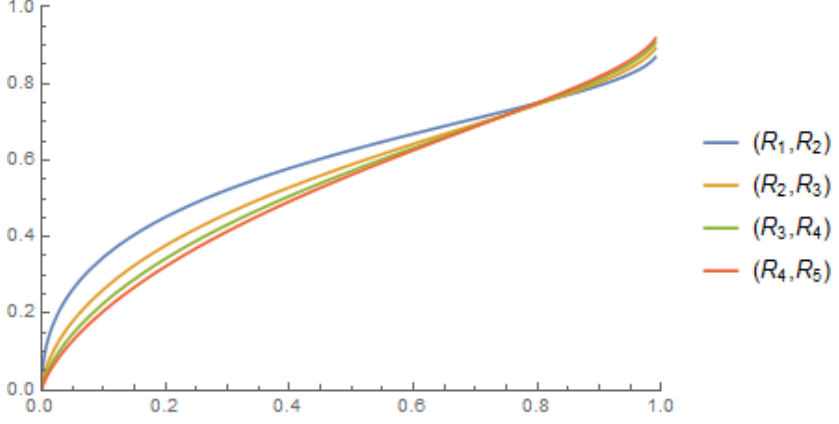


Figure 3.15. Function $v \mapsto \text{PELCoV}_v$ of a vector of consecutive upper records (R_n, R_{n+1}) for different values of the parameter n .

Therefore, we have that, for all $n = 1, 2, \dots$, as we can see in Figure 3.15, the PELCoV_v are continuous functions going from 0 to 1.

We can now state the following corollary.

Corollary 3.20. *Let $(X_n)_n$ be an infinite sequence of independent and identically distributed (iid) random variables and $(R_n)_n$ its sequence of upper records. Then, there does not exist any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that:*

$$\{R_{n+1} \mid R_n = x\} \geq_{st} R_{n+1} \quad \text{or} \quad \{R_{n+1} \mid R_n = x\} \leq_{st} R_{n+1}. \quad (3.22)$$

Proof. Let us assume, without loss of generality, that there exists $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\{R_{n+1} \mid R_n = x\} \geq_{st} R_{n+1}$. Then, for all $v \in (0, 1)$, $F_{n+1|n}^{-1}(v|x) > F_{n+1}^{-1}(v)$, and, necessarily, for all $v \in (0, 1)$, $\text{PELCoV}_v \leq F_n(x) < 1$, which contradicts Proposition 3.19, as $\lim_{v \rightarrow 1} \text{PELCoV}_{v,n} = 1$ for all $n = 1, 2, 3, \dots$

Intuitively, Corollary 3.20 states that the present record given that the value of the preceding record is known, will never be greater, in an stochastic sense, than the present unconditioned record value.

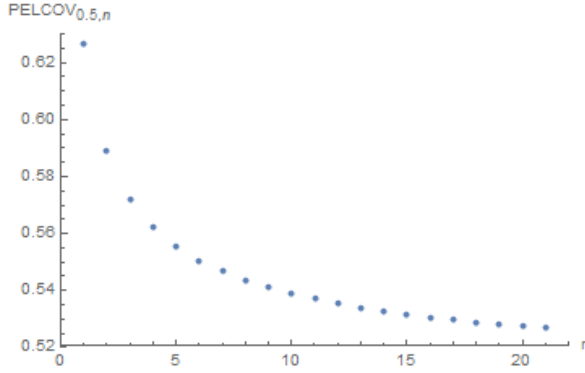


Figure 3.16. $PELCOV_{0.5,n}$ for the first values of n .

3.5.2 The $PELCOV_{0.5}$ as the threshold for increasing the predictions.

It is well-known that, given a random variable X , the median is the best prediction if the criterion is to minimize the expected absolute error. Therefore, considering the $(n + 1)$ -th upper record value, R_{n+1} , with no information about preceding record values, the prediction would be $F_{n+1}^{-1}(0.5)$. In Navarro (2022), future record values are predicted by means of the conditional median regression curves in terms of the preceding record value. That is, considering $u \in (0, 1)$, the prediction of R_{n+1} given that $R_n = F_n^{-1}(u)$, is $F_{n+1|n}^{-1}(0.5|F_n^{-1}(u))$. Therefore, for $u > PELCOV_{v,n}$, the prediction considering the preceding record value would be greater than the one without considering any additional information. That is, the $PELCOV_{v,n}$ sets the threshold of the values of the preceding record that lead to an increase in our prediction.

From (3.20), we can obtain the expression of the $PELCOV_{0.5,n}$, given by $u_{0.5,n} = q_n(2q_{n+1}^{-1}(0.5) - 1)$. We can compute numerically such values, which are plotted in Figure 3.16. For example, $PELCOV_{0.5,1} = 0.626635$. The interpretation is that if X_1 thresspases its quantile 0.627, then, our prediction about R_2 increases.

Intuitively, Figure 3.16 suggests that the $PELCOV_{0.5,n}$ will decrease in terms of $n \in \mathbb{N}$ and that it is bounded from below by 0.5. This lower bound should not be confused with the upper bound u^* discussed in this chapter. Note that the succession of $PELCOV_v$ values in Figure 3.16 is plotted in terms of $n \in \mathbb{N}$ and not in terms of v .

3.6 A real numerical example

As an empirical illustration, we apply the methodology seen in the present work with data from the IBEX35 companies. We consider the companies International Consolidated Airlines Group S.A. (formed, within others by Iberia or Vueling) and Melia Hotels International SA. As the first is an airline company and the second an hotel one, we could think that the distress of the first one could have a significant impact on the second.

We follow the common approach of considering the contagion based on the analysis of interaction among asset log returns of the losses, which are weekly logarithm price differentials. If we denote as p_t the price of an asset at week t , the log return at week t is defined by $r_t = \log(p_{t-1}/p_t)$. That is, positive log-returns represent losses. Our model considers a vector of log returns, $\mathbf{X} = (X, Y)$, where X and Y represent the log returns of International Consolidated Airlines Group S.A. and Melia Hotels International SA, respectively. Data were gathered from the public website <http://es.finance.yahoo.com> and are available from authors upon request. The sample size is $n = 521$ for each financial institution, measuring the share value from February 2012 until February 2022, and, in order to eliminate the time dependent effect, data are related to the weekly close of trading.

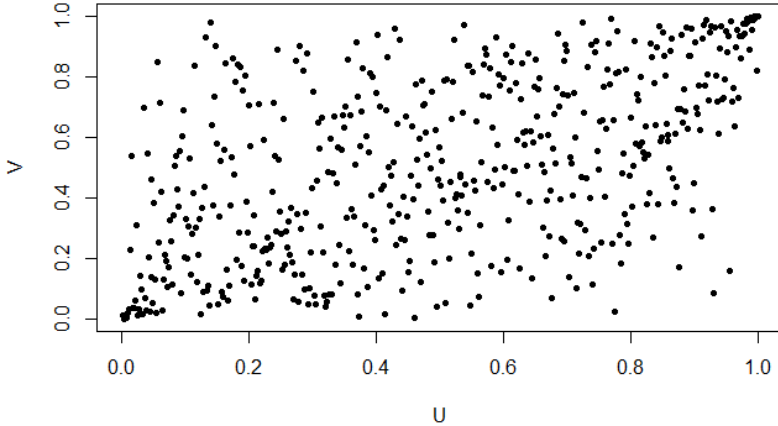


Figure 3.17. *Scatter-plot of the empirical copula of the log returns, $\mathbf{X} = (X, Y)$.*

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Figure 3.17 plots the scatterplot of the empirical copula, that is, the empirical $(U, V) = (F_X(X), F_Y(Y))$, where F_X and F_Y refer to the empirical distribution functions of X and Y , respectively. Because of the distribution and the weight in the tails we could think that it follows a Clayton copula and, consequently, we use statistical inference to check that hypothesis. We perform a goodness-of-fit test for bivariate copulas based on White's information matrix equality and a p -value of 0.23 is obtained, so we can not reject that the copula adjusts to a Clayton copula. Using the maximum likelihood estimator, we can obtain the empirical parameter of the copula, given by $\hat{\theta} = 1.168$. Therefore, we can also plot, based on the expression (3.25), the $PELCoV_v$ of such vector.

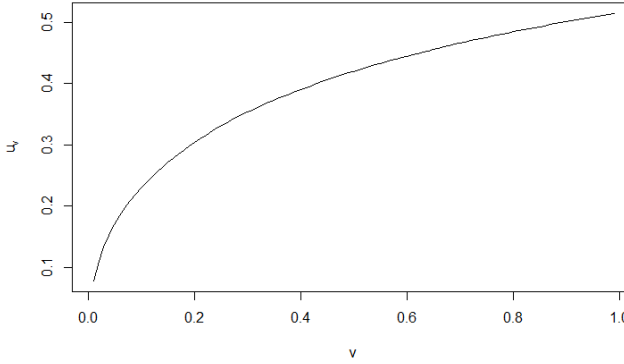


Figure 3.18. Function $v \mapsto PELCoV_v$ for the vector of log returns $\mathbf{X} = (X, Y)$.

Using the results from Example 3.4.1.3, as $u^* = (1 + \theta)^{-\frac{1}{\theta}}$, we can consider that the upper bound of the $PELCoV_v$ is $u^* = 0.5506456$. Therefore, $x^* = F_X^{-1}(u^*) = 0.0003144142$, where the quantile stands for the empirical one. That is, for any value of the log-return of International Consolidated Airlines Group S.A., we can expect that the VaR of the log returns of Melia Hotels International S.A. will not cover the risk, and we should consider the CoVaR instead.

3.7 Conclusions and further work

Given a random vector (X, Y) satisfying the regularity conditions with copula C such that $Y \uparrow_{SSI} X$, we studied the existence of the $PelCoV_v$. In addition, sufficient conditions were given for the monotonicity of the $PelCoV_v$

in terms of v , dealing with comparisons on variability between the conditioned and the unconditioned variables. Furthermore, it was observed that, if there exists $u^* < 1$ such that $\text{PELCoV}_v \leq u^*$ for all $v \in (0, 1)$, then $Y \leq_{st} [Y|X = F_X^{-1}(u)]$, $\forall u \in [u^*, 1)$. Therefore, sufficient conditions for the existence of the upper bound were provided in terms of the density of the copula.

Finally, different parametric examples, both for the existence and non-existence of the bound, were provided and summarized in Tables 3.1 and 3.2. In addition, the PelCoV_v was studied for vectors of consecutive record values. Finally, a parametric methodological approach was considered in order to provide an empirical illustration of the results, dealing with data from IBEX35 touristic companies.

Many different future research lines can arise from this chapter. The most natural step to follow this work would be to find a consistent estimator for the PelCoV_v , in order to perform the estimations under a non parametric approach. Moreover, it would be of great interest to study, in an analogous manner, the probability equivalent level of ES and CoES. However, this will not be direct (unless in simple cases where the PelCoV_v is constant, as the FGM copula), an further work is required. In addition, considering $(\text{VaR}_{u_v}[X], \text{VaR}_v[Y])$, in Fernández-Ponce and Suárez-Llorens (2003) was defined the upper corrected orthant at such point in the following manner:

$$U(u_v, v) = \{(x, y) : x \geq \text{VaR}_{u_v}[X], y \geq \text{CoVaR}_{v, F_X(x)}[Y|X]\},$$

This measure, also studied in Belzunce et al. (2008), presents interesting properties as the one that follows. Considering a random vector (X, Y) under the regularity conditions, it was shown that $P[(X, Y) \in U(u_v, v)] = (1 - u_v)(1 - v)$. Therefore, the greater the value of u_v , the lower probability of the event's occurrence. A generalization of the excess wealth function was proposed in Fernández-Ponce et al. (2011) in terms of the upper corrected orthant. The novel risk measure that is generated would need further study.

A Appendix A: Additional computations

We require the following lemma to study the analytical expressions of some bounds considered in Section 3.4.1.

Lemma 3.21 (L'Hôpital's Rule of Monotonicity (Anderson et al. (1997))). *Let $f, g : [c, d] \rightarrow \mathbb{R}$ be two continuous functions which are differentiable on (c, d) and $g' \neq 0$ in (c, d) . If $f'(x)/g'(x)$ is increasing (or decreasing) on (c, d) , then*

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the functions $(f(x) - f(c))/(g(x) - g(c))$ and $(f(x) - f(d))/(g(x) - g(d))$ are also increasing (or decreasing) on (c, d) . If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

A.1 Analytical expression of PELCOV_vs and their bounds.

A.1.1 Frank family of copulas.

Let us consider the Frank copula studied in Section 3.4.1.2 and given by $C(u, v) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-u\theta} - 1)(e^{-v\theta} - 1)}{e^{-\theta} - 1} \right)$, for $\theta > 0$. We already saw that conditions of Theorem 3.18 hold and, therefore, an upper bound exists for the PELCOV_vs. In order to find the analytical form of the PELCOV_v and the bound, let us first note that:

$$\partial_1 C(u, v) = \frac{e^\theta (1 - e^{\theta v})}{e^{\theta(u+v)} + e^\theta (1 - e^{\theta u} - e^{\theta v})},$$

so, when $\theta > 0$, if we consider a vector $\mathbf{X} = (X, Y)$ with such copula, $Y \uparrow_{SI} X$. We can see that $\partial_1 C(u, v) \leq v$ if and only if

$$\frac{e^\theta (1 - e^{\theta v})}{v} \leq e^{\theta u} e^{\theta v} + e^\theta - e^{\theta u} e^\theta - e^{\theta(1+v)}.$$

That occurs if and only if $\frac{e^\theta (1 - e^{\theta v})}{v} - e^\theta + e^{\theta u + v} \leq e^{\theta u} (e^{\theta v} - e^\theta)$, which holds when $\frac{1}{\theta} \ln \left(\frac{e^\theta (1 - e^{\theta v}) - v e^\theta + v e^{\theta(1+v)}}{v (e^{\theta v} - e^\theta)} \right) \geq u$. Therefore, the PELCOV_v = u_v is given by:

$$u_v = \frac{1}{\theta} \ln \left(\frac{e^\theta (1 - e^{\theta v}) (1 - v)}{v (e^{\theta v} - e^\theta)} \right) \quad (3.23)$$

For $\theta > 0$, as the logarithm is an increasing function, u_v increases in v if $\frac{(1 - e^{\theta v})(1 - v)}{v (e^{\theta v} - e^\theta)}$ increases in v . Considering $f(v) = (1 - e^{\theta v})(1 - v)$ and $g(v) = v (e^{\theta v} - e^\theta)$, by Lemma 3.21, it would suffice to observe that $\frac{f'(v)}{g'(v)}$ increases in $v \in (0, 1)$. Now,

$$\frac{d}{dv} \left(\frac{f'(v)}{g'(v)} \right) = \frac{e^{v\theta} \theta (2 + e^{v\theta} \theta + v\theta + e^\theta (-2 + \theta - v\theta))}{(e^\theta - e^{v\theta} (1 + v\theta))^2} \quad (3.24)$$

In order to see whether (3.24) is positive, we need to see that the numerator, given by $h(v) = 2 + e^{v\theta} \theta + v\theta + e^\theta (-2 + \theta - v\theta)$, is positive for $v \in (0, 1)$, when $\theta > 0$. h is a continuous and derivable function, whose derivative is given by $h'(v) = \theta - e^{\theta v} \theta + e^{\theta v} \theta^2$, which is zero for $v_0 = \frac{1}{\theta} \ln \left(\frac{e^\theta - 1}{\theta} \right)$. Since $h''(v_0) = (e^\theta - 1) \theta^2 > 0$, v_0 is a minimum. Therefore, we do only

need to check that $h(0)$, $h(1)$ and $h(v_0)$ are positive when $\theta > 0$. We have $h(0) = 2 + e^\theta(\theta - 2) + \theta > 0$ and $h(v_0) = 1 + e^\theta(\theta - 1) + (e^\theta - 1) \ln \left(\frac{\theta}{e^\theta - 1} \right)$.

As $\ln(x) > 1 - x$ for all $x \in (0, 1)$, taking $x = \frac{\theta}{e^\theta - 1}$, we get

$$h(v_0) > 1 + e^\theta(\theta - 1) + (e^\theta - 1) \left(1 - \frac{\theta}{e^\theta - 1} \right) = (e^\theta - 1) \theta.$$

Hence, for the Frank copula, the PELCoV_v is increasing with respect to v . Therefore, in order to find u^* , we should only check what happens when v tends to 1. Using continuity and L'Hôpital's result, as $u_v = \frac{1}{\theta} \ln \left(\frac{e^\theta(1 - e^{\theta v})(1 - v)}{v(e^{\theta v} - e^\theta)} \right)$,

$$\begin{aligned} \lim_{v \rightarrow 1} u_v &= \frac{1}{\theta} \cdot \ln \left(\lim_{v \rightarrow 1} \frac{e^\theta(1 + e^{v\theta}(-1 + \theta - v\theta))}{e^\theta - e^{v\theta}(1 + v\theta)} \right) = \frac{1}{\theta} \ln \left(\frac{e^\theta(1 - e^\theta)}{e^\theta(1 - 1 + \theta)} \right) = \\ &= \frac{1}{\theta} \ln \left(\frac{e^\theta - 1}{\theta} \right) \end{aligned}$$

Therefore, we get that $u^* = \frac{1}{\theta} \ln \left(\frac{e^\theta - 1}{\theta} \right)$.

A.1.2 Clayton family of copulas

Considering the Clayton copula given by $C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ for $\theta > 0$, we saw in Section 3.4.1.3 that the copula is PDS and, since conditions of Theorem 3.18 hold, the PELCoV_v s are bounded. For this example, we can also obtain both the expressions of the PELCoV_s and their bound u^* analytically. Let us see that

$$\partial_1 C(u, v) = u^{-1-\theta} (u^{-\theta} + v^{-\theta} - 1)^{-1-\frac{1}{\theta}}.$$

Proceeding analogously as in the previous example, we see that $\partial_1 C(u, v) \leq v$ if and only if $u^\theta (u^{-\theta} + v^{-\theta} - 1) \geq v^{-\theta/(\theta+1)}$. This inequality holds when $u^\theta (v^{-\theta} - 1) \geq v^{-\theta/(\theta+1)} - 1$, that is, when $u \geq \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1} \right)^{1/\theta}$. So, the $\text{PELCoV}_v = u_v$ is given by:

$$u_v = \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1} \right)^{1/\theta}. \quad (3.25)$$

For $\theta > 0$, u_v increases in v if $\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1}$ increases in v . Considering $f(v) = v^{-\theta/(\theta+1)}$ and $g(v) = v^{-\theta}$ and the interval $(0, 1)$, by Lemma 3.21, it would suffice to see that $\frac{f'(v)}{g'(v)}$ increases in $v \in (0, 1)$. Now,

$$\frac{d}{dv} \left(\frac{f'(v)}{g'(v)} \right) = \frac{d}{dv} \left(\frac{v^{\theta - \frac{\theta}{1+\theta}}}{1 + \theta} \right) = \frac{v^{-2+\theta + \frac{1}{1+\theta}} \theta^2}{(1 + \theta)^2}$$

which is clearly positive. Therefore, u_v given by the expression in (3.25) increases in v and, using l'Hôpital's result,

$$\begin{aligned} u^* &= \lim_{v \rightarrow 1} \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1} \right)^{\frac{1}{\theta}} = \lim_{v \rightarrow 1} \left(\frac{-v^{-1-\frac{\theta}{1+\theta}}}{-v^{-1-\theta}} \right)^{\frac{1}{\theta}} \\ &= \lim_{v \rightarrow 1} \left(\frac{v^{\frac{\theta^2}{1+\theta}}}{1+\theta} \right)^{\frac{1}{\theta}} = (1+\theta)^{-\frac{1}{\theta}}. \end{aligned} \quad (3.26)$$

A.1.3 Ali-Mikhail-Haq family of copulas

Considering the AMH copula given by $C(u, v) = \frac{uv}{(1-\theta(1-u)(1-v))}$ for $\theta \in (0, 1]$, it was shown in Section 3.4.1.4 that the PELCoV $_v$ s were upperly bounded. In order to find the analytical expressions, let us note that

$$\partial_1 C(u, v) = \frac{v(1-\theta(1-v))}{(1-\theta(1-u)(1-v))^2}.$$

The SI condition trivially holds for $\theta \in (0, 1]$. We see that $\partial_1 C(u, v) \leq v$ if and only if $(1-\theta(1-v)) \leq (1-\theta(1-u)(1-v))^2$, that is, when $1 - \frac{1-\sqrt{1-\theta(1-v)}}{\theta(1-v)} \geq u$. We have that the PELCoV $_v$ is given by:

$$u_v = \frac{\theta(1-v) - 1 + \sqrt{1-\theta(1-v)}}{\theta(1-v)}. \quad (3.27)$$

In order to see that this is an increasing function of v , let us note that

$$\frac{d}{dv} u_v = \frac{2-\theta(1-v) \left(1 + 2\sqrt{\frac{1-\theta(1-v)}{(1-v)^2\theta^2}} \right)}{2(1-v)^3\theta^2\sqrt{\frac{1-\theta(1-v)}{(1-v)^2\theta^2}}}. \quad (3.28)$$

As the denominator of such expression is positive, it would be positive if $2 \geq \theta(1-v) \left(1 + 2\sqrt{\frac{1-\theta(1-v)}{(1-v)^2\theta^2}} \right) = \theta(1-v) + 2\sqrt{1-\theta(1-v)}$.

Note that $(1 - \sqrt{1-\theta(1-v)})^2 = 1 - \theta(1-v) - 2\sqrt{1-\theta(1-v)} + 1 = 2 - \theta(1-v) - 2\sqrt{1-\theta(1-v)}$. Therefore, (3.28) is positive and u_v increases in v . As a consequence, using L'Hôpital, we get that:

$$u^* = \lim_{v \rightarrow 1} \frac{\theta(1-v) - 1 + \sqrt{1-\theta(1-v)}}{\theta(1-v)} = \lim_{v \rightarrow 1} \frac{-\theta + \frac{\theta}{2\sqrt{1-\theta(1-v)}}}{-\theta} = \frac{1}{2}.$$

“A ship in port is safe, but that’s not what ships are built for”.

Grace Hopper

4

Stochastic Comparisons of Some Distances between Random Variables

The aim of this chapter is twofold. First, we show that the expectation of the absolute value of the difference between two copies, not necessarily independent, of a random variable is a measure of its variability in the sense of Bickel and Lehmann (1979). Moreover, if the two copies are negatively dependent through stochastic ordering, this measure is subadditive. The second purpose of this chapter is to provide sufficient conditions for comparing several distances between pairs of random variables (with possibly different distribution functions) in terms of various stochastic orderings. Applications in actuarial and financial risk management are given.

4.1 Introduction

Given a bivariate random vector $\mathbf{X} = (X_1, X_2)$ with joint distribution function $\mathbf{F}$ and marginal distribution functions F_1 and F_2 , the random variable $|X_1 - X_2|$ describes the distance between X_1 and X_2 in a sense that depends on the dependence structure of the vector. Different structures assign different meanings to this random variable and lead, obviously, to different ways of computing the expectation $E(|X_1 - X_2|)$. The distance can be applied to random variables with identical and non-identical distribution functions, and we consider both cases. If $F_1 = F_2 = F$, then X_1 and X_2 are copies of the same random variable X with distribution function F , and $E(|X_1 - X_2|)$ reveals information about X . An example is the case of considering independent and identically distributed random variables, in which $E(|X_1 - X_2|)$ is the Gini’s mean difference of X_1 , a well-known measure of variability (see, for example, Yitzhaki (2003)). We show in this work that, when X_1 and X_2 are

dependent copies of the same random variable X , $E(|X_1 - X_2|)$ is still a measure of variability of X . A purpose of this paper is to study the properties of this functional in a general setting, where X_1 and X_2 are not necessarily independent.

If X_1 and X_2 are independent (or, more generally, if they are linked by a symmetric dependence structure), $|X_1 - X_2|$ treats symmetrically the events $X_1 < X_2$ and $X_2 < X_1$. However, sometimes, it is convenient to use a characteristic of proximity by treating them differently (in finance, for example, an investor evaluates differently gains and losses). The random excess of X_1 over X_2 , $(X_1 - X_2)^+$, where we recall that $x^+ = \max\{x, 0\}$ denotes the positive part of x , is useful if we are interested in measuring the extent to which one random variable exceeds the other, rather than the distance between them in a bidirectional sense. Note that the absolute value $|X_1 - X_2|$ can be split into two terms, each describing the excess of one random variable over the other, as follows:

$$|X_1 - X_2| = (X_1 - X_2)^+ + (X_2 - X_1)^+.$$

If X_1 and X_2 are copies of the same variable X , then $E((X_1 - X_2)^+)$ also reveals information about X . For example, if X_1 and X_2 are independent, $E((X_1 - X_2)^+)$ is Gini's mean semidifference.

In general, the functional $E(\phi(|X_1 - X_2|))$, where ϕ is a non-negative real function, has been largely studied in mathematics literature, mainly in the context of the Monge–Kantorovich problem (see Rachev (1985), and references therein). The interest on this and other functionals used to measure the degree of difference between two random quantities goes back at least to the 1930s and the important contributions by Gini (see Giorgi (1990)) and Hoeffding (Hoeffding (1950)).

Given two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, another purpose of this chapter is to find conditions under which

$$E(\Phi(|X_1 - X_2|)) \leq E(\Phi(|Y_1 - Y_2|)), \text{ for all } \Phi \in \Omega, \quad (4.1)$$

where Ω is a subset of increasing real functions. Different choices of Ω give rise to different stochastic orderings between $|X_1 - X_2|$ and $|Y_1 - Y_2|$. This problem was addressed in Giovagnoli and Wynn (1995); Kochar and Carrière (1997); Bassan et al. (1999) for the case where $\mathbf{X}$ and $\mathbf{Y}$ have independent components with the same marginal distribution functions (see Section 4.2 below for details). Here, we are concerned with two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, whose components are not necessarily independent nor are they required to have identical distribution functions. In this case, we

explore conditions under which

$$|X_1 - X_2| \leq_{st, icx} |Y_1 - Y_2| \text{ and } (X_1 - X_2)^+ \leq_{st, icx} (Y_1 - Y_2)^+, \quad (4.2)$$

where $\leq_{st}$ and $\leq_{icx}$ are the usual stochastic order and the increasing convex order, respectively (defined in Section 1.2.1).

This chapter is organized as follows. Section 4.2 contains preliminaries, such as background needed after. In Section 4.3, given a random variable X with distribution function F , we show that any functional of the form $\nu(X) = E(|X_1 - X_2|)$, where X_1 and X_2 are two copies of X with any type of dependence structure, is a measure of variability of X . More generally, the distribution function F_1 of X_1 is allowed to be a distortion of F (we will explain the meaning of this below). In Section 4.4, given two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, we obtain conditions (both in terms of the marginals and the copulas) to make comparisons of the form (4.2). Section 4.5 contains two applications. In Section 4.5.1, we define a general class of premium principles based on the class of variability measures studied in Section 4.3. In Section 4.5.2, in the context of portfolio risk management, we assess the inclusion of a new asset in a portfolio by using the results obtained in Section 4.4. Finally, Section 4.6 contains conclusions.

4.2 Preliminaries

We will consider bivariate random vectors and, throughout this chapter, given two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, we denote by F_1, F_2 and G_1, G_2 the respective marginal distribution functions and C, C' the respective copulas. All this functions will be considered continuous and derivable, unless it is otherwise stated.

The motivation for the study of the properties of $E(|X_1 - X_2|)$, where X_1 and X_2 are not necessarily independent, comes from the fact that some probability metrics and measures of variability that are sometimes better known under other expressions, take this form for different copulas between X_1 and X_2 . To give some examples, note that

$$E(|X_1 - X_2|) = \int_{-\infty}^{\infty} (F_1(x) + F_2(x) - 2C(F_1(x), F_2(x))) dx. \quad (4.3)$$

If X_1 and X_2 are two copies of a random variable X with distribution function $F(x)$, (4.3) becomes

$$E(|X_1 - X_2|) = 2 \int_{-\infty}^{\infty} (F(x) - C(F(x), F(x))) dx. \quad (4.4)$$

If X_1 and X_2 are two independent copies of X , then

$$E(|X_1 - X_2|) = 2(E(\max(X_1, X_2)) - E(X))$$

is the Gini's mean difference (GMD) of X , a well-known index of variability (see, for example, Yitzhaki (2003)). If X_1 and X_2 are comonotonic, then (see Dall'Aglia (1956) or Vallender (1974))

$$E(|X_1 - X_2|) = \int_0^1 |F_1^{-1}(u) - F_2^{-1}(u)| du = \int_{-\infty}^{\infty} |F_1(x) - F_2(x)| dx, \quad (4.5)$$

which is the Wasserstein distance, a well-known characteristic of proximity of two random variables (see Kantorovich and Rubinstein (1958)). If X_1 and X_2 are countermonotonic, then

$$E(|X_1 - X_2|) = \int_0^1 |F_1^{-1}(u) - F_2^{-1}(1 - u)| du \quad (4.6)$$

(see, for example, Rachev (1985)). It is easy to see that, if X_1 and X_2 are two copies of X , (4.6) can be rewritten as

$$E(|X_1 - X_2|) = 2E(|X - m_X|),$$

where m_X is the median of X . This measure is twice the median absolute deviation (MAD), another popular measure of variability.

In view of the above examples, it is natural to ask whether $E(|X_1 - X_2|)$, where X_1 and X_2 are two copies of X with a copula C , fulfills the requirements to be considered as a measure of variability of X . Recall that, in short, a measure of variability ν is a map from the set of random variables to $\mathbb{R}$, such that given a random variable X , $\nu(X)$ quantifies the variability of X . For the formal definition and required properties one can check Section 1.6.

When X_2 and Y_2 are independent copies of X_1 and Y_1 , respectively, it is well-known (see Giovagnoli and Wynn (1995) and Kocher and Carrière (1997)) that

- $X_1 \leq_{disp} Y_1$ implies $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$,
- $X_1 \leq_{ew,cx} Y_1$ implies $|X_1 - X_2| \leq_{icx} |Y_1 - Y_2|$.

The result for the convex order was extended to the so-called s-convex order in Bassan et al. (1999). In Sections 4.3 and 4.4, we extend these results to the case where X_2 and Y_2 are not necessarily independent from X_1 and Y_1 , respectively. For this, to see under which dependence framework the results can be generalized, one should recall the dependence notions seen in Section 1.3.3.

4.3 A Family of Measures of Variability

Let X_1 and X_2 be two random variables with respective continuous distribution functions F_1 and F_2 and finite expectations. If X_1 and X_2 are two independent copies of X , it is well-known (see Bickel and Lehmann (1979)) that $\nu(X) = E(|X_1 - X_2|)$ is a measure of variability in the sense of Bickel and Lehmann (that is, it satisfies properties (P0) to (P4)). Let h be a distortion function, that we recall, is a non-decreasing function from $[0, 1]$ to $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$. In this section, we show that any functional of the form $E(|X_1 - X_2|)$, where $F_2 = F$ and $F_1 = h \circ F$, is a measure of variability of X . In particular, if h is the identity function ($h(t) = t$, for all $t \in [0, 1]$) and X_1 and X_2 have a NDS copula, this measure satisfies all the properties (P1 to P7) listed in Section 1.6.

The following theorem extends a result of Giovagnoli and Wynn (1995), stated as Theorem 3.B.42 in the book Shaked and Shanthikumar (2007), in two directions: first, we consider two random vectors with the same copula instead of two random vectors with independent components; and, second, we allow the first marginal of each vector to be a distortion of the other (via the same h) instead of taking two copies of the same random variable.

Theorem 4.1. *Let X and Y be two random variables with distribution functions F and G , respectively and let h be a distortion function. Let $\mathbf{X} = (X_1, X_2)$ be a random vector with respective marginal distribution functions $F_1 = h \circ F$ and $F_2 = F$. Similarly, let $\mathbf{Y} = (Y_1, Y_2)$ be a random vector with marginal distribution functions $G_1 = h \circ G$ and $G_2 = G$, respectively. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ have the same copula C . If $X \leq_{disp} Y$, then $|X_2 - X_1| \leq_{st} |Y_2 - Y_1|$.*

Proof. Since the dispersive order is preserved by distortion functions (Theorem 13 in Sordo et al. (2015b)), we have $X_1 \leq_{disp} Y_1$ and $X_2 \leq_{disp} Y_2$. Since $\mathbf{X}$ and $\mathbf{Y}$ have the same copula, it follows from Definition 2.1 in Fernández-Ponce and Suárez-Llorens (2003) and Theorem 1 in Arias-Nicolás et al. (2005) that there exists a function Φ that maps stochastically (X_1, X_2) into (Y_1, Y_2) , i.e., $\Phi(X_1, X_2) =_{st} (Y_1, Y_2)$, defined as

$$\Phi(x_1, x_2) = (\Phi_1(x_1), \Phi_2(x_2)) = (G_1^{-1}(F_1(x_1)), G_2^{-1}(F_2(x_2))),$$

where $\Phi_i(\cdot)$, $i = 1, 2$, is an increasing function that satisfies

$$|\Phi_i(x) - \Phi_i(x')| \geq |x - x'|, \quad \forall x, x' \in \mathbb{R}. \quad (4.7)$$

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It follows from the assumptions that $\Phi_2(x) = \Phi_1(x) = G^{-1}F(x)$ for all x . Therefore,

$$\begin{aligned} |Y_2 - Y_1| &=_{st} |\Phi_2(X_2) - \Phi_1(X_1)|, \\ &=_{st} |\Phi_1(X_2) - \Phi_1(X_1)|, \\ &\geq_{st} |X_2 - X_1|, \end{aligned} \quad (4.8)$$

where the first and second equality in (4.8) follow, respectively, from the facts that $\Phi(X_1, X_2) =_{st} (Y_1, Y_2)$ and $\Phi_2(\cdot) = \Phi_1(\cdot)$. The inequality follows from (4.7) by using Theorem 1.A.1 in Shaked and Shanthikumar (2007).

By taking $h(x) = x$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let X_2 and Y_2 be two copies of X_1 and Y_1 , respectively, such that $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ have the same copula. If $X_1 \leq_{disp} Y_1$, then $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$.*

Example 4.3. *In order to illustrate Corollary 4.2, we consider two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, where $X_1, X_2 \sim N(0, \sigma_X)$, $Y_1, Y_2 \sim N(0, \sigma_Y)$, both have the copula 9 in Table 4.1 in Nelsen (2007), given, for $\theta \in (0, 1]$, by:*

$$C_\theta(u, v) = uv e^{-\theta \text{Log}(u) \text{Log}(v)}, \quad (4.9)$$

which verifies the NDS property. Then, if $\sigma_X \leq \sigma_Y$, it follows $X_1 \leq_{disp} Y_1$ (see Table 2.2 in Belzunce et al. (2015)) and, therefore, by Corollary 4.2, we have $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$. In Figure 4.1, we can see $\bar{F}_{|X_1 - X_2|}$ considering different values of σ_X .

Remark 4.4. *Given two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ with the same copula, the condition $X_i \leq_{disp} Y_i$, $i = 1, 2$ is equivalent to say that the bivariate random vectors $\mathbf{X}$ and $\mathbf{Y}$ are ordered in a multivariate dispersion sense, see Belzunce et al. (2008).*

Now, we can prove the following result.

Theorem 4.5. *Let X be a random variable with strictly increasing distribution function F and let h be a strictly increasing distortion function. Let X_1 and X_2 be two random variables with copula C and marginal distribution functions F_1 and F_2 , respectively. Let $\nu_C(X) = E(|X_1 - X_2|)$.*

- (i) *If $F_1 = h \circ F$ and $F_2 = F$, then $\nu_C(X)$ is a comonotonic additive measure of variability in the sense of Bickel and Lehmann, that is, it satisfies properties (P0)–(P4) and (P6).*

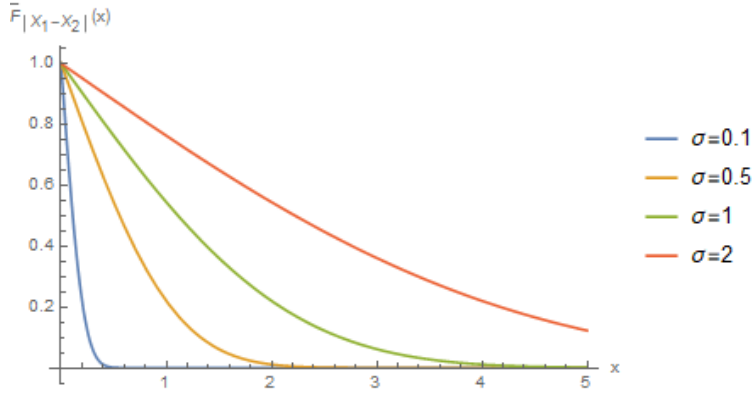


Figure 4.1. Survival function of the absolute difference $\bar{F}_{|X_1 - X_2|}(x)$, considering a vector $\mathbf{X} = (X_1, X_2)$ with copula $C_{0.5}$, as given in (4.9) and $X_1, X_2 \sim N(0, \sigma)$, for different values of σ .

(ii) If $F_1 = F_2 = F$ and the copula C is NDS, then $\nu_C(X)$ satisfies all the properties (P0) to (P7).

Proof. We first prove (i). Let C be the copula of X_1 and X_2 . From (4.3), we have

$$\begin{aligned} \nu_C(X) &= \int_{-\infty}^{\infty} (h(F(x)) + F(x) - 2C(h(F(x)), F(x))) dx \\ &= \int_0^1 (h(u) + u - 2C(h(u), u)) dF^{-1}(u). \end{aligned} \quad (4.10)$$

Clearly, $\nu_C(X) = 0$ if X is degenerated at $c \in \mathbf{R}$. This, together with the fact that F^{-1} is non-decreasing, $F_{X+k}^{-1}(x) = F_X^{-1}(x) + k$, for all k and $F_{\lambda X}^{-1}(x) = \lambda F_X^{-1}(x)$, for all $\lambda > 0$ (see Parzen (1979)), ensures that $\nu_C(X)$ satisfies properties (P0) to (P3). Since, given two random variables Z_1 and Z_2 , the condition $Z_1 \leq_{st} Z_2$ implies that $E(Z_1) \leq E(Z_2)$, property (P4) (consistency of $\nu_C(X)$ with respect to the dispersive order) is a direct consequence of Theorem 4.1. Property (P6) follows from the fact that, if Z_1 and Z_2 are comonotonic, then $F_{Z_1+Z_2}^{-1}(u) = F_{Z_1}^{-1}(u) + F_{Z_2}^{-1}(u)$, for all $u \in (0, 1)$ (see Wang (1996)). Under the assumptions in (ii), we have

$$\nu_C(X) = 2 \int_0^1 (u - C(u, u)) dF^{-1}(u). \quad (4.11)$$

Standard arguments show that

$$\lim_{u \rightarrow i} F^{-1}(u) (u - C(u, u)) = 0, \quad i = 0, 1.$$

Therefore, integrating (4.11) by parts, we get

$$\nu_C(X) = 2 \int_0^1 F^{-1}(u) d(C(u, u) - u).$$

Since C is componentwise convex, (P5) follows from Theorem 8. (ii) in Sordo (2008) and (P7) follows from Theorem 2.1 in Furman et al. (2017).

Example 4.6. Two functionals satisfying the assumptions of part (i) are $\nu_{C_1}(X) = GMD(X)$ and $\nu_{C_2}(X) = \int_{-\infty}^{\infty} |F_h(x) - F(x)| dx$, which is the Wasserstein distance between F and its distortion $F_h = h \circ F$, a variability measure introduced by López-Díaz et al. (2012). Note that $\nu_{C_2}(X) = E(|X_1 - X_2|)$, where $F_1 = h \circ F$, $F_2 = F$ and C_2 is the Fréchet-Hoeffding upper bound copula (see (4.5)).

Example 4.7. Using (4.6), it follows from Theorem 4.5 (ii) that $\nu_C(X) = E(|X - m_X|)$, where m_X is the median of X , satisfies all the properties (P0) to (P7) listed above. This measure can be written in the form $\frac{1}{2}E(|X_1 - X_2|)$ where $F_1 = F_2 = F$ and where C is the Fréchet-Hoeffding lower bound copula (see (4.6) and the paragraph below it), which is an example of NDS copula (see Li and Li (2018) for this and other examples of NDS copulas).

4.4 Other Stochastic Comparisons

To begin this section, we consider two random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ with the same marginals. Denote by $\mathcal{R}(F_1, F_2)$ the space of bidimensional random vectors with marginal distribution functions F_1 and F_2 .

Theorem 4.8. Let $\mathbf{X} = (X_1, X_2) \in \mathcal{R}(F_1, F_2)$ and $\mathbf{Y} = (Y_1, Y_2) \in \mathcal{R}(F_1, F_2)$ be two random vectors with copulas C and C' , respectively. If $C' \prec C$, then:

- (i) $|X_1 - X_2| \leq_{icx} |Y_1 - Y_2|$.
- (ii) $(X_1 - X_2)^+ \leq_{icx} (Y_1 - Y_2)^+$.

Proof. Under the assumptions, it follows from Theorem 4 of Müller (2000) that $X_1 - X_2 \leq_{icx} Y_1 - Y_2$. This means that $E(\Phi(X_1 - X_2)) \leq E(\Phi(Y_1 - Y_2))$ for all increasing convex Φ . Since $\Phi(t) = \Psi(|t|)$ is increasing and convex for any increasing convex function Ψ , it follows that $E(\Psi|X_1 - X_2|) \leq E(\Psi|Y_1 - Y_2|)$ for all increasing and convex Ψ , which proves (i). The proof of (ii) is similar using the function $\Phi(t) = \Psi((t)^+)$.

Remark 4.9. An alternative proof of Theorem 4.8 can be given by using Theorem 1 in Cambanis et al. (1976), which provides conditions to ensure, under the above assumptions, that $E(k(X_1, Y_2)) \geq E(k(Y_1, Y_2))$ for certain classes of functions k . The proof is based on proving that the functions $k(x, y) = \phi|x - y|$ and $k(x, y) = \phi(x - y)^+$, with ϕ increasing and convex, satisfy those conditions.

A more general type of comparison can be made between two random vectors with possibly different (but stochastically ordered) marginals. The following two results provide conditions to compare two random excesses. The first result is given in terms of the usual stochastic order and the second result in terms of the increasing convex order.

Theorem 4.10. Let $\mathbf{X} = (X_1, X_2)$ be a random vector with respective marginal distribution functions F_1 and F_2 . Similarly, let $\mathbf{Y} = (Y_1, Y_2)$ be a random vector with marginal distribution functions G_1 and G_2 , respectively. If $\mathbf{X}$ and $\mathbf{Y}$ have the same copula C , $X_1 \leq_{st} Y_1$ and $X_2 \geq_{st} Y_2$, then $(X_1 - X_2)^+ \leq_{st} (Y_1 - Y_2)^+$.

Proof. For $x \geq 0$,

$$\begin{aligned} \bar{F}_{(X_1 - X_2)^+}(x) &= 1 - \int_0^1 P[X_1 \leq F_2^{-1}(p) + x | X_2 = F_2^{-1}(p)] dp \\ &= 1 - \int_0^1 \partial_2 C(F_1(F_2^{-1}(p) + x), p) dp, \quad x > 0. \end{aligned}$$

Therefore, given $x \geq 0$,

$$\begin{aligned} &\bar{G}_{(Y_1 - Y_2)^+}(x) - \bar{F}_{(X_1 - X_2)^+}(x) \\ &= \int_0^1 (\partial_2 C(F_1(F_2^{-1}(p) + x), p) - \partial_2 C(G_1(G_2^{-1}(p) + x), p)) dp \geq 0, \end{aligned}$$

Since $G_1(x) \leq F_1(x)$ for all x , $G_2^{-1}(p) \leq F_2^{-1}(p)$ for all $p \in (0, 1)$ and $\partial_2 C$ increases in the first argument (since $\partial_2 C(u, p)$ is the distribution function of the random variable $[U|V = p]$). Therefore, $\bar{F}_{(X_1 - X_2)^+}(x) \leq \bar{G}_{(Y_1 - Y_2)^+}(x)$ for all $x \geq 0$, which ends the proof.

Theorem 4.11. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors with copulas C and C' , and marginal distribution functions F_1, F_2 and G_1, G_2 , respectively. If C is NDS, $C' \prec C$, $X_1 \leq_{icx} Y_1$ and $X_2 \geq_{icv} Y_2$, then $(X_1 - X_2)^+ \leq_{icx} (Y_1 - Y_2)^+$.

Proof. Let us consider a vector $\mathbf{Y}^* = (Y_1^*, Y_2^*)$ with copula C and such that $Y_i^* =_{st} Y_i$ for $i = 1, 2$. From the assumptions, it follows that $X_2 \geq_{icv} Y_2^*$, which is equivalent to say that $-X_2 \leq_{icx} -Y_2^*$ (Theorem 4.A.1 in Shaked and Shanthikumar (2007)). Since $\mathbf{X}$ and $\mathbf{Y}^*$ have the same copula C , the random vectors $\hat{\mathbf{X}} = (X_1, -X_2)$ and $\hat{\mathbf{Y}} = (Y_1^*, -Y_2^*)$ have the same copula $\hat{C}(u, v) = u - C(u, 1 - v)$. Moreover, since C is NDS (that is, componentwise convex), then $\hat{C}(u, v)$ is PDS (that is, componentwise concave). It follows from Corollary 2.7 in Balakrishnan et al. (2012) that $X_1 - X_2 \leq_{icx} Y_1^* - Y_2^*$. Since $C' \prec C$, it follows from Theorem 4.8 that $Y_1^* - Y_2^* \leq_{icx} Y_1 - Y_2$. The result follows by using the fact that the increasing convex order is transitive and is preserved by the increasing convex transformation $\phi(t) = t^+$ (see Theorem 4.A.8(a) in Shaked and Shanthikumar (2007)).

Remark 4.12. *In particular, Theorem 4.11 holds when $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ have the same NDS copula C . In this case, $X_1 \leq_{icx} Y_1$ and $X_2 \geq_{icv} Y_2$ imply $(X_1 - X_2)^+ \leq_{icx} (Y_1 - Y_2)^+$.*

Lemma 4.13. *Let X and Y be two random variables that are symmetric about 0. Then:*

- (i) *If $X^+ \leq_{st} Y^+$, then $|X| \leq_{st} |Y|$.*
- (ii) *If $X^+ \leq_{icx} Y^+$, then $|X| \leq_{icx} |Y|$.*

Proof. Let $\bar{F}_{X^+}$ and $\bar{F}_{|X|}$ be the tail functions of X^+ and $|X|$, respectively. If X and Y are symmetric about 0, it is easy to see that $\bar{F}_{|X|}(t) = h(\bar{F}_{X^+}(t))$ for all t , where h is the concave distortion function

$$h(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 1/2 \\ 1 & \text{if } 1/2 < t \leq 1. \end{cases}$$

Now (i) and (ii) follow, respectively, from Theorem 2.6 (i) and Theorem 2.6 (v) in Navarro et al. (2013).

The following corollaries extend Lemma 2.2 in Kochar and Carrière (1997) from the case of two random vectors with independent components to the case of two random vectors with the same symmetric NDS copula.

Corollary 4.14. *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors with the same symmetric NDS copula C and with marginal distribution functions F_1, F_2 and G_1, G_2 , respectively. We consider $F_1 = F_2$ and $G_1 = G_2$. If $X_1 \leq_{cx} Y_1$, then $|X_1 - X_2| \leq_{icx} |Y_1 - Y_2|$.*

Proof. The assumption $X_1 \leq_{cx} Y_1$ implies $X_1 \leq_{icx} Y_1$. Since $X_2 \leq_{cx} Y_2$ holds if and only if $-X_2 \leq_{cx} -Y_2$ (Theorem 3.A.12 in Shaked and Shanthikumar (2007)), it follows $-X_2 \leq_{icx} -Y_2$. This is equivalent to write $X_2 \geq_{icv} Y_2$; therefore, as $X_1 - X_2$ and $Y_1 - Y_2$ are symmetric about 0, the result follows from Lemma 4.13.

Corollary 4.15. *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors with the same symmetric NDS copula, such that $X_2 =_{st} X_1$ and $Y_1 =_{st} Y_2$, all variables having finite means. If $X_1 \leq_{ew} Y_1$, then $|X_1 - X_2| \leq_{icx} |Y_1 - Y_2|$.*

Proof. It is well-known (see (3.C.7) in Shaked and Shanthikumar (2007)) that $X_1 \leq_{ew} Y_1$ implies $X_1 - E(X_1) \leq_{cx} Y_1 - E(Y_1)$. The result follows from Corollary 4.14.

4.5 Applications

4.5.1 An Application in Actuarial Science

In actuarial theory, a premium principle is a decision rule used by the insurer in order to determine the price for a risk to be insured. More formally, given a random variable X describing an insurance risk, a premium principle T assigns to X a number $T(X)$ which is the premium to be charged for making acceptable the risk X (see Young (2004) for an overview). The simplest premium principle is the net premium $T(X) = E(X)$, which does not load for risk. More general premium principles are obtained by adding a load to the net premium that reflects the danger associated with the risk. Since the danger is often interpreted in terms of variability, a number of premium principles are obtained by adding to the net premium a risk load proportional to a specific measure of variability. Examples include the following:

- $T_1(X) = E(X) + \lambda \sqrt{\text{Var}[X]}$, with $\lambda > 0$ (standard deviation premium principle)
- $T_2(X) = E(X) + \lambda GMD(X)$, with $\lambda > 0$ (Gini's premium principle)
- $T_3(X) = E(X) + \lambda E(|X - m_X|)$, with $\lambda > 0$ (Denneberg's premium principle)
- $T_4(X) = E(X) + \int_{-\infty}^{\infty} |F_h(x) - F(x)|^p dx$, where $p \geq 0$ and h is a distortion function (see López-Díaz et al. (2012) and Yang et al. (2014)).

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Other examples can be found in Sordo et al. (2016a). Following this schema, we define a general class of premium principles based on distances between random variables.

Definition 4.16. *Given a risk X with distribution function F , let $\mathfrak{F}$ be the family of premium principles of the form*

$$T_C(X) = E(X) + \lambda E(|X_1 - X_2|), \text{ for some } \lambda > 0,$$

where X_1 and X_2 are two copies of X such that X_1 and X_2 have copula C .

From the results in Section 2, we see that T_2, T_3 and T_4 (for $p = 1$) are premium principles that belong to the family $\mathfrak{F}$ for different choices of the copula C . Moreover, it follows from Theorem 4.5 that a premium principle $T_C \in \mathfrak{F}$ satisfies the following properties:

- (a) Risk loading: $T_C(X) \geq E(X)$.
- (b) Non unjustified risk loading: If $X = k \geq 0$ ($k \in \mathbb{R}$), then $T_C(X) = k$.
- (c) Translation invariance: $T_C(X + k) = T_C(X) + k$ for all constant k .
- (d) Scale invariance: $T_C(bX) = bT_C(X)$ for all constant $b > 0$.
- (e) Comonotonic additivity: if X and Y are comonotonic random variables, then $T_C(X + Y) = T_C(X) + T_C(Y)$.
- (f) Subadditivity: if C is NDS, then $T_C(X + Y) \leq T_C(X) + T_C(Y)$ for all X and Y .

A premium principle satisfying the above properties that does not follow the schema $T(X) = E(X) + \lambda D(X)$, where D is a measure of variability of X , is the distortion premium principle (Wang (1996)), defined by

$$T_h(X) = - \int_{-\infty}^0 (1 - h[\bar{F}(x)])dx + \int_0^{\infty} h[\bar{F}(x)]dx,$$

where h is a concave distortion function. T_h corresponds to the distorted expectation defined in (1.10). When h is concave, T_h satisfies risk loading, non unjustified risk loading, translation invariance, scale invariance, comonotonic additivity and subadditivity.

Our next result is related to a property of T_h . Recall that, given a random variable Z with distribution function F_Z , the family

$$\Pi_Z \equiv \{X = \mu + \sigma Z : \mu \in \mathbb{R}, \sigma > 0\}$$

is called a location-scale family of random variables. In Young (1999), it is shown that T_h reduces to T_1 (the standard deviation premium principle) for

location-scale families of distributions. Next, we give a similar result involving the premium principle T_C given in Definition 4.16.

Theorem 4.17. *Consider a location scale family Π_Z and let $T_C \in \mathfrak{F}$. Then, T_C reduces to T_1 (standard deviation premium principle) or, equivalently, to T_h (distortion premium principle) on Π_Z .*

Proof. Since T_1 is a special case of T_h on location-scale families of distributions (Young (1999)), it suffices to prove that T_C reduces to T_h on Π_Z . Let $X \in \Pi_Z$. Since $F_X(x) = F_Z(\frac{x-\mu}{\sigma})$ for all x , if X_1 and X_2 are two copies of X with copula C , we have

$$\begin{aligned} T_C(X) &= E(X) + \lambda E[|X_1 - X_2|] \\ &= \mu + \sigma E(Z) + 2\lambda \int_{-\infty}^{\infty} (F_X(x) - C(F_X(x), F_X(x))) dx \\ &= \mu + \sigma E(Z) + 2\lambda \int_{-\infty}^{\infty} (F_Z(z) - C(F_Z(z), F_Z(z))) \sigma dz \\ &= \mu + \sigma E(Z) + 2\lambda \sigma E[|Z_1 - Z_2|], \end{aligned}$$

where Z_1 and Z_2 are two copies of Z with copula C (here, we have used that copulas are invariant for strictly monotone transformations of the random variables). Now, we equate this expression with $T_h(X)$ to obtain

$$\lambda = \frac{T_h(Z) - E(Z)}{E[|Z_1 - Z_2|]}$$

where we have used that $T_h(X) = \mu + \sigma T_h(Z)$, because T_h is scale and translation invariant. Observe that λ is independent of μ and σ ; therefore, we conclude that T_C reduces to T_h on C .

4.5.2 An Application in Portfolio Risk Management

In portfolio risk management, investors diversify portfolios to reduce market risk. On average, a portfolio with several assets exhibits, unless they are perfectly correlated, less variability in returns than a portfolio with only one asset. To illustrate the results in Section 4, let us consider an investor whose portfolio has only one asset A with log-return X_2 . During the diversification process, the investor is concerned with the risk of two assets B and C with log-returns X_1 and Y_1 , respectively, that might be included in her/his portfolio. Here, the risk of these two assets B and C must be considered in relation to the asset A, which acts as a hedge.

One way to assess the impact of each of these two assets is by comparing the distances $(X_1 - X_2)^+$ and $(Y_1 - Y_2)^+$ in some stochastic sense. The

study of these random excesses has an appealing role in risk management and quantitative finance. If $X_1(t)$ and $X_2(t)$ are the prices of two risky assets at time t , the payoff of the option that gives the buyer the right to exchange the second asset for the first at the expiry time t (called exchange option) is $(X_1(t) - X_2(t))^+$. If assets B and C move in the opposite direction as hedge A, greater distance suggests a more significant effect in counteracting small values of the hedged asset. An alternative method to assess the impact of B and C on the portfolio is by using measures of contagion (see Section 4 in Ortega-Jiménez et al. (2021)).

The log return of an asset at week t is defined by $r_t = \log(p_t/p_{t-1})$, where p_t is the price of the asset at week t . For our empirical example, we work with log-returns of three stocks included in Nasdaq Composite index: Zoom Video Communications (X_1), Moderna (Y_1) and Booking Holdings Inc ($X_2 = Y_2$) (we have selected three companies that were affected differently at the beginning of the financial crisis of COVID-19). The study is based on samples of size $n = 64$ for each financial institution ($\{x_{1i}\}$, $\{y_{1i}\}$ and $\{x_{2i} = y_{2i}\}$, for $i = 1, \dots, 64$), measuring the share value from 23 December 2019 until 15 March 2021. Data were gathered from the public website <http://es.finance.yahoo.com>, (accessed on March 22, 2021), and are related to the weekly close of trading to eliminate the time dependent effect. Suppose that, initially, the whole portfolio of our investor consists of stocks in only one company: Booking Holdings, Inc. To reduce risk, the investor plans to invest either in Zoom or Moderna and faces the problem of which of them should be chosen. A method that helps to make a decision, as explained above, is to compare the excesses of the new component with respect to the hedge considering the random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$.

Figure 4.2 plots the empirical distribution functions of sample excesses $(x_{1i} - x_{2i})^+$ between Zoom and Booking (green curve) and $(y_{1i} - y_{2i})^+$ between Moderna and Booking (blue curve), for $i = 1, \dots, 64$. Both curves start equal or with the blue one lying above; at some point around $x = 0.05$, it crosses from above to below the green curve and, after that point, it seems to be everywhere below. The graphic is consistent with a model where $(X_1 - X_2)^+$ and $(Y_1 - Y_2)^+$ are ordered in the increasing convex order. Next, we give statistical significance to this conclusion.

We first perform some tests to study the marginal distributions of X_1, X_2 and Y_1 . In order to check randomness, the classical runs test is performed with p -values 0.6143, 0.3134 and 0.2077, respectively. Symmetry is tested using the symmetry test by Miao et al. (2006), obtaining p -values 0.388, 0.81, and 0.174,

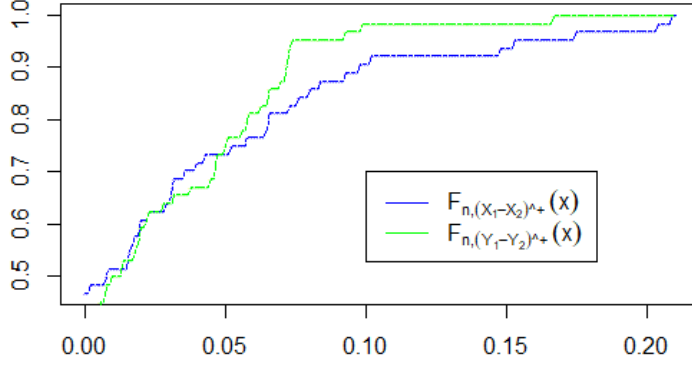


Figure 4.2. Distribution functions of empirical absolute differences. The green curve corresponds to $F_{n,|X_1-X_2|}^+(x)$ and the blue curve to $F_{n,|Y_1-Y_2|}^+(x)$.

respectively. The Kolmogorov–Smirnov test for normality gives, respectively, the p -values 0.5424, 0.9154, and 0.3486. Therefore, there is not significant evidence to reject the hypothesis that the three log return distributions are random, symmetric, and normal.

A unilateral F-test for paired data, performed for testing the hypothesis of equality of variances against $\sigma_{X_1} < \sigma_{Y_1}$, gives a p -value of 0.000493, showing significant evidence that $\sigma_{X_1} < \sigma_{Y_1}$. The p -value of the t -test for testing $\mu_{X_1} = \mu_{Y_1}$ against $\mu_{X_1} \neq \mu_{Y_1}$ is 0.7124, so we can not reject the equality of means. From the assumptions of normality, $\mu_{X_1} = \mu_{Y_1}$ and $\sigma_{X_1} < \sigma_{Y_1}$, it follows $X_1 \leq_{icx} Y_1$ (see Table 2.2 in Belzunce et al. (2015)).

The copulas C and C' are adjusted by using the goodness of fit test based on Kendall's process (Wang and Wells (2000); Genest et al. (2006)). Considering a bivariate normal (BN) copula, we obtain p -values 0.74 and 0.79, respectively; therefore, there is not statistical evidence to reject that C and C' are BN. Since the bivariate normal copula parameter is the Pearson correlation coefficient $\rho_{\mathbf{X}}$, we perform the Williams's Test (bilateral) (Steiger (1980); Williams (1959)) for testing the hypothesis $\rho_{\mathbf{X}} = \rho_{\mathbf{Y}}$ against $\rho_{\mathbf{X}} \neq \rho_{\mathbf{Y}}$ when the vectors share one component ($X_2 = Y_2$). The p -value, 0.4622, indicates that we cannot reject the equality, which leads us to admit that $C = C' \sim BN(\rho)$. Since the sample estimate of the Pearson's correlation coefficient is negative, we test $\rho = 0$ against $\rho < 0$ by running the test for association between paired samples using Pearson's correlation coefficient for the vector $\mathbf{X}$. The

p -value, 0.0442, suggests that $\rho < 0$, which means that the copulas are NDS (see Example 4.1 in Block et al. (1985)).

To conclude: the assumptions (1) $X_1 \leq_{icx} Y_1$ and that (2) the copulas C and C' are equal and are NDS, are supported by statistical significance. It follows from Remark 17 that $(X_1 - X_2)^+ \leq_{icx} (Y_1 - Y_2)^+$, which indicates that Moderna would lead to a more significant effect than Zoom in counteracting small values of Booking.

4.6 Conclusions

Given two random variables X_1 and X_2 that are not necessarily independent, we have provided several results concerning the distances $|X_1 - X_2|$, $(X_1 - X_2)^+$ and their expectations. The most remarkable results of this study can be summarized as follows:

- (a) If X is a random variable with strictly increasing distribution function F and X_1 and X_2 are two random variables with a NDS (negative dependent through stochastic ordering) copula C and with marginal distribution functions $F_1 = F_2 = F$, then $\nu(X) = E(|X_1 - X_2|)$ is a variability measure satisfying the following properties: law invariance, translation invariance, positive homogeneity, non-negativity, consistency with the dispersive order, consistency with the excess wealth order, comonotonic additivity, and subadditivity. An example is the median absolute deviation $\nu(X) = E(|X - m_X|)$, where m_X is the median of X , which can be written in the form $\frac{1}{2}E(|X_1 - X_2|)$, where C is the Fréchet–Hoeffding lower bound copula.
- (b) Given two random vectors (X_1, X_2) and (Y_1, Y_2) with possibly different marginals and copulas, we have given conditions, in terms of several stochastic orders, under which $|X_1 - X_2| \leq_{st, icx} |Y_1 - Y_2|$ and $(X_1 - X_2)^+ \leq_{st, icx} (Y_1 - Y_2)^+$.

Two applications have been provided. In actuarial science, given a risk X , we have proposed a general class of premium principles of the form $T_C(X) = E(X) + \lambda E(|X_1 - X_2|)$, for some $\lambda > 0$, where X_1 and X_2 are two copies of X with copula C . In portfolio risk management, we have assessed the inclusion of a new asset in a portfolio by comparing absolute values of differences. It is a question of future research to determine the circumstances under which the criterion used in Section 4.5.2 to include a new asset in a portfolio gives rise to a portfolio with smaller realized variance.

“The definition of a good mathematical problem is the mathematics it generates rather than the problem itself.”

Andrew Wiles

5

A minimization problem of distances between random variables with proportional reversed hazard rate functions.

Let X be a continuous random variable with distribution function F and let $\mathcal{F}_X$ be the family of proportional reversed hazard rate distribution functions associated to F . Given the random vector (X, Y) with copula C and respective marginal distribution functions F and $G \in \mathcal{F}_X$, we obtain sufficient conditions for the existence of $G \in \mathcal{F}_X$ that minimizes $E_C|X - Y|$. Parametric examples of the expression of such G are provided for different families of copulas and random variable X .

5.1 Introduction and motivation

Given a random variable X , several location measures can be defined as the argument that minimizes a variability functional of X . Examples of such measures are:

- The expectation, μ_X , which can be defined as the value that minimizes the mean square error of X ,

$$\mu_X = \operatorname{argmin}_{t \in \mathbb{R}} E[(X - t)^2].$$

- The median, Me_X , defined as the value that minimizes the mean absolute deviation:

$$Me_X = \operatorname{argmin}_{t \in \mathbb{R}} E|X - t|.$$

- The α -expectile, $\rho_\alpha(X)$, which are defined in Krätschmer and Zähle (2017) as the value that minimizes the following linear combination

of expected square excesses:

$$\rho_\alpha(X) = \operatorname{argmin}_{t \in \mathbb{R}} \left\{ \alpha E \left[((X - t)^+)^2 \right] + (1 - \alpha) E \left[((t - X)^+)^2 \right] \right\}.$$

- Analogously, the α -quantiles, $q_\alpha(X)$, can be defined in a similar way:

$$q_\alpha(X) = \operatorname{argmin}_{t \in \mathbb{R}} \left\{ \alpha E \left[(X - t)^+ \right] + (1 - \alpha) E \left[(t - X)^+ \right] \right\}.$$

Therefore, it is natural to wonder whether we can proceed analogously with a different variability functional of X , that is, if the functional can be minimized in order to have a measure that gives information about X .

In order to address the problem stated above, let us firstly establish our framework. Throughout the chapter, we will consider that all variables are absolutely continuous and that all distribution functions and copulas are continuously differentiable. We consider a random vector $\mathbf{X} = (X, Y)$ with marginal distribution functions F and G and joint distribution function K . We will consider K, F, G continuously differentiable. As seen in the introductory chapter, by the Sklar theorem, the joint distribution K can be written as $K(x, y) = C(F(x), G(y))$, where C is the copula of the vector $\mathbf{X}$ and, under the given assumptions, it is unique and continuously differentiable.

In this chapter, we will use the notions of stochastically increasing (see Definition 1.29) and the aging notion IFR (see Definition 1.9). Let us recall that, since $P[X > x \mid Y = y] = 1 - \partial_2 C(F(x), G(y))$, $X \uparrow_{SI} Y$ if $\partial_2 C(u, v)$ decreases in v for all u . And, therefore, we will say that the copula C is *PDS* if the vector is *PDS*.

Let us now consider a random variable X with strictly increasing distribution function F and $h : [0, 1] \rightarrow [0, 1]$, a strictly increasing distortion function. If we take (X, Y) , a random vector with copula C and marginal distribution functions F and $G = h(F)$ respectively, in Chapter 4 was shown that $\nu(X) = E_C |X - Y|$, where

$$E_C |X - Y| = \int_{-\infty}^{\infty} (F(x) + G(x) - 2 C(F(x), G(x))) dx, \quad (5.1)$$

is a comonotonic additive measure of variability (see Section 1.6) in the sense of Bickel and Lehmann (1979), that is, it is a measure that satisfies the following properties: (P0) Law invariance; (P1) Translation invariance; (P2) Positive homogeneity; (P3) Non-negativity; (P4) Consistency with dispersive order and (P5) Comonotonic additivity. Note that the subscript C in $E_C |X - Y|$ addresses the fact that the copula is given and fixed, and therefore, the distance that we are considering is between the distribution functions of X and Y .

Let us consider the distortion given by the power function $h(t) = t^\alpha, \alpha > 0$ which characterizes the proportional reversed hazard rate (PRHR) model. A variable satisfies such a model if its reversed hazard rate function (recall that it is given by $\tilde{r}(t) = \frac{f(t)}{F(t)}$) is proportional to the baseline reversed hazard rate function. Given a random variable X , we will denote as

$$\mathcal{F}_X = \{X_\alpha : F_{X_\alpha}(x) = F(x)^\alpha, \alpha > 0\} \quad (5.2)$$

the family of all random variables that satisfy the PRHR model. The interest on considering such distortion function lies in the importance of the PRHR model, which, as mentioned in the introductory chapter, has applications in various areas, among which we can highlight insurance risk or reliability (see Gupta and Gupta (2007) and Psarrakos and Sordo (2019)).

Following the approach that initialized the chapter, we can now consider the following problem. Fixed the copula C and the marginal distribution function F , we study sufficient conditions for the existence of a distribution function G of $Y \in \mathcal{F}_X$ as defined in (5.2), such that the distance (5.1) is minimal.

The first false intuition may suggest, at least when X and Y have the same support, that the smallest value of $E_C|X - Y|$ is reached when $F = G$, that is, when $Y =_{st} X$. This is not necessarily true, as we can see in the following counterexample. Considering C the independence copula and $X \sim U(0, 1)$, it is easy to see that, considering any $\alpha \in (1, 2)$, if Y_1 has a distribution function $G(u) = u^\alpha$ for $u \in [0, 1]$ and $Y_2 \sim U(0, 1)$, then $E_I|X - Y_1| < E_I|X - Y_2|$. The minimum is reached when $\alpha = \sqrt{2}$. It may even happen that there is not a minimizer α for the function. If we consider the independence copula and $X \sim Weibull(k, 1)$, we can see that, for some values of k , the minimizer exists and for some others it does not. For $k = 1$ the minimum is reached in $\alpha = 0.390$ and for $k = 1.3$ is reached in $\alpha = 0.714$. Although, it can be checked that, for example, for $k = 0.7$ there is not $\alpha > 0$ that minimizes such functional (see Figure 5.1).

Addressing the stated problem, the rest of the chapter is organized as follows. Section 5.2 contains the main result of the chapter. Given a random variable X , a copula C and the family of variables that satisfy the PRHR model, $\mathcal{F}_X$, in this section we obtain sufficient conditions for the existence of $Y^* \in \mathcal{F}_X$ that minimizes $E_C|X - Y|$ among all $Y \in \mathcal{F}_X$. Finally, Section 5.3 includes examples of the result, both for existence and nonexistence of such minimizer. Parametric examples of the expression of the minimizer are also provided for particular cases. A second step in this research would be to obtain

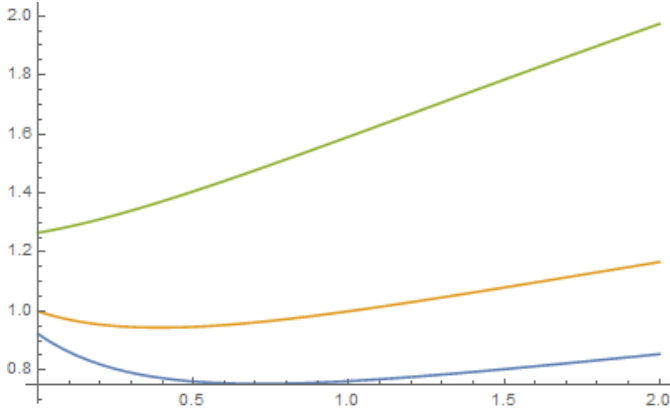


Figure 5.1. $E_I|X - X_\alpha|$ in terms of α , considering the random variable $X \sim \text{Weibull}(k, 1)$ with $k = 0.7$ (green), $k = 1$ (orange) and $k = 1.3$ (blue).

analytically the minimizer α and to provide a plausible interpretation. Such work remains for future research.

5.2 Determining $\min_{\{Y \in \mathcal{F}_X\}} E_C|X - Y|$ with X and C fixed.

We will see that, under some conditions on the fixed copula, if the variable X is IFR, then $\min_{\{Y \in \mathcal{F}_X\}} E_C|X - Y|$ is reached. We need some results to prove it. From now on, we will denote

$$\varphi_0(u) = \lim_{v \rightarrow 0} \partial_2 C(u, v) \quad \text{and} \quad \varphi_1(u) = \lim_{v \rightarrow 1} \partial_2 C(u, v).$$

Such limits exist for all $u \in (0, 1)$. When the copula is PDS, $\partial_2 C(u, v)$ decreases in v . Since $\int_0^1 \partial_2 C(u, v) dv = u$, for all $u \in (0, 1)$, it follows that that $\varphi_0(u) \geq u \geq \varphi_1(u)$.

Lemma 5.1. *Let C be an absolutely continuous copula, and let X be an IFR random variable with F, f and r its respective distribution, density and hazard rate functions. Assume that $r(F^{-1}(0)) \neq 0$. Then:*

$$\lim_{\alpha \rightarrow 0} \int_0^1 \frac{1 - 2 \partial_2 C(u, u^\alpha)}{f(F^{-1}(u))} u^\alpha \log(u) du = \int_0^1 \frac{1 - 2\varphi_1(u)}{f(F^{-1}(u))} \log(u) du, \quad (5.3)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^1 \frac{1 - 2 \partial_2 C(u, u^\alpha)}{f(F^{-1}(u))} \log(u) du = \int_0^1 \frac{1 - 2\varphi_0(u)}{f(F^{-1}(u))} \log(u) du. \quad (5.4)$$

5.2. Determining $\min_{\{Y \in \mathcal{F}_X\}} E_C |X - Y|$ with X and C fixed.

Proof. Let us we define, for all $u \in (0, 1)$ and all $n \in \mathbb{N}$, the functions $g_n(u)$, $g(u)$, $h_n(u)$ and $h(u)$, given by:

$$\begin{aligned} g_n(u) &= \frac{1 - 2 \partial_2 C(u, u^{1/n})}{f(F^{-1}(u))} u^{1/n} \log(u), \\ g(u) &= \frac{1 - 2\varphi_1(u)}{f(F^{-1}(u))} \log(u), \\ h_n(u) &= \frac{1 - 2 \partial_2 C(u, u^n)}{f(F^{-1}(u))} \log(u), \\ h(u) &= \frac{1 - 2\varphi_0(u)}{f(F^{-1}(u))} \log(u). \end{aligned}$$

Then, for all $u \in (0, 1)$ $\lim_{n \rightarrow \infty} g_n(u) = g(u)$ and $\lim_{n \rightarrow \infty} h_n(u) = h(u)$. As $|g_n(u)|$ and $|h_n(u)|$ are bounded by $\frac{-\log(u)}{f(F^{-1}(u))}$, by the Dominated Convergence Theorem, if $\frac{-\log(u)}{f(F^{-1}(u))}$ is a positive integrable function, (5.3) and (5.4) hold. By the Cauchy-Schwarz inequality and considering the hazard rate function r :

$$\begin{aligned} \int_0^1 \frac{-\log(u)}{f(F^{-1}(u))} du &= \int_0^1 \frac{-\log(u)}{1-u} \frac{1}{r(F^{-1}(u))} du \leq \\ &\leq \left(\int_0^1 \left(\frac{-\log(u)}{1-u} \right)^2 du \right)^{1/2} \left(\int_0^1 \left(\frac{1}{r(F^{-1}(u))} \right)^2 du \right)^{1/2} \end{aligned} \quad (5.5)$$

Note that $\int_0^1 \left(\frac{-\log(u)}{1-u} \right)^2 du = \frac{\pi^2}{3}$ and, since r increases and $r(F^{-1}(0)) \neq 0$, $\frac{1}{r(F^{-1}(u))}$ is bounded. Therefore, the last term in (5.5) is finite and the assertion follows.

Lemma 5.2. *Let X be an IFR random variable with F , f and r its respective distribution, density and hazard rate functions. Then:*

$$\int_0^1 \frac{1-2u}{f(F^{-1}(u))} (-\log(u)) du \quad (5.6)$$

is strictly positive and finite.

Proof. First we will show that

$$b(s) = \int_0^s \left(\frac{1-2u}{1-u} (-\log(u)) \right) du > 0 \text{ for all } s \in [0, 1].$$

Since $b'(s) = \frac{1-2s}{1-s} (-\log(s))$, $b(s)$ increases if $s < 1/2$ and decreases if $s > 1/2$. Moreover, $b(0) = 0$ and $b(1) = 2 - \pi^2/6 > 0$, therefore $b(s) > 0$ for all $s \in (0, 1]$.

We can rewrite (5.6) in the following form:

$$\int_0^1 \left(\frac{1-2u}{1-u} (-\log(u)) \right) \frac{1}{r(F^{-1}(u))} du. \quad (5.7)$$

As X is IFR, $1/r(F^{-1}(u))$ decreases in u . By Lemma 4.7.1 in Barlow and Proschan (1975), if, for all $s \in [0, 1]$, $b(s) = \int_0^s \left(\frac{1-2u}{1-u} (-\log(u)) \right) du$ is positive,

then (5.7) is also positive. $b'(s) = \frac{1-2s}{1-s}(-\log(s))$, so $b(s)$ increases if $s < 1/2$ and decreases if $s > 1/2$. As $b(0) = 0$ and $b(1) = 2 - \pi^2/6 > 0$, $b(s) > 0$ for all $s \in (0, 1]$ and (5.7) is strictly positive. Also, by Cauchy-Schwarz inequality, (5.7) is smaller or equal than:

$$\left(\int_0^1 \left(\frac{1-2u}{1-u} (-\log(u)) \right)^2 du \right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{r(F^{-1}(u))^2} du \right)^{\frac{1}{2}}.$$

Integrating by parts and considering the Spence's function, given by $Li_2(u) = \int_0^u \frac{-\log(t)}{1-t} dt$, we can see that $\int_0^1 \left(\frac{-\log(u)(1-2u)}{1-u} \right)^2 du \leq 1 + Li_2(1) < 3$ (we can obtain, computationally, that the value is approximately 1.2936). As we already saw in Lemma 5.1 that the second element is finite, (5.6) is finite.

We can now move on to the main result:

Theorem 5.3. *Let X be a random variable with F and f its respective distribution and density function. Let us consider the family of random variables $\mathcal{F}_X$ described above. Let us consider the random vector $\mathbf{X} = (X, Y)$ with PDS copula C , such that $\partial_2 C(u, u^n)$ increases in u for all $n \in \mathbb{N}$ and $\lim_{u \rightarrow 1} \varphi_1(u) = 1$. If X is IFR, there exists $\alpha_0 > 0$ such that $X_{\alpha_0} \in \mathcal{F}_X$ and:*

$$E_C|X - X_{\alpha_0}| \leq E_C|X - Y| \text{ for all } Y \in \mathcal{F}_X.$$

Proof. Given X , we can consider the function $\alpha \mapsto E_C|X - X_\alpha|$, given by:

$$\int_{-\infty}^{\infty} (F(x) + F(x)^\alpha - 2C(F(x), F(x)^\alpha)) dx = \int_0^1 \frac{u + u^\alpha - 2C(u, u^\alpha)}{f(F^{-1}(u))} du$$

This is a continuous and derivable function for $\alpha > 0$, and

$$\partial_\alpha E_C|X - X_\alpha| = \int_0^1 \frac{1 - 2\partial_2 C(u, u^\alpha)}{f(F^{-1}(u))} u^\alpha \log(u) du$$

In order to see that there exists α_0 that minimizes $E_C|X - X_\alpha|$, it would be enough to see that

1. $\lim_{\alpha_1 \rightarrow 0} (\partial_\alpha E_C|X - X_\alpha|_{\alpha=\alpha_1}) < 0$, and
2. $\lim_{\alpha_2 \rightarrow +\infty} (\partial_\alpha E_C|X - X_\alpha|_{\alpha=\alpha_2}) > 0$.

It would mean that there exists at least one value $\alpha_0 > 0$ where $E_C|X - X_\alpha|$ attains a local minimum, and necessarily one of them will be the global one.

By Lemma 5.1, $\lim_{\alpha_1 \rightarrow 0} (\partial_\alpha E_C |X - X_\alpha|_{\alpha=\alpha_1}) = - \left(\int_0^1 \frac{1-2\varphi_1(u)}{f(F^{-1}(u))} (-\log(u)) du \right)$ and, as $u \geq \varphi_1(u)$, by Lemma 5.4,

$$\lim_{\alpha_1 \rightarrow 0} (\partial_\alpha E_C |X - X_\alpha|_{\alpha=\alpha_1}) \leq - \left(\int_0^1 \frac{1-2u}{f(F^{-1}(u))} (-\log(u)) du \right) < 0.$$

In order to study $\lim_{n \rightarrow \infty} \partial_\alpha E_C |X - X_\alpha|_{\alpha=c_n}$, let us note that, as $\partial_2 C(u, u^n)$ increases in u , for each n there exists $c_n \in [0, 1)$ such that, for $u < c_n$, $\partial_2 C(u, u^n) \leq \frac{1}{2}$ and for $u > c_n$, $\partial_2 C(u, u^n) \geq \frac{1}{2}$. Note also that $\lim_{n \rightarrow \infty} c_n < 1$ (such limit exists due to the smoothness of C); otherwise, if $\lim_{n \rightarrow \infty} c_n = 1$, then $\varphi_1(u) \leq \frac{1}{2}$ for all $u \in (0, 1)$, which contradicts the fact that $\lim_{u \rightarrow 1} \varphi_1(u) = 1$. Taking this into consideration, we have that, for all $n \in \mathbb{N}$:

$$\begin{aligned} \partial_\alpha E_C |X - X_\alpha|_{\alpha=c_n} &= \int_0^1 \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} u^n \log(u) du = \\ &= \int_0^{c_n} \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} u^n \log(u) du + \int_{c_n}^1 \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} u^n \log(u) du > \\ &= c_n^n \int_0^{c_n} \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} \log(u) du + c_n^n \int_{c_n}^1 \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} \log(u) du > \\ &> (c_n)^n \int_0^1 \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} \log(u) du \quad (5.8) \end{aligned}$$

Taking limits in (5.8), $\lim_{n \rightarrow \infty} (c_n)^n = 0$ and by Lemma 5.3,

$$\lim_{\alpha \rightarrow \infty} \int_0^1 \frac{1-2\partial_2 C(u, u^\alpha)}{f(F^{-1}(u))} \log(u) du = \int_0^1 \frac{1-2\varphi_0(u)}{f(F^{-1}(u))} \log(u) du$$

Note that, as $u \leq \varphi_0(u) \leq 1$,

$$\int_0^1 \frac{1-2u}{f(F^{-1}(u))} \log(u) du \leq \int_0^1 \frac{1-2\varphi_0(u)}{f(F^{-1}(u))} \log(u) du \leq \int_0^1 \frac{-\log(u)}{f(F^{-1}(u))} du$$

In Lemmas 5.3 and 5.4, we saw that both the bounds are finite, so we can conclude that $\lim_{n \rightarrow \infty} \left((c_n)^n \int_0^1 \frac{1-2\partial_2 C(u, u^n)}{f(F^{-1}(u))} (\log(u)) du \right) = 0$, and, therefore $\lim_{\alpha_2 \rightarrow +\infty} (\partial_\alpha E_C |X - X_\alpha|_{\alpha=\alpha_2}) > 0$. This concludes the proof.

5.3 Examples

Let us give some examples of *PDS* copulas that satisfy both that $\partial_2 C(u, u^n)$ increases in u for all $n \in \mathbb{N}$ and that $\lim_{u \rightarrow 1} \varphi_1(u) = 1$:

1. Independence, $C(u, v) = uv$.
2. Farlie-Gumbel-Morgenstern copula, $C_\theta(u, v) = uv(1 + \theta(1-u)(1-v))$, for $\theta \in [0, 1]$.
3. Frank copula, $C_\theta(u, v) = -\frac{1}{\theta} \log \left(1 + \frac{(e^{-u\theta} - 1)(e^{-v\theta} - 1)}{e^{-\theta} - 1} \right)$, for $\theta > 0$.

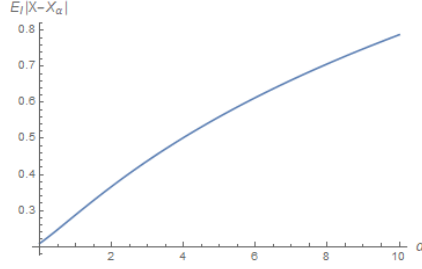


Figure 5.2. $E_I|X - X_\alpha|$ in terms of α , considering the independence copula and the random variable X determined by its hazard rate function $r(x) = x^{-0.5} + x^{-0.2}$.

4. Copula 17 in Table 4.1 in Nelsen (2007), for $\theta > 1$,

$$C_\theta(u, v) = \left(1 + \frac{[(1+u)^{-\theta} - 1][(1+v)^{-\theta} - 1]}{2^{-\theta} - 1} \right)^{-1/\theta} - 1.$$

Note that there are many *PDS* copulas that do not verify such conditions. For example, there are copulas that satisfy $\lim_{u \rightarrow 1} \varphi_1(u) = 1$ but do not verify that $\partial_2 C(u, u^n)$ increases in u for all $n \in \mathbb{N}$. Examples of this are the following copulas in Table 4.1 in Nelsen (2007): 1 (Clayton), 3 (Ali-Mikhail-Haq), 13 and 19. Also, it can be checked that, for some *PDS* copulas, $\lim_{u \rightarrow 1} \varphi_1(u) = 0$ and therefore, do not satisfy conditions on Theorem 5.3. Examples of this are the Gaussian copula for $\rho > 0$ or the following copulas in Table 4.1 in Nelsen (2007): 2, 4 (Gumbel-Hougaard), 6, 12 and 14.

Let us also recall the importance of the property *IFR* in X . If we consider (X, Y) with the independence copula and $X \sim Weibull(k, 1)$, the *IFR* assumption fails for $k < 1$. Figure 5.1 shows that for $k = 0.7$, $E_I|X - X_\alpha|$ increases in α and the minimizer does not exist. A different example can arise if we consider a random vector (X, X_α) with the independence copula and with X having a hazard rate given by $r(x) = x^{-a} + x^{-b}$ for $x > 0$ with $0 < a, b < 1$. One can check that the density of X is given by:

$$f(x) = r(x)e^{-\int_0^x r(t)dt} = (x^{-a} + x^{-b})e^{-x(\frac{x^{-a}}{1-a} + \frac{x^{-b}}{1-b})}.$$

Taking $a = 0.5, b = 0.2$, one can check that the variable is not *IFR* and that it also occurs that the functional $\alpha \mapsto E_I|X - X_\alpha|$ is increasing, there is not a minimizer, as it is illustrated in Figure 5.2.

As already stated, it remains for future work to obtain, in general, the analytical expression of the minimizer α . Although, we can study the behavior

of the minimizer for simple random variables and copulas that verify conditions of Theorem 5.3. Note that, in the following examples, we will consider uniform and exponential random variables. These are widely used distributions, *IFR* and which present nice properties. Due of their simplicity, we will see that, in both cases, the minimizer α will not depend on the parameters of shape or location of the variables. Although, it must be pointed out that this can not be generalized. Again, in Figure 5.1, we could see that the shape parameter k of the Weibull distribution $Weibull(k, 1)$ influences not only on the value of the minimizer but also on its existence.

5.3.1 Uniform marginals $X \sim U(a, b)$

In this section we will consider $X \sim U(a, b)$ with $b > a$. Then, for $x \in [a, b]$, $F(x) = \frac{x-a}{b-a}$. Let us now consider the independence and the FGM copula:

5.3.1.1 Independence copula

If we consider $C(u, v) = uv$, one can see that:

$$\begin{aligned} E_I |X - X_\alpha| &= \int_a^b \left(\frac{x-a}{b-a} + \left(\frac{x-a}{b-a} \right)^\alpha - 2 \left(\frac{x-a}{b-a} \right)^{\alpha+1} \right) dx \\ &= \frac{(b-a)(2+\alpha+\alpha^2)}{2(1+\alpha)(2+\alpha)}, \end{aligned}$$

and

$$\partial_\alpha E_I |X - X_\alpha| = \frac{(b-a)(\alpha^2 - 2)}{(\alpha^2 + 3\alpha + 2)^2}.$$

Clearly, the α value that minimizes $E_I |X - X_\alpha|$ does not depend on the parameters a, b and, by solving $\partial_\alpha E_I |X - X_\alpha| = 0$, one can obtain that such minimizer is $\alpha = \sqrt{2}$. Since the shape of $E_I |X - X_\alpha|$ only depends on $(b-a)$ and α , for illustrative purposes, we can fix $a = 0$ and consider different values of $b = 0$ to illustrate the plot of $E_I |X - X_\alpha|$ in terms of α . Such functions are plotted in Figure 5.3, where the dashed line represents the minimizer $\alpha = \sqrt{2}$.

5.3.1.2 Farlie-Gumbel-Morgenstern copula

If we consider $X \sim U(a, b)$, with $b > a \geq 0$ and the copula given by $C_\theta(u, v) = uv(1 + \theta(1-u)(1-v))$, for $\theta \in [0, 1]$, one can compute, using (5.1) that:

$$E_{C_\theta} |X - X_\alpha| = \frac{(b-a)(18 + \alpha(27 + \alpha(22 + \alpha(11 + 2\alpha) - 6\theta) - 10\theta))}{2(1+\alpha)(2+\alpha)(3+\alpha)(3+2\alpha)},$$

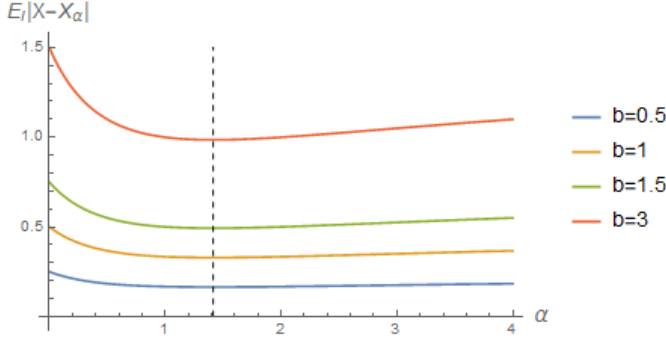


Figure 5.3. $E_I|X - X_\alpha|$ in terms of α , considering the random variable $X \sim U(a, b)$ with $a = 0$ and different values of b .

and $\partial_\alpha E_{C_\theta} |X - X_\alpha|$ is given by

$$K \left((3 + \alpha)^2 (3 + 2\alpha)^2 (\alpha^2 - 2) + (\alpha(\alpha(65 + 3\alpha(50 + \alpha(25 + 4\alpha)))) - 108) - 90 \right) \theta$$

where

$$K = \frac{(b - a)}{(1 + \alpha)^2 (2 + \alpha)^2 (3 + \alpha)^2 (3 + 2\alpha)^2}.$$

Again, the α value that minimizes $E_{C_\theta} |X - X_\alpha|$ does not depend on the parameters a, b (and, therefore, from now on, we assume $X \sim U(0, 1)$), but it does depend on θ . We can not obtain an analytical expression of the minimizer in terms of θ , but we can do it numerically. As expected, since, for $\theta = 0$, the copula is the independence, the minimizer is a function that decreases from $\alpha = \sqrt{2}$. In Figure 5.4 we plot the value of α that minimizes $E_{C_\theta} |X - X_\alpha|$ in terms of θ and in Figure 5.5 we illustrate $E_{C_\theta} |X - X_\alpha|$ in terms of α for different values of the parameter θ . Again, the minimizer is represented by dashed lines for each value of θ .

5.3.2 Exponential marginals $X \sim \text{Exp}(\lambda)$

Instead of uniform random variables, we can consider that $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$. Then, $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Let us consider again the computation of $E_C |X - X_\alpha|$ when C is the independence or the FGM copula.

In the following examples, $H_\alpha = \int_0^1 \frac{1-x^\alpha}{1-x} dx$, which is the Harmonic number if α is an integer. In addition, ψ' stands for the derivative of the function $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. The function Γ is the well-known Euler Gamma function given by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

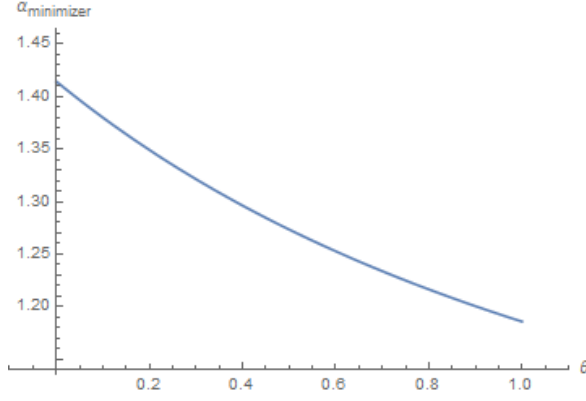


Figure 5.4. Value of α that minimizes $E_{C_\theta}|X - X_\alpha|$ in terms of θ , considering the random variable $X \sim U(0, 1)$.

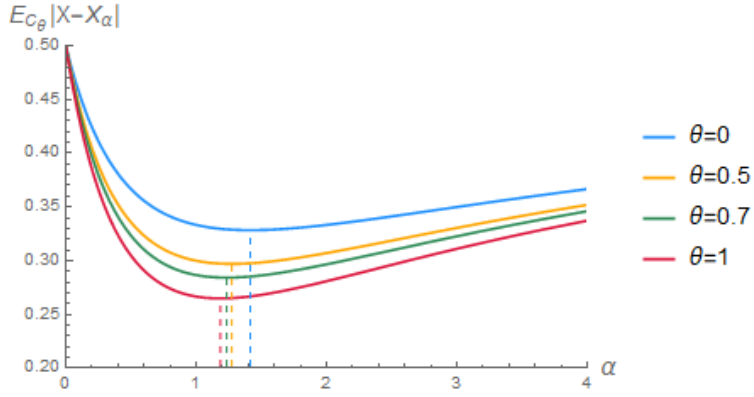


Figure 5.5. $E_{C_\theta}|X - X_\alpha|$ in terms of α , considering the random variable $X \sim U(0, 1)$ with different values of θ .

5.3.2.1 Independence copula

If we consider $C(u, v) = uv$, taking expression (5.1) into account, one can compute that:

$$E_I |X - X_\alpha| = \frac{1}{\lambda} \left(\frac{1 - \alpha + H_\alpha + \alpha H_\alpha}{(1 + \alpha)} \right)$$

and

$$\partial_\alpha E_I |X - X_\alpha| = \frac{1}{\lambda} \left(\psi'(1 + \alpha) - \frac{2}{(1 + \alpha)^2} \right),$$

where H_α and ψ' were defined above.

Again, if we solve numerically $\partial_\alpha E_I |X - X_\alpha| = 0$, we can find the α value that minimizes $E_I |X - X_\alpha|$. We can see that it does not depend on λ , the shape parameter of the exponential distribution, and that the value of such minimizer is $\alpha \simeq 0.389592$. This is illustrated in Figure 5.6, where again $E_I |X - X_\alpha|$ is plotted in terms of α , and the dashed line represents the value of α where the minimum value of $E_I |X - X_\alpha|$ is reached.

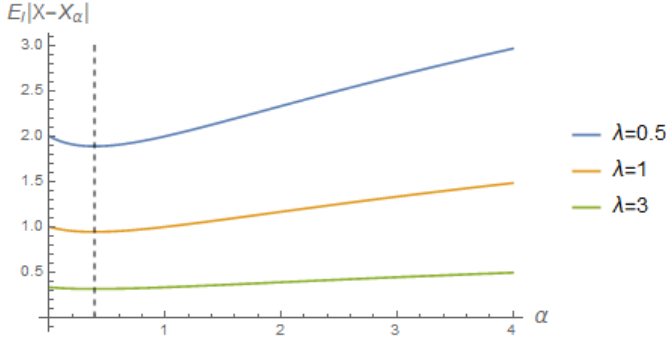


Figure 5.6. $E_I |X - X_\alpha|$ in terms of α , considering the random variable $X \sim \text{Exp}(\lambda)$ with different values of λ .

5.3.2.2 Farlie-Gumbel-Morgenstern copula

We can proceed analogously as above, considering $X \sim \text{Exp}(\lambda)$ ($\lambda > 0$) and the copula given by $C_\theta(u, v) = uv(1 + \theta(1 - u)(1 - v))$, for $\theta \in [0, 1]$. We can compute again the following.

$$E_{C_\theta} |X - X_\alpha| = \frac{1}{\lambda} \left(H_\alpha + \frac{2 - \alpha(1 + \alpha + \theta)}{(1 + \alpha)(2 + \alpha)} \right),$$

and

$$\partial_\alpha E_{C_\theta} |X - X_\alpha| = \frac{1}{\lambda} \left(\frac{\theta(\alpha^2 - 2)}{(1 + \alpha)^2(2 + \alpha)^2} - \frac{2}{(1 + \alpha)^2} + \psi'(1 + \alpha) \right),$$

with H_α and ψ' the functions defined above.

It is easy to see that, if we consider $Y \sim \text{Exp}(1)$ and $X \sim \text{Exp}(\lambda)$, one gets $E_{C_\theta} |X - X_\alpha| = \frac{1}{\lambda} E_{C_\theta} |Y - Y_\alpha|$. Hence, the parameter of shape λ does not affect the minimizer and for the following graphics we will assume $\lambda = 1$. Although, the minimizer does depend on the dependence parameter $\theta \in [0, 1]$. We can expect again that, if we compute numerically and plot the minimizer depending on such parameter, we get a function starting from 0.389592. It is the case, as we can see in Figure 5.7. However, it is now an increasing function.

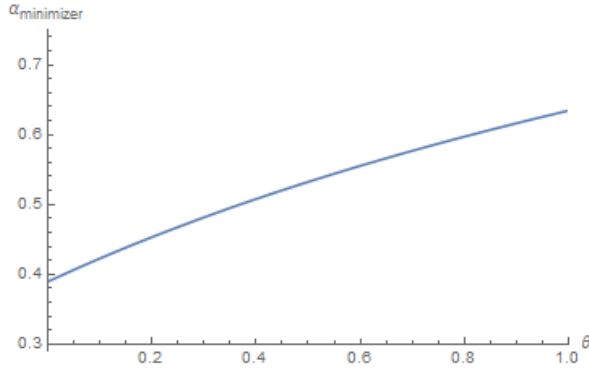


Figure 5.7. Value of α that minimizes $E_{C_\theta}|X - X_\alpha|$ in terms of θ , considering the random variable $X \sim \text{Exp}(1)$.

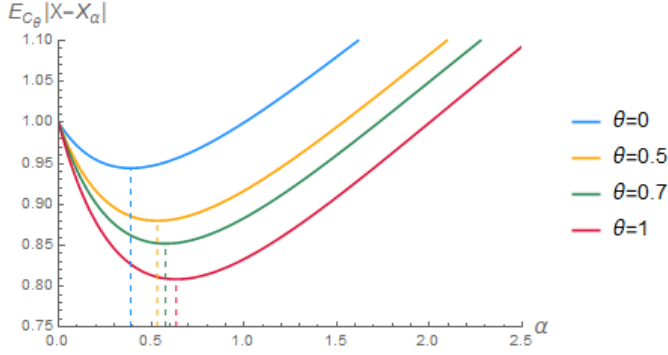


Figure 5.8. $E_{C_\theta}|X - X_\alpha|$ in terms of α , considering the random variable $X \sim \text{Exp}(1)$ with different values of θ .

In Figure 5.8 we illustrate how $E_{C_\theta}|X - X_\alpha|$ behaves in terms of α , considering $X \sim \text{Exp}(1)$ and it can be seen how the minimizer exists and it increases in terms of the dependence parameter θ .

5.4 Conclusions

In this chapter, we studied sufficient conditions for the existence of a value $\alpha_0 > 0$ that minimizes the variability functional $X \mapsto E_C|X - X_\alpha|$, where the copula C and the distribution function F of X are fixed and X_α verifies the proportional reversed hazard rate model associated to F . Sufficient conditions were provided, in terms of the copula and aging notions of X , for the existence of such minimizer.

Several examples were provided, both for the existence and non-existence of the minimizer. In addition, assuming uniform and exponential distributions, the minimizer was determined when the independence or the FGM copula were assumed. Although, those examples highlighted that, when X is fixed, the concordance order does not imply the order of the minimizers. When we consider the FGM copula with parameter θ , the effect of a greater value of θ was not the same on the minimizer when $X \sim \text{Exp}(\lambda)$ as it was on the minimizer when $X \sim U(a, b)$. Assuming $X \sim \text{Exp}(\lambda)$, an increasing of θ implies an increasing of the minimizer. However, if we consider $X \sim U(a, b)$, the implication holds in the opposite way, an increasing of θ implies that the minimizer decreases. This fact motivates the need on further work on finding sufficient conditions for the ordering of the minimizer. In addition, future working lines are to find an analytical expression of the minimizer, as well as to provide a plausible interpretation.

Note that $\min_{\alpha > 0} E_C|X - X_\alpha|$ is a particular problem on $\min_h E_C|X - X_h|$, where h is a distortion function. This is, again a simplification of the problem $\min_Y E_C|X - Y|$. Therefore, methodologies which approach one of the more general problems would be also welcome.

“Imagination is the Discovering Faculty. It is that which penetrates into the unseen worlds around us, the worlds of Science”.

Ada Lovelace



Note on Efron’s monotonicity property under given copula structures.

Given a multivariate random vector, Efron’s marginal monotonicity (EMM) refers to the stochastic monotonicity of the variables given the value of their sum. Recently, based on the notion of total positivity of the joint density of the vector, Pellerey and Navarro (2022) obtained sufficient conditions for EMM when the monotonicity is in terms of the likelihood ratio order. We provide in this chapter new sufficient conditions based on properties of the marginals and the copula. Moreover, parametric examples are provided for some of the results included in Pellerey and Navarro (2022).

6.1 Introduction and background

Given a random vector of independent continuous marginals with log-concave densities, Efron (1965) studied the stochastic monotonicity of the marginals given the value of their sum, obtaining the following result.

Proposition 6.1 (Efron (1965)). *Let $X_1, X_2, \dots, X_n$ be n independent random variables with ILR densities, let $S = \sum_{i=1}^n X_i$ be their sum, and let $\Phi(x_1, x_2, \dots, x_n)$ be a real measurable function, increasing in each of its arguments. Then, the function $s \mapsto E(\Phi(X_1, X_2, \dots, X_n) \mid S = s)$ is increasing.*

Note that in this chapter, as well as we have already done throughout the dissertation, the term “increasing” is used for “non-decreasing” and “decreasing” is used for “non-increasing”. We recall that, as seen in Definition 1.10, a continuous random variable X having density f is said to have the ILR property if $\frac{f(x+y)}{f(x)}$ decreases in x for all $y \geq 0$, i.e., if $\log f(x)$ is concave and the IPLR property if $f(\lambda x)/f(x)$ is increasing in x for any positive constant $\lambda < 1$ or if $x \eta(x)$ increases in x , where $\eta(x) = -f'(x)/f(x)$. In

addition, we recall that, given X and Y two continuous random variables with respective distribution functions F, G and densities f, g , respectively, as defined in Section 1.2.1 $X \leq_{st} Y$ if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$ and $X \leq_{lr} Y$ if $f(x)/g(x)$ is decreasing in the union of the supports. Finally, we recall that given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$, in relation with Definition 1.31, if the second order partial derivative $\frac{\partial^2}{\partial x \partial y} \log(f(x, y))$ exists, then f is TP_2 if and only if $\frac{\partial^2}{\partial x \partial y} \log(p(x, y)) \geq 0$.

From now on, based in Proposition 6.1, we will refer as “Efron's strong monotonicity” (ESM) to the monotonicity of $s \mapsto \{(X_1, X_2, \dots, X_n) \mid S = s\}$ in terms of any stochastic order, and “Efron's marginal monotonicity” (EMM) to the monotonicity of $s \mapsto \{X_i \mid S = s\}$. Efron's monotonicity and its subsequent generalizations have been of great interest in different areas, as economics, combinatorial probability, dependence modeling and statistical theory. For a list of references on its applications, the interested reader may consult Saumard and Wellner (2018); Pellerey and Navarro (2022).

In the ensuing years, the results obtained in Efron (1965) were extended in several ways. Lehmann (1966) showed that the conditions stated in Proposition 6.1 imply EMM in terms of the lr -order. Denuit and Robert (2021) considered the generalization of ESM when sums involve a large number of terms satisfying some regularity conditions. More recently, Saumard and Wellner (2018) extended Efron's results to bivariate vectors with non-independent variables, providing conditions in terms of the second derivatives of $-\log f(x, y)$, which imply ESM and EMM in the usual stochastic order. In the same framework, Oudghiri (2021) provided sufficient conditions for a stronger ESM and EMM assumption, considering not only that $\phi(s) = E(\Phi(X_1, X_2, \dots, X_n) \mid S = s)$ increases in s , but also that $\alpha(s)\phi(s)$ increases in s , for some functions α . Also in the bivariate setting, for non-independent vectors, Pellerey and Navarro (2022) give sufficient conditions which imply ESM in the usual stochastic order. They also generalize the result in Lehmann (1966), connecting the EMM monotonicity in the lr -order to the notion of total positivity.

Pellerey and Navarro (2022) provide the following results.

Proposition 6.2 (Pellerey and Navarro (2022)). *Let the vector (X_1, X_2) have a joint density f . Then, the following conditions are equivalent:*

1. *The function $f(x, s - x)$ is TP_2 in (x, s) ;*
2. *$[X_1 \mid S = s_1] \leq_{lr} [X_1 \mid S = s_2]$ whenever $s_1 \leq s_2$;*
3. *$[S \mid X_1 = x_1] \leq_{lr} [S \mid X_1 = x_2]$ whenever $x_1 \leq x_2$.*

Proposition 6.3 (Pellerey and Navarro (2022)). *Let the vector (X_1, X_2) have a joint density f . If $f(x_1, x_2)$ is TP_2 in (x_1, x_2) and logconcave in x_2 (respectively, x_1) for every x_1 (respectively, x_2), then $f(x, s - x)$ ($f(s - x, x)$) is TP_2 in (x, s) .*

Let $\mathbf{X} = (X_1, X_2)$ be a random vector with joint density f and survival copula $\hat{C}$. Let $\hat{c}$ the second mixed partial derivative of $\hat{C}$ (here $\hat{c}$ is referred to as the density of the survival copula $\hat{C}$). As pointed out in Example 2.4 in Pellerey and Navarro (2022), $f(x_1, x_2)$ is TP_2 in (x_1, x_2) if and only if $\hat{c}(u_1, u_2)$ is TP_2 . However, the condition $\hat{c}(u, z - u)$ is TP_2 in (u, z) does not imply that $f(x, s - x)$ is TP_2 in (x, s) . This suggests to find conditions on $\hat{c}$ and the marginals implying that $f(x, s - x)$ is TP_2 in (x, s) . This is what we do in Section 6.2 below. In Section 6.3, we provide a list of copulas that, when joined to exponential or uniform marginals, imply that $f(x, s - x)$ is TP_2 in (x, s) . The final Section 6.4, instead, is devoted to an application of Proposition 6.3 in the context of Multivariate Normal Distributions and Generalized Order Statistics (GOSs).

6.2 Efron's marginal monotonicity in terms of the copula

Given a random vector (X_1, X_2) with exponential marginals and joint density f , our first result provides a conditions on the density of the corresponding survival copula, $\hat{c}$, that ensures that $f(x, y - x)$ is TP_2 in (x, y) . Note that $f(x_1, x_2) = \hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))f_1(x_1)f_2(x_2)$.

Proposition 6.4. *Let the vector (X_1, X_2) , with $X_1, X_2 \sim \text{Exp}(\lambda)$ have joint density f and density of the survival copula $\hat{c}$. If $\hat{c}(u, v/u)$ is TP_2 in (u, v) , for all $0 < u < 1, 0 < v < u$, then $f(x, y - x)$ is TP_2 in (x, y) .*

Proof. The function $f(x, s - x)$ is TP_2 in (x, s) if

$$\frac{f(x, s_2 - x)}{f(x, s_1 - x)} = \frac{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_2 - x))f_1(x)f_2(s_2 - x)}{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_1 - x))f_1(x)f_2(s_1 - x)} = \frac{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_2)/\bar{F}_2(x))}{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_1)/\bar{F}_2(x))}$$

is increasing in x for all $s_1 < s_2$. The monotonicity follows from the fact that $\bar{F}_2(s - x) = \frac{\bar{F}_2(s)}{\bar{F}_2(x)} \cdot \frac{f_2(s_2 - x)}{f_2(s_1 - x)}$ is constant in x and $\frac{\hat{c}(u, v_1/u)}{\hat{c}(u, v_2/u)}$ is decreasing in u for $v_1 < v_2$. This last condition is equivalent to say that $\hat{c}(u, v/u)$ is TP_2 in (u, v) , for all $0 < u < 1, 0 < v < u$.

In Proposition 6.3, the joint density function $f(x_1, x_2)$ is required to be logconcave in x_2 for every x_1 , which is equivalent to say that $\{X_2|X_1 = x_1\}$ is *ILR* for all x_1 . This condition can be weakened when it is expressed in terms of the density of the survival copula whenever the marginal X_2 is exponential.

Proposition 6.5. *Let (X_1, X_2) be a random vector with joint density f and density of the survival copula $\hat{c}$. Let $X_2 \sim \text{Exp}(\lambda)$. If $\hat{c}(u, v)$ is TP_2 in (u, v) and $[U_2|U_1 = u_1]$ is $IPLR$, where $U_i \sim \bar{F}_i(X_i)$ for $i = 1, 2$ for all u_1 , then $f(x, y - x)$ is TP_2 in (x, y) .*

Proof. Since $\hat{c}(u, v)$ is TP_2 in (x_1, x_2) , then $f(x_1, x_2)$ is TP_2 in (x_1, x_2) . By Proposition 6.3, it remains to see that $f(x_1, x_2)$ is logconcave in x_2 for all x_1 . This is the same as proving that

$$\frac{f(x_1, x_2 + y)}{f(x_1, x_2)} = \frac{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2 + y))f_2(x_2 + y)}{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))f_2(x_2)} = \frac{\hat{c}(\bar{F}_1(x_1), e^{-\lambda y}\bar{F}_2(x_2))}{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))}$$

is decreasing in x_2 for all x_1 and $y \geq 0$, where we have used that $X_2 \sim \text{Exp}(\lambda)$. Taking into account that $f_{U_2|U_1=v}(u) = \hat{c}(v, u)$, this follows from the fact that $[U_2|U_1 = u_1]$ is $IPLR$ for all u_1 , which is equivalent to say that $f_{U_2|U_1=v}(\alpha u)/f_{U_2|U_1=v}(u)$ increases in u for all v and any $0 < \alpha \leq 1$.

It should be noted that the sufficient conditions in Proposition 6.4 neither imply, nor are implied, by the conditions in Proposition 6.5. For example, if $X_1, X_2 \sim \text{Exp}(\lambda)$ and $\hat{C}$ is the Ali-Mikhail-Haq copula with parameter $\theta \leq 1$ (see Subsection 6.3.3) then (X_1, X_2) satisfies the conditions on Proposition 6.4 but not those in Proposition 6.5. Similarly, if $X_2 \sim \text{Exp}(\lambda)$, $X_1 \sim \text{Exp}(\lambda)$ and $\hat{C}$ is a Clayton copula (see Subsection 6.3.2) then (X_1, X_2) satisfies the conditions on Proposition 6.5 but not those in Proposition 6.4.

6.3 Examples of copulas

Given a copula C (joint distribution function of (U_1, U_2)) with density c , we consider the following properties:

- (P1) $c(u, s - u)$ is TP_2 in (u, s) for $0 < u < 1, u < s < 1 + u$.
- (P2) $c(u, v)$ is TP_2 in (u, v) for $0 < u < 1, 0 < v < 1$.
- (P3) $c(u, v)$ is logconcave in v for all u .
- (P4) $c(u, v/u)$ is TP_2 in (u, v) for $0 < u < 1, 0 < v < u$.
- (P5) $v\eta_{U_2|U_1=u}(v) = -(v\frac{\partial}{\partial v}c(u, v))/c(u, v)$ increases in v for all u .

Given $\mathbf{X} = (X_1, X_2)$ with copula C and survival copula $\hat{C}$, if any of these conditions hold:

- $X_1, X_2 \sim U(0, 1)$ and C verifies (P1) (Proposition 6.2),
- $X_1, X_2 \sim U(0, 1)$ and C verifies (P2) and (P3) (Proposition 6.3),
- $X_1, X_2 \sim \text{Exp}(\lambda)$ and $\hat{C}$ verifies (P4) (Proposition 6.4),

- $X_1 \sim \text{Exp}(\lambda)$ and $\hat{C}$ verifies (P2) and (P5) (Proposition 6.5),

then $f(x, y - x)$ is TP_2 in (x, y) . Next, we provide examples of parametric families of copulas satisfying the property (P2) that also satisfy some of the other properties. Note that this is not the general case, for example, copulas (4.1.4) (Gumbel-Hougaard copula) and (4.1.2) in Nelsen (2007) satisfy (P2) but not (P1), (P3) (P4) or (P5).

6.3.1 Farlie-Gumbel-Morgenstern copula

Let consider the copula given by $C_\theta(u, v) = uv(1 + \theta(1 - u)(1 - v))$, for $\theta \in [-1, 1]$, $u, v \in [0, 1]$. The copula density, given by $c_\theta(u, v) = 1 + \theta(1 - 2v)(1 - 2u)$, is TP_2 for $\theta \in [0, 1]$, thus for such values (P2) holds. In addition:

$$\frac{d}{ds} \frac{d}{du} (\log(c_\theta(u, s - u))) = \frac{4\theta(1 + (1 - 2u)^2\theta)}{(1 + (1 - 2u)(1 + 2u - 2s)\theta)^2}$$

So, trivially, for $\theta \in [0, 1]$, we have that $\frac{d}{ds} \frac{d}{du} (\log(c_\theta(u, s - u))) \geq 0$ for all $0 < u < 1, u < s < u + 1$ and, therefore (P1) holds for all $\theta \in [0, 1]$. Moreover $\frac{d^2}{dv^2} (\log(c_\theta(u, v))) = 0$, and, therefore c_θ verifies (P3). Analogously, as considering $\theta \in [0, 1]$,

$$\frac{d}{dv} \frac{d}{du} (\log(c_\theta(u, v/u))) = \frac{2\theta(1 + (1 - 2u)^2\theta)}{(u + (1 - 2u)(u - 2v)\theta)^2} \geq 0$$

we have that (P4) is verified for all $\theta \in [0, 1]$. Although, one can compute

$$\frac{d}{dv} \left(\frac{-v \frac{\partial}{\partial v} c(u, v)}{c(u, v)} \right) = \frac{2(1 - 2u)\theta(1 + (1 - 2u)\theta)}{(1 + (-1 + 2u)(-1 + 2v)\theta)^2}.$$

Here $\frac{d}{dv} \left(\frac{-v \frac{\partial}{\partial v} c(u, v)}{c(u, v)} \right) \geq 0$ if and only if $(1 - 2u)\theta \geq 0$. Hence, (P5) does not hold for all u and $\theta \neq 0$ (Note that $\theta = 0$, would be the independence).

6.3.2 Clayton copula

Let consider the copula 1 in Table 4.1 in Nelsen (2007), given by $C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ for $\theta > 0$ (note that, when θ tends to zero, this is the independence copula, which trivially verifies (P1)-(P5), as the density is 1). The density is given by $c_\theta(u, v) = u^{-1-\theta} v^{-1-\theta} (u^{-\theta} + v^{-\theta} - 1)^{-2-\frac{1}{\theta}} (1 + \theta)$.

It can be seen that, whether the density is TP_2 (P2), $c(u, s - u)$ is not. That is, for $\theta > 0$, (P2) holds, but (P1) does not. Consequently, (P3) also fails to be satisfied (because (P2) and (P3) imply (P1)). It can be easily computed

that, considering $\theta > 0$:

$$\frac{d}{dv} \frac{d}{du} (\log(c_\theta(u, v/u))) = \frac{u^{-2+\theta} (2 - u^\theta) \left(\frac{v}{u}\right)^{-1+\theta} \theta(1+2\theta)}{\left(u^\theta + \left(\frac{v}{u}\right)^\theta - v^\theta\right)^2} \geq 0.$$

Therefore, (P4) holds for all $\theta > 0$. Analogously, for $\theta > 0$

$$\frac{d}{dv} \left(\frac{-v \frac{\partial}{\partial v} c(u, v)}{c(u, v)} \right) = \frac{u^\theta (1 - u^\theta) v^{-1+\theta} \theta(1+2\theta)}{(v^\theta + u^\theta - (uv)^\theta)^2} \geq 0.$$

Since the derivative is positive, (P5) is also verified for all $\theta > 0$. Here, we can see an example of the fact that *IPLR* does not imply *ILR*. Considering a vector (U_1, U_2) with a Clayton copula, there are values of u_1 for which $[U_2|U_1 = u_1]$ is *IPLR* but it is not *ILR*.

6.3.3 Ali-Mikhail-Haq copula

Let us now consider the copula 3 in Table 4.1 in Nelsen (2007), given by $C_\theta(u, v) = \frac{uv}{(1-\theta(1-u)(1-v))}$ for $\theta \in [-1, 1)$. The density of the copula, given by

$$c_\theta(u, v) = \frac{1 + \theta(u + v + uv - 2 + (1-u)(1-v)\theta)}{(1 - (1-u)(1-v)\theta)^3}$$

is *TP₂* for $\theta \in [0, 1)$. Computing $\frac{d}{ds} \frac{d}{du} (\log(c_\theta(u, s-u)))$ and $\frac{d^2}{dv^2} (\log(c_\theta(u, v)))$, we see that, for all $\theta \in [0, 1)$, $c(u, s-u)$ is *TP₂* in (u, s) but $c(u, v)$ is not logconcave in v for all u . It can be also shown that $\frac{d}{dv} \frac{d}{du} (\log(c_\theta(u, v/u)))$ is positive for all $0 < u \leq 1$ and $0 \leq v \leq u$ if and only if $\theta \in [0, 1/2]$ (which means that (P4) holds whenever $\theta \in [0, 1/2]$). Finally, one can check that (P5) does not hold.

6.3.4 Frank copula

Given the copula $C_\theta(u, v) = -\frac{1}{\theta} \log \left(1 + \frac{(e^{-u\theta} - 1)(e^{-v\theta} - 1)}{e^{-\theta} - 1} \right)$, for $\theta \in \mathbb{R} \setminus \{0\}$ (copula 5 in Table 4.1 in Nelsen (2007)), the density is given by

$$c_\theta(u, v) = \frac{e^{(1+u+v)\theta} (e^\theta - 1) \theta}{(e^{(u+v)\theta} - e^\theta (e^{u\theta} + e^{v\theta} - 1))^2}.$$

This function is *TP₂* in (u, v) if $\theta \geq 0$ and, for such values of the parameter, since

$$\frac{d^2}{dv^2} (\log(c_\theta(u, v))) = \frac{2e^{\theta+v\theta} (e^\theta - e^{u\theta}) (1 - e^{u\theta}) \theta^2}{(e^{(u+v)\theta} - e^\theta (e^{u\theta} + e^{v\theta} - 1))^2} < 0,$$

(P3) holds, and, therefore, also does (P1) for $\theta \geq 0$. It can be verified that $\frac{d}{dv} \frac{d}{du} (\log(c_\theta(u, v/u)))$ is positive (and therefore (P4) holds) for all $0 < u \leq 1$

and $0 \leq v \leq u$ if and only if $\theta \in [0, 1]$. Although, (P5) do not hold whatever θ -value is chosen.

The results of this section can be summarized in Table 1:

	P1	P2	P3	P4	P5
FGM copula	$\theta \in [0, 1]$	$\theta \in [0, 1]$	$\theta \in [-1, 1]$	$\theta \in [0, 1]$	$\theta > 0$
Clayton copula		$\theta > 0$		$\theta > 0$	
AMH copula	$\theta \in [0, 1]$	$\theta \in [0, 1]$		$\theta \in [0, \frac{1}{2}]$	
Frank copula	$\theta > 0$	$\theta > 0$	$\theta \in \mathbb{R} \setminus \{0\}$	$\theta \in (0, 1]$	

Table 6.1. Values of the parameters under which properties P1 to P5 are satisfied.

6.4 Other examples

6.4.1 Multivariate Normal Distribution

In Corollary 3.1, Pellerey and Navarro (2022) studied Efron's monotonicity for multivariate normal distributed vectors. The authors states that, given a multivariate normal distributed vector $\mathbf{X} = (X_1, X_2, \dots, X_n) \sim \mathcal{N}(\bar{\mu}, \Sigma)$, if $\sum_{j \neq i} \text{Cov}(X_i, X_j) \geq 0$, then $\{X_i \mid S = s\}$ is non-decreasing in the lr -order. In the bivariate case, that holds if $\text{Cov}(X_1, X_2) \geq 0$. Although, these are not the only cases. Since $\mathbf{X} \sim \mathcal{N}(\bar{\mu}, \Sigma)$, by closure properties of normal distributions the vector (X_i, S) has a bivariate normal distribution and it satisfies the TP_2 property if $\text{Cov}(X_i, S) \geq 0$, that is, if $\sum_{i=1}^n \text{Cov}(X_1, X_j) \geq 0$ (with $\text{Cov}(X_1, X_1)$ included).

In particular, in the bivariate framework, the result by Pellerey and Navarro (2022) states that Efron's monotonicity holds in the likelihood ratio order when $\rho > 0$. Although, here we saw that it holds even if the variables have a negative correlation ρ , as far as $\rho \geq -\frac{\sqrt{\text{Var}(X_1)}}{\sqrt{\text{Var}(X_2)}}$.

6.4.2 Generalized Order Statistics (GOSs)

In this Section, we aim to provide further examples of parametric families where Proposition 3 can be applied. With this purpose, we introduce the notion of generalized order statistics (GOSs) (see Kamps (1995)).

Definition 6.6. Let $n \in \mathbb{N}, k \geq 1, m_1, \dots, m_{n-1} \in \mathbb{R}, M_r = \sum_{j=r}^{n-1} m_j, 1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in$

$\{1, \dots, n-1\}$, and let $\tilde{m} = (m_1, \dots, m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). If the random variables $U_{(r,n,\tilde{m},k)}$, $r = 1, \dots, n$, possess a joint density of the form

$$h(u_1, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} (1 - u_j)^{m_j} \right) (1 - u_n)^{k-1}$$

defined on $0 \leq u_1 \leq \dots \leq u_n \leq 1$, then they are called GOSs. For a given distribution function F , the random variables $X_{(r,n,\tilde{m},k)} = F^{-1}(U_{(r,n,\tilde{m},k)})$, for $r = 1, \dots, n$, are called the GOSs based on F .

Several models of ordered random variables are included in this model:

- Considering $m_i = 0$ for all $i = 1, \dots, n-1$ and $k = 1$, we get the order statistics from a distribution F .
- Taking $m_i = -1$ for all $i = 1, \dots, n-1$ and $k = 1$, we get the first n record values from a sequence of random variables with distribution F (or the first n epoch times of a nonhomogeneous Poisson process).
- A generalization of the previous model is the case in which $k \in \mathbb{N}$, resulting in the so-called k -records.
- GOS also includes some other models of interest such as sequential order statistics and progressively type-II censored order statistics.

Proposition 6.7. *Let F be an absolutely continuous distribution function with logconcave density f . Let $(X_{(i,n,\tilde{m},k)}, X_{(j,n,\tilde{m},k)})$, $1 \leq i < j \leq n$, a bivariate random vector of GOSs from F . Assume that any of the following conditions is satisfied:*

(a) $k \geq 1, m_i \geq 0$.

(b) $k > 0, m_i \geq -1$ and the failure rate function $\lambda(\cdot)$ of F is log-concave.

Then, $\{X_{(i,n,\tilde{m},k)} | X_{(i,n,\tilde{m},k)} + X_{(j,n,\tilde{m},k)} = t\}$ increases in t in the likelihood ratio order.

Proof. It is well-known that any random vector of GOS is MTP_2 (Belzunce et al. (2005)). Then, the bivariate vector $(X_{(i,n,\tilde{m},k)}, X_{(j,n,\tilde{m},k)})$, $1 \leq i \leq j \leq n$, is TP_2 . In order to obtain the result, by Proposition 2.4 in Pellerey and Navarro (2022), it is sufficient to prove that $\{X_{(j,n,\tilde{m},k)} | X_{(i,n,\tilde{m},k)} = t\}$ has a logconcave density function (or it is ILR). It is known that

$$\{X_{(j,n,\tilde{m},k)} | X_{(i,n,\tilde{m},k)} = t\} \simeq_{\text{st}} \{X_{(j-i,n-i,\tilde{m}',k)} | X_{(1,n-r+1,\tilde{m}',k)} > t\},$$

where $\tilde{m}' = (m'_1, \dots, m'_{n-r})$ is such that $m'_j = m_{n-j}$ for $j = 1, \dots, n-r$. If condition (a) or (b) holds, then $X_{(j-i, n-i, \tilde{m}', k)}$ has a logconcave density (Chen et al. (2009)) and it is easy to see that logconcavity is preserved by right truncations, that is, $\{X_{(j-i, n-i, \tilde{m}', k)} \mid X_{(1, n-r+1, \tilde{m}', k)} > t\}$ has also a logconcave density and the assertion follows.

In particular:

- If f is logconcave and $X_{i:n}, X_{j:n}$ are two order statistics with $1 \leq i < j \leq n$ then, $\{X_{i:n} \mid X_{i:n} + X_{j:n} = s\}$ increases in s in the lr -ratio order. In particular, it holds for $\{\min(X_1, X_2) \mid X_1 + X_2 = s\}$. It can be also checked that, when $n = 2$, $\{\max(X_1, X_2) \mid X_1 + X_2 = s\}$ increases in s in the lr -ratio order.
- If $X_{L_n}, X_{L_m}, n < m$ are two record values of F and f and the failure rate function $\lambda(t)$ are both logconcave, then $\{X_{L_n} \mid X_{L_n} + X_{L_m} = s\}$ increases in s in the likelihood ratio order.
- If $X_{L_n^k}, X_{L_m^k}, n < m$ are two k -record values of F , and $f(\cdot)$ and $\lambda(\cdot)$ are both logconcave, then $\{X_{L_n^k} \mid X_{L_n^k} + X_{L_m^k} = s\}$ increases in s in the lr -ratio order.

6.5 Conclusions

In this chapter, sufficient conditions for Efron's marginal monotonicity (EMM) were studied in terms of the copula when assuming uniform or exponential distributed marginals. These conditions implied the sufficient conditions given in Pellerey and Navarro (2022) and based on the notion of total positivity of the joint density of the vector. Several examples were provided where the conditions hold. In addition, examples on GOS and the multivariate normal distribution were provided for some of the results included in Pellerey and Navarro (2022).

*“My interest is in the future because I am
going to spend the rest of my life there.”*

Charles Kettering



Conclusions and Future Work

In this last chapter, we present the conclusions drawn from this research work, together with a list of the publications and communications in conferences obtained during the Ph.D. thesis. Future research lines complement this closing chapter.

7.1 Conclusions of the Ph. D. Thesis

This Ph.D. dissertation has been devoted to the study of mathematical methodologies applied to Financial Risk and Actuarial Theory, under the Stochastic Orders and Copula modeling framework. Our contributions can be categorized according to the division in chapters. The range of parametric examples studied in each of the chapters facilitates the management of the stated results in a given data set where parametric estimation was performed. The considered examples illustrate the usefulness of such approach.

Chapter 2 addresses the consistency, with respect to a wide set of univariate and multivariate stochastic orders, of the comparison of co-risk measures. These results extend, to the multivariate general case, comparisons of conditional values at risk ($\text{CoVaR}^>$) and conditional expected shortfalls (CoES) in the bivariate framework. Novel conditions are given for these comparisons. In addition, sufficient conditions are provided for the ordering of the marginal mean excess (MME). These results serve as tools which support decision making and help to define preferences between two n -dimensional portfolios of dependent risks. The core content of this chapter has been published in Ortega-Jiménez et al. (2021):

Chapter 7. Conclusions

P. Ortega-Jiménez, M. A. Sordo, and A. Suárez-Llorens. Stochastic orders and multivariate measures of risk contagion. Insurance: Mathematics and Economics, 96:199-207, 2021.

Chapter 3 is devoted to the study of an “alert” value, denoted by PELCoV_v . In this chapter, we tackle the problem of determining the capital risk of a random variable Y , traditionally performed by means of the VaR. Under very mild regularity conditions and under some kind of positive dependence structure, the PELCoV_v sets the first value of a second variable X which leads to $\text{CoVaR}_{v,u}[Y|X] \geq \text{VaR}_v[Y]$. Under this scenario, the spillover effect has led to higher risk than the one considered by the unconditional VaR. The usefulness of such alert value is discussed and sufficient conditions are provided, both for its monotonicity and boundness. Determine whether the PELCoV_v s are bounded takes special relevance since the existence of $u^* < 1$ such that, for all $v \in (0, 1)$, $\text{PELCoV}_v \leq u^*$ ensures that $Y \leq_{st} [Y|X = x]$ for all $x \leq F_X^{-1}(u^*)$. A parametric methodological approach was considered in order to provide an empirical illustration of the results, dealing with data from IBEX35 companies. The contents of this chapter have been submitted.

In Chapter 4, given pairs of random variables (not necessarily independent or identically distributed), sufficient conditions were provided with the aim of ordering several distances between each pair. These comparisons are performed in terms of several stochastic orderings. Some of the results extend comparisons that were already addressed considering independent and equally distributed components. Moreover, the main contribution of the chapter dealt with variability measures. Considering a random vector (X, Y) , where the second component is a distortion of the first ($Y = X_h$) and the dependence structure (C) is fixed, the functional $X \mapsto E_C|X - X_h|$ is proved to be a measure of variability in the sense of Bickel and Lehmann (1979). This is a subadditive measure when the components are negatively dependent. A general family of premium principles is defined by adding to the net premium a risk load proportional to the established measure of variability (considering $h(u) = u$). In addition, this premium principle reduces to the well-known standard deviation premium principle when considering a location scale family. Particular applications were also provided in actuarial and financial risk management in order to illustrate the usefulness of the results. The contents of this chapter have been published in Ortega-Jiménez et al. (2021):

P. Ortega-Jiménez, M.A. Sordo and A. Suárez-Llorens. Stochastic comparison of some distances between random variables. Mathematics, 9(9):981, 2021.

In view of the previous chapter and motivated by the fact that some location measures can be defined as the argument which minimizes a variability measure, in Chapter 5 we study a minimization problem of the variability functional $X \mapsto E_C|X - X_h|$. Assuming that the copula C and the distribution function F of X are known, we look for $h^* = \arg \min_{h: F_h \in \mathcal{F}_X} E_C|X - X_h|$. Here, $\mathcal{F}_X$ denotes the family of proportional reversed hazard rate distribution functions associated to F . The main contributions of the chapter are to highlight the fact that such minimizer does not always exist and to state that, if X is IFR and the copula verifies some conditions, we can ensure the existence of the minimizer. The content of this chapter has been published as a proceeding in Ortega-Jiménez et al. (2023a):

P. Ortega-Jiménez, F. Pellerey, M.A. Sordo, and A. Suárez-Llorens. A minimizing problem of distances between random variables with Proportional Reversed Hazard Rate functions. In International Conference on Soft Methods in Probability and Statistics, pages 311-318. Springer, 2023.

To conclude, in Chapter 6, we study a well-known dependence monotonicity notion in terms of the copula. The study of the stochastic monotonicity of $[X|X + Y = s]$ is reduced, under given assumptions, to study well-known positive dependence notions on the copula. Even if the context appears to be completely different to the previous chapters, this theoretical methodology has numerous possibilities of applications, among which, Financial Risk and Actuarial Theory can be listed. Results in this chapter have been also published as a conference proceeding in Ortega-Jiménez et al. (2023b).

P. Ortega-Jiménez, F. Pellerey, M.A. Sordo, and A. Suárez-Llorens. Note on Efron's Monotonicity property under given copula structures. In International Conference on Soft Methods in Probability and Statistics, pages 303-310. Springer, 2023.

7.2 Future work

The area studied in this dissertation, as well as the developed research lines, look promising in terms of future work. Among the different future research lines that can arise from this work, we list some open problems which are reviewed hereafter:

- Regarding the contagion index based on the MME which was defined in Section 2.4, an exhaustive study of its properties, consistency with respect to other contagion indexes, as well as providing a plausible

interpretation and application is a work that remains for future research.

- Following the approach of Chapter 3, instead of considering the VaR, one would naturally think on the ES, measure that overcomes an important limitation of the VaR, which is not a coherent risk measure. A natural step in this research line is to consider the ES and its dependence-adjusted version, known as CoES^π . Hence, it is also a question of future research to determine the probability equivalent level of ES and $CoES$. However, due to the nature of such measures (which represent averages of quantiles), the study would require further methodologies.
- Regarding the interpretability of the PELCoV_v (denoted by u_v), we can consider the following extreme events

$$U(u_v, v) = \{(x, y) : x \geq \text{VaR}_{u_v}[X], y \geq \text{CoVaR}_{v, F_X(x)}[Y|X]\},$$

which corresponds to the upper corrected orthant at the point $(x, y) = (\text{VaR}_{u_v}[X], \text{VaR}_v[Y])$ defined in Fernández-Ponce and Suárez-Llorens (2003) and also studied in Belzunce et al. (2008). Considering a random vector (X, Y) under the regularity conditions, the authors showed that the cumulative probability in the upper corrected orthant is given by $P[(X, Y) \in U(u_v, v)] = (1 - u_v)(1 - v)$. Therefore, greater values of u_v decrease the probability of the event's occurrence. Based on $U(u_v, u)$, an interesting generalization of the excess wealth function was considered in Fernández-Ponce et al. (2011). This generalization provides a novel risk measure that needs further study.

- In Section 3.5.2, we recalled that, given a vector of consecutive record values $(R_n, R_{n+1})_{n \in \mathbb{N}}$, we studied the succession $(\text{PELCoV}_{0.5, n})_{n \in \mathbb{N}}$, where $\text{PELCoV}_{0.5, n}$ is the unique value $u \in (0, 1)$ such that

$$F_{R_{n+1}|R_n=F_{R_n}^{-1}(u)}^{-1}(0.5) = F_{R_{n+1}}^{-1}(0.5).$$

Intuition suggested that the succession was decreasing and has the value 0.5 as lower bound. Although, the formal proof of such intuition needs further work. In addition, it remains for future research to extent this work analyzing the PELCoV_v of the vector (R_n, R_m) , where R_n, R_m are record values verifying $n, m \in \mathbb{N}$ with $n < m$.

- It would also be interesting to analyze how does the PELCoV_v behaves in vectors of Generalized Order Statistics (see Definition 6.6). The interest on this topic comes motivated by the following example.

Example 7.1. Let us consider two iid random variables X_1, X_2 with distribution function F and the vector $(X_{(1)}, X_{(2)})$ where $X_{(1)}$ stands for $\min(X_1, X_2)$ and $X_{(2)}$ for $\max(X_1, X_2)$. It is direct to verify that $F_{X_{(i)}}(x) = g_i(F(x))$ for $i = 1, 2$ where $g_1(u) = 1 - (1-u)^2$ and $g_2(u) = u^2$. Let us recall that $[X_{(2)}|X_{(1)} = x] =_{st} [X_1|X_1 > x]$. Then, for $v \in (0, 1)$, if we denote by u_v the PELCoV $_v$ of the vector $(X_{(1)}, X_{(2)})$, we obtain that:

$$F_{X_{(2)}|X_{(1)}=F_{X_{(1)}}^{-1}(u_v)}^{-1}(v) = F_{X_{(2)}}^{-1}(v) \iff v = \frac{g_2^{-1}(v) - g_1^{-1}(u_v)}{1 - g_1^{-1}(u_v)}.$$

We can directly obtain the analytical expression of the PELCoV $_v$:

$$u_v = g_1\left(\frac{g_2^{-1}(v) - v}{1 - v}\right) = 1 - \left(1 - \left(\frac{\sqrt{v} - v}{1 - v}\right)^2\right).$$

Here, we can check that the PELCoV $_v$ increases in terms of v and that $\lim_{v \rightarrow 1} u_v = \frac{3}{4} < 1$. Hence, we have that:

$$X_{(2)} \leq_{st} [X_{(2)}|X_{(1)} = F_{X_{(1)}}^{-1}(u)], \text{ for all } u \geq \frac{3}{4}. \quad (7.1)$$

Note that, since $F_{X_{(1)}}^{-1}(u) = F^{-1}(1 - \sqrt{1-u})$, statement (7.1) means that $X_{(2)} \leq_{st} [X_1|X_1 > F^{-1}(u)]$ for all $u \geq \frac{1}{2}$.

This example highlights the fact that not all vectors consisting on Generalized Order Statistics have the same condition with respect to the boundness of the PELCoV $_v$. Therefore, further work on determining sufficient conditions for the existence of such bound in these families of vectors would be interesting.

- With respect to the examples included in the dissertation dealing with real datasets, all estimations were obtained under a parametric methodological approach. With the aim of providing non parametric alternatives, some further work on finding consistent estimators for the defined indicators or measures would be welcome. With respect to the non parametric estimation of the PELCoV $_v$, a candidate of the estimator would come as a natural extension to the estimators of $F_{Y|X=F_X^{-1}(u)}^{-1}(v)$ or $\partial_1 C(u, v)$ given in Fuchs and Trutschnig (2020). We propose, for $n \in \mathbb{N}$, the following estimator:

$$\text{PELCoV}_{v,n} = \inf \left\{ u : K_{\mathcal{CB}_{[\sqrt{n}]}(\hat{E}_n)}(u, [0, v]) \leq v \right\},$$

where $K_C : [0, 1] \times \mathcal{B}([0, 1]) \rightarrow [0, 1]$ is the Markov Kernel of C , $\mathcal{CB}_N(C)$ is the checkerboard approximation of C and $\hat{E}_n$ is the empirical copula. The interested reader may consult Fuchs and Trutschnig

(2020) for further details on the concepts. Note that the Markov kernel verifies that for all $v \in (0, 1)$, $\partial_1 C(u, v) = K_C(u, [0, v])$ holds for almost all $u \in (0, 1)$. A continuation on this research line requires further discussion on continuity results of PELCoV_v (in terms of the copula) that lead to consistent estimators when plugging in the so-called empirical checkerboard copulas.

- By expression (5.1) in Chapter 5, one can easily see that, given $X =_{st} Y$, $E|X - Y| = 0$ only if the copula verifies $C(u, u) = u$ (for example, the comonotonic copula). Therefore, if the copula C is fixed, $\nu_C(X, Y) = E_C|X - Y|$ in general is not a metric. Hence, given C and X , it is natural to wonder which is the distribution function G of Y for which $E_C|X - Y|$ is minimized. A first approach to the problem is to take G as a distortion of the distribution function of X . Furthermore, a particular case of this minimization problem is to consider a particular family of distortions (in Chapter 5, the power functions were considered). To study the existence of such minimizers, as well as plausible interpretations, remains as work that can lead to future research.

Note that, when the marginal distributions are fixed, minimizing $E|X - Y|$ is a particular case of the Monge-Kantorovich problem, which seeks to find a probability measure K on $X \times Y$ that minimizes $E(c(X, Y))$ where $c : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is the cost function. That is, to find

$$\inf \left\{ \int_{X \times Y} c(x, y) dK(x, y) : K \in \Gamma(F, G) \right\},$$

where $\Gamma(F, G)$ denotes the collection of all probability measures on $X \times Y$ with marginal measures F on X and G on Y . Considering $c(x, y) = |x - y|$, the problem has been widely studied and the minimum coincides with the Wasserstein metric, which is the expected absolute difference between two random variables X and Y when the copula of (X, Y) is the comonotonicity. The approach that we suggested can be intuitively seen as a modified version of this problem, since we seek to find

$$\inf \left\{ \int_{X \times Y} |x - y| dC(F(x), G(y)) : C, F \text{ are fixed} \right\}.$$

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