Exact general solution and first integrals of a remarkable static Euler-Bernoulli beam equation

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Abstract

A static fourth-order Euler-Bernoulli beam equation, corresponding to a negative fractional power law for the applied load, has been completely integrated in this paper. For this equation the Lie symmetry and the Noether symmetry algebras are maximal and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Due to this algebra is nonsolvable, the symmetry reductions that have been employed so far in the literature fail to obtain the complete solution of the equation. A new strategy to obtain a third-order reduction has been performed, which provides, by direct integration, one of the first integrals of the equation. This first integral leads to a one-parameter family of third-order equations which preserves $\mathfrak{sl}(2,\mathbb{R})$ as symmetry algebra. From these equations, three remaining functionally independent first integrals have been computed in terms of solutions to a linear second-order equation and, as a consequence, the exact general solution has been obtained. As far as we know, this has not been previously reported in the literature. The general solution can be expressed in parametric form in terms of a fundamental set of solutions to a one-parameter family of Schrödinger-type equations.

Keywords: Euler-Bernoulli beam equation, nonsolvable symmetry algebra, first integrals, exact solution, Schrödinger-type equation.
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1. Introduction

The problem of the transverse motion of an elastic thin beam bearing a load was formulated by Daniel Bernoulli and Leonard Euler. If $y(t,x)$ denotes the transverse displacement at time t and position x, then the deflection of the beam can be modelled [1] by the following fourth-order partial differential equation:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+\mu\frac{\partial^2 y}{\partial t^2}=f(y), \tag{1}$$

where E is the elastic modulus (also known as the Young's modulus), I is the second moment of area, μ the mass per unit length and the function f, which will be assumed to be smooth, describes the applied load.

Equation (1) is known as the dynamic Euler-Bernoulli beam equation and it is of great interest due to its applications in many engineering fields, such as mechanical engineering and civil engineering. For this reason, the development of new techniques to obtain exact solutions to some particular cases of equation (1) is of great importance. One of the most powerful tools in this regard is the Lie symmetry approach [2, 3]. In [4], the authors performed a complete classification of the Lie symmetries of equation (1) assuming that E, I and μ are constant. The Lie symmetries of equation (1) for the case of variable lineal mass density were studied in [5].

In the static case, assuming that E and I are constant, equation (1) reduces (up to constants) to the fourth-order ordinary differential equation (ODE)

$$y_4=f(y), \tag{2}$$

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where $y_4 = \frac{d^4 y}{dx^4}$. In [6], the complete symmetry classification of equation (2) was carried out and it was proved that

$$y_4 = \delta y^{-5/3}, \quad \delta = \pm 1, \tag{3}$$

corresponds to the case admitting a maximum number of symmetries. Specifically, the Lie symmetry algebra of equation (3) is three-dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Since the Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ is nonsolvable, the standard Lie reduction method cannot be used to stepwise reduce the order of (3). In [6], the authors used a solvable two-dimensional subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ to reduce the order of equation (3) by two. As a result, a second-order equation admitting a nonlocal symmetry was obtained, although it could not be used to reduce further the equation. This difficulty motived the consideration of other approaches; for instance, in [6] it was also proved that equation (3) is the Euler-Lagrange equation associated to a variational problem that inherits $\mathfrak{sl}(2,\mathbb{R})$ as Noether symmetry algebra. As a consequence, a two-parameter family of solutions was obtained by using the first integrals provided by Noether's theorem. In [7], those first integrals were analysed further and some two-parameter and three-parameter families of exact solutions to (3) were found in implicit form. However, all these solutions involve primitives that cannot be evaluated. Although in [8, 9] an apparently three-parameter family of solutions was obtained in explicit form by means of the characteristic method [3], we prove in this paper that such family of solutions depends, in fact, on only two parameters.

Therefore, all those methods used to study the integrability of equation (3) fail to provide the complete solution of the equation. The goal of this paper is to give a definitive response to the integrability of equation (3) by computing its exact four-parameter general solution.

With this aim, in Section 2, we use a new strategy based on reducing the order of equation (3) by three by means of the joint invariants of the algebra $\mathfrak{sl}(2,\mathbb{R})$. After such order reduction, we obtain a first-order equation that can be explicitly solved, which provides by quadrature a first integral of equation (3).

Each level set of this first integral provides a third-order ODE that preserves a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. For this type of equations, the Lie method of reduction leads to a first-order Riccati equation which does not allow to recover a closed form expression for the general solution [10]. This problem has been recently solved: in [11] it was proven the existence of a solvable structure [12, 13] that can be constructed from the generators of a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and, as a consequence, a complete set of first integrals and parametric general solutions were explicitly obtained. These first integrals and solutions were expressed in terms of fundamental set of solutions to second-order linear equations for the four inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ [14]. In Section 3 we apply these previous results to the above-mentioned one-parameter family of third-order equations admitting $\mathfrak{sl}(2,\mathbb{R})$, in order to obtain a complete set of four functionally independent first integrals to equation (3). These first integrals are expressed in terms of solutions to a related second-order linear equation.

Moreover, in Section 4, the four functionally independent first integrals are used to obtain the (four-parameter) general solution to equation (3). The solution is given in parametric form and in terms of solutions to a one-parameter family of Schrödinger-type equations. This expression for the general solution is, as far as we know, new in the literature and completes all the previous works on the integrability of the Euler-Bernoulli beam equation (3).

Finally, in Section 5, a singular case that arises in our approach is analysed. This provides a two-parameter family of solutions which includes the explicit solution reported in [8, 9]. In this last section we also obtain new three-parameter families of solutions in parametric form which can be expressed in terms of elementary functions.

2. Reduction of order and one first integral of equation (3)

In [6], it was proved that the Lie symmetry algebra of equation (3) is three-dimensional, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and spanned by the vector fields

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 3xy\partial_y, \quad \mathbf{X}_3 = x\partial_x + \frac{3}{2}y\partial_y. \tag{4}$$

In this section we focus on reducing as far as possible the order of equation (3) by using the joint differential invariants of the algebra. Such invariants can be deduced from [14, Table 2] for the four inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$. Although (4) does not exactly correspond with any one of the four inequivalent realizations considered in

[14], it can be checked that by means of the transformation $u = y^{2/3}$ the symmetry generators (4) are respectively mapped into

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{X}_3 = x\partial_x + u\partial_u, \tag{5}$$

which corresponds with the basis elements of $\mathfrak{sl}(2,\mathbb{R})$ given in [14, Case 2 in Table 1]. This allows us to use directly the results of [14] at the cost of considering the corresponding transformed equation

$$24u^3u_4 + 48u_3u_1u^2 + 36u_2^2u^2 - 36u_2u_1^2u + 9u_1^4 - 16\delta = 0, \qquad \delta = \pm 1. \tag{6}$$

We assume that equation (6) is defined for (x,u) in some open set $M \subset \mathbb{R}^2$ and denote by $M^{(4)}$ the corresponding fourth-order jet space [2].

A set of joint invariants $\{s, w\}$ for the involutive system of vector fields $\left\{\mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)}, \mathbf{X}_3^{(3)}\right\}$ is given by

$$s = u_1^2 - 2uu_2 \quad \text{and} \quad w = u^2u_3, \tag{7}$$

where $\mathbf{X}_i^{(3)}$ stands for the third-order prolongation [2] of the vector field $\mathbf{X}_i$, for $i = 1,2,3$. If we consider the invariant obtained by derivation

$$w_1 = \frac{dw}{ds} = -\frac{2u_3u_1 + uu_4}{u_3}, \tag{8}$$

then equation (6) can be expressed in terms of the invariants $\{s, w, w_1\}$ as the following reduced equation:

$$2w_1w = \frac{3}{8}s^2 - \frac{2}{3}\delta. \tag{9}$$

This is a separable equation that can be integrated by quadratures and whose solutions satisfy:

$$w(s)^2 = \frac{1}{8}s^3 - \frac{2}{3}\delta s + K, \qquad K \in \mathbb{R}. \tag{10}$$

By isolating K in (10) and by writing the resulting expression in terms of the coordinates $\{x, u, u_1, u_2, u_3\}$ by means of (7), we obtain the following first integral $I_0 = I_0(x, u, u_1, u_2, u_3)$ of the fourth-order equation (6):

$$I_0 = u^4u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\left(u_1^2 - 2uu_2\right). \tag{11}$$

The reconstruction of the general solution of equation (6) can be carried out by solving the third-order ODEs $I_0 = K_0, K_0 \in \mathbb{R}$, i.e:

$$u^4u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\left(u_1^2 - 2uu_2\right) = K_0, \qquad K_0 \in \mathbb{R}. \tag{12}$$

Since these equations can be written in terms of the joint differential invariants $\{s, w\}$, they inherit a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and spanned by (5).

By writing the first integral I_0 given in (11) in terms of the original variables $\{x, y, y_1, y_2, y_3\}$ by means of

$$u = y^{2/3},\ u_1 = \frac{2}{3y^{1/3}}y_1,\ u_2 = \frac{-2}{9y^{4/3}}(y_1^2 - 3y_2y),\ u_3 = \frac{2}{27y^{7/3}}(4y_1^3 - 9y_1y_2y + 9y_3y^2), \tag{13}$$

we obtain a first integral $J_0 = J_0(x, y, y_1, y_2, y_3)$ of the original Euler-Bernoulli beam equation (3). Furthermore, if $\mathbf{A} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + y_3\partial_{y_2} + \delta y^{-5/3}\partial_{y_4}$ denotes the vector field associated to equation (3), then J_0 is a common first integral to the set of vector fields $\left\{\mathbf{A}, \mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)}, \mathbf{X}_3^{(3)}\right\}$. It should be remarked that such first integral has been obtained directly by quadrature. The explicit expression of the function J_0 is presented in the following theorem:

Theorem 1. *A first integral of the Euler-Bernoulli beam equation (3), which can be obtained by quadrature, is given by*

$$J_0(x,y,y_1,y_2,y_3) = 9y^2y_3^2 + \left(-18yy_1y_2 + 8y_1^3\right)y_3 + 6y_2^3y - 18\delta y_2y^{1/3} - 3y_1^2y_2^2 + 12\delta y_1^2y^{-2/3}. \tag{14}$$

Besides, the general solution of equation (3) can be found by solving the third-order ODEs in the family

$$J_0(x,y,y_1,y_2,y_3) = K_0, \quad K_0 \in \mathbb{R}, \tag{15}$$

which inherit a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ *and spanned by (4).*

In what follows we address the study of the third-order equations in the family (15). With the goal of applying directly the results obtained in [14] we consider, instead of (15), the transformed equation (12), which can be locally written as the canonical $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE given in [14, Case 2 of Table 2]:

$$u_3 = -\frac{1}{8u^2 C(s;K_0)} \qquad (s = u_1^2 - 2uu_2), \tag{16}$$

where the function $C = C(s;K_0)$ satisfies

$$C(s;K_0)^2 = \frac{1}{8\left(s^3 - \dfrac{16}{3}\delta\, s + 8K_0\right)}. \tag{17}$$

Explicit expressions of the first integrals of the canonical $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE (16) were obtained in [14], in terms of the fundamental set of solutions to a second-order linear equation. In Section 3 we compute for equation (16) such expressions in order to obtain a complete set of first integrals of the Euler-Bernoulli beam equation (3). As far as we know, this has not previously appeared in the literature. Throughout Section 3 it will be also assumed that expression (17) is well-defined. The possible solutions of equation (6) for which (17) is not defined are analysed separately in Section 5.

3. Complete set of first integrals

In this section we prove that a complete system of four functionally independent first integrals to equation (3) can be explicitly computed in terms of solutions to a second-order linear equation.

Firstly, we analyse the first integrals of equation (12), locally written as in (16). These first integrals can be expressed in terms of a fundamental set of solutions to the second-order linear equation (see [14, Case 2 in Table 6]):

$$\psi''(s) - \frac{C'(s;K_0)}{C(s;K_0)}\psi'(s) - 4C(s;K_0)^2\, s\,\psi(s) = 0. \tag{18}$$

Since for equation (16) the function $C = C(s;K_0)$ satisfies (17), equation (18) becomes

$$\left(s^3 - \frac{16}{3}\delta\, s + 8K_0\right)\psi''(s) + \frac{1}{2}\left(3s^2 - \frac{16}{3}\delta\right)\psi'(s) - \frac{1}{2}s\,\psi(s) = 0. \tag{19}$$

If $\psi_1 = \psi_1(s;K_0)$ and $\psi_2 = \psi_2(s;K_0)$ are two linearly independent solutions to equation (19) then a complete set of first integrals $I_i = I_i(x,u,u_1,u_2;K_0)$, for $i = 1,2,3$, to equation (16) are (see [14, Case 2 in Table 7]):

$$\begin{aligned} I_1 &= \frac{2u_1 C(s;K_0)\psi_1(s;K_0) + \psi_1'(s;K_0)}{2u_1 C(s;K_0)\psi_2(s;K_0) + \psi_2'(s;K_0)}, \qquad I_2 = \frac{2(u_1x - 2u)C(s;K_0)\psi_1(s;K_0) + x\psi_1'(s;K_0)}{2(u_1x - 2u)C(s;K_0)\psi_2(s;K_0) + x\psi_2'(s;K_0)},\\ I_3 &= \frac{\big(C(s;K_0)2(u_1x - 2u)\psi_2(s;K_0) + x\psi_2'(s;K_0)\big)^2}{4C(s;K_0)uW(\psi_1,\psi_2)(s;K_0)}, \end{aligned} \tag{20}$$

where $W(\psi_1,\psi_2)(s;K_0) = \psi_1(s;K_0)\psi_2'(s;K_0) - \psi_1'(s;K_0)\psi_2(s;K_0)$ stands for the Wronskian determinant of ψ_1 and ψ_2, $C = C(s;K_0)$ satisfies (17) and $s = u_1^2 - 2uu_2$.

By substituting K_0 by I_0, where I_0 is given by (11), into the expressions of I_1, I_2 and I_3 given in (20) we obtain three functionally independent first integrals to the fourth-order equation (6) which, together with I_0, complete the integration of this equation.

Liouville's formula for the linear equation (19) implies that $W(\psi_1,\psi_2)(s;I_0)$ and $C(s;I_0)$ are proportional, where, according to (16), $C(s;I_0) = -\frac{1}{8u^2u_3}$. This allows us to simplify the expressions of the first integrals (20) after replacing K_0 by I_0. The explicit expressions of the complete set of (adequately rescaled) first integrals to equation (6) are presented in the following lemma:

Lemma 2. *A complete set of first integrals* $\{I_0, I_1, I_2, I_3\}$ *to equation (6) is given by*

$$\begin{aligned} I_0(x,u,u_1,u_2,u_3) &= u^4 u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\left(u_1^2 - 2uu_2\right), \\ I_1(x,u,u_1,u_2,u_3) &= \frac{u_1\psi_1(s;I_0) - 4u^2u_3\psi_1'(s;I_0)}{u_1\psi_2(s;I_0) - 4u^2u_3\psi_2'(s;I_0)}, \\ I_2(x,u,u_1,u_2,u_3) &= \frac{(u_1x-2u)\psi_1(s;I_0) - 4xu^2u_3\psi_1'(s;I_0)}{(u_1x-2u)\psi_2(s;I_0) - 4xu^2u_3\psi_2'(s;I_0)}, \\ I_3(x,u,u_1,u_2,u_3) &= \frac{1}{u}\left((u_1x-2u)\psi_2(s;I_0) - 4xu^2u_3\psi_2'(s;I_0)\right)^2, \end{aligned} \tag{21}$$

where ψ_1 *and* ψ_2 *are two linearly independent solutions to the linear second-order ODE (19) and* $s = u_1^2 - 2uu_2$.

Proof. Let $V \subset M^{(3)}$ be an open set where the first integrals (21) are well-defined and $u_3 \neq 0$. Let φ be the transformation defined in V by $\varphi(x,u,u_1,u_2,u_3) = (x, I_0, I_1, I_2, I_3)$. It can be checked that the determinant of the Jacobian matrix of φ is

$$J_\varphi(x,u,u_1,u_2,u_3) = \frac{-64u^7u_3^2W(\psi_1,\psi_2)^2(s)}{\left(u_1\psi_2(s) - 4u^2u_3\psi_2'(s)\right)^2} \qquad (s = u_1^2 - 2uu_2),$$

and it does not vanish in V, which proves the functional independence of the first integrals $\{I_0, I_1, I_2, I_3\}$ in V. □

So far, we have explicitly computed a complete set of first integrals to the transformed equation (6). A complete set of first integrals $\{J_0, J_1, J_2, J_3\}$ to equation (3) can be constructed by expressing (21) in terms of the original coordinates by means of (13). The explicit expressions of these first integrals are given below:

Theorem 3. *A complete set of first integrals* $\{J_0, J_1, J_2, J_3\}$ *to equation (3), is given by*

$$\begin{aligned} J_0(x,y,y_1,y_2,y_3,y_4) &= 9y^2y_3^2 + \left(-18yy_1y_2 + 8y_1^3\right)y_3 + 6y_2^3y - 18\delta y_2y^{1/3} - 3y_1^2y_2^2 + 12\delta y_1^2y^{-2/3}, \\ J_1(x,y,y_1,y_2,y_3,y_4) &= \frac{4y^{1/3}(4{y_1}^3 - 9y_1y_2y + 9y_3y^2)\psi_1(\tilde{s};J_0) - 9yy_1\psi_1'(\tilde{s};J_0)}{4y^{1/3}(4{y_1}^3 - 9y_1y_2y + 9y_3y^2)\psi_2(\tilde{s};J_0) - 9yy_1\psi_2'(\tilde{s};J_0)}, \\ J_2(x,y,y_1,y_2,y_3,y_4) &= \frac{4y^{1/3}x(4{y_1}^3 - 9y_1y_2y + 9y_3y^2)\psi_1(\tilde{s};J_0) - 9y(y_1x-3y)\psi_1'(\tilde{s};J_0)}{4y^{1/3}x(4{y_1}^3 - 9y_1y_2y - 9y_3y^2)\psi_2(\tilde{s};J_0) + 9y(y_1x-3y)\psi_2'(\tilde{s};J_0)}, \\ J_3(x,y,y_1,y_2,y_3,y_4) &= \frac{1}{y^{10/3}}\left(\frac{1}{9}y^{1/3}x(4{y_1}^3 - 9y_1y_2y + 9y_3y^2)\psi_2(\tilde{s};J_0) - \frac{1}{4}y(y_1x-3y)\psi_2'(\tilde{s};J_0)\right)^2, \end{aligned}$$

where ψ_1 *and* ψ_2 *are two linearly independent solutions to the linear second-order ODE (19) and* $\tilde{s} = \dfrac{4}{9}\,\dfrac{2{y_1}^2 - 3y_2y}{y^{2/3}}$.

4. Parametric general solution

In this section we focus on obtaining a general parametric solution to the Euler-Bernoulli beam equation (3) by means of computing such solution for the corresponding transformed equation (6). As an immediate consequence of Lemma 2, the general solution to the fourth-order ODE (6) is implicitly defined by

$$I_1(x,u,u_1,u_2;K_0) = K_1,\ I_2(x,u,u_1,u_2;K_0) = K_2,\ I_3(x,u,u_1,u_2;K_0) = K_3, \tag{22}$$

where the expressions of I_1, I_2 and I_3 are given in (20) and $K_i \in \mathbb{R}$ for $i = 0,1,2,3$.

The elimination of u_1 and u_2 from (22) in order to obtain a closed-form solution of equation (6) seems to be impossible, because both functions ψ_1 and ψ_2 and their derivatives are evaluated in $s = u_1^2 - 2uu_2$. With the aim of overcoming such difficulty, we focus on obtaining the solution in parametric form by introducing a new parameter t such that $s = s(t)$ is determined as follows:

$$s'(t) = \frac{1}{C(s(t;K_0))}, \tag{23}$$

where the prime symbol denotes derivation with respect to t. By (17), $s = s(t)$ satisfies

$$s'(t)^2 = 8\left(s(t)^3 - \frac{16}{3}\delta\, s(t) + 8K_0\right). \tag{24}$$

The general solution of equation (24) can be expressed as $s(t; t_0, K_0) = \frac{1}{2}\wp\left(t - t_0; g_2, g_3\right)$, where $\wp(t) = \wp(t - t_0; g_2, g_3)$ denotes the Weierstrass $\wp$-function [15, 16] with invariants

$$g_2 = \frac{16^2}{3}\delta, \quad g_3 = -16^2 K_0. \tag{25}$$

Let $s(t; K_0)$ be the particular solution to equation (24) corresponding to $t_0 = 0$. If $\psi = \psi(s; K_0)$ is a solution to the linear equation (19), then $\phi(t; K_0) = \psi(s(t; K_0); K_0)$ verifies the following Schrödinger-type equation:

$$\phi''(t; K_0) - 2\wp\left(t; \frac{16^2}{3}\delta, -16^2 K_0\right)\phi(t; K_0) = 0. \tag{26}$$

Therefore, if $\psi_1 = \psi_1(s; K_0)$ and $\psi_2 = \psi_2(s; K_0)$ are two linearly independent solutions to equation (19) then $\phi_1(t; K_0) = \psi_1(s(t); K_0)$ and $\phi_2(t; K_0) = \psi_2(s(t); K_0)$ is a fundamental set of solutions to equation (26) and the implicit general solution (22) can be expressed as follows:

$$\begin{aligned} &\frac{2u_1\phi_1(t; K_0) + \phi_1'(t; K_0)}{2u_1\phi_2(t) + \phi_2'(t; K_0)} = K_1, \qquad \frac{2(u_1 x - 2u)\phi_1(t; K_0) + x\phi_1'(t; K_0)}{2(u_1 x - 2u)\phi_2(t; K_0) + x\phi_2'(t; K_0)} = K_2, \\ &\frac{(2(u_1 x - 2u)\phi_2(t; K_0) + x\phi_2(t; K_0))^2}{4uW(\phi_1, \phi_2)(t; K_0)} = K_3. \end{aligned} \tag{27}$$

Let us observe that, by Liouville's formula for the linear equation (26), the Wronskian $W(\phi_1, \phi_2)(t; K_0)$ appearing in (27) is a constant. By means of a convenient choice of ϕ_1 and ϕ_2 we can assume that $W(\phi_1, \phi_2)(t; K_0) = 1$. From (27) we can eliminate u_1 to obtain the following parametrized general solution to equation (6):

$$x(t) = \frac{K_3(K_1 - K_2)\left(\phi_1(t; K_0) - K_2\phi_2(t; K_0)\right)}{\phi_1(t; K_0) - K_1\phi_2(t; K_0)}, \quad u(t) = \frac{K_3(K_1 - K_2)^2}{4\left(\phi_1(t; K_0) - K_1\phi_2(t; K_0)\right)^2},$$

where $K_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$, $K_3 > 0$, and $K_1 \neq K_2$.

As a result of the discussion carried out in this section and taking into account that $y = \pm u^{3/2}$, the next theorem has been proven:

Theorem 4. *Let $\phi_1 = \phi_1(t; K_0)$ and $\phi_2 = \phi_2(t; K_0)$ be two linearly independent solutions to the one-parameter family of Schrödinger-type equations (26) such that $W(\phi_1, \phi_2)(t; K_0) = 1$. Then an exact four-parameter solution to the static Euler-Bernoulli beam equation (3) is given in parametric form by*

$$x(t) = K_3(K_1 - K_2)\frac{\phi_1(t; K_0) - K_2\phi_2(t; K_0)}{\phi_1(t; K_0) - K_1\phi_2(t; K_0)}, \qquad y(t) = \pm\left(\frac{K_3^{1/2}\,(K_1 - K_2)}{2\left(\phi_1(t; K_0) - K_1\phi_2(t; K_0)\right)}\right)^3, \tag{28}$$

where $K_i \in \mathbb{R}$ for $i = 1, 2, 3$, $K_3 > 0$, and $K_1 \neq K_2$.

5. Analysis of some families of solutions

In this section we obtain some two-parameter and three-parameter families of solutions to the beam equation (3). The two-parameter family arises as a singular case in our method and corresponds to the explicit solution reported in [8, 9]. The two new three-parameter families of solutions are obtained in parametric form and correspond to the case in which the discriminant of the Weierstrass $\wp$-function appearing in (26) is equal to zero. Remarkably, all the solutions obtained in this section are expressed in terms of elementary functions.

5.1. *A singular two-parameter familiy of solutions: comparison with other results*

In [8, 9] the following particular solution to equation (3) was obtained by using different methods:

$$y(x) = \left(\frac{\delta}{9(\beta^2-\alpha\gamma)^2}\right)^{3/8} (\gamma x^2 + 2\beta x + \alpha)^{3/2}, \quad \text{where} \quad \beta^2 - \alpha\gamma \neq 0. \tag{29}$$

In this subsection we prove that this solution appears in our procedure as a singular case in which the function $C(s;K_0)^2$ given in (17) is not defined. That situation occurs if $s = s_0(K_0)$, being $s_0(K_0)$ one of the roots of the polynomial $s^3 - \frac{16}{3}\delta\, s + 8K_0 = 0$. According to the value of s given in (16), $s = s_0(K_0)$ yields the following second-order equation $u_1^2 - 2uu_2 = s_0(K_0)$, whose solutions $u = f(x)$ are of the form

$$f(x) = ax^2 + bx + c, \tag{30}$$

where the constants a, b and c satisfy the condition $s_0(K_0) = b^2 - 4ac$.

It is clear that (30) satisfies equation (12), i.e. $I_0 - K_0 = 0$. It can be checked that $D_x(I_0 - K_0) = \mu\Delta$, where $\mu = \mu(x, u, u_1, u_2, u_3) = 2u^4u_3$ and Δ denotes the left-hand side of equation (6). Since μ is null for $u = f(x) = ax^2 + bx + c$, such function might not be a solution of $\Delta = 0$. In fact, it can be checked that (30) satisfies the equation $\Delta = 0$ if and only if

$$3(b^2 - 4ac)^2 - \frac{16}{3}\delta = 0. \tag{31}$$

Solutions (30) verifying (31) correspond to solutions of (3) of the form

$$y(x) = \pm(ax^2 + bx + c)^{3/2}, \tag{32}$$

when the constants a, b and c satisfy (31).

Apparently, the family of solutions (29) reported in [8, 9] seems to be three-parameter because no constraints on α, β and γ are imposed. However, it can be checked that (29) can be expressed in terms of only two or one parameters as follows:

- If $\gamma \neq 0$ then $y(x) = \left(\dfrac{(2ax+b)^2 + \frac{4}{3}\delta}{4a}\right)^{3/2}$ where $a = \dfrac{\sqrt{3}}{3}\,\dfrac{\delta^{1/4}\gamma}{(\alpha\gamma - \beta)^{1/2}}$ and $b = \dfrac{2\sqrt{3}}{3}\,\dfrac{\delta^{1/4}\beta}{(\alpha\gamma - \beta)^{1/2}}$.
- If $\gamma = 0$ then $y(x) = \left(\dfrac{2\sqrt{3}}{3}\delta^{1/4}x + c\right)^{3/2}$ where $c = \dfrac{\sqrt{3}}{3}\,\dfrac{\delta^{1/4}\alpha}{\beta}$ and $\beta \neq 0$.

The previous discussion proves that (29) is actually a two-parameter family of solutions of the form (32) where the constant a, b and c satisfy (31).

We observe that condition (31) implies that $s_0(K_0) = b^2 - 4ac$ is a double root of the polynomial $s^3 - \frac{16}{3}\delta\, s + 8K_0 = 0$. Hence, if $s_1(K_0)$ denotes the other remaining root of the polynomial, we have the following decomposition: $s^3 - \frac{16}{3}\delta\, s + 8K_0 = \big(s - s_0(K_0)\big)^2(s - s_1(K_0))$. After a straightforward computation and by identifying coefficients we deduce that $\delta = 1$, $s_0(K_0) = \pm\frac{4}{3}$, $s_1(K_0) = -2s_0(K_0)$ and

$$K_0 = \pm\frac{16}{27}. \tag{33}$$

In the next subsection we find new solutions of the equation (3) that correspond to (28) for the values of K_0 given in (33).

5.2. *New three-parameter families of solutions corresponding to* $K_0 = \pm\dfrac{16}{27}$

For the invariants g_2 and g_3 that appear in (25), we observe that the values of K_0 given in (33) correspond precisely to the cases in which the discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function $\wp(t) = \wp(t; g_2, g_3)$ is equal to zero. In such a case the invariants g_2 and g_3 can be related and the Weierstrass $\wp$-function can be expressed in terms of elementary functions [15, pag. 651] (see also [16, sections 22.5(ii) and 23.6(ii)]). This fact allows us to find explicitly a fundamental set of solutions to the Schrödinger-type equation (26). We distinguish two possible cases:

- If $K_0 = -\frac{16}{27}$ then $\wp(t; g_2, g_3) = -\frac{8}{3} + 8\csc^2\left(2\sqrt{2}t\right)$ and it can be checked that two linearly independent solutions to the corresponding equation (26) verifying $W(\phi_1, \phi_2)(t) = 1$ become

  $$\begin{aligned} \phi_1(t) &= \frac{\sqrt{3}}{8}\cot\left(2\sqrt{2}t\right)\sin\left(\alpha(t)\right) - \frac{\sqrt{2}}{8}\cos\left(\alpha(t)\right), \\ \phi_2(t) &= \csc\left(2\sqrt{2}t\right)\left((3+\sqrt{6})\cos\left(\beta_1(t)\right) + (3-\sqrt{6})\cos\left(\beta_2(t)\right)\right), \end{aligned} \tag{34}$$

  where

  $$\alpha(t) = \frac{4\sqrt{2}t - \pi}{\sqrt{6}}, \quad \beta_1(t) = \frac{4(\sqrt{2}-\sqrt{3})t - \pi}{\sqrt{6}}, \quad \beta_2(t) = \frac{4(\sqrt{2}+\sqrt{3})t - \pi}{\sqrt{6}}. \tag{35}$$

- If $K_0 = \frac{16}{27}$ then $\wp(t; g_2, g_3) = \frac{8}{3} + 8\,\mathrm{csch}^2\left(2\sqrt{2}t\right)$ and two linearly independent solutions to the corresponding equation (26) satisfying $W(\phi_1, \phi_2)(t) = 1$ become

  $$\begin{aligned} \phi_1(t) &= \mathrm{csch}\left(2\sqrt{2}t\right)\left((3+\sqrt{6})\cosh\left(\beta_1(t)\right) + (3-\sqrt{6})\cosh\left(\beta_2(t)\right)\right), \\ \phi_2(t) &= \frac{\sqrt{3}}{8}\coth\left(2\sqrt{2}t\right)\sinh\left(\alpha(t)\right) - \frac{\sqrt{2}}{8}\cosh\left(\alpha(t)\right), \end{aligned} \tag{36}$$

  where $\alpha(t), \beta_1(t)$ and $\beta_2(t)$ are given in (35).

In consequence, the functions ϕ_1 and ϕ_2 given in (34) and (36) provide, through (28), two three-parameter family of solutions to the beam equation (3). As far as we know, these families of solutions are new in the literature and, remarkably, are expressed in terms of elementary functions.

Concluding remarks

A complete set of four first integrals and an exact four-parameter family of solutions to the static Euler-Bernoulli beam equation (3) have been obtained. As far as we know, these results have not been reported before in the literature. The expression of the first integrals are explicitly given in terms of two solutions of a linear second-order ODE. As a consequence, the general solution of the equation is parametrized in terms of the solutions to a one-parameter family of Schrödinger-type equations. The connection of the static Euler-Bernoulli beam equation (3) with Schrödinger-type equations seems also to be new.

Scalar higher-order equations that admit $\mathfrak{sl}(2,\mathbb{R})$ are of great importance because they arise in diverse physical problems. In this paper we have considered the simplest case of fourth-order ODE admitting $\mathfrak{sl}(2,\mathbb{R})$, but a similar approach could be also applied to more complicated equations or even to higher-order $\mathrm{SL}(2,\mathbb{R})$-invariant equations. A work in this line is currently in progress.

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