On the integrability of Liénard I-type equations via λ-symmetries and solvable structures

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Abstract

For Liénard I-type equations it is proved the existence of a family of $\lambda-$symmetries such that any of them lets the computation by quadratures of a time-dependent first integral of the equation. This is achieved by using a solvable structure constructed out of the $\lambda-$symmetry and one Lie point symmetry. The first integral obtained by quadratures and the first integral associated to the Lie symmetry generator are always functionally independent and they can be therefore used to integrate completely the Liénard I-type equation.

The method is illustrated by examples of wide classes of Liénard I-type equations. These classes contain but not limited to generalized force-free Duffing-Van der Pol oscillator equations. Analytical solutions are explicitly provided in closed form for both oscillatory and nonoscillatory types of equations.

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1. Introduction

An important family of ordinary differential equations (ODEs) from both mathematical and physical points of view is the Liénard I-type equation [1, 2]

$$x_2 + a_1(x)x_1 + a_0(x) = 0, \tag{1}$$

where a_1 and a_0 are arbitrary smooth functions of the dependent variable x and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, 2$. Equation (1) models a wide class of nonlinear oscillators such

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as the Emden-type equation, the Van der Pol equation, the Duffing oscillator, etc [3, 4]; different classes of these oscillator equations have been intensively studied in the recent literature [5, 6, 7]. Besides, equations of the form (1) can be also found as reductions of nonlinear partial differential equations (PDEs), such as the Fisher equation [8], the Burgers-Korteweg-de Vries equation [9], and the Burgers-Huxley equation [10] (see also [11]). The general solution of equation (1) is not known and usually numerical methods have to be used to obtain some information on the solutions of these problems (see, for instance, [12, 13] and references therein). Therefore, the development of new methods to solve analytically subclasses of Liénard I-type equations is of great importance.

The Lie symmetry approach [14, 15, 16] is one of the most powerful methods to reduce and integrate differential equations. A detailed symmetry analysis of equation (1) was performed in [17, 18] and families of equations which can be either integrated or linearized by the Lie method were characterized. However there are many equations in (1) admitting only the symmetry generator ∂_t, which is insufficient to complete the integration. It is well known that this symmetry generator can be used to reduce equation (1) to an Abel equation of the first kind (see equation (3) below). Equation (1) cannot be solved if either the general solution of this Abel equation is unknown or the quadrature necessary to reconstruct the solution of (1) cannot be evaluated.

Some integrable subclasses of equations (1) were found in [19, 20] by using the Chiellini integrability condition (see [21]) for the reduced Abel equation. Other recently introduced methods have been employed to find new integrable equations of the form (1), such as the Prelle-Singer method [22] and the use of nonlocal transformations that generalize the Sundman transformations [23, 24, 25] to linearlize the given equations [26]. The two last mentioned methods have been recently connected [27, 28, 29, 30, 31] with the existence of $\lambda-$symmetries [32]. In [33] the $\lambda-$symmetry method was used to identify an integrable family of Liénard I-type equations including many well-known oscillator equations and to explicitly construct nonlocal transformations to linearize them. The use of generalized Sundman transformations to connect the mixed Liénard type equations with Painlevé-Gambier equations has been investigated in [11, 34, 35, 36] and new integrability conditions were also obtained.

In this paper a novel $\lambda-$symmetry method to identify integrable classes of Liénard I-type equations and to solve them analytically is presented. Although $\lambda-$symmetries of (1) do always exist, they can be difficult to calculate. Even when a $\lambda-$symmetry has been found, the calculation of its associated first integral [37] can be a difficult task. One of the objectives of this paper is to determine classes of $\lambda-$symmetries of equation (1) whose associated first integral can be calculated by quadratures.

This problem is addressed in Section 2. We firstly characterize (Proposition 1) the class of the $\lambda-$symmetries of equation (1) for which a solvable structure [38, 39, 40] can be constructed by using the vector field associated to equation (1) and the symmetry generator ∂_t. Later, in Theorem 1 it is proved that any of these $\lambda-$symmetries provides a first integral of (1) of the form

$$I(t, x, x_1) = t + F(x, x_1), \tag{2}$$

that can be calculated by quadratures. Furthermore, this first integral and the first integral associated to the symmetry generator ∂_t are always functionally independent. Consequently, both first integrals provide directly the implicit solution for the given Liénard I-type equation.

The characterized $\lambda-$symmetries can be found by using particular solutions of a determining equation (see equation (14)), which is a first order quasilinear PDE. Such particular solutions may be difficult to find for arbitrary Liénard I-type equations (1). However, in Section 3 the class of equations (1) for which the determining equation admits solutions only depending on x is characterized in terms of their coefficients a_0, a_1. It turns out that they define a subclass of a family of second order ODEs whose first integrals and properties of linearization (under both local and generalized Sundman transformations) have been extensively investigated in the contemporary literature [27, 28, 41].

Several additional examples have been included in order to illustrate how the method works in practice. One of the equations presented in Subsection 4.1 does not satisfy the integrability conditions established in [11, 34, 35] therefore, to the best of our knowledge, it is a new integrable case. In consequence, this enlarges the subclass of integrable Liénard I-type equations characterized by other authors [11, 19, 20, 34, 35]. For the rest of equations, we compare our method with the results obtained in works [11, 20, 34, 35, 41]. The examples include nonlinear oscillators equations such as generalized force-free Duffing-Van der Pol equations. General solutions are obtained in closed form for all the presented examples.

2. On the integrability of the Liénard I-type equation

Throughout this paper the variables (t, x) of a Liénard I-type equation (1) belong to some open and simply connected set $M \subset \mathbb{R}^2$ and $\mathbf{A} = \partial_t + x_1\partial_x - (a_1(x)x_1 + a_0(x))\partial_{x_1}$ stands for the vector field associated to (1).

We refer the reader to [14, 15, 16] for the basics concerning Lie symmetries and to [32, 37] for the foundations of λ-symmetries.

2.1. Classical reduction of order and λ-symmetries of Liénard I-type equations

It is well known that the vector field $\mathbf{X}_1 = \partial_t$ is a Lie point symmetry of equation (1) and the associated reduced equation by means of the transformation $w = 1/x_1$ becomes

$$w'(x) = a_1(x)\,w(x)^2 + a_0(x)\,w(x)^3. \tag{3}$$

Equation (3) is an Abel equation of the first kind and its integrability by quadratures cannot be guaranteed in general. If $J_1 = J_1(t,w)$ denotes a first integral of (3) then the function J_1 written in terms of the original variables

$$I_1(x,x_1) = J_1\left(x, \frac{1}{x_1}\right) \tag{4}$$

is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_1\}$. In this case we say that I_1 is a first integral of equation (1) associated to the Lie point symmetry $\mathbf{X}_1 = \partial_t$.

In order to recover the general solution of equation (1) w has to be isolated from the implicit solution to equation (3), $J_1(t,w) = K_1$, where $K_1 \in \mathbb{R}$, to obtain an explicit solution $w = H(t,K_1)$ of equation (3). The general solution of equation (1) can be obtained after evaluating the quadrature

$$\int \frac{dx}{H(x,K_1)} = t + K_2, \quad K_2 \in \mathbb{R}. \tag{5}$$

In many cases this procedures fails to solve equation (1) because it is impossible either to obtain an explicit solution of the Abel equation (3) or to calculate a primitive of the function $1/H(x,K_1)$, which is required in (5).

With the aim of overcoming these difficulties we focus on obtaining a first integral I_2 of $\mathbf{A}$ functionally independent with I_1, the first integral defined in (4). According to the results derived in [37], a first integral $I_2 = I_2(t,x,x_1)$ of equation (1) is always associated to a λ-symmetry of the equation of the form (∂_x, λ), where $\lambda = \lambda(t,x,x_1) \in \mathcal{C}^\infty(M^{(1)})$ is a particular solution of the following determining equation (see equation (7) in [37]):

$$\lambda_t + \lambda_x x_1 - \lambda_{x_1}\left(a_1 x_1 + a_0\right) + \lambda^2 = -a_1' x_1 - a_0' - a_1 \lambda. \tag{6}$$

Moreover, the first integral I_1, given in (4), and the first integral I_2, associated to (∂_x, λ), are functionally independent if and only if $\lambda \neq \mathbf{A}(Q_1)/Q_1$, where $Q_1 = -x_1$

is the characteristic of the vector field $\mathbf{X}_1$ [37]. In other words, the required function λ must be a particular solution of (6) such that

$$\lambda \neq -\frac{a_1 x_1 + a_0}{x_1}. \tag{7}$$

By using the two functionally independent first integrals, I_1 and I_2, the general solution of equation (1) is implicitly defined by

$$I_1(x, x_1) = K_1, \quad I_2(t, x, x_1) = K_2, \tag{8}$$

where $K_1, K_2 \in \mathbb{R}$. By eliminating x_1 in (8) the solution of the Liénard I-type equation (1) is obtained in closed form.

A general procedure to compute a first integral I_2 associated to a λ-symmetry of the form $\left(\partial_x, \lambda\right)$ is described in [37]. In the next section we prove that any Liénard I-type equation admits a specific λ-symmetry with the property that an associated first integral I_2 can be computed by quadrature.

2.2. First integrals of Liénard I-type equations obtainable by quadratures

Let $\lambda \in \mathscr{C}^\infty(M^{(1)})$ be any particular solution of the determining equation (6) satisfying (7). Then the pair $\left(\partial_x, \lambda\right)$ is a λ-symmetry of equation (1) and the corresponding first order λ-prolongation of ∂_x [32], i.e. the vector field

$$\mathbf{X}_2 = (\partial_x)^{[\lambda,(1)]} = \partial_x + \lambda \partial_{x_1} \tag{9}$$

verifies the following commutation relation

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2. \tag{10}$$

Besides, it is easy to check that $\mathbf{X}_1 = \partial_t$ satisfies

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A}\right). \tag{11}$$

We recall [38, 39, 40] that the ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:

(a) $[\mathbf{X}_2, \mathbf{A}] \in \operatorname{span}\{\mathbf{X}_2\}$,
(b) $[\mathbf{A}, \mathbf{X}_1] \in \operatorname{span}\{\mathbf{X}_2, \mathbf{A}\}$,
(c) $[\mathbf{X}_1, \mathbf{X}_2] \in \operatorname{span}\{\mathbf{X}_2, \mathbf{A}\}$.

Relations (a) and (b) follow immediately from (10) and (11). By (11), relation (c) holds if and only if $\lambda_t = 0$. Therefore the next proposition has been proved:

Proposition 1. *Consider the vector field* $\mathbf{A}$ *associated to equation (1), the symmetry generator* $\mathbf{X}_1 = \partial_t$, *and the vector field* $\mathbf{X}_2 = \partial_x + \lambda\partial_{x_1}$, *where* λ *is any particular solution of the determining equation (6) satisfying (7).*

Then the ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ *is a solvable structure with respect to* $\mathbf{X}_2$ *if and only if* $\lambda_t = 0$.

Let $\lrcorner$ denote the interior product and let $\mathbf{\Omega} = dt \wedge dx \wedge dx_1$ be the volume form on $M^{(1)}$. According to [39, Theorem 3.15], if $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ then the differential 1-form

$$\boldsymbol{\omega}_2 = \frac{\mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}}{\mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}} \tag{12}$$

is locally exact, and a function I_2 such that $dI_2 = \boldsymbol{\omega}_2$ is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_2\}$. In this case we say that I_2 is a first integral of equation (1) associated to the λ−symmetry $\left(\partial_x, \lambda\right)$.

It can be checked that the expression of $\boldsymbol{\omega}_2$ in local coordinates (t, x, x_1) is

$$\boldsymbol{\omega}_2 = dt - \frac{\lambda}{(\lambda + a_1)x_1 + a_0} dx + \frac{1}{(\lambda + a_1)x_1 + a_0} dx_1.$$

A primitive $I_2 = I_2(t, x, x_1)$ of $\boldsymbol{\omega}_2$ verifies

$$\frac{\partial I_2}{\partial t} = 1, \quad \frac{\partial I_2}{\partial x} = -\frac{\lambda}{(\lambda + a_1)x_1 + a_0}, \quad \frac{\partial I_2}{\partial x_1} = \frac{1}{(\lambda + a_1)x_1 + a_0}. \tag{13}$$

Relations (13) clearly imply that:

- The function $\nu(x, x_1) = \left((\lambda + a_1)x_1 + a_0\right)^{-1}$ is an integrating factor of equation (1).

- An associated first integral is of the form $I_2(t, x, x_1) = t + F(x, x_1)$, where $F = F(x, x_1)$ satisfies $F_x = -\lambda\nu$ and $F_{x_1} = \nu$.

Previous discussion proves that any Liénard I-type equation admits a λ-symmetry whose associated first integral can be calculated by quadratures. It follows that the function λ does not depend on t, i.e., $\lambda = \lambda(x, x_1)$. Therefore, this function λ is a particular solution of the determining equation

$$\lambda_x x_1 - \lambda_{x_1}\left(a_1 x_1 + a_0\right) + \lambda^2 + a_1\lambda + a_1' x_1 + a_0' = 0 \tag{14}$$

verifying condition (7). These results are collected in the following theorem.

Theorem 1. *Any Liénard I-type equation (1) admits a specific λ-symmetry of the form $\left(\partial_x, \lambda\right)$ with the property that an associated first integral can be computed by quadratures. The corresponding function λ does not depend on t, i.e., $\lambda = \lambda(x, x_1)$ and it is any particular solution of the determining equation (14) verifying condition (7).*

Any of these λ-symmetries lets the computation by quadratures of a first integral of the equation as follows:

1. *The function*
$$v(x, x_1) = \frac{1}{(\lambda(x, x_1) + a_1(x))x_1 + a_0(x)} \tag{15}$$
is an integrating factor of equation (1).
2. *The associated first integral is of the form*
$$I_2(t, x, x_1) = t + F(x, x_1), \tag{16}$$
where $F = F(x, x_1)$ can be calculated by quadratures by solving the system
$$F_x = -\lambda v, \quad F_{x_1} = v. \tag{17}$$

In the next section we characterize a subclass of Liénard I-type equations which have special properties in the context of linearization procedures by both point and nonlocal transformations [27, 28, 30, 33, 42].

3. Liénard I-type equations admitting $\boldsymbol{\lambda}$-symmmetries of the form $(\boldsymbol{\partial_x}, \boldsymbol{\beta(x)})$

By Theorem 1 a first integral for any Liénard I-type equation can be calculated by quadratures, provided that a particular solution of the determining equation (14) satisfying (7) is known. For an arbitrary equation (1), such solution may be difficult to find, although it locally exits. In this section we characterize the class of Liénard I-type equations for which (14) admits solutions of the simple form $\lambda = \beta(x)$. They are included in the wider class of ODEs admitting $\lambda-$symmetries defined by functions linear in x_1, which have been connected with the second order ODEs that can be linearized by point transformations [43] and generalized Sundman tranformations [23, 24, 25, 41].

If we search for particular solutions of (6) of the form $\lambda = \beta(x)$, then the general equation (14) splits into the following system of determining equations:

$$\begin{aligned} \beta_x + a_1' &= 0, && (18) \\ \beta^2 + a_1\beta + a_0' - a_0\alpha &= 0. && (19) \end{aligned}$$

Equation (18) yields $\beta(x) = -a_1(x) + c_1$, where c_1 is a constant. Equation (19) gives the following compatibility condition:

$$c_1^2 - c_1 a_1 + a_0' = 0. \tag{20}$$

We distinguish two possible cases:

CASE 1: If $c_1 = 0$ then (20) yields $a_0 = c_2$, $c_2 \in \mathbb{R}$. Therefore the family of equations

$$x_2 + a_1(x)x_1 + c_2 = 0 \tag{21}$$

admits the λ-symmetry $(\partial_x, -a_1(x))$.

CASE 2: If $c_1 \neq 0$ then (20) yields $a_1(x) = a_0'(x)/c_1 + c_1$. Therefore the following family of equations

$$x_2 + \left(\frac{a_0'(x)}{c_1} + c_1 \right) x_1 + a_0(x) = 0 \tag{22}$$

admits the λ-symmetry $(\partial_x, -a_0'(x)/c_1)$.

Next we apply Theorem 1 to compute by quadratures a first integral of the families of equations (21) and (22). We also present two particular examples of Liénard I-type equations in each one of these two families that can be completely solved. In both cases ∂_t is the unique Lie point symmetry admitted by the equations and the equations cannot be integrated by the classical Lie method.

3.1. Case 1

We consider the family of equations (21), for which the pair $(\partial_x, -a_1(x))$ defines a λ−symmetry. The function $\lambda = -a_1(x)$ satisfies condition (7) if and only if $c_2 \neq 0$. The case $c_2 = 0$ is trivial because the corresponding reduced equation (3) becomes $w'(x) = a_1(x)w(x)^2$ which is clearly integrable by quadrature.

If $c_2 \neq 0$, by Theorem 1, $v(x, x_1) = 1/c_2$ is an integrating factor of equation (21). It can be checked that the corresponding first integral (16) becomes

$$I_2(t, x, x_1) = t + \frac{x_1 + A_1(x)}{c_2}, \tag{23}$$

where $A_1'(x) = a_1(x)$.

A functionally independent first integral of equation (21) can be derived by considering the corresponding reduced equation (3) associated to the symmetry generator ∂_t. Next we consider the particular case

$$x_2 + \exp(x)x_1 + 1 = 0, \tag{24}$$

corresponding to $c_2 = 1$ and $a_1(x) = \exp(x)$, in order to illustrate how the procedure works in practice. It can be checked that ∂_t is the only Lie point symmetry generator admitted by the equation. A first integral of the corresponding reduced equation (3)

$$w'(x) = \exp(x)\,w(x)^2 + w(x)^3 \tag{25}$$

becomes

$$J_1(x,w) = \exp\left(-\frac{1}{2}\Big(\exp(x) + \frac{1}{w}\Big)^2 - x\right) + \frac{\pi}{\sqrt{2}}\Psi\left(\frac{1}{\sqrt{2}}\Big(\exp(x) + \frac{1}{w}\Big)\right),$$

where Ψ stands for the error function $\Psi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-s^2)\,ds$.

The function $I_1(x,x_1) = J_1(x, 1/x_1)$ and the corresponding function (23), which becomes $I_2(t,x,x_1) = t + x_1 + \exp(x)$, are two functionally independent first integrals of equation (24). From $I_2(t,x,x_1) = K_2$, for $K_2 \in \mathbb{R}$, we get

$$\frac{1}{x_1} = \frac{1}{K_2 - t - \exp(x)}$$

and by substituting this value into $J_1(x,1/x_1) = K_1, K_1 \in \mathbb{R}$, it is obtained the following implicit solution of equation (25):

$$\exp\left(-\frac{1}{2}\Big(K_2 - t\Big)^2 - x(t)\right) + \frac{\pi}{\sqrt{2}}\Psi\left(\frac{1}{\sqrt{2}}\Big(K_2 - t\Big)\right) = K_1.$$

3.2. Case 2

The family of Liénard I-type equations obtained in Case 2

$$x_2 + \left(\frac{a_0'(x)}{c_1} + c_1\right) x_1 + a_0(x) = 0, \tag{26}$$

where $c_1 \in \mathbb{R}$, $c_1 \neq 0$, admits the λ-symmetry $(\partial_x, -a_0'(x)/c_1)$. By Theorem 1 the function

$$\nu(x,x_1) = \frac{1}{c_1 x_1 + a_0(x)}$$

is an integrating factor of equation (26). An associated first integral (16) can be calculated by quadratures by using the corresponding system (17), which provides:

$$I_2(t,x,x_1) = t + \frac{1}{c_1}\ln|c_1 x_1 + a_0(x)|. \tag{27}$$

A functionally independent first integral can be obtained from the reduced Abel equation (3) associated to the Lie point symmetry $\mathbf{X}_1 = \partial_t$:

$$w_1 = \left(\frac{a_0'(x)}{c_1} + c_1 \right) w^2 + a_0(x) w^3, \tag{28}$$

Once a first integral $J_1 = J_1(x, w)$ of the Abel equation (28) has been obtained, the general solution of equation (26) is given by (8).

Next we study a nontrivial particular case for which such first integral can be explicitly provided. For $c_1 = 1$ and $a_0(x) = 1/x$ equation (26) becomes

$$x_2 + \left(1 - \frac{1}{x^2} \right) x_1 + \frac{1}{x} = 0. \tag{29}$$

It can be checked that the Lie symmetry algebra of equation (29) is one-dimensional and it is spanned by $\mathbf{X}_1 = \partial_t$. A first integral of the corresponding reduced Abel equation (28)

$$w_1 = \left(1 - \frac{1}{x^2} \right) w^2 + \frac{w^3}{x} \tag{30}$$

can be expressed in terms of the error function in the form

$$J_1(x, w) = \frac{\sqrt{\pi}}{2} \Psi \left(\frac{\sqrt{2}(x + w + x^2 w)}{2xw} \right) - \frac{\sqrt{2} x w}{2(x + w)} \exp \left(\frac{(x + w + y^2 w)^2}{2x^2 w^2} \right). \tag{31}$$

It should be noted that the general solution of (29) cannot be recovered from the expression $J_1(x, 1/x_1) = K_1, K_1 \in \mathbb{R}$. This difficulty can be overcome by using the corresponding first integral (27) associated to the $\lambda-$symmetry $(\partial_x, 1/x^2)$:

$$I_2(t, x, x_1) = t + \ln \left| x_1 + \frac{1}{x} \right|.$$

By eliminating x_1 from $J_1(x, 1/x_1) = K_1, \exp(I_2(t, x, x_1)) = K_2$, where $K_1, K_2 \in \mathbb{R}$, the following implicit solution of equation (29) is obtained:

$$\frac{\sqrt{\pi}}{2} \Psi \left(\frac{\sqrt{2}}{2} (K_2 \exp(-t) + x(t)) \right) - \frac{\sqrt{2}}{K_2} \exp \left(t + \frac{1}{2} K_2^2 \exp(-2t) + x(t) K_2 \exp(-t) + \frac{1}{2} x(t)^2 \right) = K_1.$$

3.3. Generalized Sundman transformations

According to [42], the families of equations (21)-(22) are the Liénard type-I equations which can be transformed into $X_{TT} = 0$ by means of a generalized Sundman transformation of the form

$$X = \varphi(t,x), \quad dT = \psi(t,x)dt. \tag{32}$$

Linearizability conditions by means of (32) and a procedure for determining the functions φ and ψ were presented for the first time in [41] (see also [37, 42]). For a first integral of the form $I(t,x,x_1) = A(t,x)x_1 + B(t,x)$, a generalized Sundman transformation can be obtained by using the following conditions [42]:

$$B\varphi_x - A\varphi_t = 0, \quad \psi = \frac{\varphi_x}{A}. \tag{33}$$

It can be checked that a transformation of the form (32) for the particular examples considered in this section are determined by:

- Equation (24): $\varphi(t,x) = \exp\left(\frac{t^2}{2} - x\right) + \frac{\pi}{\sqrt{2}}\Psi\left(\frac{-t}{\sqrt{2}}\right), \quad \psi(t,x) = -\exp\left(\frac{t^2}{2} - x\right).$
- Equation (29): $\varphi(t,x) = \frac{x^2}{2} + t, \quad \psi(t,x) = x\exp(-t).$

Although the presented nonlocal transformations permit to linearize the equations, the reconstruction of the general solutions in closed form seems to be impossible due to the nonlocal term appearing in (32). This problem can be avoided by means of our approach, which permits to obtain the general solutions of equations (24) and (29) in closed form.

4. Some examples

The procedure presented in this paper can be applied to any Liénard type I equation (1) admitting some λ−symmetry $\left(\partial_x, \lambda\right)$ such that λ verifies (7). This is the case of the family of equations

$$x_2 + (xf'(x) + 2f(x) + \gamma)x_1 + x(f^2(x) + \gamma f(x) + \mu) = 0, \tag{34}$$

where $\gamma, \mu \in \mathbb{R}$, and f is an arbitrary smooth function. It can be checked [33] that equation (34) admits the λ−symmetry $\left(\partial_x, \lambda\right)$ where

$$\lambda(x, x_1) = \frac{x_1}{x} - xf'(x). \tag{35}$$

Well-known equations representing physically important oscillator systems are particular cases of equation (34): for $f(x) = kx$ (34) is a generalized modified Emden type equation and for $f(x) = kx^2$ (34) is the generalized force-free Duffing-Van der Pol type equation [3, 4].

By Theorem 1 the function

$$v(x,x_1) = \frac{1}{\left(\frac{x_1}{x} + 2f(x) + \gamma\right)x_1 + x(f^2(x) + \gamma f(x) + \mu)}$$

is an integrating factor of (34). An associated first integral of equation (39) can be computed by quadratures by solving the corresponding system (17). We have obtained the following expressions for such first integral:

Table 1: First integral $I_2(t,x,x_1)$ of Eq. (39)

$\rho = \gamma^2 - 4\mu$	$I_2(t,x,x_1)$
$\rho > 0$	$t - \frac{2}{\sqrt{\rho}}\operatorname{arctanh}\left(\frac{2x_1 + 2xf(x) + \gamma x}{x\sqrt{\rho}}\right)$
$\rho < 0$	$t + \frac{2}{\sqrt{-\rho}}\arctan\left(\frac{2x_1 + 2xf(x) + \gamma x}{x\sqrt{-\rho}}\right)$
$\rho = 0$	$t - \frac{2x}{2x_1 + 2xf(x) + \gamma x}$

In the next subsection we study particular cases of equation (34) corresponding some of them to generalized force-free Duffing-Van der Pol equations. In all cases a first integral of the corresponding reduced Abel equation (3) can be explicitly obtained and it is used to integrate completely the corresponding equations.

4.1. An example with a root square nonlinearity

Let us consider $f(x) = \sqrt{x}$, $\gamma = 0$ and $\mu = 1$. Equation (34) becomes in this particular case

$$x_2 + \frac{5}{2}\sqrt{x}x_1 + (x+1)x = 0. \tag{36}$$

We have that that $\gamma^2 - 4\mu = -4 < 0$ therefore, according to Table 3, a first integral of equation (36) is given by

$$I_2(t,x,x_1) = t + \arctan\left(\frac{x_1 + x\sqrt{x}}{x}\right). \tag{37}$$

The Abel equation 3 obtained after reducing the order of (36) by means of the Lie point symmetry ∂_t is

$$w_1=\frac{5}{2}\sqrt{x}\,w^2+x(x+1)\,w^3. \tag{38}$$

It can be checked that a first integral of equation (38) can be expressed as follows:

$$J_1(x,w)=\frac{1}{\sqrt{x}\Big(\big(\sqrt{x}+\frac{1}{xw}\big)^2+1\Big)^{1/4}}+\frac{1}{2xw}\big(wx^{3/2}+1\big)\,{}_2\mathrm{F}_1\left(a_1,a_2;b_1;\frac{-(wx^{3/2}+1)^2}{xw^2}\right),$$

where ${}_2\mathrm{F}_1(a_1,a_2;b_1;z)$ denotes the generalized hypergeometric function of parameters $a_1=\frac{1}{2}$, $a_2=\frac{5}{2}$ and $b_1=\frac{3}{2}$. From $J_1(x,1/x_1)=K_1$, $I_2(t,x,x_1)=K_2$, with $K_1,K_2\in\mathbb{R}$, we have been able to obtain the following explicit solution to equation (36):

$$x(t)=\frac{4\cos^3(K_2-t)}{\big(2K_1\cos(K_2-t)+\sin(K_2-t)\,{}_2\mathrm{F}_1\big(a_1,a_2;b_1;-\tan^2(K_2-t)\big)\big)^2}.$$

4.2. Generalized force-free Duffing-Van der Pol equations

For $f(x)=kx^2$, where $k\in\mathbb{R}$, equation (34) becomes the following family of generalized force-free Duffing-Van der Pol equations

$$x_2+(4kx^2+\gamma)x_1+k^2x^5+k\gamma x^3+\mu x=0. \tag{39}$$

A first integral of equation (39) associated to the $\lambda-$symmetry (35), which for equation (39) becomes $\left(\partial_x,\dfrac{x_1}{x}-2kx^2\right)$, can be derived by setting $f(x)=kx^2$ in Table 1.

The Abel equation (3) derived from the symmetry generator ∂_t becomes

$$w'(x)=(4kx^2+\gamma)\,w(x)^2+(k^2x^5+k\gamma x^3+\mu x)\,w(x)^3. \tag{40}$$

Next we study some particular examples for which (39) can be completely integrated.

4.2.1. Generalized force-free Duffing-Van der Pol equation for $k=1/4$, $\gamma=-1$ and $\mu=3/16$

Equation (39) becomes

$$x_2+(x^2-1)x_1+\frac{x^5}{16}-\frac{x^3}{4}+\frac{3x}{16}=0. \tag{41}$$

In this particular case we have that $\gamma^2 - 4\mu = \frac{1}{4} > 0$. According to Table 3, a first integral of equation (41) is

$$I_2(t,x,x_1) = t - 4\operatorname{arctanh}\left(\frac{4x_1 + x^3 - 2x}{x}\right). \tag{42}$$

It can be checked that a first integral of the corresponding Abel equation (40) becomes

$$J_1(x,w) = \frac{\left(35wx(x^2-3)+420\right)^7 w^{14}}{(wx(x^2-3)+4)^{18}\left(7wx(x^2-3)+28\right)^3}. \tag{43}$$

From $J_1(x,1/x_1) = K_1, I_2(t,x,x_1) = K_2$ we have obtained the following solution of equation (41) in closed form:

$$\frac{2x(t)^2 + 3\tanh\left(\frac{1}{4}(K_2-t)\right) - 3}{x(t)^2\left(1+\tanh\left(\frac{1}{4}(K_2-t)\right)\right)^3} = \widetilde{K}_1, \tag{44}$$

where the new constant $\widetilde{K}_1$ has been adequately rescaled.

4.2.2. Genelarized force-free Duffing-Van der Pol equation for $k = 1, \gamma = 0$ and $\mu = 1$

Equation (39) becomes

$$x_2 + 4x^2x_1 + (x^4+1)x = 0. \tag{45}$$

In this case $\rho = \gamma^2 - 4\mu = -4 < 0$ therefore, according to Table 3, a first integral of equation (45) is

$$I_2(t,x,x_1) = t + \arctan\left(\frac{x_1 + x^3}{x}\right). \tag{46}$$

It can be checked that a first integral of the corresponding Abel equation (40) is given by

$$J_1(x,w) = \frac{w\left(x + w(1+x^4)\right)}{x^2w^2 + (1+x^3w)^2} + \arctan\left(\frac{1+x^3w}{xw}\right). \tag{47}$$

By eliminating x_1 from $J_1(x,1/x_1) = K_1, I_2(t,x,x_1) = K_2$, the following solution of equation (45) is obtained:

$$x(t)^2 = \frac{4\cos^2(K_2-t)}{\widetilde{K_1} - 4(K_2-t) - 2\sin(2(K_2-t))},$$

where $\widetilde{K}_1 = 4K_1 + 2\pi$, and $K_1, K_2 \in \mathbb{R}$.

4.2.3. Genelarized force-free Duffing-Van der Pol equation for $k = 1, \gamma = 2$ and $\mu = 2$

Equation (39) becomes

$$x_2 + (4x^2 + 2)x_1 + (2 + 2x^2 + x^4)x = 0. \tag{48}$$

Since $\gamma^2 - 4\mu = -4 < 0$, by Table 3, a first integral of equation (48) is

$$I_2(t, x, x_1) = t + \arctan\left(\frac{x_1 + x^3 + x}{x} \right). \tag{49}$$

A first integral $J_1 = J_1(x, w))$ of the Abel equation (40) is given by

$$J_1 = \frac{\left(18\,x^2 + 12 + 3\,x^6 + 12\,x^4\right) w^2 + \left(12\,x + 6\,x^3\right) w + 3}{\left(1 + \left(x^6 + 2\,x^4 + 2\,x^2\right) w^2 + \left(2\,x + 2\,x^3\right) w\right) \exp\left(2 \arctan\left(\frac{1 + \left(x^3 + 2\,x\right) w}{w x^3 + 1}\right)\right)}.$$

From $J_1(x, 1/x_1) = K_1, I_2(t, x, x_1) = K_2$ the following general solution can be obtained:

$$x(t)^2 = \frac{4e^{2\beta(t)}\Big(3K_1\left(1 + \tan^2(K_2 - t)\right) - 9e^{2\beta(t)}\left(\tan^2(K_2 - t) + 2\tan(K_2 - t) + 3\right)\Big)}{\Big(3e^{2\beta(t)}\left(\tan^2(K_2 - t) + 2\tan(K_2 - t) + 3\right) - K_1\left(1 + \tan^2(K_2 - t)\right)\Big)^2},$$

where

$$\beta(t) = -\arctan\left(\frac{1 + \tan(K_2 - t)}{\tan(K_2 - t) - 1} \right), \quad K_1, K_2 \in \mathbb{R}.$$

4.2.4. Genelarized force-free Duffing-Van der Pol equation for $k = 1$, $\gamma = 1$ and $\mu = 0$

Equation (39) becomes

$$x_2 + (4x^2 + 1)x_1 + x(x^4 + x^2) = 0. \tag{50}$$

In this case $\rho = \gamma^2 - 4\mu = 1 > 0$, therefore a first integral of equation (50) is, by Table 3:

$$I_2(t, x, x_1) = t - 2\text{arctanh}\left(\frac{2x_1 + 2x^3 + x}{x} \right) \tag{51}$$

A first integral of the reduced Abel equation (40) becomes

$$J_1(x,w)=\frac{(x^3w+1)\exp\left(\dfrac{w(2x^4w+3x^2w+2x+w)}{2(x^3w+xw+1)^2}\right)}{x^3w+xw+1}. \tag{52}$$

From $J_1(x,1/x_1)=K_1, I_2(t,x,x_1)=K_2$, we have obtained the following general expression of the solution of equation (50):

$$x(t)^2=\frac{(e^{K_2-t}+1)^2}{\widetilde{K}_1-e^{2(K_2-t)}-4e^{K_2-t}+2\ln\left|\dfrac{\tanh\left(\frac{1}{2}(K_2-t)\right)-1}{\tanh\left(\frac{1}{2}(K_2-t)\right)+1}\right|},$$

where $\widetilde{K}_1=K_1-3$, and $K_1,K_2\in\mathbb{R}$.

4.3. Comparison with some recent methods

If the parameters k,γ and μ of equation (40) satisfy

$$\gamma=-4k \quad \text{and} \quad \mu=3k^2, \tag{53}$$

then the reduced Abel equation (40) verifies the Chiellini's Lemma [21] and equation (39) satisfies the integrability conditions established in [20].

The unique example that satisfies (53) is equation (41). The integration method based on the Chiellini's integrability condition presented in [20] provides a parametric solution of equation (41) given by

$$t-t_0=\int\frac{ds}{(x(s)^2-1)\left(s^2+s+\frac{3}{16}\right)}, \tag{54}$$

where $x(s)$ is a solution of the algebraic equation

$$(4s+1)^{3/2}\left(\frac{x(s)^5}{16}-\frac{x(s)^3}{4}+\frac{3x(s)}{16}\right)=C^{-1}(x(s)^2-1)(4s+3)^{1/2}s, \tag{55}$$

for $C\in\mathbb{R}$, $C\neq 0$. The application of our method leads to the exact solution (44) in closed form for equation (41), which improves the expression given in (54) because it is neither necessary to solve the algebraic equation (55) nor the evaluation of the primitive required in (54).

On the other hand, in [11], it was proved that the Chiellini's integrability condition is equivalent to certain linearizability conditions via nonlocal transformations. In consequence, equation (41) also satisfies the condition presented in [11, Theorem 2], and therefore it can be linearized to the equation

$$w_{\zeta\zeta} + \frac{4}{\sqrt{3}} w_{\zeta} + w = 0$$

by means of the nonlocal transformation

$$w(x) = \frac{x^3}{3} - x, \quad d\zeta = \frac{\sqrt{3}}{4}(x^2 - 1)\,dt. \tag{56}$$

By using the strategy provided is such paper, the solution can be expressed as follows:

$$\frac{x^3}{3} - x = C_1 e^{-\sqrt{3}\zeta} + C_2 e^{-\sqrt{3}/3\zeta}, \quad \zeta = \int \frac{\sqrt{3}}{4}(x^2 - 1)\,dt \quad (C_1, C_2 \in \mathbb{R}). \tag{57}$$

The general solution (44) of equation (41) obtained by means of our procedure does not involve nonlocal terms, which is an advantage over (57).

Equations (45), (48) and (50) satisfy the integrability condition established in [35, Theorem 2], therefore they can be transformed into the Painlevé–Gambier equation XXVII by means of the transformation

$$w = \frac{1}{4x^2}, \quad \zeta = \frac{1}{2} t.$$

For equation (45) we obtain equation XXVII (see [45, pag. 338]) at $m = -2$, $\phi = f = 0$ and $\psi = 2$. For equations (48) and (50) we also get equation XXVII in [45, pag. 338] at $m = -2$, $\phi = -2$, $f = 0$, $\psi = 4$ and $m = -2$, $\phi = -1$, $f = 0$, $\psi = 0$, respectively. In consequence, the general solutions in closed form of these equations could also be obtained by solving the associated Painlavé-Gambier equations.

Finally, it is important to remark that equation (36) does not satisfy the integrability conditions established in [11, 34, 35] therefore, to the best of our knowledge, it is a new integrable case. New integrable cases could be generated by considering different choices of the arbitrary function f appearing in (34).

5. Concluding remarks

It has been demonstrated the existence of a family of $\lambda-$symmetries that allows the computation by quadratures of a time-dependent first integral of any Liénard I-type equation. A time-independent first integral can be found by means of a first integral of an associated Abel equation. Both first integrals are functionally independent and they can be used to provide exact solutions of the equations under study. The method successes for Liénard I-type equations for which the associated Abel equations do not satisfy the Chiellini's integrability condition. This enlarges the class of the integrable Liénard I-type equations studied in the literature [20].

Interesting well-known equations have been completely integrated by using simple solutions of the determining equation (14). It is expected that the investigation of different $\lambda-$symmetries could provide in future new classes of integrable Liénard type-I equations.

Finally, we would like to mention that the theoretical background is not limited to Liénard I-type equations: the combine use of Lie point symmetries and $\lambda-$symmetries to construct solvable structures can be very useful to compute by quadratures first integrals of other nonlinear equations.

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