# First integrals and parametric solutions of third-order ODEs with Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$

**A. Ruiz**[1] **& C. Muriel**[1]

[1]Department of Mathematics, University of Cádiz, 11510 Puerto Real, Spain

E-mail: `adrian.ruiz@uca.es; concepcion.muriel@uca.es`

October 2016

**Abstract.** A complete set of first integrals for any third-order ordinary differential equation admitting a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ is explicitly computed. These first integrals are derived from two linearly independent solutions of a linear second-order ODE without additional integration. The general solution in parametric form can be obtained by using the computed first integrals. The study includes a parallel analysis of the four inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ and it is applied to several particular examples. They include the generalized Chazy equation as well as an example of an equation which admits the most complicated of the four inequivalent realizations.

PACS numbers: 02.30.Hq, 02.20.Sv

*Keywords*: First integral, $\mathscr{C}^\infty$-symmetry, nonsolvable algebra, Chazy equation

Submitted to: *J. Phys. A: Math. Gen.*

## 1. Introduction

The order of an $n$th-order ordinary differential equation (ODE) admitting an $n$-dimensional solvable Lie symmetry algebra can be stepwise reduced and, consequently, the equation can be integrated by quadratures [1, 2, 3, 4]. If the symmetry algebra is nonsolvable, this step-by-step method of reduction is no longer applicable and the integrability by quadratures cannot be warranted [5]. The study of different strategies to integrate completely this type of equations has been of great interest in the literature [1, 6, 7, 8, 9, 10].

It is well known [11] that there are three inequivalent nonsingular local actions of the three-dimensional special linear or unimodular group SL$(2,\mathbb{C})$, with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, on any two-dimensional complex manifold and that an additional fourth realization appears for $\mathfrak{sl}(2,\mathbb{R})$ (Cases 1, 2, 3 and 4 in Table 1). Most of the studies [1, 6, 7] on ordinary differential equations invariant under the unimodular group correspond to the first three cases. The fourth case has not been so deeply studied. One of the main objectives of this paper is to provide a unified treatment for the four cases, in order to obtain new integration strategies for the corresponding invariant equations.

Some of the milestones on the study of ODEs invariant under the unimodular group are the following:

- In [6], Clarkson and Olver studied the cases 1, 2 and 3. In this seminal paper they proved that the three inequivalent actions of $\mathrm{SL}(2,\mathbb{C})$ can be connected via the standard prolongation process. As a consequence, the integration methods for Cases 2 and 3 are closely related to the integration method for the basic unimodular action (Case 1).
- Ibragimov and Nucci [7] focused on $\mathrm{SL}(2,\mathbb{C})$-invariant third-order ODEs (Cases 1, 2 and 3). A two-dimensional subalgebra $\mathscr{L}_2$ of $\mathfrak{sl}(2,\mathbb{C})$ was used to reduce the given ODE to a first-order equation, which cannot be integrated by quadrature, but can be transformed into a Riccati equation by using a nonlocal symmetry [12, 13, 14]. Provided that the general solution of this Riccati equation is known, such solution, once written in terms of the original variables, becomes a second-order ODE which can be integrated by quadratures by using $\mathscr{L}_2$.
- In [8] it is shown that the order of any given ODE invariant under the unimodular group (Cases 1, 2, 3 and 4) can be reduced step-by-step but, since the symmetry algebra is not solvable, some of generators have to be recovered as $\mathscr{C}^\infty$-symmetries (or $\lambda$-symmetries [15]) for the reduced equations. The general solution of the original equation can be obtained through two successive quadratures from the solution of the reduced equation. Starting with a third-order equation, the reduction process led to a first-order ODE, which was a Riccati equation for Cases 1, 2 and 3 but for Case 4 it was not apparently a Riccati equation.

In our unified approach to integrate a $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE corresponding to any of the four cases, the theoretical background is the existence of a solvable structure [16, 17, 18, 19, 20, 21] that can be computed from the generators of the symmetry algebra, as it has been recently proved in [22]. A brief summary of some theoretical results in [22] that are used in this paper is included in Section 2.

In Section 3 we reduce as far as possible the order of a $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE corresponding to any of the four cases. If an appropriate basis element of the symmetry algebra is firstly used then two remaining symmetry generators can be inherited as $\lambda$-symmetries for the reduced second-order equation [8]. In order to calculate two functionally independent first integrals for this reduced equation, both $\lambda$-symmetries are used separately to reduce the order again. In Section 3, it is proved that, written by means of appropriate systems of invariants, both reduced equations are exactly the same Riccati equation (Theorem 3.2). This result unifies the study performed in [8], in which the reduced equation for Case 4 was not of the Riccati type. Remarkably these Riccati equations or the associated linear second-order ODEs can be constructed directly from the canonical forms of the $\mathrm{SL}(2,\mathbb{R})$-invariant ODEs (see Tables 2, 5 and 6). The used approach is the same for the four cases, which are studied in parallel. A fundamental consequence of these results is that two first integrals of the reduced second-order (nonlinear) ODE can be explicitly expressed in terms of the solutions of a second-order ODE that is linear.

In Section 4 we prove, by using the results derived in [22], that these two first integrals and the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ permit the construction of a complete set of first integrals of the original third-order equation, without any kind of quadrature. Consequently, explicit expressions of such first integrals for each canonical form of the $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODEs can be directly expressed in terms of the solutions of an associated linear second-order ODE (see Table 6). As far as we know, such explicit expressions for the first integrals of $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODEs had not been provided so far in the literature; the connection of the equations corresponding to Case 4 with linear second-order ODEs seems also to be new.

In Section 5 we analyze the solutions of $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODEs. The complete sets of first integrals given in Tables 7-8 lead directly (without quadratures) to the implicit solutions of the canonical forms of the $\mathrm{SL}(2,\mathbb{R})$-invariant ODEs. For Case 1 an explicit expression of the general solution can be easily found. For each one of the three remaining cases we prove that a parametric solution can be directly expressed in terms of two linearly independent solutions of a linear second-order ODE (related via a change of independent variable to the linear second-order ODE mentioned above). This completes and improves the results derived in [6], where the most complicated realization of $\mathfrak{sl}(2,\mathbb{R})$ was not included. We want also to emphasize that our procedure provides directly the first integrals and the parametric solutions from the solutions to linear second-order ODEs without any additional quadrature.

In Section 6 the results are applied to different examples, corresponding to each one of the four cases. The well-known generalized Chazy equation [6, 23], which corresponds to Case 3, has been included for illustration purposes of the unified procedure and for the seek of comparison with other methods in the literature. The method in this paper provides directly three functionally independent first integrals, which can be expressed in terms of either Legendre or Bessel functions, depending on the values of the parameters of the equation. The second example corresponds to the most complicated realization of $\mathfrak{sl}(2,\mathbb{R})$ (Case 4). The first integrals and the parametric solution can be expressed in terms of two linearly independent solutions to a Schrödinger-type equation. The equations corresponding to the example of Case 1 (resp. Case 2) can be completely solved by giving the explicit (resp. parametric) expressions of the solutions in terms of elementary functions.

## 2. First integrals of third-order ordinary differential equations with Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$

In this section we recall some theoretical results obtained by the authors in [22] on the study of a third-order ODE

$$u_3 = \phi(x, u, u_1, u_2) \tag{1}$$

that admits a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

We assume that equation (1) is defined for points of the corresponding third-order jet space whose projections to the second-order jet belong to the domain of $\phi$. Let $M \subset \mathbb{R}^2$ be an open set of the projection of this domain to the zero-order jet space. In what follows $M^{(k)}$ will denote the corresponding jet space of order $k \geq 1$. A basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of the Lie symmetry algebra satisfying the commutation relations

$$[\mathbf{v}_1, \mathbf{v}_3] = \mathbf{v}_1, \quad [\mathbf{v}_1, \mathbf{v}_2] = 2\mathbf{v}_3, \quad [\mathbf{v}_3, \mathbf{v}_2] = \mathbf{v}_2, \tag{2}$$

can always be chosen. The procedure followed in [22] uses the Lie point symmetry $\mathbf{v}_3$ to reduce the order of equation (1). For that, canonical coordinates $\{y, \alpha\}$ are introduced for $\mathbf{v}_3$, i.e., a local change of variables

$$\varphi(x,u) = (y(x,u), \alpha(x,u)) \tag{3}$$

such that $\mathbf{v}_3 = \partial_\alpha$ (we have used the same letter $\mathbf{v}_3$ for the respective vector field written in variables $\{y, \alpha\}$, which should not cause misunderstanding).

Through the local change of variables (3), equation (1) can be written as

$$\alpha_3 = \widetilde{\phi}(y, \alpha_1, \alpha_2). \tag{4}$$

We denote by $\mathbf{A}_{(x,u)}$ and $\mathbf{A}_{(y,\alpha)}$ the vector fields associated to equations (1) and (4), respectively. It can be easily checked that $\mathbf{A}_{(y,\alpha)} = \frac{1}{\mathbf{D}_x(y)}\mathbf{A}_{(x,u)}$.

By setting $w = \alpha_1 = \frac{d\alpha}{dy}$, $w_i = \alpha_{i+1} = \frac{d^i w}{dy^i}$, for $i \geq 1$, equation (1) can be written in terms of the invariants $\{y, w, w_1, w_2\}$ of $\mathbf{v}_3^{(3)}$ as a reduced equation:

$$w_2 = \widetilde{\phi}(y, w, w_1). \tag{5}$$

Let $M_1 \subset \mathbb{R}^2$ be an open set of the projection of the domain of $\widetilde{\phi}$ to the corresponding zero-order jet space. For $k \in \mathbb{N}$, $k \geq 1$, we consider the projection

$$\begin{array}{rccc} \pi^{(k)}_{\mathbf{v}_3}: & \varphi^{(k)}(M^{(k)}) & \to & M_1^{(k-1)} \\ & (y, \alpha, w, \ldots, w_{k-1}) & \mapsto & (y, w, w_1, \ldots, w_{k-1}). \end{array}$$

A vector field $\mathbf{V}$ on $M^{(k)}$ is called $\pi^{(k)}_{\mathbf{v}_3}$-projectable [24] if $[\mathbf{v}_3^{(k)}, \mathbf{V}] = f\mathbf{v}_3^{(k)}$, for some function $f \in \mathscr{C}^\infty(M^{(k)})$. If $\mathbf{V}$ is $\pi^{(k)}_{\mathbf{v}_3}$-projectable then $\mathbf{V}$ must be of the form

$$b_0(y, w, \ldots, w_{k-1})\partial_y + a_0(y, \alpha, w, \ldots, w_{k-1})\partial_\alpha + \cdots + a_k(y, w, \ldots, w_{k-1})\partial_{w_{k-1}}$$

and the projection of $\mathbf{V}$ on $M_1^{(k-1)}$ is the vector field

$$(\pi^{(k)}_{\mathbf{v}_3})_*(\mathbf{V}) = b_0(y, w, \ldots, w_{k-1})\partial_y + a_1(y, w, \ldots, w_{k-1})\partial_w + \cdots + a_k(y, w, \ldots, w_{k-1})\partial_{w_{k-1}}.$$

It is clear that $\mathbf{A}_{(y,\alpha)}$ is $\pi^{(2)}_{\mathbf{v}_3}$-projectable and its projection is the vector field $\mathbf{A}_{(y,w)}$ associated to equation (5).

By (2), $\mathbf{v}_1^{(1)}$ and $\mathbf{v}_2^{(2)}$ are not $\pi^{(1)}_{\mathbf{v}_3}$-projectable. Nevertheless, it can be checked [22] that, if a nonvanishing function $\varsigma_1 \in \mathscr{C}^\infty(M)$ satisfies $\mathbf{v}_3(\varsigma_1) = \varsigma_1$, then $[\mathbf{v}_3^{(k)}, \varsigma_1\mathbf{v}_1^{(k)}] = 0$ and $\varsigma_1\mathbf{v}_1^{(k)}$ is $\pi^{(k)}_{\mathbf{v}_3}$-projectable. Similarly, $\varsigma_2\mathbf{v}_2^{(k)}$ is $\pi^{(k)}_{\mathbf{v}_3}$-projectable for the function $\varsigma_2 = \frac{1}{\varsigma_1}$. In variables $\{y, \alpha\}$ such pair of functions can be easily found:

$$\varsigma_1 = e^\alpha, \varsigma_2 = e^{-\alpha}. \tag{6}$$

For $i=1,2$, let

$$\overline{\mathbf{v}}_i=(\pi^{(1)}_{\mathbf{v}_3})_*(\varsigma_i\mathbf{v}_i^{(1)})=\xi_i(y,w)\partial_y+\eta_i(y,w)\partial_w \tag{7}$$

denote the corresponding projections. The vector fields $\overline{\mathbf{v}}_1$ and $\overline{\mathbf{v}}_2$ are not Lie point symmetries of the reduced equation (5) but they can be recovered as $\lambda$-symmetries [8, Theorem 3]: the pairs $(\overline{\mathbf{v}}_1,\lambda_1)$ and $(\overline{\mathbf{v}}_2,\lambda_2)$ are $\lambda$-symmetries of equation (5), where

$$\lambda_1=-\frac{\mathbf{A}_{(y,\alpha)}(\varsigma_1)}{\varsigma_1}=-w \quad\text{and}\quad \lambda_2=-\frac{\mathbf{A}_{(y,\alpha)}(\varsigma_2)}{\varsigma_2}=w. \tag{8}$$

For $i=1,2$, the respective $\lambda$-prolongation [15] (see also [8])

$$\overline{\mathbf{v}}_i^{[\lambda_i,(1)]}=\xi_i\partial_y+\eta_i\partial_w+\big((\mathbf{A}_{(y,w)}+\lambda_i)(\eta_i)-(\mathbf{A}_{(y,w)}+\lambda_i)(\xi_i)\,w_1\big)\partial_{w_1} \tag{9}$$

verifies the following condition

$$\left[\,\overline{\mathbf{v}}_i^{[\lambda_i,(1)]},\mathbf{A}_{(y,w)}\right]=\lambda_i\overline{\mathbf{v}}_i^{[\lambda_i,(1)]}-(\mathbf{A}_{(y,w)}+\lambda_i)(\xi_i)\mathbf{A}_{(y,w)}. \tag{10}$$

Let us denote

$$\mathbf{Y}_i=\overline{\mathbf{v}}_i^{[\lambda_i,(1)]},\quad i=1,2. \tag{11}$$

By (11), expression (10) can be rewritten as follows:

$$[\mathbf{Y}_i,\mathbf{A}_{(y,w)}]=\lambda_i\mathbf{Y}_i+\rho_i\mathbf{A}_{(y,w)},\quad i=1,2, \tag{12}$$

where $\rho_i=-(\mathbf{A}_{(y,w)}+\lambda_i)(\mathbf{Y}(y))$. By Frobenius' Theorem, for $i=1,2$, there exists a first integral $I_i=I_i(y,w,w_1)$ common to the involutive system $\{\mathbf{A}_{(y,w)},\mathbf{Y}_i\}$. The functions $I_1,I_2$ written in terms of the original variables $\{x,u,u_1,u_2\}$ :

$$I_i=I_i(y(x,u),w(x,u,u_1),w_1(x,u,u_1,u_2)),\quad i=1,2, \tag{13}$$

provide two functionally independent first integrals of the third-order ODE (1) [25].

A fundamental result proved in [22] establishes that a functionally independent first integral respect to $I_1$ and $I_2$ can be computed without additional integration. In fact, any of the functions

$$F_1=\frac{1}{\mathbf{v}_1^{(2)}(I_2)},\quad F_2=\frac{1}{\mathbf{v}_2^{(2)}(I_1)} \tag{14}$$

is a first integral of the original equation and $\{I_1,I_2,F_1\}$ (resp. $\{I_1,I_2,F_2\}$) is a complete set of first integrals of $\mathbf{A}_{(x,u)}$ [22, Lemma 4.2].

In consequence, once explicit expressions for the first integrals (13) have been obtained, the complete solution of any $SL(2,\mathbb{R})$-invariant third-order ODE can be derived without any additional quadrature. With the aim of computing explicit expressions for the first integrals (13), we exploit the inherited $\lambda$-symmetries $(\overline{\mathbf{v}}_1,\lambda_1)$ and $(\overline{\mathbf{v}}_2,\lambda_2)$ to reduce again the order of equation (5).

## 3. Use of $\lambda$-symmetries to reduce equation (5)

### 3.1. First-order reduced equations

In this section we use the $\lambda$-symmetries $(\overline{\mathbf{v}}_1,\lambda_1)$ and $(\overline{\mathbf{v}}_2,\lambda_2)$ defined by (7)-(8) to reduce the order of equation (5). Before that, we need to prove that the vector fields $\mathbf{Y}_1$ and $\mathbf{Y}_2$ given in (11) are in involution.

**Proposition 3.1.** *Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ be the vector fields given in (11). Then there exist two smooth functions, $c_1=c_1(y)$ and $c_2=c_2(y)$, such that*

$$[\,\boldsymbol{Y}_1,\boldsymbol{Y}_2]=c_1(y)\boldsymbol{Y}_1+c_2(y)\boldsymbol{Y}_2. \tag{15}$$

*Proof.* Let $\varsigma_1,\varsigma_2$ be the functions defined in (6), which clearly satisfy $\varsigma_1\varsigma_2=1$. From (2) and the properties of the Lie bracket it follows that

$$\begin{aligned}[\varsigma_1\mathbf{v}_1^{(2)},\varsigma_2\mathbf{v}_2^{(2)}] &= 2\varsigma_1\varsigma_2\mathbf{v}_3^{(2)}+\varsigma_1\mathbf{v}_1(\varsigma_2)\mathbf{v}_2^{(2)}-\varsigma_2\mathbf{v}_2(\varsigma_1)\mathbf{v}_1^{(2)}\\ &= 2\mathbf{v}_3^{(2)}+\varsigma_1\mathbf{v}_1\left(\frac{1}{\varsigma_1}\right)\mathbf{v}_2^{(2)}-\varsigma_2\mathbf{v}_2\left(\frac{1}{\varsigma_2}\right)\mathbf{v}_1^{(2)}\\ &= 2\mathbf{v}_3^{(2)}-\frac{1}{\varsigma_1}\mathbf{v}_1(\varsigma_1)\mathbf{v}_2^{(2)}+\frac{1}{\varsigma_2}\mathbf{v}_2(\varsigma_2)\mathbf{v}_1^{(2)}\\ &= 2\mathbf{v}_3^{(2)}-\mathbf{v}_1(\varsigma_1)\varsigma_2\mathbf{v}_2^{(2)}+\mathbf{v}_2(\varsigma_2)\varsigma_1\mathbf{v}_1^{(2)}.\end{aligned} \tag{16}$$

On the other hand, from $[\mathbf{v}_1,\mathbf{v}_3]=\mathbf{v}_1$ it follows that

$$\mathbf{v}_1(\mathbf{v}_3(\varsigma_1))-\mathbf{v}_3(\mathbf{v}_1(\varsigma_1))=\mathbf{v}_1(\varsigma_1); \tag{17}$$

since $\mathbf{v}_3(\varsigma_1)=\varsigma_1$, (17) implies that the function $\mathbf{v}_1(\varsigma_1)$ is an invariant of $\mathbf{v}_3$ that can therefore be written, in variables $\{y,\alpha\}$, in the form $\mathbf{v}_1(\varsigma_1)=c_1(y)$. A similar reasoning shows that $\mathbf{v}_2(\varsigma_2)$ is an invariant of $\mathbf{v}_3$ and can be written as $\mathbf{v}_2(\varsigma_2)=c_2(y)$.

The vector field $\left[\,\varsigma_1\mathbf{v}_1^{(2)},\varsigma_2\mathbf{v}_2^{(2)}\right]$ is $\pi^{(2)}_{\mathbf{v}_3}$-projectable and, by (11), its projection becomes $[\mathbf{Y}_1,\mathbf{Y}_2]$. By using (16), relation (15) follows immediately. □

By Frobenius' Theorem, condition (15) implies the existence of a system of coordinates

$$\left\{s(y,w,w_1),\mu_1(y,w,w_1),\mu_2(y,w,w_1)\right\} \tag{18}$$

such that

$$\left\{\begin{aligned}\mathbf{Y}_1(s)&=0,\\ \mathbf{Y}_1(\mu_1)&=0,\\ \mathbf{Y}_1(\mu_2)&\neq 0.\end{aligned}\right.\qquad \left\{\begin{aligned}\mathbf{Y}_2(s)&=0,\\ \mathbf{Y}_2(\mu_1)&\neq 0,\\ \mathbf{Y}_2(\mu_2)&=0.\end{aligned}\right. \tag{19}$$

The vector fields $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{A}_{(y,w)}$ can be written in coordinates (18) as:

$$\mathbf{A}_{(y,w)}=\mathbf{A}_{(y,w)}(s)\partial_s+\mathbf{A}_{(y,w)}(\mu_1)\partial_{\mu_1}+\mathbf{A}_{(y,w)}(\mu_2)\partial_{\mu_2}, \tag{20}$$

$\mathbf{Y}_1=\mathbf{Y}_1(\mu_2)\partial_{\mu_2}$ and $\mathbf{Y}_2=\mathbf{Y}_2(\mu_1)\partial_{\mu_1}$. We observe that $\mathbf{A}_{(y,w)}(s)\neq 0$, because a non-constant function $s$ cannot be a common invariant of the set of independent vector fields $\left\{\mathbf{A}_{(y,w)},\mathbf{Y}_1,\mathbf{Y}_2\right\}$. From (12), it follows that

$$\mathbf{Y}_i(\mathbf{A}_{(y,w)}(s))=\rho_i\mathbf{A}_{(y,w)}(s)\quad\text{and}\quad\mathbf{Y}_i(\mathbf{A}_{(y,w)}(\mu_i))=\rho_i\mathbf{A}_{(y,w)}(\mu_i), \tag{21}$$

for $i = 1,2$. Therefore,

$$\begin{aligned}\mathbf{Y}_i\left(\frac{\mathbf{A}_{(y,w)}(\mu_i)}{\mathbf{A}_{(y,w)}(s)}\right) &= \frac{1}{\mathbf{A}_{(y,w)}(s)^2}\big(\mathbf{Y}_i(\mathbf{A}_{(y,w)}(\mu_i))\mathbf{A}_{(y,w)}(s)- \\ &\quad - \mathbf{Y}_i(\mathbf{A}_{(y,w)}(s))\mathbf{A}_{(y,w)}(\mu_i)\big) \\ &= \frac{1}{\mathbf{A}_{(y,w)}(s)^2}\big(\rho_i\mathbf{A}_{(y,w)}(s)\mathbf{A}_{(y,w)}(\mu_i)- \\ &\quad - \rho_i\mathbf{A}_{(y,w)}(s)\mathbf{A}_{(y,w)}(\mu_i)\big) \\ &= 0.\end{aligned} \tag{22}$$

In consequence, for $i = 1,2$, the function $\frac{\mathbf{A}_{(y,w)}(\mu_i)}{\mathbf{A}_{(y,w)}(s)}$ is an invariant of $\mathbf{Y}_i$ and can therefore be written as

$$\frac{\mathbf{A}_{(y,w)}(\mu_i)}{\mathbf{A}_{(y,w)}(s)} = \varphi_i(s,\mu_i). \tag{23}$$

By (20) and (23), the vector field $\mathbf{Z} = \frac{1}{\mathbf{A}_{(y,w)}(s)}\mathbf{A}_{(y,w)}$, in coordinates $\{s,\mu_1,\mu_2\}$, becomes

$$\mathbf{Z} = \partial_s + \varphi_1(s,\mu_1)\partial_{\mu_1} + \varphi_2(s,\mu_2)\partial_{\mu_2}. \tag{24}$$

For $i = 1,2$, the set $\{s,\mu_i\}$ is a complete set of invariants for $\mathbf{Y}_i$; by derivation we obtain the invariant $\frac{d\mu_i}{ds}$ of $\overline{\mathbf{v}}_i^{[\lambda_i,(2)]}$ [15]. We deduce, by using (23), that

$$\mu_i'(s) = \varphi_i(s,\mu_i(s)) \tag{25}$$

is a reduced equation of (5) associated to the $\lambda$-symmetry $(\overline{\mathbf{v}}_i,\lambda_i)$. The respective associated vector field is the projection of (24) to the space of variables $\{s,\mu_i\}$, i.e. the vector field

$$\mathbf{A}_i = \partial_s + \varphi_i(s,\mu_i)\partial_{\mu_i}. \tag{26}$$

In the next subsection we explicitly determine the reduced equations (25) for any third-order ODE with Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

### *3.2. Explicit expressions of the reduced equations (25) for the canonical* $SL(2,\mathbb{R})$*-invariant third-order ODEs*

In this subsection we prove that, written in an appropriate system of coordinates (18) verifying (19), both reduced equations (25) are exactly the same Riccati equation. For that, we consider the sets of Lie point symmetries which correspond to the four inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ (see Table 1). Each one of them is a representative of an equivalence class of realizations that can be mapped to one another by a point transformation [26].

Table 1: Basis elements for the inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$

| | $\mathbf{v}_1$ | $\mathbf{v}_2$ | $\mathbf{v}_3$ |
|---|---|---|---|
| Case 1 | $\partial_x$ | $x^2\partial_x$ | $x\partial_x$ |
| Case 2 | $\partial_x$ | $x^2\partial_x+2xu\partial_u$ | $x\partial_x+u\partial_u$ |
| Case 3 | $\partial_x+\partial_u$ | $x^2\partial_x+u^2\partial_u$ | $x\partial_x+u\partial_u$ |
| Case 4 | $\partial_x$ | $(x^2-u^2)\partial_x+2xu\partial_u$ | $x\partial_x+u\partial_u$ |

In Table 2 it is shown the form of the most general third-order ODE whose Lie symmetry algebra is generated by the respective basis elements given in Table 1, where $C$ denotes an arbitrary smooth function of its argument [8], denoted by $s$, which is a differential invariant common to $\mathbf{v}_1^{(2)}$, $\mathbf{v}_2^{(2)}$ and $\mathbf{v}_3^{(2)}$.

Table 2: Canonical SL$(2,\mathbb{R})$-invariant third-order ODEs

| | Third-order ODEs | Common invariant |
|---|---|---|
| Case 1 | $u_3=\dfrac{3u_2^2}{2u_1}-2u_1^3C(s)$ | $s=u$ |
| Case 2 | $u_3=\dfrac{-1}{8u^2C(s)}$ | $s=u_1^2-2uu_2$ |
| Case 3 | $u_3=\dfrac{3u_2^2}{2u_1}-\dfrac{u_1^2}{2(u-x)^2C(s)}$ | $s=(2u_1+2u_1^2+u_2(-u+x))u_1^{-3/2}$ |
| Case 4 | $u_3=\dfrac{3u_2^2u_1}{1+u_1^2}+\dfrac{(1+u_1^2)^2}{2u^2C(s)}$ | $s=(1+u_1^2+uu_2)(1+u_1^2)^{-3/2}$ |

It should be remarked that different expressions for canonical ODEs corresponding to Cases 1-3 in Table 2 can be found in [1, 7, 27] by choosing different coordinates for the basis elements of the Lie symmetry algebra.

A system of canonical coordinates (3) and the differential invariant $w=\alpha_y$ for each vector field $\mathbf{v}_3$ in Table 1 are given by:

$$\begin{array}{ll}\text{Case 1:} & y=u,\quad \alpha=\ln(x),\quad w=\alpha_y=\dfrac{1}{xu_x};\\ \text{Cases 2-4:} & y=\dfrac{u}{x},\quad \alpha=\ln(x),\quad w=\alpha_y=\dfrac{x}{xu_x-u}.\end{array}\tag{27}$$

The corresponding $\lambda$-symmetries $(\overline{\mathbf{v}}_1,-w)$ and $(\overline{\mathbf{v}}_2,w)$ of equation (5) defined by (7)-(8) can be constructed for each case from the vector fields given in Table 3:

Table 3: Vector fields to construct the $\lambda$-symmetries $(\overline{\mathbf{v}}_1, -w)$ and $(\overline{\mathbf{v}}_2, w)$ of equation (5)

| | $\overline{\mathbf{v}}_1$ | $\overline{\mathbf{v}}_2$ |
|---|---|---|
| Case 1 | $-w\partial_w$ | $w\partial_w$ |
| Case 2 | $-y\partial_y - yw^2\partial_w$ | $y\partial_y - yw^2\partial_w$ |
| Case 3 | $-(y-1)\partial_y - (y-1)w^2\partial_w$ | $y(y-1)\partial_y - (y-1)(yw+2)w\partial_w$ |
| Case 4 | $-y\partial_y - yw^2\partial_w$ | $y(y^2+1)\partial_y - y((y^2+1)w^2 + 4yw + 2)\partial_w$ |

The associated vector fields $\mathbf{Y}_1$ and $\mathbf{Y}_2$ defined by (11) can immediately be deduced from Table 3 by using (9). For each case, now we determine a system of coordinates $\{s, \mu_1, \mu_2\}$ verifying the conditions given in (19). The function $s = s(y, w, w_1)$ written in terms of the original variables $(x, u, u_1, u_2)$ through the transformations (27) must be a second-order differential invariant common to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. For each case, such invariant is given in the second column of Table 2.

In order to simplify the computations to find $\mu_1$ and $\mu_2$ satisfying (19) we observe that, by using the following changes of variables, $\mathbf{v}_1$ is transformed into $\mathbf{v}_2$, $\mathbf{v}_2$ into $\mathbf{v}_1$ and $\mathbf{v}_3$ into $-\mathbf{v}_3$ :

$$\begin{aligned} &\text{Case 1:} \quad \bar{x} = -\frac{1}{x}, \quad \bar{u} = u; && \text{Case 2:} \quad \bar{x} = -\frac{1}{x}, \quad \bar{u} = -\frac{1}{u}; \\ &\text{Case 3:} \quad \bar{x} = -\frac{1}{x}, \quad \bar{u} = u; && \text{Case 4:} \quad \bar{x} = -\frac{u}{x^2}, \quad \bar{u} = -\frac{u}{(x^2+u^2)}. \end{aligned} \tag{28}$$

In consequence, for each case there exists a change of variables in $M_1^{(1)}$, derived by prolongation from (28), that transforms the vector field $\mathbf{Y}_2$ (resp. $\mathbf{Y}_1$) into $\mathbf{Y}_1$ (resp. $\mathbf{Y}_2$). Therefore we only need to compute an invariant $\mu_1 = \mu_1(y, w, w_1)$ for $\mathbf{Y}_1$, and determine an invariant $\mu_2 = \mu_2(y, w, w_1)$ of $\mathbf{Y}_2$ by using these changes of variables. For each case, such functions $\mu_1$ and $\mu_2$ appear in Table 4.

Table 4: Functions $\mu_1, \mu_2$ satisfying (19)

| | $\mu_1$ | $\mu_2$ |
|---|---|---|
| Case 1 | $\dfrac{w_1 + w^2}{2w}$ | $\dfrac{w_1 - w^2}{2w}$ |
| Case 2 | $\dfrac{1+wy}{2w}$ | $\dfrac{1-wy}{2w}$ |
| Case 3 | $\sqrt{\dfrac{wy+1}{w}}$ | $\dfrac{1}{y}\sqrt{\dfrac{wy+1}{w}}$ |
| Case 4 | $\tan\left(\dfrac{1}{2}\arctan\left(\dfrac{wy+1}{w}\right)\right)$ | $\tan\left(\dfrac{1}{2}\arctan\left(\dfrac{1+yw}{w}\right) - \arctan y\right)$ |

The corresponding reduced equations (5) by using the invariants $\{s, \mu_1\}$ of $\mathbf{Y}_1$ (resp. $\{s, \mu_2\}$ of $\mathbf{Y}_2$) are shown in Table 5, where $C(s)$ is the corresponding arbitrary function given in Table 2. The obtained first-order equation is, in all cases, a Riccati equation.

Table 5: Reduced Riccati equations by using $(\overline{\mathbf{v}}_i, \lambda_i)$, for $i = 1,2$

| | |
|---|---|
| Case 1 | $\mu_i'(s) = C(s) + \mu_i(s)^2$ |
| Case 2 | $\mu_i'(s) = C(s)(4\mu_i(s)^2 - s)$ |
| Case 3 | $\mu_i'(s) = C(s)(2\mu_i(s)^2 - s\mu_i(s) + 2)$ |
| Case 4 | $\mu_i'(s) = C(s)(s+1)\mu_i(s)^2 + C(s)(s-1)$ |

Thus, the following theorem has been proved:

**Theorem 3.2.** *For any of the four inequivalent realizations of* $\mathfrak{sl}(2,\mathbb{R})$ *there is a Riccati equation which is the reduced equation for the reduction processes associated to both* $\lambda$*-symmetries* $(\overline{\boldsymbol{v}}_1, \lambda_1)$ *and* $(\overline{\boldsymbol{v}}_2, \lambda_2)$.

It is well known that a given Riccati equation

$$\beta'(s) = p(s)\beta(s)^2 + q(s)\beta + r(s) \tag{29}$$

can be transformed into the linear second-order equation

$$\psi''(s) = \left( q(s) + \frac{p'(s)}{p(s)} \right) \psi'(s) - r(s)p(s)\psi(s) \tag{30}$$

through the standard transformation

$$\beta(s) = \frac{-\psi'(s)}{p(s)\psi(s)}. \tag{31}$$

The linear second-order ODEs associated to each Riccati equation in Table 5 through transformations of the type (31), are given in the following table:

Table 6: Linear second-order ODEs associated to the Riccati equations given in Table 5

| | |
|---|---|
| Case 1 | $\psi''(s)+C(s)\psi(s)=0$ |
| Case 2 | $\psi''(s)-\dfrac{C'(s)}{C(s)}\psi'(s)-4C(s)^2 s\psi(s)=0$ |
| Case 3 | $\psi''(s)-\left(\dfrac{C'(s)}{C(s)}-sC(s)\right)\psi'(s)+4C(s)^2\psi(s)=0$ |
| Case 4 | $\psi''(s)-\left(\dfrac{C'(s)}{C(s)}-\dfrac{1}{s+1}\right)\psi'(s)+C(s)^2(s^2-1)\psi(s)=0$ |

## 4. Explicit expressions of first integrals in terms of solutions of linear second-order ODEs for the canonical SL$(2,\mathbb{R})$-invariant third-order ODEs

Let us recall that our immediate goal is to compute an explicit expression for the first integrals (13) associated to the $\lambda$-symmetries $(\overline{\mathbf{v}}_1,\lambda_1)$ and $(\overline{\mathbf{v}}_2,\lambda_2)$ because once this is achieved, a third functionally independent first integral of the original equation can be directly calculated by using (14).

In Theorem 3.2 we have proved that if any of the $\lambda$-symmetries $(\overline{\mathbf{v}}_1,\lambda_1)$ or $(\overline{\mathbf{v}}_2,\lambda_2)$ is used to reduce the order of (5) then the same Riccati equation can be obtained. In the following proposition we provide an explicit expression of a first integral $I=I(s,\beta)$ of the vector field associated to the Riccati equation (29)

$$\overline{\mathbf{A}}=\partial_s+(p(s)\beta^2+q(s)\beta+r(s))\partial_\beta \tag{32}$$

in terms of two linearly independent solutions of the associated linear equation (30). It should be noted that $\beta$ is considered in $\overline{\mathbf{A}}$ and in the first integral $I=I(s,\beta)$ as an independent variable and it does not refer to the transformation (31).

**Proposition 4.1.** *Let $\psi_1=\psi_1(s)$ and $\psi_2=\psi_2(s)$ be two linearly independent solutions of the linear equation (30) associated to a given Riccati equation (29). Then the function*

$$I(s,\beta)=\frac{\beta p(s)\psi_1(s)+\psi_1'(s)}{\beta p(s)\psi_2(s)+\psi_2'(s)} \tag{33}$$

*is a first integral of the vector field associated to equation (29).*

*Proof.* Let $I$ be the function defined in (33) and consider the vector field (32) associated to the Riccati equation (29). Therefore, $\overline{\mathbf{A}}(I)=I_s+(p(s)\beta^2+q(s)\beta+r(s))I_\beta$. The partial derivative of the function $I$ defined in (33) with respect to the variable $s$ becomes

$$I_s = \frac{\beta\, p'(s)\psi_1(s) + \beta\, p(s)\psi_1'(s) + \psi_1''(s)}{\beta\, p(s)\psi_2(s) + \psi_2'(s)} - \frac{\left(\beta\, p(s)\psi_1 + \psi_1'(s)\right)\left(\beta\, p'(s)\psi_2 + \beta\, p(s)\,\psi_2'(s) + \psi_2''(s)\right)}{\left(\beta\, p(s)\psi_2(s) + \psi_2'(s)\right)^2}.$$

Since $\psi_1 = \psi_1(s)$ and $\psi_2 = \psi_2(s)$ are solutions of the linear equation (30) then $\psi_1''(s)$ and $\psi_2''(s)$ can be accordingly replaced into $I_s$, which provides

$$I_s = \frac{p(s)\left(p(s)\beta^2 + q(s)\beta + r(s)\right)\left(\psi_1'(s)\psi_2(s) - \psi_1(s)\psi_2'(s)\right)}{\left(\beta\, p(s)\psi_2(s) + \psi_2'(s)\right)^2}. \tag{34}$$

On the other hand, we have that:

$$I_\beta = \frac{p(s)\left(\psi_1(s)\psi_2'(s) - \psi_1'(s)\psi_2(s)\right)}{\left(\beta\, p(s)\psi_2(s) + \psi_2'(s)\right)^2}. \tag{35}$$

Therefore, by (32), (34) and (35), we can write

$$\begin{aligned}
\overline{\mathbf{A}}(I) &= I_s + (p(s)\beta^2 + q(s)\beta + r(s))I_\beta \\
&= \frac{p(s)\left(p(s)\beta^2 + q(s)\beta + r(s)\right)\left(\psi_1'(s)\psi_2(s) - \psi_1(s)\psi_2'(s)\right)}{\left(\beta\, p(s)\psi_2(s) + \psi_2'(s)\right)^2} \\
&\quad + \frac{(p(s)\beta^2 + q(s)\beta + r(s))\,p(s)\left(\psi_1(s)\psi_2'(s) - \psi_1'(s)\psi_2(s)\right)}{(\beta p(s)\psi_2(s) + \psi_2'(s))^2} = 0,
\end{aligned}$$

which proves the result. □

Proposition 4.1 can be applied to construct two functionally independent first integrals for any of the four canonical SL(2,ℝ)-invariant third-order ODEs by using fundamental sets of solutions of associated linear second-order ODEs. For each inequivalent realization of $\mathfrak{sl}(2,\mathbb{R})$, let $\psi_1 = \psi_1(s)$ and $\psi_2 = \psi_2(s)$ be two independent solutions of the corresponding linear second-order ODE displayed in Table 6. By Proposition 4.1, for each one of the cases

$$I_i = \frac{\mu_i\, p(s)\psi_1(s) + \psi_1'(s)}{\mu_i\, p(s)\psi_2(s) + \psi_2'(s)}, \quad i = 1,2, \tag{36}$$

is a first integral of the corresponding Riccati equation in Table 5, where the function $p(s)$ is the respective coefficient of $\mu_i^2$, $i = 1,2$. Once (36) are written in terms of the original variables $\{x, u, u_1, u_2\}$, they provide two independent first integrals of the associated third-order ODEs given in Table 2. Furthermore, if we consider any of the functions $F_1$ or $F_2$ given by (14), then the set $\{I_1, I_2, F_1\}$ (or $\{I_1, I_2, F_2\}$) is a system of functionally independent first integrals of $\mathbf{A}_{(x,u)}$ that completes the integration of (1) [22]. An important remark is that no further integration is needed once two first integrals of the reduced equation (5) are known.

The results obtained for each case are summarized in Tables 7-8. The Wronskian of $\psi_1$ and $\psi_2$ is denoted by $W(\psi_1,\psi_2)(s) = \psi_1(s)\psi_2'(s) - \psi_1'(s)\psi_2(s)$. We include below the explicit expressions of $I_1, I_2$ and $F_1$ in terms of the fundamental set of solutions to the corresponding linear second-order ODE in Table 6, where $s$ is the respective value given in the second column of Table 2.

Table 7: Complete set of first integrals of ODEs in Table 2 for Cases 1-2

| | Case 1 | Case 2 |
|---|---|---|
| $I_1$ | $\dfrac{u_2\psi_1(s)-2u_1^2\psi_1'(s)}{u_2\psi_2(s)-2u_1^2\psi_2'(s)}$ | $\dfrac{2u_1C(s)\psi_1(s)+\psi_1'(s)}{2u_1C(s)\psi_2(s)+\psi_2'(s)}$ |
| $I_2$ | $\dfrac{(2u_1+xu_2)\psi_1(s)-2xu_1^2\psi_1'(s)}{(2u_1+xu_2)\psi_2(s)-2xu_1^2\psi_2'(s)}$ | $\dfrac{2(-2u+u_1x)C(s)\psi_1(s)+x\psi_1'(s)}{2(-2u+u_1x)C(s)\psi_2(s)+x\psi_2'(s)}$ |
| $F_1$ | $\dfrac{\left((2u_1+xu_2)\psi_2(s)-2xu_1^2\psi_2'(s)\right)^2}{4u_1^3W(\psi_1,\psi_2)(s)}$ | $\dfrac{\left(C(s)(-4u+2u_1x)\psi_2(s)+x\psi_2'(s)\right)^2}{4C(s)uW(\psi_1,\psi_2)(s)}$ |

Table 8: Complete set of first integrals of ODEs in Table 2 for Cases 3-4. The value $\theta$ is given by $\theta=\dfrac{1}{2}\arctan(u_1)-\arctan\left(\dfrac{u}{x}\right)$

| | Case 3 | Case 4 |
|---|---|---|
| $I_1$ | $\dfrac{2\sqrt{u_1}C(s)\psi_1(s)+\psi_1'(s)}{2\sqrt{u_1}C(s)\psi_2(s)+\psi_2'(s)}$, | $\dfrac{\psi_1(s)(s+1)C(s)\tan\left(\frac{1}{2}\arctan(u_1)\right)+\psi_1'(s)}{\psi_2(s)(s+1)C(s)\tan\left(\frac{1}{2}\arctan(u_1)\right)+\psi_2'(s)}$ |
| $I_2$ | $\dfrac{2\sqrt{u_1}xC(s)\psi_1(s)+u\psi_1'(s)}{2\sqrt{u_1}xC(s)\psi_2(s)+u\psi_2'(s)}$ | $\dfrac{\psi_1(s)(s+1)C(s)\tan(\theta)+\psi_1'(s)}{\psi_2(s)(s+1)C(s)\tan(\theta)+\psi_2'(s)}$ |
| $F_1$ | $\dfrac{\left(2C(s)x\sqrt{u_1}\psi_2(s)+u\psi_2'(s)\right)^2}{2\sqrt{u_1}C(s)(u-x)W(\psi_1,\psi_2)(s)}$ | $\dfrac{(x^2+u^2)\left(C(s)(s+1)\psi_2(s)\sin(\theta)+\psi_2'(s)\cos(\theta)\right)^2}{uC(s)(s+1)W(\psi_1,\psi_2)(s)}$ |

## 5. Parametrization of the solutions

As a result of the study carried out in the previous section, a set of functionally independent first integrals $\{I_1,I_2,F_1\}$ of the vector field $\mathbf{A}_{(x,u)}$ has been obtained. Then, the general solution of the original third-order equation (1) is implicitly defined by

$$\left\{\begin{array}{lcl} I_1(x,u,u_1,u_2) &=& C_1,\\ I_2(x,u,u_1,u_2) &=& C_2,\\ F_1(x,u,u_1,u_2) &=& C_3,\end{array}\right. \tag{37}$$

where $C_i\in\mathbb{R}$, for $i=1,2,3$. We now obtain from (37) an explicit expression of the general solution for Case 1 and parametric general solutions for Cases 2, 3 and 4.

- Case 1.

Since $s = u$, an explicit expression of the general solution can be easily obtained by eliminating $u_1$ and $u_2$ from (37), after using the corresponding values of $I_1$, $I_2$ and $F_1$ given in Table 7:

$$x = \frac{C_3(C_1 - C_2)(C_1\psi_2(u) - \psi_1(u))}{C_2\psi_2(u) - \psi_1(u)}, \qquad C_1 \neq C_2, C_3 \neq 0. \tag{38}$$

In this expression $\psi_1$ and $\psi_2$ denotes two linearly independent solutions to the Schrödinger-type equation

$$\psi''(u) + C(u)\psi(u) = 0. \tag{39}$$

- Cases 2, 3 and 4.
  In these three cases the elimination of $u_1$ and $u_2$ from (37), by using the corresponding values of $I_1$, $I_2$ and $F_1$ given in Tables 7-8, to obtain an explicit expression of the general solution seems to be impossible, because the functions $\psi_1$ and $\psi_2$ and their derivatives are evaluated in $s = s(x, u, u_1, u_2)$. Our goal is to construct the solutions in parametric form. For that purpose, we introduce a new parameter $t$ such that $s = s(t)$ is determined from the function $C = C(s)$ associated to each equation in Table 5 as follows:

$$\text{Cases 2 and 3:} \qquad s'(t) = \frac{1}{C(s(t))}, \tag{40}$$

$$\text{Case 4:} \qquad s'(t) = \frac{1}{C(s(t))(s(t)+1)}. \tag{41}$$

If $\psi = \psi(s)$ is a solution of the corresponding linear second-order ODE for Cases 2, 3 and 4 in Table 6, then it can be checked that $\phi(t) = \psi(s(t))$, where $s(t)$ is defined by (40)-(41), is a solution of the respective linear second-order ODE that appears in Table 9:

Table 9: Linear second-order ODEs in Table 6 for $s = s(t)$ given by (40)-(41)

| | |
|---|---|
| Case 2 | $\phi''(t) - 4s(t)\phi(t) = 0$ |
| Case 3 | $\phi''(t) + s(t)\phi'(t) + 4\phi(t) = 0$ |
| Case 4 | $\phi''(t) + \left(\dfrac{s(t)-1}{s(t)+1}\right)\phi(t) = 0$ |

Let us observe that the linear second-order ODEs corresponding to Cases 2 and 4 in Table 9 are Schrödinger-type equations. Needless to say, the respective linear second-order ODE of Case 3 in Table 9 could also be transformed into a Schrödinger-type equation by means of a change of the dependent variable.

Next, we show that the changes of independent variable $s = s(t)$ defined by (40)-(41) permit the construction of parametric general solutions for Cases 2, 3 and 4. The implicit solutions (37) corresponding to the respective values of $I_1$, $I_2$ and $F_1$ given in Tables 7-8 are expressed in terms of two linearly independent solutions, $\psi_1$ and $\psi_2$, of the respective linear equations in Table 6. For $i=1,2$, we use the transformation $\phi_i(t)=\psi_i(s(t))$, where $s(t)$ is defined respectively by (40)-(41), to write such implicit solutions in terms of two linearly independent solutions of the associated linear second-order ODEs given in Table 9. This enables to eliminate $u_1$ and $u_2$ to obtain parametric general solutions. We include below the corresponding expressions of the implicit and parametric general solutions corresponding to Cases 2, 3 and 4. In all cases $C_i\in\mathbb{R}$ for $i=1,2,3$, and it is assumed that $C_1\neq C_2$ and $C_3\neq 0$.

- Case 2.
  - Implicit general solution:

$$\frac{2u_1\phi_1(t)+\phi_1'(t)}{2u_1\phi_2(t)+\phi_2'(t)}=C_1,\quad \frac{2(-2u+u_1x)\phi_1(t)+x\phi_1'(t)}{2(-2u+u_1x)\phi_2(t)+x\phi_2'(t)}=C_2,$$

$$\frac{((-4u+2u_1x)\phi_2(t)+x\phi_2'(t))^2}{4uW(\phi_1,\phi_2)(t)}=C_3. \tag{42}$$

  - Parametric general solution:

$$\begin{cases} x(t) &= \dfrac{C_3(C_1-C_2)(\phi_1(t)-C_2\phi_2(t))}{\phi_1(t)-C_1\phi_2(t)}, \\[2ex] u(t) &= \dfrac{C_3(C_1-C_2)^2W(\phi_1,\phi_2)(t)}{4(\phi_1(t)-C_1\phi_2(t))^2}. \end{cases} \tag{43}$$

- Case 3.
  - Implicit general solution:

$$\frac{2\sqrt{u_1}\phi_1(t)+\phi_1'(t)}{2\sqrt{u_1}\phi_2(t)+\phi_2'(t)}=C_1,\quad \frac{2\sqrt{u_1}x\phi_1(t)+u\phi_1'(t)}{2\sqrt{u_1}x\phi_2(t)+u\phi_2'(t)}=C_2,$$

$$\frac{\left(2x\sqrt{u_1}\phi_2(t)+u\phi_2'(t)\right)^2}{2\sqrt{u_1}(u-x)W(\phi_1,\phi_2)(t)}=C_3. \tag{44}$$

  - Parametric general solution:

$$\begin{cases} x(t) &= \dfrac{C_3(C_1-C_2)(C_2\phi_2'(t)-\phi_1'(t))}{C_1\phi_2'(t)-\phi_1'(t)}, \\[2ex] u(t) &= \dfrac{C_3(C_1-C_2)(C_2\phi_2(t)-\phi_1(t))}{C_1\phi_2(t)-\phi_1(t)}. \end{cases} \tag{45}$$

- Case 4.
  - Implicit general solution:

$$\frac{\phi_1(t)\tan\left(\dfrac{1}{2}\arctan(u_1)\right)+\phi_1'(t)}{\phi_2(t)\tan\left(\dfrac{1}{2}\arctan(u_1)\right)+\phi_2'(t)}=C_1,\quad \frac{\phi_1(t)\tan(\theta)+\phi_1'(t)}{\phi_2(t)\tan(\theta)+\phi_2'(t)}=C_2,$$

$$\frac{(x^2+u^2)\left(\phi_2(t)\sin(\theta)+\phi_2'(t)\cos(\theta)\right)^2}{uW(\phi_1,\phi_2)(t)}=C_3, \tag{46}$$

where

$$\theta = \frac{1}{2}\arctan(u_1) - \arctan\left(\frac{u}{x}\right). \tag{47}$$

– Parametric general solution:

$$\left\{\begin{array}{rcl} x(t) & = & \dfrac{(C_1-C_2)C_3\left(\zeta_1'(t)^2+\zeta_1(t)^2\right)\left(\zeta_1'(t)\zeta_2'(t)+\zeta_1(t)\zeta_2(t)\right)}{\left(\zeta_1'(t)\left(\left((C_1-C_2)\phi_2(t)+C_2\right)\phi_2'(t)-\phi_1'(t)\right)+\zeta_1(t)^2\right)^2}, \\ \\ u(t) & = & \dfrac{W(\phi_1,\phi_2)(t)(C_1-C_2)^2C_3}{\zeta_1'(t)^2+\zeta_1(t)^2}, \end{array}\right. \tag{48}$$

where

$$\zeta_i(t) = C_i\phi_2(t) - \phi_1(t), \quad i = 1,2. \tag{49}$$

It has been proved that the solution of any third-order ODE with Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ can be explicitly parametrized in terms of two linearly independent solutions of a linear second-order ODE. As far as we know, this is the first time that a parametric solution for SL(2,$\mathbb{R}$)-invariant third-order ODEs corresponding to Case 4 in Table 2 is presented in the literature. Besides, our approach provides a unified treatment of SL(2,$\mathbb{R}$)-invariant third-order ODEs which generalizes the results obtained in the literature for the case of $\mathfrak{sl}(2,\mathbb{C})$. We also want to remark that the complete set of first integrals and the parametric general solutions are directly given in terms of the solutions of linear second-order equations for each case. It should be mentioned that the apparently missing integration is due to the initial reduction of order made by using $\mathbf{v}_3$. Nevertheless, the quadratures involved in this process as well as in the determination of the functions (6) are all immediate.

## 6. Some examples

In order to facilitate the application of the method in practice, we now summarize the steps that can be followed to find three functionally independent first integrals of SL(2,$\mathbb{R}$)-invariant third-order ODEs as well as the parametric solution of the equation.

- STEP 1: Map the basis elements of $\mathfrak{sl}(2,\mathbb{R})$ into one of the cases given in Table 1.
- STEP 2: Obtain the corresponding invariants $\{s,\mu_1,\mu_2\}$, written in terms of the original coordinates $\{x,u,u_1,u_2\}$, from Table 2 (second column) and Table 4 by using (27).
- STEP 3: Express the original equation in terms of $\{s,\mu_i,\mu_i'\}$, for $i = 1,2$. A Riccati equation will be always obtained. Compare this Riccati equation with the corresponding given in Table 5 to identify the function $C(s)$.
- STEP 4: A complete set of functionally independent first integrals of the equation expressed in terms of two linearly independent solutions of a second-order linear ODE are given in Table 7 for Cases 1-2, and in Table 8 for Cases 3-4. The corresponding linear ODEs are displayed in Table 6. In the expressions of such first integrals, the value of $C(s)$ has been calculated in the step 3.

- STEP 5: For Case 1 an explicit general solution (38) is obtained in terms of the solutions of the Schrödinger-type equation (39). A parametric solution expressed in terms of two linearly independent solutions of second-order linear ODEs is given for the Cases 2,3 and 4 in (43), (45) and (48), respectively. The corresponding linear ODEs are shown in Table 9.

Next the procedure is illustrated in two examples, in which $SL(2,\mathbb{R})$-invariant third-order ODEs corresponding to Cases 3 and 4 are considered. Since the study of the four inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ has been carried out in parallel, other examples corresponding to Cases 1 and 2 (see, for instance, [22]) can be similarly treated.

### *6.1. An example of Case 3: the generalized Chazy equation*

In this section, for illustration purposes and comparison with other methods in the literature, we apply the obtained results to the generalized Chazy equation [23]. In his study of third-order ordinary differential equations having the Painlevé property [23], Chazy was led to the following remarkable family of equations:

$$v_3 - 2vv_2 + 3v_1^2 = \gamma(6v_1 - v^2)^2, \tag{50}$$

where $x$ is the independent variable, $v = v(x)$ is the dependent variable, and $v_i = \frac{d^i v}{dx^i}$ for $i = 1,2,3$. Chazy showed that when

$$\gamma = 0 \quad \text{or} \quad \gamma = \frac{4}{36 - k^2},$$

where $6 < k \in \mathbb{N}$, then the nontrivial solutions $v = f(x)$ to (50) have a moveable circular natural boundary.

It is well known [23] that the generalized Chazy equation (50) admits a three-dimensional symmetry group with infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = x\partial_x - v\partial_v, \quad \mathbf{v}_3 = x^2\partial_x - (2xv + 6)\partial_v, \tag{51}$$

which satisfy the commutation relations (2).

- STEP 1.
  By means of the map $u = x + 6/v$, the Lie algebra spanned by the vector fields (51) is mapped into the Lie algebra generated by the vector fields given in Case 3 of Table 1 and equation (50) becomes

$$-\frac{u_3}{6}(u-x)^2 = u_1 + u_2 x - 2\,{u_1}^2 + u_1 u_2 x - u_1 u_2 u + 36\gamma\,{u_1}^2 + {u_1}^3 - u_2 u. \tag{52}$$

- STEP 2.
  Local coordinates $\{s, \mu_1, \mu_2\}$ verifying the conditions given in (19) are

$$s = (2u_1 + 2u_1^2 + u_2(-u + x))\,u_1^{-3/2}, \quad \mu_1 = \sqrt{u_1}, \quad \mu_2 = \frac{x\sqrt{u_1}}{u},$$

  which come from the corresponding functions $\mu_1$ and $\mu_2$ given in Table 4 expressed in terms of the original variables by means of (27).

- STEP 3.
  In terms of $\{s, \mu_i, \mu_i'\}$, for $i = 1,2$, equation (52) becomes
  $$\mu_i'(s) = \frac{1}{3(s^2+144\gamma-16)}(2\mu_i^2 - s\mu_i + 2) \quad \text{for} \quad i = 1,2, \tag{53}$$
  which correspond to the Riccati equations appearing in Case 3 of Table 5, for the function $C(s) = \dfrac{1}{3(s^2+144\gamma-16)}$.
- STEP 4.
  In consequence, a complete set of first integrals for the generalized Chazy equation can be derived from Table 7 (Case 3) and it is given by
  $$\left\{\begin{array}{lcl} I_1 & = & \dfrac{2\sqrt{u_1}\psi_1(s)+3(s^2+144\gamma-16)\psi_1'(s)}{2\sqrt{u_1}\psi_2(s)+3(s^2+144\gamma-16)\psi_2'(s)}, \\ \\ I_2 & = & \dfrac{2\sqrt{u_1}x\psi_2(s)+3(s^2+144\gamma-16)\psi_2'(s)}{2\sqrt{u_1}x\psi_1(s)+3(s^2+144\gamma-16)\psi_1'(s)}, \\ \\ F_1 & = & \dfrac{\left(2x\sqrt{u_1}\psi_1(s)+3u(s^2+144\gamma-16)\psi_1'(s)\right)^2}{6\sqrt{u_1}(u-x)(s^2+144\gamma-16)W(\psi_1,\psi_2)(s)}, \end{array}\right. \tag{54}$$
  where $s = (2u_1 + 2u_1^2 + u_2(-u+x))u_1^{-3/2}$ and $\psi_1, \psi_2$ are two linearly independent solutions of the associated linear second-order equation of Case 3 in Table 6 for $C(s) = \dfrac{1}{3(s^2+144\gamma-16)}$. Two possible cases arise:
  - If $\gamma \neq \frac{1}{9}$ then the associated linear second-order equation in Table 6 becomes
    $$\psi''(s) + \frac{7s}{3(s^2+144\gamma-16)}\psi'(s) + \frac{4}{9(s^2+144\gamma-16)^2}\psi(s) = 0. \tag{55}$$
    If the functions $\Psi_1$ and $\Psi_2$ are two linearly independent solutions to the Legendre equation
    $$\left(16(1-9\gamma)-s^2\right)\Psi''(s) - 2s\Psi'(s) + \left(\frac{7}{36} - \frac{4\gamma}{s^2+144\gamma-16}\right)\Psi(s) = 0, \tag{56}$$
    then it can be checked that two independent solutions to equation (55) are given by
    $$\psi_1(s) = \frac{\Psi_1(s)}{(s^2+144\gamma-16)^{1/12}} \quad \text{and} \quad \psi_2(s) = \frac{\Psi_2(s)}{(s^2+144\gamma-16)^{1/12}}.$$
    In consequence, in this case, the first integrals (54) can be expressed in terms of Legendre functions and their derivatives.
  - If $\gamma = \frac{1}{9}$ then the associated linear second-order equation in Case 3 of Table 6 is
    $$\psi''(s) + \frac{7}{3s}\psi'(s) + \frac{4}{9s^4}\psi(s) = 0. \tag{57}$$
    If $\mathrm{J}_\nu$ and $\mathrm{Y}_\nu$ stand for the Bessel functions of first and second kind of order $\nu$, respectively, then two linearly independent solutions to equation (57) are given by
    $$\psi_1(s) = \frac{\mathrm{J}_\nu\left(\frac{2}{3s}\right)}{s^{2/3}} \quad \text{and} \quad \psi_2(s) = \frac{\mathrm{Y}_\nu\left(\frac{2}{3s}\right)}{s^{2/3}},$$

where $\nu = \frac{-2}{3}$. Therefore, in this case, the first integrals (54) can be written in terms of the Bessel functions and their derivatives.

- STEP 5.
  With the aim of obtaining a parametric solution to equation (52), we solve the equation (40) for $C(s) = \dfrac{1}{3(s^2+144\gamma-16)}$:
  - If $\gamma > \frac{1}{9}$ then (40) provides $s(t) = 4\sqrt{9\gamma-1}\tan\left(12t\sqrt{9\gamma-1}\right)$ . Thus, the corresponding linear second-order ODE given in Table 9 becomes
  $$\phi''(t) + 4\sqrt{9\gamma-1}\tan\left(12t\sqrt{9\gamma-1}\right)\phi'(t) + 4\phi(t) = 0. \tag{58}$$
  - If $\gamma = \frac{1}{9}$ we obtain from (40) $s(t) = -\frac{1}{3t}$, for $t \neq 0$, and the associated linear second-order ODE in Table 9 is
  $$\phi''(t) - \frac{1}{3t}\phi'(t) + 4\phi(t) = 0. \tag{59}$$
  - If $\gamma < \frac{1}{9}$ a solution to equation (40) is given by $s(t) = -4\sqrt{1-9\gamma}\tanh\left(12t\sqrt{1-9\gamma}\right)$ and the linear second-order ODE of Case 3 in Table 9 becomes
  $$\phi''(t) - 4\sqrt{1-9\gamma}\tanh\left(12t\sqrt{1-9\gamma}\right)\phi'(t) + 4\phi(t) = 0. \tag{60}$$

  Therefore, a parametric general solution to (52) in terms of two linearly independent solutions $\phi_1$ and $\phi_2$ of (58), (59) or (60) (depending on the values of $\gamma$) is given through (45). Let us denote $\Phi_2(t) = C_3(C_1-C_2)(C_2\phi_2(t)-\phi_1(t))$ and $\Phi_1(t) = C_1\phi_2(t)-\phi_1(t)$. Then $\Phi_1$ and $\Phi_2$ are also two linearly independent solutions to the corresponding equations (58), (59) or (60) and the parametric solution (45) can be rewritten as
  $$x(t) = \frac{\Phi_2'(t)}{\Phi_1'(t)}, \qquad u(t) = \frac{\Phi_2(t)}{\Phi_1(t)}. \tag{61}$$
  Since $u = x + 6/v$, a parametric general solution to the generalized Chazy equation (50) is
  $$x(t) = \frac{\Phi_2'(t)}{\Phi_1'(t)}, \qquad v(t) = \frac{-6\Phi_1(t)\Phi_1'(t)}{W(\Phi_1,\Phi_2)(t)}. \tag{62}$$

We observe that, by setting $\tau = \sin^2\left(12\sqrt{9\gamma-1}\,t\right)$ if $\gamma > \frac{1}{9}$ (resp. $\tau = -\sinh^2\left(12\sqrt{1-9\gamma}\,t\right)$ if $\gamma < \frac{1}{9}$), the linear equation (58) (resp. (60)) is transformed into the following hypergeometric equation:
$$\tau(1-\tau)\phi''(\tau) + \left(\frac{1}{2} - \frac{5}{6}\tau\right)\phi'(\tau) - \frac{1}{144(1-9\gamma)}\phi(\tau) = 0, \tag{63}$$
and the parametric solution (62) is unaffected by this change of independent variable.

The appearance of an hypergeometric equation in the study of the generalized Chazy equation (50) is not new in the literature: in [23], Chazy showed, by using methods which are not related with the symmetry analysis of (50), that the general solution to equation (50) can be parametrized in terms of the solutions of a slightly different hypergeometric equation. On the other hand, in [6], the authors also obtained a parametric general solution to (50) in terms of the solutions of either a Lamé equation ($\gamma \neq \frac{1}{9}$) or an Airy equation ($\gamma = \frac{1}{9}$). For that, they connected the Lie symmetry algebra (51) with the basic unimodular action

(Case 1) via the standard prolongation process. In [8] the authors reduced equation (50) for the particular value of $\gamma = 0$ to a Riccati equation, whose general solution was expressed in terms of Legendre functions.

*6.2. An example of Case 4*

For the third-order equation

$$u_3 = \frac{6u_2^2 u_1 u^2 + 3u_1^2 + 3u_1^4 + u_1^6 + 1}{2u^2(1+u_1^2)} \tag{64}$$

the determining equations for a Lie point symmetry $\xi(x,u)\partial_x + \eta(x,u)\partial_u$ become:

$$\xi_x = \frac{\eta}{u}, \quad \xi_u = -\eta_x, \quad \eta_{xx} = 0 \quad \text{and} \quad \eta_u = \frac{\eta}{u}. \tag{65}$$

From (65) we deduce that the Lie symmetry algebra of (64) is three-dimensional and generated by the vector fields given in Case 4 of Table 1. We consider the coordinates $\{s, \mu_1, \mu_2\}$, where $s$ is given in the second column of Table 2 and $\mu_1$ and $\mu_2$ are the functions of Case 4 in Table 4 expressed in terms of the original variables:

$$\begin{cases} s &= (1+u_1^2+uu_2)(1+u_1^2)^{-3/2}, \\ \mu_1 &= \tan\left(\dfrac{1}{2}\arctan(u_1)\right), \\ \mu_2 &= \tan\left(\dfrac{1}{2}\arctan(u_1) - \arctan\left(\dfrac{u}{x}\right)\right). \end{cases}$$

It can be checked that equation (64) in terms of $\{s, \mu_i, \mu_i'\}$, for $i = 1,2$, becomes:

$$\mu_i'(s) = (s+1)\mu_i(s)^2 + (s-1),$$

which corresponds to the Riccati equations given in Table 5 for the function $C(s) = 1$. According to Table 8, three functionally independent first integrals of equation (64) are:

$$\begin{cases} I_1 &= \dfrac{\psi_1(s)(s+1)\tan\left(\dfrac{1}{2}\arctan(u_1)\right) + \psi_1'(s)}{\psi_2(s)(s+1)\tan\left(\dfrac{1}{2}\arctan(u_1)\right) + \psi_2'(s)}, \\ I_2 &= \dfrac{\psi_1(s)(s+1)\tan(\theta) + \psi_1'(s)}{\psi_2(s)(s+1)\tan(\theta) + \psi_2'(s)}, \\ F_1 &= \dfrac{(x^2+u^2)\left((s+1)\psi_2(s)\sin(\theta) + \psi_2'(s)\cos(\theta)\right)^2}{u(s+1)W(\psi_1,\psi_2)(s)}, \end{cases}$$

where $\theta$ is given in (47), $s = (1+u_1^2+uu_2)(1+u_1^2)^{-3/2}$ and $\psi_1, \psi_2$ are two linearly independent solutions of the corresponding linear second-order equation of Case 4 in Table 6 for $C(s) = 1$:

$$\psi''(s) - \left(\frac{1}{s+1}\right)\psi'(s) + (s^2-1)\psi(s) = 0.$$

In order to obtain a parametric solution to (64), equation (41) corresponding to $C(s) = 1$ provides $s(t) = -1+\sqrt{1+2t}$, for $t > -\frac{1}{2}$. The functions $\phi_i(t) = \psi_i(s(t))$, for $i = 1,2$, are two linearly independent solutions to the Schrödinger-type equation (see Table 9):

$$\phi''(t) + \left( \frac{-2+\sqrt{1+2t}}{\sqrt{1+2t}} \right) \phi(t) = 0.$$

In terms of these linearly independent solutions, $\phi_1$ and $\phi_2$, a parametrization of the solution to the equation (64) is given through (48)-(49).

### *6.3. Other examples: Cases 1 and 2*

In this subsection we include some nontrivial examples corresponding to Cases 1 and 2 which can be completely solved by giving the expressions of the (explicit or parametric) general solutions in terms of elementary functions. The expressions of the corresponding first integrals are not written because they can be easily derived from Table 7.

**A.-Two examples of Case 1**

- Consider the third-order ODE
$$u_3 = \frac{3u_2^2}{2u_1} + \frac{2u_1^3}{u^{\frac{12}{7}}}, \tag{66}$$
which corresponds to the equation given in Case 1 of Table 2 for the function $C(s) = -s^{-12/7}$ (recall that $s = u$ for Case 1). The associated linear second-order ODE in Table 6 becomes
$$\psi''(u) - u^{-\frac{12}{7}}\psi(u) = 0, \tag{67}$$
which admits the following two linearly independent solutions:
$$\psi_1(s) = e^{7s^{\frac{1}{7}}}\left(343s^{\frac{3}{7}} - 294s^{\frac{2}{7}} + 105s^{\frac{1}{7}} - 15\right), \quad \psi_2(s) = e^{-7s^{\frac{1}{7}}}\left(343s^{\frac{3}{7}} + 294s^{\frac{2}{7}} + 105s^{\frac{1}{7}} + 15\right).$$
Therefore, according to (38), an explicit general solution $x = x(u)$ to equation (66) expressed in terms of elementary functions is given by
$$x = \frac{C_3(C_1-C_2)\left( C_1 e^{-7u^{\frac{1}{7}}}\left(343u^{\frac{3}{7}} + 294u^{\frac{2}{7}} + 105u^{\frac{1}{7}} + 15\right) - e^{7u^{\frac{1}{7}}}\left(343u^{\frac{3}{7}} - 294u^{\frac{2}{7}} + 105u^{\frac{1}{7}} - 15\right)\right)}{C_2 e^{-7u^{\frac{1}{7}}}\left(343u^{\frac{3}{7}} + 294u^{\frac{2}{7}} + 105u^{\frac{1}{7}} + 15\right) - e^{7u^{\frac{1}{7}}}\left(343u^{\frac{3}{7}} - 294u^{\frac{2}{7}} + 105u^{\frac{1}{7}} - 15\right)},$$
where $C_i \in \mathbb{R}$ for $i = 1,2,3$, $C_3 \neq 0$ and $C_1 \neq C_2$.
- The equation
$$u_3 = \frac{3u_2^2}{2u_1} - \left(\frac{1}{4} + k^2\right)\frac{2u_1^3}{u^2}, \quad k \neq 0 \tag{68}$$
corresponds to the equation given in Case 1 of Table 2 for the function $C(s) = \left(\frac{1}{4} + k^2\right)s^{-2}$. The associated linear second-order ODE in Table 6 admits the following two linearly independent solutions:
$$\psi_1(s) = \sqrt{|s|}\sin(k\ln(|s|)) \text{ and } \psi_2(s) = \sqrt{|s|}\cos(k\ln(|s|)).$$

By using the corresponding expression (38), the following explicit general solution $u = u(x)$ to equation (68) is finally obtained:

$$u(x) = \exp\left(\frac{1}{k}\arctan\left(\frac{x + C_3(C_2 - C_1)}{C_2 x + C_1 C_3(C_2 - C_1)}\right)\right), \qquad C_1 \neq C_2, C_3 \neq 0.$$

**B. -Example of Case 2**

Let us consider now the third-order ODE

$$u_3 = \frac{(u_1^2 - 2uu_2)^{7/4}}{6u^2}, \tag{69}$$

which corresponds with the equation given in Case 2 of Table 2 for the function $C(s) = \frac{-3}{4}s^{-7/4}$, being $s = u_1^2 - 2uu_2$. In this case it can be checked that the Lie symmetry algebra of equation (69) is three-dimensional and it is spanned by the vector fields given in Case 2 of Table 1. For this particular example equation (40) yields $s(t) = t^{-4/3}$, therefore the corresponding linear second-order ODE in Table 9 becomes

$$\phi''(t) - 4t^{-4/3}\phi(t) = 0,$$

which admits the following two linearly independent solutions:

$$\phi_1(t) = e^{-6t^{1/3}}(6t^{1/3} + 1) \text{ and } \phi_2(t) = e^{6t^{1/3}}(6t^{1/3} - 1).$$

According to (43) and taking that $W(\phi_1, \phi_2)(t) = 144$ into account, we conclude that a parametric solution to equation (69) expressed in terms of elementary functions is

$$\left\{\begin{array}{rcl} x(t) & = & \dfrac{C_3(C_1 - C_2)\left(e^{-6t^{1/3}}(6t^{1/3} + 1) - C_2 e^{6t^{1/3}}(6t^{1/3} - 1)\right)}{e^{-6t^{1/3}}(6t^{1/3} + 1) - C_1 e^{6t^{1/3}}(6t^{1/3} - 1)}, \\ u(t) & = & \dfrac{36C_3(C_1 - C_2)^2}{\left(e^{-6t^{1/3}}(6t^{1/3} + 1) - C_1 e^{6t^{1/3}}(6t^{1/3} - 1)\right)^2}, \end{array}\right.$$

where $C_i \in \mathbb{R}$ for $i = 1, 2, 3$, $C_1 \neq C_2$ and $C_3 \neq 0$.

## 7. Concluding remarks

A unified procedure to compute explicitly a complete set of first integrals and a parametric general solution for any SL$(2,\mathbb{R})$-invariant third-order ODE has been presented. Such first integrals and parametric solutions can be directly expressed in terms of the solutions of linear second-order ODEs, without additional quadratures.

Explicit equations of these linear second-order ODEs for the canonical SL$(2,\mathbb{R})$-invariant ODEs have been explicitly provided. The performed study includes the class of ODEs invariant under the realization of $\mathfrak{sl}(2,\mathbb{R})$ that does not appear in $\mathfrak{sl}(2,\mathbb{C})$ (Case 4 in Table 2). A connection between SL$(2,\mathbb{R})$-invariant ODEs corresponding to Case 4 in Table 2 and linear second-order ODEs has been established, as far as we know, for the first time in the literature.

Remarkably, the procedure can be used always in the same way, regardless of the inequivalent realization associated to the ODE under study. Particular examples, including an equation of the most complicated realization (Case 4), show how the first integrals and the explicit or parametric solutions can be expressed in terms of linearly independent solutions of linear second-order equations.

## Acknowledgments

The authors appreciate very much the comments and suggestions of the anonymous reviewers which greatly contribute to improve the paper.

The authors acknowledge the financial support from the University of Cádiz and from Junta de Andalucía to the research group FQM 377. A. Ruiz acknowledges the financial support from the Ministry of Education, Culture and Sport of Spain (FPU grant FPU15/02872) and from a grant of the University of Cádiz program "Movilidad Internacional, Becas UCA- Internacional-Posgrado" during his stay at the University of Minnesota, as well as Prof. Olver for his attention and comments during the stay. The authors also thank Prof. Romero from the University of Cádiz for his time and advice in the preparation of this contribution.

## References

[1] Hydon P E 2000 *Symmetry Methods for Differential Equations: A Beginner's Guide* (Cambridge Texts in Applied Mathematics: Cambridge University Press)

[2] Ibragimov N H 2004 *A Practical Course in Differential Equations and Mathematical Modelling: Classical and New Methods, Nonlinear Mathematical Models, Symmetry and Invariance Principles* (ALGA Publications)

[3] Olver P J 2000 *Applications of Lie Groups to Differential Equations* (Graduate Texts in Mathematics. Springer-Verlag)

[4] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* (Cambridge University Press)

[5] González-López A 1988 Symmetry and integrability by quadratures of ordinary differential equations *Phys. Lett. A.* **133**(4-5) 190–4 doi: 10.1016/0375-9601(88)91015-8

[6] Clarkson P and Olver P J 1996 Symmetry and the Chazy equation *J. Differ. Equations* **124**(1) 225–46 doi: 10.1006/jdeq.1996.0008

[7] Ibragimov N H and Nucci M C 1994 Integration of third order ordinary differential equations by Lie's method: equations admitting three-dimensional Lie algebras *Lie Groups Appl.* **1**(2) 49–64

[8] Muriel C and Romero J L 2001 $\mathcal{C}^\infty$-symmetries and non-solvable symmetry algebras *IMA J. Appl. Math.* **66** 477–98 doi: 10.1093/imamat/66.5.477

[9] Schmucker A and Czichowski G 1998 Symmetry algebras and normal forms of third order ordinary differential equation *J. Lie Theory* **8** 129–37

[10] Schwarz F 2007 *Algorithmic Lie Theory for Solving Ordinary Differential Equations* (Chapman & Hall/CRC Pure and Applied Mathematics. CRC Press)

[11] Olver P J 1995 *Equivalence, Invariants, and Symmetry* (Cambridge University Press. Cambridge)

[12] Adam A A and Mahomed F M 2002 Integration of ordinary differential equations via nonlocal symmetries *Nonlinear Dyn.* **30**(2) 267–75 doi: 10.1023/A:1020518129295

[13] Govinder K S and Leach P G L 1995 On the determination of non-local symmetries *J. Phys. A: Math. Gen.* **28**(18) 5349–59 doi: 10.1088/0305-4470/28/18/023

[14] Govinder K S and Leach P G L 1997 A group-theoretic approach to a class of second-order ordinary differential equations not possessing Lie point symmetries *J. Phys. A: Math. Gen.* **30**(6) 2055–68 doi: 10.1088/0305-4470/30/6/026

[15] Muriel C and Romero J L 2001 New methods of reduction for ordinary differential equations *IMA J. Appl. Math.* **66**(2) 111–25 doi: 10.1093/imamat/66.2.111.

[16] Barco M A and Prince G E 2001 Solvable symmetry structures in differential form applications *Acta Appl. Math.* **66**(1) 89–121 doi: 10.1023/A:1010609817442

[17] Basarab-Horwath P 1991 Integrability by quadratures for systems of involutive vector fields *Ukr. Math. J.* **43**(10) 1236–42 doi: 10.1007/BF01061807

[18] Hartl T and Athorne C 1994 Solvable structures and hidden symmetries *J. Phys. A: Math. Gen.* **27** 3463–74 doi: 10.1088/0305-4470/27/10/022

[19] Prince G and Sherring J 1992 Geometric aspects of reduction of order *Trans. Amer. Math. Soc.* **334** 433–53 doi: 10.1090/S0002-9947-1992-1149125-6

[20] Catalano Ferraioli D and Morando P 2009 Applications of solvable structures to the nonlocal symmetry-reduction of ODEs *Nonlinear Math. Phys.* **16**(suppl.1) 27–42 doi: 10.1142/S1402925109000303

[21] Catalano Ferraioli D and Morando P 2009 Local and nonlocal solvable structures in the reduction of ODEs *J. Phys. A: Math. Theor.* **42**(3) 035210 doi: 10.1088/1751-8113/42/3/035210

[22] Ruiz A and Muriel C 2016 Solvable structures associated to the non-solvable symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ *SIGMA* **077** 18 pages doi: 10.3842/SIGMA.2016.077

[23] Chazy J 1911 Sur les équations differentielles du troisiéme ordre et d'ordre supérier dont l'intégrale générale a ses points critiques fixes *Acta Math.* **34** 317–85

[24] Sardanashvily G 2013 *Advanced Differential Geometry for Theoreticians* (Lap Lambert Academic Publishing)

[25] Ruiz A and Muriel C 2016 Applications of $\mathcal{C}^\infty$−symmetries in the construction of solvable structures In *Trends in Differential Equations and Applications*, vol 8 (SEMA SIMAI Springer Series) 387-403 doi: 10.1007/978-3-319-32013-7

[26] Kamran N, Olver P J and González-López A 1992 Lie algebras of the vector fields in the real plane it*Proc. Lond. Math. Soc.* **64** 339-68 doi: 10.1112/plms/s3-64.2.339

[27] Gat O 1992 Symmetry algebras of third-order ordinary differential equations *J. Math. Phys.* **33**(9) 2966-71 doi: 10.1063/1.529566