

# DISTRIBUTIONAL CHAOS FOR THE FORWARD AND BACKWARD CONTROL TRAFFIC MODEL

XAVIER BARRACHINA, J. ALBERTO CONEJERO, MARINA MURILLO-ARCILA,  
AND JUAN B. SEOANE-SEPÚLVEDA

ABSTRACT. The interest in car-following models has increased in the last years due to its connection with vehicle-to-vehicle communications and the development of driverless cars. Some non-linear models such as the Gazis-Herman Rothery model were already known to be chaotic. We consider the linear Forward and Backward Control traffic model for an infinite number of cars on a track. We show the existence of solutions with a chaotic behaviour by using some results of linear dynamics of  $C_0$ -semigroups. In contrast, we also analyze which initial configurations lead to stable solutions.

## 1. INTRODUCTION

In this paper we study chaotic traffic patterns described by car-following models. Several notions of chaos can be considered, such as the one of Devaney [19] or the one of distributional chaos by Schweizer and Smítal [41]. Devaney chaos consists of 3 ingredients: transitivity, existence of a dense set of periodic points, and sensitive dependence on the initial conditions. Many ways have been used in order to explain this last notion. Here we refer to the one by Ethan Hunt due its connection with traffic. *For instance, when you hit the brakes for a second, just tap them on the freeway, you can literally track the ripple effect of that action across a 200-mile stretch of road, because traffic has a memory*, see [1]. In other words, when considering a number of cars on a track, the behavior of one of them can be transmitted and propagated to the ones in front (and behind) of it. The mathematical models used to describe these interactions are known as *car-following* models.

The first ones were due to Greenshields [28, 29] in the 1930's. Car-following models were perfected in the 50's and 60's by taking into account considerations involved in driving a motor vehicle on a lane [17], such as the difference between the velocities of a car and the car in front of it, a distance of a car respect to the preceding one, or the driver's reaction time, see for instance [26, 39]. An interested reader can find a historical evolution of these models in [14]. Recently, these models have attracted on the interest of researchers thanks to the development of vehicle-to-vehicle (V2V) and of vehicle-to-infrastructure (V2I) communications [31]. These models contribute not only to the study the possibility of allowing vehicles to talk or communicate with each other, but also to increase the efficiency of the communication of the vehicles with the communication networks.

One of the simplest models is the *Quick-Thinking Driver* (QTD) model, which states that the acceleration of a car depends on its distance respect to the car in front of it. With just two cars, with one of them following the other, one can even find chaos relating its dynamics with certain the solutions of the the logistic equation [35]. Nevertheless, it has already been known for decades that chaotic behaviors exist in traffic flow systems. Gazis, Herman, and Rothery developed for

---

*Date:* April 18, 2019.

The first three authors were supported by MTM2013-47093-P. The first author was supported by the ERC grant HEVO n. 277691. The third author was also supported by MTM2013-47093-P and by a grant from the FPU program of MEC. The fourth author was supported by CNPq Grant 401735/2013-3 (PVE-Linha 2).

General Motors a generalized car-following model, known as the (GHR) model. The discontinuous behavior of some of their solutions and the nonlinearity presented there suggested the existence of chaotic solutions for a certain range of input parameters [27, 40]. Later on, Disbro and Frame [20] showed the presence of chaos for (GHR) model without taking into account signals, bottlenecks, intersections, etc. or with a coordinated signal network. Chaos was also observed for a platoon of vehicles described by the traditional (GHR) model modified by adding a nonlinear inter-car separation dependent term [2, 3].

However, when taking an infinite number of cars on a lane, each of them following one another, a linear simplification of these models can even show chaotic phenomena. In [15] the authors show the existence of some Devaney and distributional chaotic solutions for the *Infinite Quick-Thinking-Driver (IQTD)* model. This situation can be represented by the following infinite system of ordinary differential equations,

$$(1) \quad u'_i(t) = \lambda_i(u_{i+1}(t) - u_i(t)) \text{ for } i \in \mathbb{N},$$

in which  $u_1$  stands for the velocity of the car 1,  $u_2$  for the velocity of the car in front of car 1, namely car 2,  $t_1$  denotes the reaction time of driver 1, and the positive number  $\lambda_1$  is a sensitivity coefficient that measures how strong driver 1 responds to the acceleration of the car in front of her. Usually  $\lambda_i$  lies between  $0.3 - 0.4s^{-1}$  [14]. We also assume that the velocities at  $t = 0$ ,  $(u_i(0))_i$ , are given and belong to  $\ell^1$ . Such a behavior is obtained by relating this model with the one of gene amplification–deamplification processes with cell proliferation. These models have been widely studied by Banasiak et al. [6, 7, 8, 10] (see also [5, 18, 30]). Such a behavior can also be found on certain size structured cell populations, c.f. [22, 23] and analyzing the growth of a cell population where cellular development is characterized by cellular size [33].

We want to emphasize that the models we shall study, which are based on simple linear equations, cannot describe all the highly complex situations that, as a matter of fact, occur on a roadway every morning on the way to work. These theoretical models work better on long stretches of road with dense traffic [15]. In this note we introduce the infinite version of the *Forward and Backward Control (FBC)* model. The (FBC) model was developed by [32] for General Motors. The infinite coupled system of ordinary differential equations that is required to model the behavior of these vehicles can be represented as a linear operator on a suitable infinite-dimensional separable Banach space. Then, using some results of linear dynamics of  $C_0$ -semigroups we can prove the existence of different chaotic behaviors for the solutions of these equations.

The paper is organized as follows: In Section 2 we describe the Infinite Forward and Backward Control (IFBC) model and introduce all the preliminaries required on linear dynamics and  $C_0$ -semigroups. Its representation as a  $C_0$ -semigroup and the main results are contained in Section 3. Finally, in Section 4 we deal with the stability of certain solutions.

## 2. PRELIMINARIES

In the basic formulation of the (FBC) car-following model, there is a relation between the acceleration of a car and the speeds of the cars that go in front and behind of it. We consider this model for an infinite number of cars that circulate on a road, then the corresponding model for these cars is given by an infinite system of first-order differential equations. We will refer to it as the Infinite Forward and Backward Control (IFBC) model.

**Definition 2.1** (The (IFBC) traffic model). *Let us consider  $(u_i)_i$ , the vector of speeds for an infinite number of cars, where  $u_i$  stands for the speed of the car  $i$ ,  $u_{i+1}$  for the speed of the car*

behind the car  $i$ , and  $u_{i-1}$  for the car in front of it. The acceleration of each car  $u_i$ ,  $i \geq 2$ , is given as a linear combination of the differences of speed of the  $i$  car respect to cars  $i - 1$  and  $i + 1$ .

$$(2) \quad \begin{aligned} u_1'(t) &= -\mu_1 u_1(t) + \mu_2(u_2(t) - u_1(t)), \\ u_i'(t) &= \mu_1(u_{i-1}(t) - u_i(t)) + \mu_2(u_{i+1}(t) - u_i(t)), \text{ for all } i \geq 2. \end{aligned}$$

with control constants  $\mu_1, \mu_2 > 0$ ,  $\mu_1 < \mu_2$ .

The vector of speeds  $(u_i)_i$  will be considered in a weighted space of summable sequences. In particular, we will consider  $\ell_1(s)$ , with  $0 < s \leq 1$ , the weighted space of summable sequences defined as

$$(3) \quad \ell_1(s) = \left\{ (v_i)_{i \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|(v_i)_{i \in \mathbb{N}}\|_s = \sum_{i \in \mathbb{N}} |v_i| s^i < \infty \right\}.$$

. If  $s = 1$ , we will simply denote it as  $\ell_1$ . If  $s < 1$ , then any vector representing the velocities of all the cars in the (IFBC) model clearly belongs to  $\ell_1(s)$ . We point out that such a choice of weights gives more importance to the speeds of cars with low index  $i$ .

Let  $X$  be a separable infinite-dimensional Banach space. We assume that the reader is familiar with the terminology of  $C_0$ -semigroups on Banach spaces, see for instance [38, 25]. Next, let us recall some basic definitions on linear dynamics of  $C_0$ -semigroups. A  $C_0$ -semigroup  $\{T_t\}_{t \geq 0}$  on  $X$  is said to be *hypercyclic* if there exists  $x \in X$  such that the set  $\{T_t x : t \geq 0\}$  is dense in  $X$ . An element  $x \in X$  is called a *periodic point* for the semigroup  $\{T_t\}_{t \geq 0}$  if there exists some  $t > 0$  such that  $T_t x = x$ . A semigroup  $\{T_t\}_{t \geq 0}$  is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in  $X$ . We point out that these two requirements also yield the sensitive dependence on the initial conditions, as it was seen by Banks et al [11, 30]. Further information on the linear dynamics of  $C_0$ -semigroups can be found in [30, Ch. 7].

Sometimes these properties do not hold in the whole space but they do on a closed subspace of  $X$ . Following the terminology of [9], we say that a  $C_0$ -semigroup  $\mathcal{T} = \{T_t\}_{t \geq 0}$  is called *sub-chaotic* (*sub-hypercyclic*) if there exists a closed subspace  $\tilde{X}$  invariant under  $\mathcal{T}$ , with  $\{0\} \neq \tilde{X} \subset X$ , such that  $\tilde{\mathcal{T}} := \{T_t|_{\tilde{X}}\}_{t \geq 0}$  is chaotic (hypercyclic) as a semigroup on  $\tilde{X}$ .

Another variation of the definition of chaos is the notion of *distributional chaos* introduced by Schweizer and Smítal [41], see also [34, 37] for its presentation in the infinite-dimensional linear setting. A  $C_0$ -semigroup  $\{T_t\}_{t \geq 0}$  on  $X$  is said to be *distributionally chaotic* if there exists an uncountable subset  $S \subset X$  and  $\delta > 0$  such that, for each pair of distinct points  $x, y \in S$  and for every  $\varepsilon > 0$ , we have  $\overline{\text{Dens}}(\{s \geq 0; \|T_s x - T_s y\| > \delta\}) = 1$  and  $\overline{\text{Dens}}(\{s \geq 0; \|T_s x - T_s y\| < \varepsilon\}) = 1$ , where  $\overline{\text{Dens}}$  stands for the upper density of a set of real positive numbers. The semigroup is said to be *densely distributionally chaotic* if  $S$  is dense on  $X$ .

### 3. CHAOS FOR THE FORWARD AND BACKWARD MODEL

In order to solve the infinite system of equations in (2), we pose the following abstract Cauchy problem on  $\ell_1(s)$ :

$$(4) \quad \begin{cases} u'(t) = Au(t), \\ u(0) = (u_i(0))_{i \in \mathbb{N}}. \end{cases}$$

Here the operator  $A$  is defined as

$$(5) \quad (Au(t))_i = \begin{cases} au_i(t) + du_{i+1}(t), & i = 1, \\ bu_{i-1}(t) + au_i(t) + du_{i+1}(t), & i \geq 2, \end{cases}$$

with  $b = \mu_1$ ,  $a = -\mu_1 - \mu_2$ ,  $d = \mu_2$ ; for  $u = (u_i(t))_{i \in \mathbb{N}} \in \ell_1(s)$  for  $t \geq 0$ , being  $(u_i(0))_{i \in \mathbb{N}}$  the vector of speeds of the cars at  $t = 0$ .

The solution to (2) can be represented by a  $C_0$ -semigroup  $\{T_t\}_{t \geq 0}$  on  $\ell^1(s)$  whose infinitesimal generator is  $A$ . If  $A \in L(X)$ , then the operators in the  $C_0$ -semigroup can be represented as  $T_t = e^{tA} = \sum_{k=0}^{\infty} (tA)^k / k!$  for all  $t \geq 0$ , see for instance [25, Ch. I, Prop. 3.5].

The problem of determining Devaney chaos for the  $C_0$ -semigroup generated by  $A$  on  $\ell_1(s)$  was analysed by Banasiak and Moszynski in [10], when studying the exponential decay of the drug resistant population of cells. There,  $u_i(t)$ ,  $i \geq 1$ , stands for the number of copies of the drug resistant gene in the  $i$ -th subpopulation of cells.

**Theorem 3.1.** [10, Th. 4] *If  $0 < |b| < |d|$  and  $|a| < |b + d|$  hold, then  $\{e^{tA}\}_{t \geq 0}$  is chaotic on  $\ell_1$ .*

We will study the existence of distributional chaos for the solutions to this general model, and we will analyze, as a particular case, its consequences for the (IFBC) car-following model. In order to do this, we will use the following criterion which ensures distributional chaos and can be found in [4].

**Criterion 3.2. Dense Distributionally Irregular Manifold Criterion** *Let  $\mathcal{T} = \{T_t\}_{t \geq 0}$  be a  $C_0$ -semigroup in  $L(X)$  such that there exist a dense subset  $X_0 \subset X$  such that  $\lim_{t \rightarrow \infty} T_t x = 0$ , for each  $x \in X_0$ , and a Lebesgue measurable set  $A \subseteq [0, \infty)$  with  $\overline{\text{Dens}}(A) = 1$  satisfying*

- (i) *either  $\int_B \frac{1}{\|T_t\|} dt < \infty$ ,*
- (ii) *or  $X$  is a complex Hilbert space and  $\int_B \frac{1}{\|T_t\|^2} dt < \infty$ .*

*Then  $\mathcal{T}$  has a dense distributionally irregular manifold. In particular,  $\mathcal{T}$  is densely distributionally chaotic.*

First, let us denote by  $A_s$  the following matrix, which will permit to relocate our problem into  $\ell_1$ .

$$(6) \quad A_s = \begin{pmatrix} a & d/s & & & \\ sb & a & d/s & & \\ & sb & a & d/s & \\ & & sb & a & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let us define the linear and continuous operator  $A_s := aI + C_s$ ,  $s > 0$ , on  $\ell_1$ , where  $C_s$  is the linear and continuous operator also defined on  $\ell_1$  as

$$(7) \quad C_s = \begin{pmatrix} 0 & d/s & & & \\ sb & 0 & d/s & & \\ & sb & 0 & d/s & \\ & & sb & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

The following lemma will be helpful in the proof of our main theorem in order to compute the powers of the operator  $C_s$ .

**Lemma 3.3** (See Lemma 1 in [10]). *We have*

$$(8) \quad (C_s^k u)_n = \sum_{i=0}^k \left[ \binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left( \frac{d}{s} \right)^i u_{n-k+2i},$$

where  $u = (u_i)_i$ ,  $f_i = 0$  for  $i \leq 0$ , and the Newton symbol is also 0 for negative entries.

We will first prove that  $\mathcal{T}_s = \{e^{tA_s}\}_{t \geq 0}$  is distributionally chaotic on  $\ell_1$  and then via conjugation we will obtain the analogous result for  $\mathcal{T} = \{e^{tA}\}_{t \geq 0}$  on  $\ell_1(s)$ . We recall that the operator  $A$  can be represented by the infinite matrix

$$(9) \quad A = \begin{pmatrix} a & d & & & \\ b & a & d & & \\ & b & a & d & \\ & & b & a & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

This matrix  $A$  is tridiagonal with constant coefficients and thus it represents a bounded operator on  $\ell_1(s)$  for any  $s > 0$ . We are interested in studying distributional chaos for the (IFBC) where  $a = -\mu_1 - \mu_2 < 0$ , so we will focus our attention on the case when  $a < 0$ . The main result of this paper is the following :

**Theorem 3.4.** *The  $C_0$ -semigroup  $\mathcal{T}_s = \{e^{tA_s}\}_{t \geq 0}$  is distributionally chaotic on  $\ell_1$  for all  $s > 0$  provided that  $a, b, d, s$  satisfy:*

$$(10) \quad 0 < b < d, \quad a < 0,$$

$$(11) \quad 0 < a + bs + \frac{d}{s}.$$

Its proof is based on an application of the Dense Distributionally Irregular Manifold Criterion (Criterion 3.1). The condition on the existence of a dense set of elements whose orbits by the  $C_0$ -semigroup tend to 0 is implicit in the proof of [10, Th. 4]. This proof is a consequence of an application of Criterion 1.2 and 1.3 from [9]. Before presenting these results, let us introduce some notation.

Let  $(\Omega, \mu)$  be a measure space and  $f : \Omega \rightarrow X$ . Given a non-empty set  $U \subset \Omega$ , we denote  $\mathcal{L}(f, U) := \overline{\text{span } f(U)}$  and  $\mathcal{L}(f) := \mathcal{L}(f, \Omega)$ . If  $U$  is a measurable set in  $\Omega$  we define

$$(12) \quad \mathcal{L}_{ess}(f, U) := \bigcap_{\substack{\Omega' \subset U, \\ \mu(\Omega')=0}} \mathcal{L}(f, U \setminus \Omega'),$$

and, as before,  $\mathcal{L}_{ess}(f) = \mathcal{L}_{ess}(f, \Omega)$ .

The next criterion permits to find subspaces of chaoticity and hypercyclicity for a  $C_0$ -semigroup. We recall that  $f$  is a selection of eigenvectors  $A$  in  $\Omega$ , that means  $f(\lambda) \in \text{Dom}(A)$  and  $Af(\lambda) = \lambda f(\lambda)$  for any  $\lambda \in \Omega$ .

**Criterion 3.5.** [9, Crit. 1.2]. *Suppose that there exists a measurable subset  $I \subseteq \mathbb{R}$  and a strongly measurable selection  $f$  of eigenvectors of  $A$  on  $iI$  which is not almost everywhere equal to zero. Then*

$$(13) \quad \mathcal{L}_{ess}(f) \neq \{0\},$$

$\mathcal{T}$  is sub-hypercyclic, and  $\mathcal{L}_{ess}(f)$  is a space of hypercyclicity for  $\mathcal{T}$ .

**Criterion 3.6.** [9, Crit. 1.3]. *Suppose that  $I$  is an interval of  $\mathbb{R}$  of non-zero length and  $f$  is a weakly continuous selection of eigenvectors of  $A$  on  $iI$  which is not constantly equal to zero. Then (13) holds,  $\mathcal{T}$  is sub-chaotic, and  $\mathcal{L}_{ess}(f)$  is a space of chaoticity for  $\mathcal{T}$ . Moreover*

$$\mathcal{L}_{ess}(f) = \mathcal{L}(f).$$

Finally, let us proceed with the proof of Theorem 3.4.

*Proof of Theorem 3.4.* We denote by  $\{e_m\}_m$  the canonical basis of  $\ell_1$ . As in [13], we compute the  $\ell^1$ -norm of  $C_s^k$  acting over a sequence  $e_m$  with  $m > k$  then,

$$(14) \quad \|C_s^k e_m\|_{\ell^1} = \sum_{n=0}^{\infty} \left| \sum_{i=0}^k \left[ \binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left( \frac{d}{s} \right)^i \delta_{n-k+2i,m} \right|.$$

Since  $\delta_{n-k+2i,m} = 0$  for  $n < m - k$  or  $n > m + k$ , then,

$$(15) \quad \|C_s^k e_m\|_{\ell^1} = \sum_{n=m-k}^{m+k} \left| \sum_{i=0}^k \left[ \binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left( \frac{d}{s} \right)^i \delta_{n-k+2i,m} \right|.$$

Making  $j = k - i$  we obtain

$$(16) \quad \sum_{n=m-k}^{m+k} \left| \sum_{j=0}^k \left[ \binom{k}{j} - \binom{k}{j-n} \right] (sb)^j \left( \frac{d}{s} \right)^{k-j} \delta_{n+k-2j,m} \right|.$$

Changing, also,  $n' = n + k - m$ , we have

$$(17) \quad \sum_{n'=0}^{2k} \left| \sum_{j=0}^k \left[ \binom{k}{j} - \binom{k}{j+k-n'-m} \right] (sb)^j \left( \frac{d}{s} \right)^{k-j} \delta_{n'+m-2j,m} \right|.$$

If  $n'$  is odd,  $\delta_{n'+m-2j,m} = 0$ , then we are left with the even terms, getting

$$(18) \quad \sum_{j=0}^k \left| \left[ \binom{k}{j} - \binom{k}{k-j-m} \right] (sb)^j \left( \frac{d}{s} \right)^{k-j} \right|.$$

Since  $m > k$ , we only have

$$(19) \quad \sum_{j=0}^k \binom{k}{j} \left| (sb)^j \left( \frac{d}{s} \right)^{k-j} \right|,$$

which is  $\left( sb + \frac{d}{s} \right)^k$ . Therefore,  $\|C_s^k\| \geq \left( sb + \frac{d}{s} \right)^k$ .

With the estimates above, we can also approximate the norm of  $e^{tC_s}$  on  $L(\ell_1)$ .

$$(20) \quad \|e^{tC_s}\| = \left\| \sum_{k=0}^{\infty} \frac{(tC_s)^k}{k!} \right\|.$$

Since  $C_s$  is a positive operator, for every  $m > 0$  we have

$$(21) \quad \left\| \sum_{k=0}^{\infty} \frac{(tC_s)^k}{k!} \right\| \geq \left\| \sum_{k=0}^{m-1} \frac{(tC_s)^k}{k!} \right\| \geq \left\| \sum_{k=0}^{m-1} \frac{(tC_s)^k}{k!} e_m \right\| = \sum_{k=0}^{m-1} \frac{t^k \left( sb + \frac{d}{s} \right)^k}{k!}.$$

Therefore, taking the supremum over  $m$  we get  $\|e^{tC_s}\| \geq e^{t\left( sb + \frac{d}{s} \right)}$ , and hence

$$(22) \quad \frac{1}{\|e^{tA_s}\|} \leq \frac{1}{e^{t\left( a + sb + \frac{d}{s} \right)}}.$$

By (11) we have

$$(23) \quad \int_{\mathbb{R}^+} \frac{1}{\|e^{tA_s}\|} < \infty.$$

In order to apply Criterion 3.2 it only remains to show that there exists a dense subset  $X_0 \subset X$  such that  $\lim_{t \rightarrow \infty} T_t x = 0$  for each  $x \in X_0$ . Let us also denote  $W_0(\mathcal{T}) := \{x \in X : \lim_{t \rightarrow \infty} T_t x = 0\}$ .

The proof of Theorem 3.1 is based on Criterion 3.6. As it is indicated in [10, pg. 74] we can find a selection of eigenvectors of  $A$  defined as  $f : iS(b, d, a) \rightarrow \ell^1$ , where  $S(b, d, a)$  is the set of the values  $y \in ]c, c[$ , with  $c = \frac{|bs - \frac{d}{s}|}{|bs + \frac{d}{s}|} \sqrt{(bs + \frac{d}{s})^2 - a^2}$ , such that  $(iy - a)^2 - 4bd \neq 0$ , see also [9, p. 579-580]. Taking  $S'$  as an arbitrary non-empty connected component of  $S(b, d, a)$ , it can be seen that the set  $f(iS')$  is linearly dense in  $\ell^1$ . Therefore :

$$(24) \quad \ell^1 = \mathcal{L}(f, iS') \subseteq \mathcal{L}(f) \subseteq \ell^1.$$

Now, proceeding with the same technique as in the proof of Criteria 3.5 and 3.6 in [9], let us re-scale the selection  $f$  by defining  $\tilde{f} := \rho \cdot f$ , where  $\rho : iS' \rightarrow \mathbb{R}$  is given by  $\rho(\lambda) := [(1 + \|f(\lambda)\|)(1 + |\lambda|^2)]^{-1}$ , for  $\lambda \in iS'$ . Let us also define  $F : \mathbb{R} \rightarrow \ell^1$  as

$$(25) \quad F(t) := \int_{S'} e^{its} \tilde{f}(is) ds, \quad t \in \mathbb{R}.$$

Denote  $Y_F := \text{lin}(F(\mathbb{R}))$ . We have, by Criterion 3.6,  $\overline{Y_F} = \mathcal{L}(F)$ ,  $Y_F \subset W_0(\mathcal{T}_s)$ , and  $\mathcal{L}(F) = \mathcal{L}_{ess}(\mathbf{f})$ . Moreover we obtain that  $\mathcal{L}_{ess}(\mathbf{f}) = \mathcal{L}(\mathbf{f})$  and, hence, we have found  $Y_F$ , a set of points which orbits by the semigroup tend to 0, dense in  $\ell^1$ , that is

$$(26) \quad \overline{Y_F} = \mathcal{L}(F) = \mathcal{L}_{ess}(\mathbf{f}) = \mathcal{L}(\mathbf{f}) = \ell^1.$$

Therefore, it only remains to apply the Dense Distributionally Irregular Manifold Criterion (Criterion 3.2) and we get the conclusion.  $\square$

**Remark 3.7.** *In the proof of the previous result the existence of  $W_0$  was deduced using techniques that requires the use of complex numbers. Nevertheless, for the case when  $a, b, d$  where real numbers, the same result holds just taking the restriction of  $W_0$  to the real numbers. For further details we refer the reader to the proof of Criterion 3.2, see [4, 12, 36]. This approach can be compared with [24, Th. 3.7].*

This result can be transferred to the  $C_0$ -semigroup via the conjugation lemma.

**Corollary 3.8.** *The  $C_0$ -semigroup  $\{e^{tA_s}\}_{t \geq 0}$  is distributionally chaotic on  $\ell_1$  if and only if  $\{e^{tA}\}_{t \geq 0}$  is distributionally chaotic on  $\ell^1(s)$ .*

*Proof.* Let us define the operator  $U_s : \ell^1 \rightarrow \ell^1(s)$ , for all  $s > 0$ , as

$$(27) \quad U_s u := \left( \frac{u_n}{s_n} \right)_{n \geq 1} \quad \text{for every } u = (u_n)_n \in \ell^1.$$

This is an isometry from  $\ell_1$  onto  $\ell_1(s)$  and it holds that

$$(28) \quad A_s = U_s^{-1} \tilde{A} U_s,$$

By an application of the conjugation lemma for distributional chaos, see for instance [34, Th. 2], we get the distributional chaos for the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  from the  $C_0$ -semigroup  $\{e^{tA_s}\}_{t \geq 0}$ .  $\square$

As a consequence, we have the expected results for the weighted  $\ell^1(s)$ -spaces.

**Corollary 3.9.** *The solution  $C_0$ -semigroup of (4),  $\mathcal{T} = \{e^{tA}\}_{t \geq 0}$  is distributionally chaotic on  $\ell_1(s)$  for each  $s > 0$  provided that  $a, b, d, s$  satisfy (10) and (11).*

In particular, for the (IFBC) car-following model we have

**Corollary 3.10.** *Let  $0 < \mu_1 < \mu_2$  the coefficients of the (IFBC) model in (2). The solution  $C_0$ -semigroup to the (IFBC) model is distributionally chaotic on  $\ell_1(s)$  with  $s > 0$  provided that*

$$(29) \quad (\mu_1 + \mu_2) < (s\mu_1 + \frac{\mu_2}{s}).$$

#### 4. STUDY OF STABILITY

In this section we analyze the stability of the (IFBC) car-following model in our setting of weighted spaces of summable sequences. We recall that a  $C_0$ -semigroup of the form  $\{e^{tA}\}_{t \geq 0}$  defined on a Banach space  $X$  is *exponentially stable*, [25, p. 296], if there exists  $\varepsilon > 0$  such that

$$(30) \quad \lim_{t \rightarrow \infty} e^{\varepsilon t} \|e^{tA}\| = 0,$$

and *uniformly stable* if

$$(31) \quad \lim_{t \rightarrow \infty} \|e^{tA}\| = 0.$$

In fact, Eisner show that both notions are equivalent [21]. We recall that a vector  $x \in X$  is said to be *distributionally irregular* for the  $C_0$ -semigroup  $\mathcal{T}$  if the following holds: for every  $\delta > 0$

$$(32) \quad \limsup_{t \rightarrow \infty} \frac{\mu(\{s \in [0, t] : \|T_s x\| < \delta\})}{t} = 1,$$

$$(33) \quad \limsup_{t \rightarrow \infty} \frac{\mu(\{s \in [0, t] : \|T_s x\| \geq \delta\})}{t} = 1.$$

The existence of a distributionally irregular vector is equivalent to the existence of distributional chaos [4]. So, it is clear that under the conditions expressed in (10) and (11), the  $C_0$ -semigroup will not be exponentially stable.

Nevertheless, a weaker version of stability can also be considered, in the same way as it has been done for considering sub-chaos and sub-hypercyclicity. We say that  $\{e^{tA}\}_{t \geq 0}$  is *exponentially stable on a subspace*  $Y \subset X$  if there exists  $\varepsilon > 0$  such that for any  $y \in Y$  we have

$$(34) \quad \lim_{t \rightarrow \infty} e^{\varepsilon t} \|e^{tA} y\| = 0.$$

Such analysis has been already performed when studying chaos of  $C_0$ -semigroups in [10, 16].

**Theorem 4.1.** *The solution  $C_0$ -semigroup of the (IFBC) model where  $\mu_1, \mu_2$  and  $s_0$  satisfy conditions in corollary 3.10, is exponentially stable on the subspace  $Y_\delta := \text{span}\{y : T_t y = \mu y, \mu \in \mathbb{K}, \Re(\mu) < \delta\}$ , for every  $\delta < 0$ .*

*Proof.* Fix  $0 < \varepsilon < -\delta$  and  $y \in Y_\delta$  of the form  $y = \sum_{i=1}^k \alpha_i y_{\mu_i}$ . Define  $\delta_y = \max\{\Re(\mu_i) : 1 \leq i \leq k\}$ . Clearly,  $\varepsilon + \delta_y < 0$ , then

$$\begin{aligned} e^{\varepsilon t} \|e^{tA} y\|_s &= e^{\varepsilon t} \left\| \sum_{i=1}^k \alpha_i e^{t\mu_i} y_{\mu_i} \right\|_s \leq e^{\varepsilon t} \left( \sum_{i=1}^k e^{t\Re(\mu_i)} \|\alpha_i y_{\mu_i}\|_s \right) \\ &< e^{t(\varepsilon + \delta_y)} \left( \sum_{i=1}^k \|\alpha_i y_{\mu_i}\|_s \right), \end{aligned}$$

which tends to 0 when  $t$  tends to  $\infty$ . □

**Remark 4.2.** *This analysis of stability can be compared with the one carried out in [32], where the condition for asymptotic stability is given by*

$$(35) \quad \frac{(\mu_1 - \mu_2)^2}{\mu_1 + \mu_2} < \frac{1}{2}.$$

## 5. FINAL COMMENTS

In this note we point out that the forward and backward traffic models can be related with birth-and-death process of cell proliferation. It is fair to note that the chaotic properties reported in this paper refer to the whole-space of solutions whereas only positive solutions make sense. Therefore, being important, e.g., in the analysis of the stability of numerical schemes, does not necessarily say anything important about the actual behaviour of the physical trajectories of the cars. To sum up,

in dense traffic drivers follow one another very closely and small disturbances such as acceleration or deceleration of one vehicle might be passed over or amplified along the line of vehicles on the road. Chaotic flow appears and this can lead to an unpredictable dynamical behaviour and, in some cases, to accidents.

## REFERENCES

- [1] D. J. Abrams. *Mission: Impossible 3*. Prod. Paramount Pictures in association with Cruise/Wagner productions, 2006.
- [2] P. Addison and D. Low. A novel nonlinear car-following model. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 8(4):791–799, 1998.
- [3] P. Addison, J. McCann, D. Low, and J. Currie. Order and chaos in the dynamics of vehicle platoons. *Traffic Engineering and Control*, 37:456–459, 1996.
- [4] A. A. Albanese, X. Barrachina, E. M. Mangino, and A. Peris. Distributional chaos for strongly continuous semigroups of operators. *Commun. Pure Appl. Anal.*, 12(5):2069–2082, 2013.
- [5] J. Aroza and A. Peris. Chaotic behaviour of birth-and-death models with proliferation. *J. Difference Equ. Appl.*, 18(4):647–655, 2012.
- [6] J. Banasiak and M. Lachowicz. Chaos for a class of linear kinetic models. *C. R. Acad. Sci. Paris Série II*, 329:439–444, 2001.
- [7] J. Banasiak and M. Lachowicz. Topological chaos for birth-and-death-type models with proliferation. *Math. Models Methods Appl. Sci.*, 12(6):755–775, 2002.
- [8] J. Banasiak, M. Lachowicz, and M. Moszyński. Topological chaos: when topology meets medicine. *Appl. Math. Lett.*, 16(3):303–308, 2003.
- [9] J. Banasiak and M. Moszyński. Hypercyclicity and chaoticity spaces of  $C_0$  semigroups. *Discrete Contin. Dyn. Syst.*, 20(3):577–587, 2008.
- [10] J. Banasiak and M. Moszyński. Dynamics of birth-and-death processes with proliferation—stability and chaos. *Discrete Contin. Dyn. Syst.*, 29(1):67–79, 2011.
- [11] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney’s definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992.
- [12] T. Bermúdez, A. Bonilla, F. Martínez-Giménez, and A. Peris. Li-Yorke and distributionally chaotic operators. *J. Math. Anal. Appl.*, 373(1):83–93, 2011.
- [13] A. Bobrowski and M. Kimmel. Asymptotic behaviour of an operator exponential related to branching random walk models of DNA repeats. *Journal of Biological Systems*, 7:33–43, 1999.
- [14] M. Brackstone and M. McDonald. Car-following: a historical review. *Transportation Research Part F: Traffic Psychology and Behaviour*, 2(4):181–196, 1999.
- [15] J. A. Conejero, M. Murillo-Arcila, and J. B. Seoane-Sepúlveda. Linear chaos for the quick-thinking-driver model. *Preprint*, 2014.
- [16] J. A. Conejero, F. Rodenas, and M. Trujillo. Chaos for the hyperbolic bioheat equation. *Discrete Contin. Dyn. Syst.*, To appear in 2015.
- [17] R. Cumming. The analysis of skills in driving. *Journal of the Australian Road Research Board*, 1(9):4–14, March 1964.
- [18] R. deLaubenfels, H. Emamirad, and V. Protopopescu. Linear chaos and approximation. *J. Approx. Theory*, 105(1):176–187, 2000.
- [19] R. L. Devaney. *An introduction to chaotic dynamical systems*. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, second edition, 1989.
- [20] J. Disbro and M. Frame. Traffic flow theory and chaotic behavior. *Transportation Research Record*, 1225:109–115, 1989.
- [21] T. Eisner. *Stability of operators and operator semigroups*, volume 209 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2010.
- [22] S. El Murchid, G. Metafune, A. Rhandi, and J. Voigt. On the chaotic behaviour of size structured cell populations. *J. Math. Anal. Appl.*, 339(2):918–924, 2008.
- [23] S. El Murchid, S. Murchid, A. Rhandi, H. Vogt, and J. Voigt. A sharp condition for the chaotic behaviour of a size structured cell population. *Differential Integral Equations*, 22(7-8):797–800, 2009.

- [24] H. Emamirad, G. R. Goldstein, and J. A. Goldstein. Chaotic solution for the Black-Scholes equation. *Proc. Amer. Math. Soc.*, 140(6):2043–2052, 2012.
- [25] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [26] T. Forbes. Human factor considerations in traffic flow theory. *Highway Research Board Record*, 15:60–66, 1963.
- [27] D. C. Gazis, R. Herman, and R. W. Rothery. Nonlinear follow-the-leader models of traffic flow. *Operations Res.*, 9(4):545–567, August 1961.
- [28] B. D. Greenshields. The photographic method of studying traffic behavior. In *Proceedings of the 13th Annual Meeting of the Highway Research Board*, pages 382–399, 1934.
- [29] B. D. Greenshields. A study of traffic capacity. In *Proceedings of the 14th Annual Meeting of the Highway Research Board*, pages 448–477, 1935.
- [30] K.-G. Grosse-Erdmann and A. Peris Manguillot. *Linear chaos*. Universitext. Springer, London, 2011.
- [31] J. Harding, G. Powell, R. Yoon, J. Fikentscher, C. Doyle, D. Sade, M. Lukuc, J. Simons, and J. Wang. Vehicle-to-Vehicle communications: Readiness of V2V technology for application. Report no. dot hs 812 014, U.S. Department of Transportation, Washington, DC: National Highway Traffic Safety Administration, 2014.
- [32] R. Herman, E. W. Montroll, R. B. Potts, and R. W. Rothery. Traffic dynamics: analysis of stability in car following. *Operations Res.*, 7:86–106, 1959.
- [33] K. E. Howard. A size structured model of cell dwarfism. *Discrete Contin. Dyn. Syst. Ser. B*, 1(4):471–484, 2001.
- [34] F. Martínez-Giménez, P. Oprocha, and A. Peris. Distributional chaos for backward shifts. *J. Math. Anal. Appl.*, 351(2):607–615, 2009.
- [35] M. McCartney and S. Gibson. On the road to chaos. *Teaching Mathematics and its Applications*, 23(2):89–96, 2004.
- [36] V. Müller and J. Vršovský. Orbits of linear operators tending to infinity. *Rocky Mountain J. Math.*, 39(1):219–230, 2009.
- [37] P. Oprocha. A quantum harmonic oscillator and strong chaos. *J. Phys. A*, 39(47):14559–14565, 2006.
- [38] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [39] L. A. Pipes. An operational analysis of traffic dynamics. *J. Appl. Phys.*, 24:274–281, 1953.
- [40] R. W. Rothery. *Revised monograph on traffic flow theory*, chapter Car following models. Federal Highway Administration, update and expansion of the transportation research board (trb) special report 165, “traffic flow theory,” published in 1975 edition, 1992.
- [41] B. Schweizer and J. Smítal. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.*, 344(2):737–754, 1994.

DIPARTIMENTO DI MATEMATICA,  
UNIVERSITÀ DI ROMA 'TOR VERGATA',  
00133, ROMA, ITALY.  
*E-mail address:* `barrachi@mat.uniroma2.it`

INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA,  
UNIVERSITAT POLITÈCNICA DE VALÈNCIA,  
46022, VALÈNCIA, SPAIN.  
*E-mail address:* `aconejero@upv.es`

INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA,  
UNIVERSITAT POLITÈCNICA DE VALÈNCIA,  
46022, VALÈNCIA, SPAIN.  
*E-mail address:* `mamuar1@posgrado.upv.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,  
FACULTAD DE CIENCIAS MATEMÁTICAS,  
PLAZA DE CIENCIAS 3,  
UNIVERSIDAD COMPLUTENSE DE MADRID,  
MADRID, 28040, SPAIN.  
*E-mail address:* `jseoane@mat.ucm.es`