Cantor sets, Bernoulli shifts and linear dynamics

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Abstract Our purpose is to review some recent results on the interplay between the symbolic dynamics on Cantor sets and linear dynamics. More precisely, we will give some methods that allow the existence of strong mixing measures invariant for certain operators on Fréchet spaces, which are based on Bernoulli shifts on Cantor spaces. Also, concerning topological dynamics, we will show some consequences for the specification properties.

1 Introduction

Our framework is the study of the dynamics of a linear operator $T : X \to X$ on a metrizable and complete topological vector space (in short, F-space) X. Moreover, we will assume that X is separable.

We recall that T is said to be *hypercyclic* if there is a vector x in X such that its *orbit* $\mathrm{Orb}(x,T) = \{x, Tx, T^2x, \dots\}$ is dense in X. The recent books [8] and [20] contain the theory and most of the recent advances on hypercyclicity and linear dynamics.

Here we want to focus on some measure-theoretic properties and notions from topological dynamics. In recent years the study of the (chaotic) dynamics of linear operators has experienced a great development. This review article pretends to focus

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on the interplay between the dynamics of the Bernoulli shift on the Cantor set and the dynamics of certain operators. More precisely, we will focus on the strong specification property, which is a concept from topological dynamics, and the existence of strong mixing measures, a concern in measure-theoretic dynamics.

We also recall that a continuous map on a separable metric space is said to be chaotic in the sense of Devaney if it is topologically transitive (i.e., within our framework, it admits points with dense orbit) and the set of periodic points is dense. A notion of chaos (in the topological sense) stronger than Devaney's definition is the so called specification property. It was introduced by Bowen [12] and several versions of this property (see [27]) have been studied for the dynamics on compact metric spaces. We follow here the approach given in [4]. Some recent works on the specification properties are [23, 24, 21]. A continuous map $f : X \to X$ on a compact metric space (X,d) has the strong specification property if, for any $\delta > 0$, there is a positive integer N_δ such that for each integer $s \geq 2$, for any set $\{y_1,\dots,y_s\} \subset X$, and for every tuple of integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1,\dots,s-1$, there is a point $x \in X$ such that, for each $r \leq s$ and for all $i \in \mathbb{N} \cup \{0\}$ with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$
$$f^n(x) = x, \quad \text{where } n = N_\delta + k_s.$$

Compact dynamical systems with the specification property are mixing and Devaney chaotic, among other basic properties (see, e.g., [14]). We will consider a notion of the strong specification property for operators as it was introduced in [2].

With respect to the measure-theoretic properties, we recall that ergodic theory was introduced in the dynamics of linear operators by Flytzanis [16] and Rudnicki [25]. It was only in recent years that it deserved special attention thanks to the work of Bayart and Grivaux [5, 6]. The papers [1, 7, 9, 13, 18, 26] contain recent advances on the subject.

The notion of frequently hypercyclicity was introduced by Bayart and Grivaux [6] as a way to measure the frequency of hitting times in an arbitrary non-empty open set for a dense orbit. In [6] a first version of a Frequent Hypercyclicity Criterion was given. We will work with the corresponding formulation of Bonilla and Grosse-Erdmann [11] for operators on separable F-spaces. Another (probabilistic) version of it was given by Grivaux [17].

After a section containing some preliminaries and basic notions, we will present the results on topological dynamics concerning the strong specification property. The last section deals with the measure-theoretic dynamics of operators with the existence of strong mixing probability measures with full support. In both cases the Frequent Hypercyclicity Criterion will play a key role. At the end it will allow us to reduce the problem to the Bernoulli shift on countable unions of Cantor sets. The particular case of weighted shifts on sequence F-spaces is also presented, since we can go a bit further in this context and they serve as a typical test ground for linear dynamics.

2 Notation and preliminaries

From now on, $T : X \to X$ will be an operator defined on a separable F-space X. We introduce first the necessary notions from topological dynamics for operators.

Definition 1. An operator $T : X \to X$ on a separable F-space X has the strong specification property (SSP) if there exists an increasing sequence $(K_m)_m$ of T-invariant compact sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SSP.

Shifts on sequence spaces will be a matter of study in this paper. By a *sequence space* we mean a topological vector space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K}$. A *sequence F-space* is a sequence space that is also an F-space. Given a sequence $w = (w_n)_n$ of positive weights, the associated *unilateral* (respectively, *bilateral*) *weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \dots) = (w_2x_2, w_3x_3, \dots)$ (respectively, $B_w : \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$ is defined by $B_w(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, w_0x_0, w_1x_1, w_2x_2, \dots)$). If a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem. Chapter 4 of [20] contains more details about dynamical properties of weighted shifts on Fréchet sequence spaces.

We recall that a series $\sum_n x_n$ in X *converges unconditionally* if it converges and, for any 0-neighbourhood U in X, there exists some $N \in \mathbb{N}$ such that $\sum_{n\in F} x_n \in U$ for every finite set $F \subset \{N, N+1, N+2, \dots\}$.

The results on Devaney chaos for shift operators given in [20] (we refer the reader to [19] for the original results) remain valid for a unilateral (respectively, bilateral) weighted backward shift $T = B_w : X \to X$ on a sequence F-space X in which the canonical unit vectors $(e_n)_{n\in\mathbb{N}}$ (respectively, $(e_n)_{n\in\mathbb{Z}}$) form an unconditional basis. In particular, B_w is chaotic if, and only if, $\sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$ (respectively, $\sum_{n=-\infty}^{0} \left(\prod_{\nu=n+1}^{0} w_\nu\right) e_n + \sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$) converges unconditionally.

For a weight sequence $(v_i)_i$ the following Banach sequence spaces are considered

$$\ell^p(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p v_i \right)^{1/p} < \infty \right\},\ 1 \le p < \infty,$$

$$c_0(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \lim_{i\to\infty} |x_i| v_i = 0,\ \|x\| := \sup_i |x_i| v_i \right\}.$$

In this situation, the required condition to have $B_w : \ell^p(v) \to \ell^p(v)$ bounded is

$$\sup_{i\in\mathbb{N}} \left|w_{i+1}^p\right| \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

For a weight sequence $(v_i)_{i\in\mathbb{Z}}$ indexed on the set of integers, we consider the Banach sequence spaces

$$\ell^p(v,\mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \|x\| := \left(\sum_{i=-\infty}^{\infty} |x_i|^p v_i \right)^{1/p} < \infty \right\},\ 1 \leq p < \infty,$$

$$c_0(v,\mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \lim_{|i|\to\infty} |x_i| v_i = 0,\ \|x\| := \sup_i |x_i| v_i \right\}.$$

The condition that characterizes that $B_w : \ell^p(v,\mathbb{Z}) \to \ell^p(v,\mathbb{Z})$ is bounded is

$$\sup_{i\in\mathbb{Z}} \left| w_{i+1}^p \right| \frac{v_i}{v_{i+1}} < \infty.$$

Finally, for measure-theoretic dynamics, let $(X, \mathfrak{B}, \mu)$ be a probability space, where X is a topological space and $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X. We say that a Borel probability measure μ has *full support* if for all non-empty open set $U \subset X$ we have $\mu(U) > 0$. A measurable map $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is called a *measure-preserving* transformation if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$). T is *ergodic* if $T^{-1}(A) = A$ for certain $A \in \mathfrak{B}$ necessarily implies that $\mu(A)(1-\mu(A)) = 0$. T is said to be *strongly mixing* with respect to μ if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}),$$

and it is *exact* if given $A \in \bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B}$ then either $\mu(A)(1-\mu(A)) = 0$. We refer to [28, 15] for a detailed account on these properties.

Given $A \subset \mathbb{N}$, its *lower density* is defined by

$$\underline{\mathrm{dens}}(A) = \liminf_{n\to\infty} \frac{\mathrm{card}(A \cap [1,n])}{n},$$

An operator T is *frequently hypercyclic* (see [6]) if there exists $x \in X$ such that for every non-empty open subset U of X, the set $\{n \in \mathbb{N} \ ; \ T^n x \in U\}$ has positive lower density.

The following version of the so called Frequently Hypercyclicity Criterion given by Bonilla and Grosse-Erdmann [11] (see also Theorem 9.9 and Remark 9.10 in [20]) gives a sufficient condition for frequent hypercyclicity.

Theorem 1 ([11]). *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*
(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally, and*
(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m} x$ *if* $n > m$,

then the operator T is frequently hypercyclic.

3 Cantor sets and the Strong Specification Property

This section is devoted to a dynamical property in the topological sense, namely, the SSP. We first study the dynamics on certain Cantor sets of the backward shift on a sequence F-space X, by following an approach slightly different from the one in [2]. Later in this section we present a more general argument that yields the SSP for operators on F-spaces satisfying the Frequent Hypercyclicity Criterion.

Theorem 2. *Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then the following conditions are equivalent:*

(i) *B_w is chaotic;*
(ii) *the series*

$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

converges in X;
(iii) *B_w has a nontrivial periodic point;*
(iv) *B_w has the SSP.*

Proof. The equivalence of (i)-(ii)-(iii) was given in [19] (See also Chapter 4 of [20]). Obviously, (iv) implies (iii). For the converse, we fix a countable set $M = \{z_n\ ;\ n \in \mathbb{N}\}$ of pairwise different scalars which form a dense set in $\mathbb{K}$ with $z_1 = 0$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$ converges, there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} \alpha_k \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k \in U_{n+1}, \text{ if } \alpha_k \in \{z_1,\dots,z_n\}, \text{ for each } n \in \mathbb{N}. \qquad (1)$$

We define $A_m = \{z_1,\dots z_m\}$ for $m \in \mathbb{N}$. $A_m^{\mathbb{N}}$ is a compact space when endowed with the product topology inherited from $M^{\mathbb{N}}$, $m \in \mathbb{N}$. Now we define the map $\Phi : \bigcup_{m=1}^{\infty} A_m^{\mathbb{N}} \to X$ given by

$$\Phi((\alpha_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_k \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k.$$

Φ is well-defined and $\Phi|_{A_m^{\mathbb{N}}}$ is continuous for each $m \in \mathbb{N}$ by (1). We have that $K_m := \Phi(A_m^{\mathbb{N}})$ is a compact Cantor subset of X, invariant under B_w, and such that $B_w|_{K_m}$ is conjugated to $\sigma|_{A_m^{\mathbb{N}}}$ via Φ, for each $m \in \mathbb{N}$. It is well-known that $\sigma|_{A_m^{\mathbb{N}}}$ satisfies the SSP (see, e.g., [27]), and by conjugacy we obtain that $B_w|_{K_m}$ satisfies the SSP too, for every $m \in \mathbb{N}$. Finally, by density of M in the scalar field $\mathbb{K}$ we conclude that $\bigcup_{m\in\mathbb{N}} K_m$ is dense in X, therefore B_w satisfies the SSP. □

A similar argument, by considering the bilateral shift on the sets $A_m^{\mathbb{Z}}$, $m \in \mathbb{N}$, and the results of [19] (See also Chapter 4 of [20]), yields the analogous characterizations for bilateral weighted shifts on sequence F-spaces.

Theorem 3. *Let $B_w : X \to X$ be a bilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then the following conditions are equivalent:*

(i) *B_w is chaotic;*

(ii) *the series*

$$\sum_{n=-\infty}^{0} \Big(\prod_{\nu=n+1}^{0} w_\nu \Big) e_n + \sum_{n=1}^{\infty} \Big(\prod_{\nu=1}^{n} w_\nu \Big)^{-1} e_n$$

converges in X;

(iii) *B_w has a nontrivial periodic point;*

(iv) *B_w has the SSP.*

Particular cases on which the above results are interesting are the (unilateral and bilateral) weighted shifts on (weighted or not) ℓ^p-spaces and c_0 (see [2]). Moreover, F. Bayart and I. Z. Ruzsa [10] recently proved that weighted shift operators on ℓ^p, $1 \le p < \infty$, are frequently hypercyclic if, and only if, they are Devaney chaotic. This fact adds a new equivalence for ℓ^p-spaces.

Corollary 1. *For a bounded weighted backward shift operator B_w defined on $X = \ell^p(v)$, $1 \le p < \infty$, (respectively, on $X = c_0(v)$) the following conditions are equivalent:*

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} \left| w_j^p \right|} < \infty$ *(respectively,* $\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j|} = 0$*),*

(ii) *B_w has SSP.*

(iii) *B_w is Devaney chaotic.*

Moreover, for $X = \ell^p(v)$, $1 \le p < \infty$, the above items are equivalent to

(iv) *B_w is frequently hypercyclic.*

Corollary 2. *For a bounded bilateral weighted backward shift operator B_w defined on $\ell^p(v,\mathbb{Z})$, $1 \le p < \infty$, (respectively, on $c_0(v,\mathbb{Z})$) the following conditions are equivalent:*

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} \left| w_j^p \right|} < \infty$ *and* $\sum_{i=1}^{\infty} \prod_{j=0}^{-i+1} \left| w_j^p \right| v_{-i} < \infty$

(respectively, $\displaystyle\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j|} = \lim_{i\to\infty} \prod_{j=0}^{-i+1} |w_j| \, v_{-i} = 0$*).*

(ii) *B_w has SSP.*

(iii) *B_w is Devaney chaotic.*

Moreover, for $X = \ell^p(v,\mathbb{Z})$, $1 \le p < \infty$, the above items are equivalent to

(iv) B_w *is frequently hypercyclic.*

In the previous results we deduced the SSP from the dynamics on certain invariant Cantor sets. Something similar can be done under the more general assumptions that the operator satisfies the Frequent Hypercyclicity Criterion (see [3]). The idea now is to work with certain invariant factors of Cantor sets. We will take the general version of the Frequent Hypercyclicity Criterion given in [20, Remark 9.10].

Theorem 4. *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*
(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally, and*
(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m} x$ *if* $n > m$,

then the operator T satisfies the SSP.

Proof. We suppose that $X_0 = \{x_n \; ; \; n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \text{ if } m_k \in \{1,\dots,n\}, \text{ for each } n \in \mathbb{N}. \quad (2)$$

We let $B_m = \{1,\dots,m\}$ and define the map $\Phi : \bigcup_{m=1}^{\infty} B_m^{\mathbb{Z}} \to X$ given by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}.$$

Φ is well-defined and $\Phi|_{B_m^{\mathbb{Z}}}$ is continuous for each $m \in \mathbb{N}$ by (2). As in Theorem 2 we have that $K_m := \Phi(B_m^{\mathbb{Z}})$ is a compact subset of X, invariant under the operator T, and such that $T|_{K_m}$ is conjugated to $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$ via Φ, for each $m \in \mathbb{N}$. Again, by conjugacy, we obtain that $T|_{K_m}$ satisfies the SSP too, for every $m \in \mathbb{N}$ and, since $\bigcup_{m\in\mathbb{N}} K_m$ is dense in X because it contains X_0, we conclude that T satisfies the SSP. □

4 Mixing measures and Bernoulli shifts

For the existence of strong mixing measures with full support, certain Cantor subsets of $\mathbb{N}^N$, with either $N = \mathbb{N}$ or $N = \mathbb{Z}$, will be needed. This time we precise some finer adjustments that involve Cantor sets larger than A^N with A finite. Actually they will be of the form $C = \prod_{n\in N} A_n$, where the cardinalities of the finite sets A_n tend to infinity as $n \to \infty$. The problem now is that these type of sets are not invariant under the shift, so actually we will use invariant sets that are countable unions of

Cantor sets. The contents of this section are based on [22]. We also recall that, recently, Bayart and Matheron gave very general conditions expressed on eigenvector fields associated to unimodular eigenvalues under which an operator T admits a T-invariant mixing measure [9].

In all the cases that we treat in this section the idea is to construct a model probability space $(Z,\overline{\mu})$ and a (Borel) measurable map $\Phi : Z \to X$, where

(a) $Z \subset \mathbb{N}^N$ is such that $\sigma(Z) = Z$ for the Bernoulli shift (either unilateral or bilateral),
(b) $\overline{\mu}$ is a σ-invariant strongly mixing measure,
(c) $Y := \Phi(Z)$ is a T-invariant dense subset of X, and
(d) the operator T is either (quasi)conjugated to σ or to σ^{-1} through Φ.

This way, the measure $\overline{\mu}$ induces by (quasi)conjugacy a Borel probability measure μ on X that is T-invariant and strongly mixing.

The model probability space $(Z,\overline{\mu})$.

We will construct $Z \subset \mathbb{N}^N$, invariant under the shift, where $N = \mathbb{N}$ or $N = \mathbb{Z}$. In all the cases we are given certain increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$. We define the compact space $L = \prod_{k\in N} A_k$ where

$$A_k = \{1,\dots m\} \text{ if } N_m < |k| \le N_{m+1},\ m \in \mathbb{N}, \text{ and } A_k = \{1\},\ \text{if } |k| \le N_1.$$

Let $L(s) := \sigma^s(L)$, $s \in N \cup \{0\}$. $L(s)$ is a subspace of $\mathbb{N}^N$, $s \in N \cup \{0\}$.

In $\mathbb{N}^N$ we define the product probability measure $\overline{\mu} = \bigotimes_{k\in N} \overline{\mu_k}$, where $\overline{\mu_k}(\{n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(\mathbb{N}) = \sum_{n=1}^{\infty} p_n = 1$, $k \in N$. The numbers $p_n \in]0,1[$, $n \in \mathbb{N}$, are so that, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j},\ j \in \mathbb{N},\ \text{then } \prod_{j=1}^{\infty} \beta_j > 0.$$

We define $Z = \bigcup_{s\in N} L(s)$, which is a countable union of Cantor sets, invariant under the shift, and satisfies

$$\overline{\mu}(Z) \ge \overline{\mu}(L) = \prod_{|k|\le N_1} \overline{\mu}_k(\{1\}) \prod_{l=1}^{\infty} \left(\prod_{N_l<|k|\le N_{l+1}} \overline{\mu}_k(\{1,\dots,l\}) \right) \ge p_1^{2N_1+1} (\prod_{l=1}^{\infty} \beta_l)^2 > 0.$$

By [28] we know that $\overline{\mu}$ is a σ-invariant strongly mixing Borel probability measure on $\mathbb{N}^N$. Since $\sigma(Z) = Z$, it has positive measure, and every strong mixing measure is ergodic, we necessarily have that $\overline{\mu}(Z) = 1$. Even more, in the case $N = \mathbb{N}$ it is known (see [28, §4.12]) that $\overline{\mu}$ is a σ-invariant exact Borel probability measure.

Once we have defined the model, as in the previous section we first analyze the situation for unilateral shifts.

Theorem 5. *Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. If B_w is chaotic then there exists a T-invariant exact Borel probability measure on X with full support.*

Proof. As in the proof of Theorem 2, we fix a countable set $M = \{z_n \; ; \; n \in \mathbb{N}\}$ of pairwise different scalars which form a dense set in $\mathbb{K}$ with $z_1 = 0$, and a basis $(U_n)_n$ of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$.

Again, although not so immediate, the fact that $\sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$ converges, implies the existence of an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} \alpha_k \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k \in U_{n+1}, \text{ if } \alpha_k \in \{z_1, \ldots, z_{2m}\}, \text{ for } N_m < k \le N_{m+1},\ m \ge n. \tag{3}$$

We set $Z = \bigcup_{s\ge 0} L(s) \subset \mathbb{N}^{\mathbb{N}}$ the σ-invariant set and the exact probability measure $\overline{\mu}$ considered in the model above, and we define the map $\Phi : Z \to X$ given by

$$\Phi((n(k))_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_{n(k)} \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k.$$

Φ is well-defined and $\Phi|_{L(s)}$ is continuous for each $s \ge 0$ by (3). We also have that $Y := \Phi(Z)$ is a countable union of Cantor subsets of X, invariant under B_w, and such that $B_w|_Y$ is conjugated to $\sigma|_Z$ via Φ. Since we know that $\overline{\mu}$ is exact on Z, the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is well-defined on X, it is B_w-invariant and exact, so it only remains to show that it has full support. Indeed, given a non-empty open set U in X, we pick $y = \Phi((n(k))_k) \in Y$ satisfying $y + U_n \subset U$. Thus

$$\begin{aligned} &\mu(U) \\ &\ge \mu\Big(\Big\{x = y + \sum_{k>N_n} \frac{\alpha_k}{\prod_{\nu=1}^{k} w_\nu} e_k \; ; \; \alpha_k \in \{z_1, \ldots, z_{2m}\}, \text{ for } N_m < k \le N_{m+1},\ m \ge n\Big\}\Big) \\ &\ge \prod_{k=1}^{N_n} \overline{\mu}_k(\{n(k)\}) \prod_{l=n}^{\infty} \Bigg(\prod_{N_l<k\le N_{l+1}} \overline{\mu}_k(\{1, \ldots, 2l\}) \Bigg) > \prod_{k=1}^{N_n} p_{n(k)} \Big(\prod_{l=n}^{\infty} \beta_l\Big)^2 > 0, \end{aligned}$$

by (3) and by the selection of the sequence $(p_n)_n$ in the model. Therefore we conclude the result. □

As in the previous section, a general result can be obtained for operators satisfying the Frequent Hypercyclicity Criterion.

Theorem 6. *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,
(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and
(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

Proof. As in Theorem 4 we suppose that $X_0 = \{x_n\ ;\ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$, and $(U_n)_n$ is a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. Again, with a more subtle argument, by (i) and (ii) we have the existence of an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \text{ if } m_k \leq 2l, \text{ for } N_l < k \leq N_{l+1},\ l \geq n. \tag{4}$$

Actually, this is a consequence of the completeness of X and the fact that, for each 0-neighbourhood U and for all $l \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\sum_{k\in F} T^k x \in U$ and $\sum_{k\in F} S_k x \in U$ for any finite subset $F \subset]N, +\infty[$ and for each $x \in \{x_1, \ldots, x_{2l}\}$.

We fix now our model space $(Z, \overline{\mu})$ with $Z = \bigcup_{s\in\mathbb{Z}} L(s) \subset \mathbb{N}^{\mathbb{Z}}$ and the measure $\overline{\mu}$ associated with the increasing sequence $(N_n)_n$.

1.-The map Φ.

We define the map $\Phi : Z \to X$ by

$$\Phi((n(k))_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n(k)} + x_{n(0)} + \sum_{k>0} T^k x_{n(k)}. \tag{5}$$

Φ is well-defined since, given $(n(k))_{k\in\mathbb{Z}} \in L(s)$, and for $l \geq |s|$, we have $n(k) \leq 2l$ if $N_l < |k| \leq N_{l+1}$, which shows the convergence of the series in (5) by (4).

$\Phi|_{L(s)}$ is also continuous for each $s \in \mathbb{Z}$. Indeed, let $(\gamma_j)_j$ be a sequence of elements of $L(s)$ that converges to $\gamma \in L(s)$ and fix any $n \in \mathbb{N}$ with $n > |s|$. We will find $n_0 \in \mathbb{N}$ such that $\Phi(\gamma_j) - \Phi(\gamma) \in U_n$ for $n \geq n_0$. To do this, by definition of the topology of $L(s)$ there exists $n_0 \in \mathbb{N}$ such that

$$\gamma(k)_j = \gamma(k) \text{ if } |k| \leq N_{n+1} \text{ and } j \geq n_0.$$

By (4) we have

$$\Phi(\gamma_j) - \Phi(\gamma) = \sum_{k<-N_{n+1}} S_{-k}(x_{\gamma(k)_j} - x_{\gamma(k)}) + \sum_{k>N_{n+1}} T^k (x_{\gamma(k)_j} - x_{\gamma(k)}) \in U_n$$

for all $j \geq n_0$. This shows the continuity of $\Phi : L(s) \to X$ for every $s \in \mathbb{Z}$.

The map $\Phi : Z \to X$ is then measurable (i.e., $\Phi^{-1}(A) \in \mathfrak{B}(Z)$ for every $A \in \mathfrak{B}(X)$).

2.-The measure μ on X.

$Y := \Phi(Z)$ is a T-invariant Borel subset of X because it is a countable union of compact sets and $\Phi\sigma^{-1} = T\Phi$. As before, the measure μ on X is defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$, which is well-defined, T-invariant, and a strongly mixing Borel probability measure. As in Theorem 5, it only remains to show that μ has full support. Indeed, given a non-empty open set U in X, we fix $n \in \mathbb{N}$ such that $x_n + U_n \subset U$. Therefore,

$$\begin{aligned}
&\mu(U) \\
&\geq \mu(\{x = x_n + \sum_{k>N_n} T^k x_{m(k)} + \sum_{k>N_n} S_k x_{m(-k)} \; ; \; m(k) \leq 2l \text{ for } N_l < |k| \leq N_{l+1},\, l \geq n\}) \\
&\geq \overline{\mu}_0(\{n\}) \prod_{0<|k|\leq N_n} \overline{\mu}_k(\{1\}) \prod_{l=n}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,2l\}) \right) > p_n p_1^{2N_n} (\prod_{l=n}^{\infty} \beta_l)^2 > 0,
\end{aligned}$$

and we obtain that μ has full support. □

As a consequence, since every chaotic bilateral shift on a sequence F-space in which the natural basis $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis satisfies the Frequent Hypercyclicity Criterion (see, e.g., Chapter 9 in [20]), we obtain strong mixing measures for these shifts.

Corollary 3. *Let $T : X \to X$ be a chaotic bilateral weighted shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.*

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