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On proportional hybrid operators in the discrete setting

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In this article, we introduce a new nonlocal operator H^{α} defined as a linear combination of the discrete fractional Caputo operator and the fractional sum operator. A new dual operator R^{α} is also introduced by replacing the discrete fractional Caputo operator with the discrete fractional Riemann–Liouville operator. It is shown that it corresponds to a natural discretization of a proportional hybrid operator defined by the Riemann–Liouville operator instead of Caputo hybrid operator. We then analyze the most important properties of these operators, such as their inverse operator and the Z-transform, among others. As an application, we solve difference equations equipped with these operators and obtain explicit solutions for them in terms of trivariate Mittag-Leffler sequences.

KEYWORDS
fractional Caputo and Riemann–Liouville operators, Laplace, discrete Laplace, and Poisson transformation, Mittag-Leffler functions and sequences, proportional hybrid operators, Toeplitz operators

MSC CLASSIFICATION
47B39, 47A62, 47B35, 33E12

1 | INTRODUCTION

A few years ago, Baleanu et al. [1] defined a new fractional operator convolving the constant proportional derivative

$$^{CP}D_t^{\alpha} f(t) := K_1(\alpha) f(t) + K_0(\alpha) f'(t), \tag{1.1}$$

with the kernel

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{1.2}$$

where $0 < \alpha < 1$. In (1.1), K_0 and K_1 are functions of the parameter α that satisfy the following conditions:

$$\lim_{\alpha\to 0} K_0(\alpha) = 0,\ \lim_{\alpha\to 1} K_0(\alpha) = 1,\ K_0(\alpha) \neq 0;$$
$$\lim_{\alpha\to 0} K_1(\alpha) = 1,\ \lim_{\alpha\to 1} K_1(\alpha) = 0,\ K_1(\alpha) \neq 0.$$

The new type of fractional operator was called the constant proportional Caputo hybrid operator, and it can be seen as a linear combination of the Riemann–Liouville integral $J^{1-\alpha}$ and the Caputo derivative ${}^C D_t^\alpha$, as follows:

$$^{CPC}D_t^\alpha f(t) := K_1(\alpha) J_t^{1-\alpha} f(t) + K_0(\alpha)\, {}^C D_t^\alpha f(t). \tag{1.3}$$

This operator has been proven to be very useful in applications and is today the subject of intense research; see below for a bibliographic discussion. However, as far as we know, there is no discrete counterpart of (1.3). Our objective in this article is to contribute to filling this important gap.

To start with, we observe that discrete counterpart of the Caputo derivative using convolution as part of the definition can be found in a number of articles, see, for example, [2, 3], as follows:

$$_C\Delta^\alpha(u)(n) := (k^{1-\alpha} * \Delta u)(n),\ n \in \mathbb{N}_0, \tag{1.4}$$

where $\Delta v(n) := v(n+1) - v(n)$ is the forward Euler operator and

$$k^\beta(n) := \frac{\Gamma(\beta+n)}{\Gamma(\beta) n!},\ \beta > 0,\ n \in \mathbb{N}_0, \tag{1.5}$$

see [3].

It happens that at the basis of the definition (1.4), there is a bounded linear operator $\mathcal{P} : L^1(\mathbb{R}_+) \to \ell^1(\mathbb{N}_0)$, called the Poisson transformation [4], which relates (1.2) to (1.5) by the following relevant identity:

$$\mathcal{P}(g_\alpha)(n) = k^\alpha(n),\ n \in \mathbb{N}_0, \tag{1.6}$$

see [3]. Moreover, it is not difficult to check that

$$\mathcal{P}(f')(n+1) = \Delta\mathcal{P}(f)(n),\ n \in \mathbb{N}_0, \tag{1.7}$$

for sufficiently regular functions f. Taking into account (1.6) and (1.7) and taking the Poisson transformation to (1.3), we arrive at the following discrete counterpart of the constant proportional Caputo hybrid operator:

$$H^\alpha(u)(n) := K_1(\alpha)\Delta^{-(1-\alpha)}(u)(n+1) + K_0(\alpha)\, {}_C\Delta^\alpha(u)(n),\ n \in \mathbb{N}_0,$$

where $\Delta^{-(1-\alpha)}u := k^{1-\alpha} * u$ is the fractional sum of order α, that is, the discrete analog for the Riemann–Liouville integral.

However, after examining some properties of H^α, and as a consequence of our analysis in this article, we will conclude that from the point of view of Poisson transformation, it gives more sense that a constant proportional hybrid operator should be defined using the Riemann–Liouville derivative instead of the Caputo derivative. Because of this, we also introduce in this article the following definition of constant proportional Riemann–Liouville hybrid operator in the discrete setting:

$$R^\alpha(u)(n) := K_1(\alpha)\Delta^{-(1-\alpha)}u(n+1) + K_0(\alpha)\Delta^\alpha(u)(n),\ n \in \mathbb{N}_0,$$

where $\Delta^\alpha(u)(n) := \Delta(k^{1-\alpha} * u)(n)$, and we prove that

$$\mathcal{P}\left({}^{PRL}D_t^\alpha f\right)(n+1) = R^\alpha(u)(n), \qquad n \in \mathbb{N}_0,$$

being

$$^{PRL}D_t^\alpha f(t) := K_1(\alpha)J_t^{1-\alpha} f(t) + K_0(\alpha)D_t^\alpha f(t),$$

the Riemann–Liouville counterpart of (1.3). To relate R^α and H^α, we find the following interesting identity:

$$R^\alpha u(n) = H^\alpha u(n) + K_0(\alpha)k^{1-\alpha}(n+1)u(0),\ n \in \mathbb{N}_0.$$

Before describing the main sections and contents of this work, we will briefly summarize some of the relevant literature related to the constant proportional Caputo hybrid operator, after the appearance of the seminal work [1]. We also refer the reader to monographies [5, 6] for more information about fractional calculus.

In [7], the constant proportional Caputo hybrid operator is used for a comparative study with other fractional order operators in the analysis of solutions of economic models based on market equilibrium. In the article [8], the same operator is applied to the heat transfer of clay–water base nanofluids over an infinite vertical surface moving with constant velocity. In such article, the authors show that this operator is better at exhibiting decay of the velocity of the fluid. An analogous analysis was conducted in the article [9] for an unsteady and incompressible MHD viscous fluid model with heat transfer. The authors conclude that the constant proportional Caputo hybrid operator is the best choice to get the controlled velocity and temperature profiles for different values of the parameter α. In the article [10], the authors analyzed an HIV model, obtaining its solution and simulation results for the operators ${}^{C}D_t{}^\alpha$ and ${}^{CPC}D_t{}^\alpha$. In [11], investigating the calcium signaling in cardiac myocytes, a mathematical model for anomalous subdiffusion using the operator ${}^{CPC}D_t{}^\alpha$ is analyzed. In [12], natural convection and slippage flow of Newtonian SWNTs nanofluid with radiation, heat generation, and chemical reaction in the presence of magnetic force through a porous media subject to the Newtonian heating is analyzed by utilizing the ${}^{CPC}D_t{}^\alpha$ operator. From another point of view, Abbas [13] studies Ulam–Hyers stability for linear and nonlinear equations involving the operator ${}^{CPC}D_t{}^\alpha$, and Ibrahim and Baleanu [14] deal with the extension of the constant proportional Caputo hybrid operator to the complex domain and its generalization by using quantum calculus. Other recent references involving the proportional hybrid operator are [15–23], the list being not exhaustive.

The paper is organized as follows. In Section 2, we present some basic notions and results that will be necessary throughout this article, including the definition of the Poisson transformation and its main properties. In Section 3, we introduce the Caputo hybrid difference operator H^α and carry out an in-deep study of it to find its inverse operator and its Z-transform. See Theorems 3.10 and 3.11, respectively.

As an illustration, we consider two examples of difference equations with the operator H^α that can be solved after introducing the notion of trivariate Mittag-Leffler sequence, see formula (3.32). To introduce such a notion, we previously realize the surprising fact that the bivariate Mittag-Leffler sequence, recently defined by Mohammed et. al. [24], is the Poisson transformation of the bivariate Mittag-Leffler function defined by Fernandez et al. [25]. See Theorem 3.12 below for a precise description of these results.

In Section 4, we define and perform a similar study for the Riemann–Liouville hybrid difference operator R^α. We show the notable fact that this new operator is the Poisson transformation of the constant proportional Riemann–Liouville hybrid operator, see Definition 4.1 and Theorem 4.4. Furthermore, we prove that R^α is a Toeplitz operator, see Theorem 4.4, in contrast to H^α which does not have this property. Finally, we illustrate our results by considering two difference equations endowed with R^α, obtaining explicit solutions and comparing them with the continuous case. Once again, these examples illustrate the fact that R^α is closer to being a consistent notion of proportional hybrid operator for the discrete setting, see our final Remark 4.10.

2 | PRELIMINARIES

Let denote $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$. As usual, $*$ denotes the convolution of two functions on $\mathbb{R}_+$ as

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)d\tau,\ f,g \in L^1(\mathbb{R}_+).$$

Definition 2.1. The Caputo fractional derivative of order $0 < \alpha < 1$ of a differentiable function f is given by

$$^{C}D_t^\alpha f(t) := (g_{1-\alpha} * f')(t),\ t > 0. \tag{2.1}$$

It can be easily seen that the Caputo fractional derivative can be defined in terms of the fractional integral defined by

$$J_t^{\beta} f(t) := (g_{\beta} * f)(t),\ t > 0,$$

for $\beta > 0$ and f a differentiable function. We now recall the notion of Riemann–Liouville fractional derivative, see [26].

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ of a function $f \in L^1(0,T), T > 0$ such that $g_{1-\alpha} * f$ is differentiable, which is given by

$$D_t^{\alpha} f(t) := \frac{d}{dt}(g_{1-\alpha} * f)(t),\ t > 0.$$

In [1], the authors introduced the proportional Caputo hybrid operator as a linear combination of the Riemann–Liouville integral and the Caputo derivative as follows:

$$^{CPC}D_t^{\alpha} f(t) = K_1(\alpha) J_t^{1-\alpha} f(t) + K_0(\alpha)\, {}^{C}D_t^{\alpha} f(t),\ t > 0, \tag{2.2}$$

where $K_0(\alpha)$ and $K_1(\alpha)$ are functions of the variable $\alpha \in (0,1]$ that satisfy

$$\lim_{\alpha \to 0^+} K_0(\alpha) = 0,\ \lim_{\alpha \to 1^-} K_0(\alpha) = 1,\ K_1(\alpha) \neq 0 \text{ and} \tag{2.3}$$

$$\lim_{\alpha \to 0^+} K_1(\alpha) = 1,\ \lim_{\alpha \to 1^-} K_1(\alpha) = 0,\ K_0(\alpha) \neq 0. \tag{2.4}$$

The prototypical case is $K_0(\alpha) := \alpha$ and $K_1(\alpha) := 1 - \alpha$.

This operator is a generalization of the constant proportional operator defined in [27] and given by

$$^{CP}D_t^{\alpha} f(t) = K_1(\alpha) f(t) + K_0(\alpha) f'(t).$$

We now recall some basic definitions from discrete fractional calculus. Let $s(\mathbb{N}_0;\mathbb{C})$ denote the vector space consisting of all complex-valued sequences $u : \mathbb{N}_0 \to \mathbb{C}$. We recall that the forward Euler operator $\Delta : s(\mathbb{N}_0;\mathbb{C}) \to s(\mathbb{N}_0;\mathbb{C})$ is defined by

$$\Delta(u)(n) := f(n+1) - f(n),\ n \in \mathbb{N}_0.$$

We also recall the concept of discrete convolution, denoted also by the symbol $*$, of two sequences $f, g \in s(\mathbb{N}_0)$:

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j) g(j),\ n \in \mathbb{N}_0.$$

We define the following special kernel.

Definition 2.3. For any $\alpha \in \mathbb{R}$, we define

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)},\ j \in \mathbb{N}_0,\ \alpha \in \mathbb{R}\backslash\{0, -1, -2, \dots\},$$

and $k^0(n) = \delta_0(n)$, $k^{-j}(n) = (\delta_0(n) - \delta_1(n))^{*j},\ n \in \mathbb{N}_0,\ j = 1, 2, \dots$, where $\delta_k(n)$ is the Kronecker delta, and the symbol $\bullet^{*j}$ denotes (discrete) convolution j-times.

For instance, $k^{-1}(n) = \delta_0(n) - \delta_1(n)$ and $k^{-2}(n) = \delta_0(n) - 2\delta_1(n) + \delta_2(n),\ n \in \mathbb{N}_0$.

Remark 2.4. The sequence $k^{\alpha}(n)$ originates from the generation formula:

$$\sum_{j=0}^{\infty} k^{\alpha}(j) z^j = \frac{1}{(1-z)^{\alpha}},$$

valid for all $z \in \mathbb{C}$ except in the case $\alpha > 0$ where it is valid only for $|z| < 1$. Note that this, in passing, justifies the definition of the values $k^{-j}(n)$ for $j = 0, 1, 2, \ldots$. This remarkable sequence has been obtained and studied by several authors, see, for example, [2, 28, 29] for an overview.

We next recall the concept of Z-transform of a sequence $f \in s(\mathbb{N}_0; \mathbb{C})$, which is defined by

$$\hat{f}(z) := \sum_{j=0}^{\infty} z^{-j} f(j)$$

where z is a complex number. Note that convergence of the series is given for $|z| > R$ with R sufficiently large. It is well known [2] that

$$\widehat{k^{\alpha}}(z) = \frac{z^{\alpha}}{(z-1)^{\alpha}}, \ \alpha \in \mathbb{R}, \tag{2.5}$$

for $|z| > 1$.

We define $\Delta^0 = I$. The following definition of fractional sum was originally introduced in [3], after the previous work by Atici, Eloe, and Abdeljawad (see [30–32]).

Definition 2.5. Let $\alpha > 0$ be given and $u : \mathbb{N}_0 \to \mathbb{C}$. We define the fractional sum of order α as follows:

$$\Delta^{-\alpha}(u)(n) = (k^{\alpha} * u)(n) = \sum_{j=0}^{n} k^{\alpha}(n-j)u(j), \ n \in \mathbb{N}_0. \tag{2.6}$$

Now, we recall from [3] the discrete analogous concept to the definition of a fractional derivative in the sense of Riemann–Liouville, see also [31]. In that paper, it is shown their strong connection, by means of the Poisson transformation, with the Riemann–Liouville fractional derivative on $\mathbb{R}_+$. We recall that the Poisson transformation of a function $u : L^1(\mathbb{R}_+) \to \ell^1(\mathbb{N}_0)$ is defined by

$$\mathcal{P}(u)(n) := \int_0^{\infty} p_n(t)u(t)dt, \ n \in \mathbb{N}_0,$$

where $p_n(t) := e^{-t}\frac{t^n}{n!}$, see [3] and [4, Section 4] for their main properties. Also, we recall from [3] that the following important property holds:

$$k^{\alpha}(n) = \int_0^{\infty} p_n(t)g_{\alpha}(t)dt, \ n \in \mathbb{N}_0, \ \alpha > 0, \ n \in \mathbb{N}_0, \tag{2.7}$$

and from [3, Theorem 3.1], the following remarkable result

$$\widehat{\mathcal{P}(f)}(z) = \mathcal{L}(f)\left(\frac{z-1}{z}\right), \ |z| > 1, \tag{2.8}$$

where $\mathcal{L}$ denotes the Laplace transform of $f : \mathbb{R}_+ \to \mathbb{R}$. This result allows to calculate easily the Z-transform of some sequences.

For example, for the function $h(t) = e^{-at}$ where $a \in \mathbb{R}$, it is well known that $\mathcal{L}(h)(s) = \frac{1}{a+s}$. On the other hand, it is easy to see that

$$\mathcal{P}(h)(n) = \left(\frac{1}{a+1}\right)^{n+1}, \ n \in \mathbb{N}_0.$$

Therefore, we conclude from (2.8) that for the sequence $\gamma(n) := \left(\frac{1}{a+1}\right)^{n+1}$, the following identity holds

$$\hat{\gamma}(z) = \frac{1}{a+s} = \frac{1}{a+1}\frac{z}{\left(z - \frac{1}{a+1}\right)}. \tag{2.9}$$

Definition 2.6. The fractional difference operator of order $\alpha > 0$ (in the sense of Riemann–Liouville) is defined by

$$\Delta^{\alpha}(u)(n) := \Delta^{m}(\Delta^{-(m-\alpha)}u)(n), \ n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

In other words, to a given complex-valued sequence, first fractional summation and then integer difference are applied.

Definition 2.7. Let $\alpha > 0$. The αth fractional Caputo-like difference operator is defined by

$$ {}_C\Delta^{\alpha}(u)(n) := \Delta^{-(m-\alpha)}(\Delta^m u)(n), \ n \in \mathbb{N}_0. \tag{2.10}$$

For further use, we note the following relation between the Caputo and Riemann–Liouville fractional difference operators of order $0 < \alpha < 1$ which can be found in [2, Theorem 2.9].

Theorem 2.8. *Let $0 < \alpha < 1$. Then, the following assertion holds for every $u \in s(\mathbb{N}_0, \mathbb{C})$:*

$$ {}_C\Delta^{\alpha}(u)(n) = \Delta^{\alpha}(u)(n) - k^{1-\alpha}(n+1)u(0), \ n \in \mathbb{N}_0.$$

3 | THE CAPUTO HYBRID DIFFERENCE OPERATOR

We introduce the corresponding analog of the operator defined in (2.2).

Definition 3.1. Let $0 < \alpha < 1$ be given and $u : \mathbb{N}_0 \to \mathbb{C}$. The Caputo hybrid difference operator $H^{\alpha} : s(\mathbb{N}_0; \mathbb{C}) \to s(\mathbb{N}_0; \mathbb{C})$ is defined by

$$H^{\alpha}(u)(n) := K_1(\alpha)\Delta^{-(1-\alpha)}(u)(n+1) + K_0(\alpha)\,{}_C\Delta^{\alpha}(u)(n), \ n \in \mathbb{N}_0,$$

where $K_0(\alpha)$ and $K_1(\alpha)$ are functions that satisfy (2.3) and (2.4).

Remark 3.2. An easy calculation shows that ${}_C\Delta^{\alpha}(u)(0) = u(1) - u(0)$ and $\Delta^{-(1-\alpha)}(u)(1) = (1-\alpha)u(0) + u(1)$. Hence, the value of $H^{\alpha}(u)(0)$ depends on $u(0), u(1)$ and α to be given, in general.

Remark 3.3. We note that for $K_0(\alpha) = \alpha$ and $K_1(\alpha) = 1 - \alpha$, we obtain

$$H^{\alpha}(u)(0) = (1 - 3\alpha + \alpha^2)u(0) + u(1),$$

where $\alpha^2 - 3\alpha + 1 = 0$ for $\alpha_0 := \frac{3-\sqrt{5}}{2} \in (0,1)$. Therefore, $H^{\alpha_0}(u)(0) = u(1)$, that is, the value of $H^{\alpha}(u)(0)$ could eventually depend only of $u(1)$ and α.

Remark 3.4. Observe that, formally

$$\lim_{\alpha \to 1^-} H^{\alpha}(u)(n) = u(n+1) - u(n) =: \Delta(u)(n) \text{ and}$$

$$\lim_{\alpha \to 0^+} H^{\alpha}(u)(n) = \Delta^{-1}(u)(n+1) = \sum_{j=0}^{n+1} u(j),$$

and therefore, H^{α} interpolates between these two limits.

Given $0 < \alpha \le 1$, we will now introduce the following operator.

Definition 3.5. The operator

$$H_b^{\alpha}(u)(n) := K_1(\alpha)u(n+1) + K_0(\alpha)\Delta(u)(n), \ n \in \mathbb{N}_0,$$

is called the *proportional difference operator*.

The following property of proportional difference operators will be useful.

Lemma 3.6. *For all $p \in \mathbb{N}$, we have*

$$\tau_p \circ H_b^{\alpha} = H_b^{\alpha} \circ \tau_p, \tag{3.1}$$

where τ_p denotes the translation operator given by $(\tau_p u)(n) := u(n+p)$.

Proof. For all $n \in \mathbb{N}_0$, we have

$$\begin{aligned}(\tau_1 \circ H_b^\alpha u)(n) &= \tau_1(H_b^\alpha u)(n) = K_1(\alpha)u(n+2) + K_0(\alpha)(u(n+2) - u(n+1)) \\ &= K_1(\alpha)(\tau_1 u)(n+1) + K_0(\alpha)((\tau_1 u)(n+1) - (\tau_1 u)(n)) = H_b^\alpha(\tau_1 u)(n) \\ &= (H_b^\alpha \circ \tau_1 u)(n).\end{aligned}$$

□

The proportional difference operator will play a key role when calculating the inverse operator of H^α since it can be expressed in terms of H_b^α as the next formula shows.

Lemma 3.7. *Let* $0 < \alpha < 1$. *The following identity holds:*

$$H^\alpha(u)(n) = \Delta^{-(1-\alpha)}(H_b^\alpha u)(n) + K_1(\alpha)k^{1-\alpha}(n+1)u(0), \ n \in \mathbb{N}_0. \tag{3.2}$$

Proof. Denoting $(\tau_p u)(n) := u(n+p)$, first observe that by definition,

$$\Delta^{-(1-\alpha)}(H_b^\alpha u)(n) = K_1(\alpha)\Delta^{-(1-\alpha)}(\tau_1 u)(n) + K_0(\alpha)\Delta^{-(1-\alpha)}(\tau_1 u - u)(n). \tag{3.3}$$

Also, we have using [28, Lemma 2.3] that

$$\begin{aligned}\Delta^{-(1-\alpha)}(\tau_1 u)(n) &= \tau_1(\Delta^{-(1-\alpha)}(u))(n) - k^{1-\alpha}(n+1)u(0) \\ &= \Delta^{-(1-\alpha)}(u)(n+1) - k^{1-\alpha}(n+1)u(0).\end{aligned} \tag{3.4}$$

As a consequence of (2.10), (3.3), and (3.4), we obtain

$$\begin{aligned}\Delta^{-(1-\alpha)}(H_b^\alpha u)(n) &= K_1(\alpha)\Delta^{-(1-\alpha)}(u)(n+1) - K_1(\alpha)k^{1-\alpha}(n+1)u(0) + K_0(\alpha)_C\Delta^\alpha(u)(n) \\ &= H^\alpha(u)(n) - K_1(\alpha)k^{1-\alpha}(n+1)u(0),\end{aligned} \tag{3.5}$$

and the conclusion holds. □

Remark 3.8. Comparing with the continuous case, we observe that for $u(0) = 0$, we have the identity

$$H^\alpha(u)(n) = \Delta^{-(1-\alpha)}(H_b^\alpha u)(n), \ n \in \mathbb{N}_0, \tag{3.6}$$

which is the analog to [1, Definition 1, formula (6)].

The right-side inverse of H_b^α is provided in the next lemma.

Lemma 3.9. *Let* $u \in s(\mathbb{N}_0; \mathbb{C})$*, and we define*

$$H_b^{-\alpha}(u)(n) = \begin{cases} \frac{1}{K_1(\alpha)+K_0(\alpha)}(\gamma * u)(n-1) & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n = 0. \end{cases}$$

where $\gamma(n) := \left(\frac{K_0(\alpha)}{K_1(\alpha)+K_0(\alpha)}\right)^n$, $n \in \mathbb{N}_0$. *Then, it satisfies*

$$H_b^\alpha(H_b^{-\alpha}u)(n) = u(n), \ n \in \mathbb{N}_0,$$

and

$$H_b^{-\alpha}(H_b^\alpha u)(n) = u(n) - \gamma(n)u(0),$$

for all $n \in \mathbb{N}_0$.

Proof. For all $n \in \mathbb{N}_0$, we have

$$H_b^{\alpha}(H_b^{-\alpha}u)(n) = (K_1(\alpha) + K_0(\alpha))(H_b^{-\alpha}u)(n+1) - K_0(\alpha)(H_b^{-\alpha}u)(n).$$

Using the definition, a simple computation shows that

$$H_b^{\alpha}(H_b^{-\alpha}u)(0) = (K_1(\alpha) + K_0(\alpha))(H_b^{-\alpha}u)(1) = \gamma(0)u(0) = u(0),$$

and, for all $n \in \mathbb{N}$,

$$\begin{aligned} H_b^{\alpha}(H_b^{-\alpha}u)(n) &= (\gamma * u)(n) - \frac{K_0(\alpha)}{K_1(\alpha) + K_0(\alpha)}(\gamma * u)(n-1) \\ &= \sum_{j=0}^{n} \gamma(n-j)u(j) - \gamma(1)\sum_{j=0}^{n-1} \gamma(n-1-j)u(j) \\ &= \sum_{j=0}^{n} \gamma(n-j)u(j) - \sum_{j=0}^{n-1} \gamma(n-j)u(j) = u(n), \end{aligned}$$

where we have used that $\gamma(n+j) = \gamma(n)\gamma(j)$ for all $j, n \in \mathbb{Z}$, and $\gamma(0) = 1$. On the other hand, it follows from definition that $H_b^{-\alpha}(H_b^{\alpha}u)(0) = 0$ and

$$\begin{aligned} H_b^{-\alpha}(H_b^{\alpha}u)(n) &= \frac{1}{K_1(\alpha) + K_0(\alpha)}(\gamma * H_b^{\alpha}u)(n-1) \\ &= \frac{1}{K_1(\alpha) + K_0(\alpha)}\left(\gamma * [K_1(\alpha)\tau_1 u + K_0(\alpha)(\tau_1 u - u)]\right)(n-1) \\ &= \frac{K_1(\alpha) + K_0(\alpha)}{K_1(\alpha) + K_0(\alpha)}(\gamma * \tau_1 u)(n-1) - \frac{K_0(\alpha)}{K_1(\alpha) + K_0(\alpha)}(\gamma * u)(n-1) \\ &= \sum_{j=0}^{n-1} \gamma(n-1-j)u(j+1) - \gamma(1)\sum_{j=0}^{n-1} \gamma(n-1-j)u(j) \\ &= \sum_{j=1}^{n} \gamma(n-j)u(j) - \sum_{j=0}^{n-1} \gamma(n-j)u(j) \\ &= \sum_{j=0}^{n} \gamma(n-j)u(j) - \sum_{j=0}^{n-1} \gamma(n-j)u(j) - \gamma(n)u(0) \\ &= u(n) - \gamma(n)u(0), \end{aligned}$$

for each $n \in \mathbb{N}$, proving the lemma. □

Our next result gives an explicit formula for the right-side inverse of the operator H^{α}.

Theorem 3.10. *Let* $0 < \alpha < 1$, $u \in s(\mathbb{N}_0; \mathbb{C})$, *and we define*

$$H^{-\alpha}(u)(n) = \begin{cases} \frac{1}{K_1(\alpha)+K_0(\alpha)}(\gamma * \Delta^{1-\alpha}u)(n-1) & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n = 0. \end{cases} \tag{3.7}$$

Then, the following identities holds:

$$H^{\alpha}(H^{-\alpha}u)(n) = u(n), \; n \in \mathbb{N}_0, \tag{3.8}$$

and

$$\begin{aligned} H^{-\alpha}(H^{\alpha}u)(n+1) &= u(n+1) - \gamma(n)u(1) \\ &+ \frac{K_1(\alpha)}{K_0(\alpha) + K_1(\alpha)}u(0)[(\gamma * k^{\alpha})(n) - (\gamma * k^{\alpha})(n+1) + \gamma(n+1) + (\alpha - 1)\gamma(n)], \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}_0$ *and* $H^{-\alpha}(H^{\alpha}u)(0) = 0$.

Proof. First, note that by definition, the following identity holds:

$$H^{-\alpha}(u)(n+1) = (H_b^{-\alpha} \circ \Delta^{1-\alpha} u)(n), \; n \in \mathbb{N}_0. \tag{3.10}$$

Also, we observe that for any $f \in s(\mathbb{N}_0, \mathbb{C})$ and $\alpha > 0$,

$$\begin{aligned} \Delta^{-\alpha}(\Delta f)(n) &= \sum_{j=0}^{n} k^{\alpha}(n-j)\Delta f(j) = \sum_{j=0}^{n} k^{\alpha}(n-j) f(j+1) - \sum_{j=0}^{n} k^{\alpha}(n-j) f(j) \\ &= \sum_{j=1}^{n+1} k^{\alpha}(n+1-j) f(j) - \sum_{j=0}^{n} k^{\alpha}(n-j) f(j) \\ &= \sum_{j=0}^{n+1} k^{\alpha}(n+1-j) f(j) - \sum_{j=0}^{n} k^{\alpha}(n-j) f(j) - k^{\alpha}(n+1) f(0) \\ &= \Delta(\Delta^{-\alpha} f)(n) - k^{\alpha}(n+1) f(0). \end{aligned} \tag{3.11}$$

Using Definition 2.6 and the fact that $0 < 1-\alpha < 1$, we obtain the identity $\Delta^{1-\alpha}(u)(n) = \Delta(\Delta^{-\alpha} u)(n)$. Therefore, from (3.11), we conclude that the following identities holds:

$$\Delta^{1-\alpha}(u)(n) = \Delta(\Delta^{-\alpha} u)(n) = \Delta^{-\alpha}(\Delta u)(n) + \tau_1(k^{\alpha})(n) u(0). \tag{3.12}$$

Using the above equality, we conclude that

$$\begin{aligned} \Delta^{-(1-\alpha)}(\Delta^{1-\alpha} u)(n) &= \Delta^{-(1-\alpha)}[\Delta(\Delta^{-\alpha} u)](n) = \Delta^{-(1-\alpha)}[\Delta^{-\alpha}(\Delta u)(n) + \tau_1(k^{\alpha})(n) u(0)] \\ &= \Delta^{-(1-\alpha)}[\Delta^{-\alpha}(\Delta u)](n) + \Delta^{-(1-\alpha)}[\tau_1(k^{\alpha})](n) u(0). \end{aligned} \tag{3.13}$$

Consequently, using [28, Corollary 2.4(a)], we obtain

$$\begin{aligned} \Delta^{-(1-\alpha)}(\Delta^{1-\alpha} u)(n) &= \Delta^{-1}(\Delta u)(n) + \sum_{j=0}^{n} k^{1-\alpha}(n-j)(\tau_1 k^{\alpha})(j) \\ &= u(n+1) - u(0) + \sum_{j=1}^{n+1} k^{1-\alpha}(n+1-j) k^{\alpha}(j) \\ &= u(n+1) - u(0) + [(k^{1-\alpha} * k^{\alpha})(n) - k^{1-\alpha}(n+1) k^{\alpha}(0)] u(0). \end{aligned}$$

Since $k^{1-\alpha} * k^{\alpha}(n) = k^1(n) = 1$ and $k^{\alpha}(0) = 1$, we finally obtain the identity

$$\Delta^{-(1-\alpha)}(\Delta^{1-\alpha} u)(n) = u(n+1) - k^{1-\alpha}(n+1) u(0). \tag{3.14}$$

On the other hand, we obtain from (3.12) and (3.11) with $f = \Delta^{-(1-\alpha)} u$ the following identities:

$$\begin{aligned} \Delta^{1-\alpha}(\Delta^{-(1-\alpha)} u)(n) &= \Delta^{-\alpha}(\Delta(\Delta^{-(1-\alpha)} u))(n) + (\tau_1 k^{\alpha})(n)(\Delta^{-(1-\alpha)} u)(0) \\ &= \Delta(\Delta^{-\alpha}(\Delta^{-(1-\alpha)} u))(n) - (\tau_1 k^{\alpha})(n)(\Delta^{-(1-\alpha)} u)(0) \\ &\quad + (\tau_1 k^{\alpha})(n)(\Delta^{-(1-\alpha)} u)(0) \\ &= \Delta(\Delta^{-1} u)(n), \end{aligned}$$

where in the last equality, we have used [28, Corollary 2.4(a)]. We conclude that

$$\Delta^{1-\alpha}(\Delta^{-(1-\alpha)} u)(n) = u(n+1) = \tau_1 u(n). \tag{3.15}$$

Also, observe that from [28, Lemma 2.3] and the fact that $H^{-\alpha}(u)(0) = 0$, we have

$$\begin{aligned}
&\tau_1(\Delta^{-(1-\alpha)} \circ H_b^\alpha \circ H^{-\alpha}u)(n) = (k^{1-\alpha} * (\tau_1(H_b^\alpha \circ H^{-\alpha}u)))(n) + k^{1-\alpha}(n+1)(H_b^\alpha \circ H^{-\alpha}u)(0) \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) + k^{1-\alpha}(n+1)(H_b^\alpha \circ H^{-\alpha}u)(0) \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) \\
&\quad + k^{1-\alpha}(n+1)[K_1(\alpha)H^{-\alpha}(u)(1) + K_0(\alpha)(H^{-\alpha}(u)(1) - H^{-\alpha}(u)(0))] \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) + k^{1-\alpha}(n+1) + k^{1-\alpha}(n+1)(K_1(\alpha) + K_0(\alpha))H^{-\alpha}(u)(1) \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) + k^{1-\alpha}(n+1)\gamma(0)\Delta^{-(1-\alpha)}u(0) \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) + k^{1-\alpha}(n+1)u(0).
\end{aligned} \tag{3.16}$$

Now, using (3.1), (3.10), (3.15), (3.16), Lemma 3.9, and the fact that $H^{-\alpha}(u)(0) = 0$, we obtain for every $n \in \mathbb{N}_0$:

$$\begin{aligned}
H^\alpha(H^{-\alpha}u)(n+1) &= \Delta^{-(1-\alpha)}H_b^\alpha(H^{-\alpha}u)(n+1) + K_1(\alpha)k^{1-\alpha}(n+2)(H^{-\alpha}u)(0) \\
&= (\tau_1 \circ \Delta^{-(1-\alpha)} \circ H_b^\alpha \circ H^{-\alpha}u)(n) \\
&= \Delta^{-(1-\alpha)}(\tau_1 \circ H_b^\alpha \circ H^{-\alpha}u)(n) + k^{1-\alpha}(n+1)u(0) \\
&= (\Delta^{-(1-\alpha)} \circ \tau_1 \circ H_b^\alpha)(\tau_{-1} \circ H_b^{-\alpha} \circ \Delta^{(1-\alpha)}u)(n) + k^{1-\alpha}(n+1)u(0) \\
&= (\Delta^{-(1-\alpha)} \circ \tau_1 \circ \tau_{-1} \circ H_b^\alpha \circ H_b^{-\alpha})(\Delta^{(1-\alpha)}u)(n) + k^{1-\alpha}(n+1)u(0) \\
&= \Delta^{-(1-\alpha)}(\Delta^{1-\alpha}u)(n) + k^{1-\alpha}(n+1)u(0) \\
&= u(n+1) - k^{1-\alpha}(n+1)u(0) + k^{1-\alpha}(n+1)u(0) = u(n+1).
\end{aligned} \tag{3.17}$$

Moreover,

$$\begin{aligned}
H^\alpha(H^{-\alpha}u)(0) &= K_1(\alpha)H^{-\alpha}(u)(1) + K_0(\alpha)(H^{-\alpha}(u)(1) - H^{-\alpha}(u)(0)) \\
&= (K_1(\alpha) + K_0(\alpha))H^{-\alpha}(u)(1) \\
&= \gamma(0)\Delta^{-(1-\alpha)}u(0) = u(0).
\end{aligned} \tag{3.18}$$

It proves the identity (3.8).

On the other hand, using (3.1), (3.2), (3.10), (3.15), and Lemma 3.9, we have for $n \in \mathbb{N}$:

$$\begin{aligned}
H^{-\alpha}(H^\alpha u)(n+1) &= \left(H_b^{-\alpha} \circ \Delta^{1-\alpha}\right)\left(\Delta^{-(1-\alpha)}(H_b^\alpha u)(n) + K_1(\alpha)k^{1-\alpha}(n+1)u(0)\right) \\
&= H_b^{-\alpha}(\tau_1(H_b^\alpha u))(n) + K_1(\alpha)u(0)H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) \\
&= (H_b^{-\alpha} \circ H_b^\alpha)(\tau_1 u)(n) + K_1(\alpha)u(0)H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) \\
&= [(\tau_1 u)(n) - \gamma(n)(\tau_1 u)(0)] + K_1(\alpha)u(0)H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) \\
&= u(n+1) - \gamma(n)u(1) + K_1(\alpha)u(0)H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n).
\end{aligned} \tag{3.19}$$

Now, from definition (2.6) and [28, Lemma 2.3], we have

$$\begin{aligned}
\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) &= \Delta(\Delta^{-\alpha}(\tau_1 k^{1-\alpha}))(n) = \tau_1(k^\alpha * (\tau_1 k^{1-\alpha}))(n) - (k^\alpha * (\tau_1 k^{1-\alpha}))(n) \\
&= (k^\alpha * k^{1-\alpha})(n+2) - k^\alpha(n+2)k^{1-\alpha}(0) - (k^\alpha * k^{1-\alpha})(n+1) \\
&\quad + k^\alpha(n+1)k^{1-\alpha}(0) \\
&= k^1(n+2) - k^\alpha(n+2) - k^1(n+1) + k^\alpha(n+1) \\
&= k^\alpha(n+1) - k^\alpha(n+2) \\
&= (\tau_1 k^\alpha)(n) - (\tau_2 k^\alpha)(n),
\end{aligned} \tag{3.20}$$

where we have used $k^1(n) = 1$ for all $n \in \mathbb{N}$ and $k^\beta(0) = 1$ for every $\beta > 0$. As a result, we get

$$\begin{aligned} H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) &= H_b^{-\alpha}[(\tau_1 k^\alpha)(n) - (\tau_2 k^\alpha)(n)] \\ &= \frac{1}{K_0(\alpha)+K_1(\alpha)}[(\gamma * (\tau_1 k^\alpha))(n-1) - (\gamma * (\tau_2 k^\alpha))(n)] \\ &= \frac{1}{K_0(\alpha)+K_1(\alpha)}[(\gamma * k^\alpha)(n) - (\gamma * k^\alpha)(n+1) + \gamma(n+1) + (\alpha-1)\gamma(n)], \end{aligned} \tag{3.21}$$

using again [28, Lemma 2.3]. Consequently, for every $n \in \mathbb{N}$, identity (3.19) leads to

$$\begin{aligned} H^{-\alpha}(H^\alpha u)(n+1) &= u(n+1) - \gamma(n)u(1) + K_1(\alpha)u(0)H_b^{-\alpha}(\Delta^{1-\alpha}(\tau_1(k^{1-\alpha}))(n) \\ &= u(n+1) - \gamma(n)u(1) \\ &\quad + \frac{K_1(\alpha)}{K_0(\alpha)+K_1(\alpha)}u(0)[(\gamma * k^\alpha)(n) - (\gamma * k^\alpha)(n+1) + \gamma(n+1) + (\alpha-1)\gamma(n)]. \end{aligned} \tag{3.22}$$

Now, if $n = 0$, we get

$$H^{-\alpha}(H^\alpha u)(0) = 0,$$

since by definition $H^{-\alpha}(v)(0) := 0$. It proves the theorem. □

The following result describes the Z-transform of H^α. It is useful for treating difference equations and can be considered the discrete analog to [1, Theorem 1].

Theorem 3.11. *The Z-transform of the operator H^α is given by*

$$\widehat{H^\alpha u}(z) = [K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - \left[zK_1(\alpha) + \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}K_0(\alpha)\right]u(0), \quad |z| > 1. \tag{3.23}$$

Proof. First, let denote the translation operator $(\tau_1 u)(n) := u(n+1)$. By definition (2.10), we have

$$\begin{aligned} {}_C\Delta^\alpha(u)(n) &= \Delta^{-(1-\alpha)}(\Delta u)(n) = \Delta^{-(1-\alpha)}(\tau_1 u)(n) - \Delta^{-(1-\alpha)}u(n) \\ &= (k^{1-\alpha} * \tau_1 u)(n) - (k^{1-\alpha} * u)(n). \end{aligned} \tag{3.24}$$

Applying now [28, Lemma 2.3] to the last expression in (3.24), we arrive to

$${}_C\Delta^\alpha(u)(n) = (k^{1-\alpha} * u)(n+1) - k^{1-\alpha}(n+1)u(0) - (k^{1-\alpha} * u)(n). \tag{3.25}$$

Applying the Z-transform to (3.25), we obtain

$$\begin{aligned} \widehat{{}_C\Delta^\alpha u}(z) &= z\widehat{k^{1-\alpha}}(z)\hat{u}(z) - z(k^{1-\alpha} * u)(0) - [z\widehat{k^{1-\alpha}}(z) - zk^{1-\alpha}(0)]u(0) - \widehat{k^{1-\alpha}}(z)\hat{u}(z) \\ &= (z-1)\frac{z^{1-\alpha}}{(z-1)^{1-\alpha}}\hat{u}(z) - \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}u(0), \end{aligned} \tag{3.26}$$

where we have used $k^\beta(0) = 1$ for every $\beta > 0$ and identity (2.5). On the other hand, we have

$$\widehat{\Delta^{-(1-\alpha)}u}(z) = \widehat{k^{1-\alpha}}(z)\hat{u}(z) = \frac{z^{1-\alpha}}{(z-1)^{1-\alpha}}\hat{u}(z). \tag{3.27}$$

Consequently, using (3.26) and (3.27) and the properties of the Z-transform for a translation, we immediately obtain

$$\begin{aligned}\widehat{H^{\alpha}u}(z) &= K_1(\alpha)\left[z\frac{z^{1-\alpha}}{(z-1)^{1-\alpha}}\widehat{u}(z) - z\Delta^{-(1-\alpha)}u(0)\right] \\ &\quad + \left[K_0(\alpha)(z-1)\frac{z^{1-\alpha}}{(z-1)^{1-\alpha}}\widehat{u}(z) - K_0(\alpha)\frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}u(0)\right] \\ &= [K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\widehat{u}(z) - \left[zK_1(\alpha) + \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}K_0(\alpha)\right]u(0),\end{aligned}$$

proving the theorem. □

Next, we will solve some difference equations involving the hybrid fractional difference operator.

Some previous comments are in order: From [24, Definition 2.3] with $a = -1$, we obtain that the discrete Laplace transform of a sequence $f : \mathbb{N}_0 \to \mathbb{R}$ satisfies

$$\mathcal{L}_{-1}(f)(s) = \sum_{j=0}^{\infty}(1-s)^j f(j) = \widehat{f}\left(\frac{1}{1-s}\right).$$

Therefore, replacing $z = \frac{1}{1-s}$, we obtain the identity

$$\widehat{f}(z) = \mathcal{L}_{-1}(f)\left(\frac{z-1}{z}\right). \tag{3.28}$$

Next, we recall that the bivariate Mittag-Leffler function was recently defined for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $Re(\alpha) > 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying $|\lambda_1| < 1$ and $|\lambda_2| < 1$ as follows [24, Definition 3.1]:

$$E^{\delta}_{\alpha,\beta,\gamma}(\lambda_1, \lambda_2; z) := \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\lambda_1^{j_1}\lambda_2^{j_2}\frac{(\delta)_{j_1+j_2}}{\Gamma(\alpha j_1 + \beta j_2 + \gamma)}\frac{z^{\overline{\alpha j_1 + \beta j_2 + \gamma - 1}}}{j_1!j_2!},\ z \in \mathbb{C},$$

where $z^{\overline{\xi}} := \frac{\Gamma(\xi+z)}{\Gamma(z)}$ and $(\delta)_n := \delta(\delta+1)\ldots(\delta+n-1)$ is the Pochhammer symbol.

Note that $\frac{(\delta)_n}{n!} = k^{\delta}(n)$ and $(n+1)^{\overline{\xi}} = \frac{\Gamma(\xi+n+1)}{n!}$. Therefore, for $\xi = \alpha j_1 + \beta j_2 + \gamma - 1$ and $z = n+1$, we obtain for each $n \in \mathbb{N}_0$:

$$E^{\delta}_{\alpha,\beta,\gamma}(\lambda_1, \lambda_2; n+1) := \sum_{j_1,j_2=0}^{\infty}\lambda_1^{j_1}\lambda_2^{j_2}\frac{(\delta)_{j_1+j_2}}{\Gamma(\alpha j_1 + \beta j_2 + \gamma)}\frac{\Gamma(\alpha j_1 + \beta j_2 + \gamma + n)}{j_1!j_2!n!}. \tag{3.29}$$

Denote $b_{ML}(n) := E^{\delta}_{\alpha,\beta,\gamma}(\lambda_1, \lambda_2; n+1)$, $n \in \mathbb{N}_0$. From [24, Theorem 4.3] and using (3.28), we conclude that

$$\widehat{b_{ML}}(z) = \frac{1}{s^{\gamma}}\left(1 - \frac{\lambda_1}{s^{\alpha}} - \frac{\lambda_2}{s^{\beta}}\right)^{-\delta}, \tag{3.30}$$

where $s := \frac{z-1}{z}$. Even more, we have the following remarkable result.

Theorem 3.12. *The discrete bivariate Mittag-Leffler sequence b_{ML} is the Poisson transformation of the bivariate Mittag-Leffler function defined as*

$$E^{\delta}_{\alpha,\beta,\gamma}(\lambda_1, \lambda_2, t) := \sum_{j_1,j_2=0}^{\infty}\frac{\lambda_1^{j_1}\lambda_2^{j_2}}{j_1!j_2!}\frac{(\delta)_{j_1+j_2}}{\Gamma(\alpha j_1 + \beta j_2 + \gamma)}t^{\alpha j_1 + \beta j_2 + \gamma - 1}. \tag{3.31}$$

Proof. Denote $B_{ML}(t) := E^{\delta}_{\alpha,\beta,\gamma}(\lambda_1, \lambda_2, t)$. Then, observing that by definition $g_{\alpha j_1+\beta j_2+\gamma}(t) = \frac{t^{\alpha j_1+\beta j_2+\gamma-1}}{\Gamma(\alpha j_1+\beta j_2+\gamma)}$, we obtain using (2.7)

$$\mathcal{P}(B_{ML})(n) = \sum_{j_1,j_2=0}^{\infty} \frac{\lambda_1^{j_1}\lambda_2^{j_2}}{j_1!j_2!}(\delta)_{j_1+j_2}\mathcal{P}(g_{\alpha j_1+\beta j_2+\gamma})(n) = \sum_{j_1,j_2=0}^{\infty} \frac{\lambda_1^{j_1}\lambda_2^{j_2}}{j_1!j_2!}(\delta)_{j_1+j_2}k^{\alpha j_1+\beta j_2+\gamma}(n)$$
$$= \sum_{j_1,j_2=0}^{\infty} \frac{\lambda_1^{j_1}\lambda_2^{j_2}}{j_1!j_2!}(\delta)_{j_1+j_2}\frac{\Gamma(\alpha j_1+\beta j_2+\gamma+n)}{\Gamma(\alpha j_1+\beta j_2+\gamma)n!},$$

which coincides with (3.29). This finishes the proof. □

Remark 3.13. Definition (3.31) is the one-parameter version of the Mittag-Leffler function applied to two variables proposed by Fernandez et al. [25]. The precise form can be found, for example, in [33, Definition 2.4].

Since by [33, Lemma 2.5] we have $\mathcal{L}(B_{ML})(s) = s^{-\gamma}(1-\lambda_1 s^{-\alpha} - \lambda_2 s^{-\beta})^{-\delta}$, we retrieve immediately from (2.8) the formula (3.30). This remarkable relation motivates the following result that allows to define a trivariate Mittag-Leffler sequence in a very natural way.

Theorem 3.14. *Let $\alpha, \beta, \gamma, \delta, \rho > 0$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. The Poisson transform of the trivariate Mittag-Leffler function defined by*

$$T_{ML}(t) := E^{\rho}_{\alpha,\beta,\gamma,\delta}(\lambda_1,\lambda_2,\lambda_3,t) := \sum_{j_1,j_2,j_3=0}^{\infty} \frac{\lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}}{j_1!j_2!j_3!}\frac{(\rho)_{j_1+j_2+j_3}}{\Gamma(\alpha j_1+\beta j_2+\gamma j_3+\delta)}t^{\alpha j_1+\beta j_2+\gamma j_3+\delta-1}$$

is given by

$$\tau_{ML}(n) := E^{\rho}_{\alpha,\beta,\gamma,\delta}(\lambda_1,\lambda_2,\lambda_3,n+1) := \sum_{j_1,j_2,j_3=0}^{\infty} \frac{\lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}}{j_1!j_2!j_3!}(\rho)_{j_1+j_2+j_3}\frac{\Gamma(\alpha j_1+\beta j_2+\gamma j_3+\delta+n)}{\Gamma(\alpha j_1+\beta j_2+\gamma j_3+\delta)n!} \quad (3.32)$$

that will be called a trivariate Mittag-Leffler sequence.

Proof. The proof follows the same steps of the above theorem, and therefore, it is omitted. □

The next important result follows again from identity (2.8) and [33, Lemma 2.8].

Theorem 3.15. *The following identity holds:*

$$\widehat{\tau_{ML}}(z) = \frac{1}{s^{\delta}}\left(1 - \frac{\lambda_1}{s^{\alpha}} - \frac{\lambda_2}{s^{\beta}} - \frac{\lambda_3}{s^{\gamma}}\right)^{-\rho},$$

where $s := \frac{z-1}{z}$.

We now consider the following example.

Example 1. Let $0 < \alpha < 1$ and consider the difference equation given by

$$H^{\alpha}u(n) = 0, \quad u(0) = C.$$

Applying Z-transform to both sides and using Theorem 3.11, we arrive to

$$(K_1(\alpha)z + K_0(\alpha)(z-1))\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - \left(K_1(\alpha)z + \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}K_0(\alpha)\right)u(0) = 0.$$

Letting $s = \frac{z-1}{z}$ and observing that $1 - s = \frac{1}{z}$, the above identity becomes

$$(K_1(\alpha)z + K_0(\alpha)z - K_0(\alpha))\frac{1}{s^{1-\alpha}}\hat{u}(z) = K_1(\alpha)zu(0) + K_0(\alpha)z\frac{1}{s^{1-\alpha}}u(0), \quad (3.33)$$

where

$$\begin{aligned}(K_1(\alpha)z + K_0(\alpha)z - K_0(\alpha))\frac{1}{s^{1-\alpha}} &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{1-\alpha}} - K_0(\alpha)\frac{1}{zs^{1-\alpha}}\right] \\ &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{1-\alpha}} - K_0(\alpha)(1-s)\frac{1}{s^{1-\alpha}}\right] \\ &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{-\alpha}}\right].\end{aligned} \tag{3.34}$$

Replacing identity (3.34) into the left-hand side of identity (3.33) and simplifying z, we obtain

$$\begin{aligned}\hat{u}(z) &= K_1(\alpha)u(0)\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{-\alpha}}\right]^{-1} + K_0(\alpha)u(0)\frac{1}{s^{1-\alpha}}\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{-\alpha}}\right]^{-1} \\ &= K_1(\alpha)u(0)s^{\alpha-1}[K_1(\alpha) + K_0(\alpha)s]^{-1} + K_0(\alpha)u(0)[K_1(\alpha) + K_0(\alpha)s]^{-1} \\ &= \frac{K_1(\alpha)}{K_0(\alpha)}u(0)s^{\alpha-1}\left[\frac{K_1(\alpha)}{K_0(\alpha)} + s\right]^{-1} + u(0)\left[\frac{K_1(\alpha)}{K_0(\alpha)} + s\right]^{-1},\end{aligned}$$

which leads to (see 2.9):

$$u(n) = \frac{K_1(\alpha)}{K_0(\alpha)}(k^{1-\alpha} * \gamma)(n)u(0) + \gamma(n)u(0),\ n \in \mathbb{N}_0,$$

with $\gamma(n) := \left(\frac{1}{\frac{K_1(\alpha)}{K_0(\alpha)}+1}\right)^{n+1}$.

For instance, for $K_0(\alpha) = \alpha$ and $K_1(\alpha) = 1-\alpha$, we obtain $\gamma(n) = \alpha^{n+1}$ and

$$u(n) = \frac{1-\alpha}{\alpha}(k^{1-\alpha} * \gamma)(n)u(0) + \gamma(n)u(0),\ n \in \mathbb{N}_0.$$

Example 2. Let $0 < \alpha < 1$ and consider the following difference equation:

$$H^{\alpha}u(n) = \lambda u(n), \quad u(0) = 1. \tag{3.35}$$

Applying Z-transform to both sides and using (3.23), we arrive to

$$(K_1(\alpha)z + K_0(\alpha)(z-1))\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - \left(K_1(\alpha)z + \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}K_0(\alpha)\right) = \lambda\hat{u}(z).$$

Denote $s = \frac{z-1}{z}$ and hence $1 - s = \frac{1}{z}$. Then, the above identity is equivalent to

$$\left[(K_1(\alpha)z + K_0(\alpha)z - K_0(\alpha))\frac{1}{s^{1-\alpha}} - \lambda\right]\hat{u}(z) = K_1(\alpha)z + K_0(\alpha)z\frac{1}{s^{1-\alpha}}, \tag{3.36}$$

where

$$\begin{aligned}\left[(K_1(\alpha)z + K_0(\alpha)z - K_0(\alpha))\frac{1}{s^{1-\alpha}} - \lambda\right] &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{1-\alpha}} - K_0(\alpha)\frac{1}{zs^{1-\alpha}} - \frac{\lambda}{z}\right] \\ &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{1-\alpha}} - K_0(\alpha)(1-s)\frac{1}{s^{1-\alpha}} - \lambda(1-s)\right] \\ &= z\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{-\alpha}} - \lambda + \lambda s\right] \\ &= z\lambda s\left[\frac{K_1(\alpha)}{\lambda}\frac{1}{s^{2-\alpha}} + \frac{K_0(\alpha)}{\lambda}\frac{1}{s^{1-\alpha}} - \frac{1}{s} + 1\right].\end{aligned} \tag{3.37}$$

Replacing (3.38) into (3.38), we obtain

$$\begin{aligned}\hat{u}(z) = &\frac{K_1(\alpha)}{\lambda}\frac{1}{s}\left[1 + \frac{K_1(\alpha)}{\lambda}\frac{1}{s^{2-\alpha}} + \frac{K_0(\alpha)}{\lambda}\frac{1}{s^{1-\alpha}} - \frac{1}{s}\right]^{-1} \\ &+ \frac{K_0(\alpha)}{\lambda}\frac{1}{s^{2-\alpha}}\left[1 + \frac{K_1(\alpha)}{\lambda}\frac{1}{s^{2-\alpha}} + \frac{K_0(\alpha)}{\lambda}\frac{1}{s^{1-\alpha}} - \frac{1}{s}\right]^{-1}.\end{aligned} \tag{3.38}$$

Hence, using (3.32), we obtain

$$u(n) = \frac{K_1(\alpha)}{\lambda} k^1 * E^1_{2-\alpha,1-\alpha,1,2-\alpha}\left(\frac{-K_1(\alpha)}{\lambda}, \frac{-K_0(\alpha)}{\lambda}, 1, \bullet + 1\right) + \\ + \frac{K_0(\alpha)}{\lambda} k^{2-\alpha} * E^1_{2-\alpha,1-\alpha,1,2-\alpha}\left(\frac{-K_1(\alpha)}{\lambda}, \frac{-K_0(\alpha)}{\lambda}, 1, \bullet + 1\right), \tag{3.39}$$

which is the solution of (3.36).

Remark 3.16. We refer the reader to [1, Example 2] for the continuous analog of Equation (3.36) where the bivariate Mittag-Leffler function is shown to be a solution. More concretely, they showed the solution of equation

$$^{CPC}D_t^{\alpha} f(t) = \lambda f(t), \; f(0) = 1, \tag{3.40}$$

is given by

$$f(t) = E^1_{\alpha,1,1}\left(\frac{\lambda}{K_0(\alpha)} t^{\alpha}, -\frac{K_1(\alpha)}{K_0(\alpha)} t\right). \tag{3.41}$$

4 | THE RIEMANN–LIOUVILLE HYBRID DIFFERENCE OPERATOR

In contrast with the continuous case, we now introduce the corresponding analog of hybrid difference operator when considering the Riemann–Liouville notion instead of Caputo.

We will see that this operator so defined is, in some sense, more natural. For that, we consider the constant proportional Riemann–Liouville hybrid operator that we define as

$$^{PRL}D_t^{\alpha} f(t) := K_1(\alpha) J_t^{1-\alpha} f(t) + K_0(\alpha) D_t^{\alpha} f(t).$$

We do not know if this operator has been considered previously in the literature.

Definition 4.1. Let $0 < \alpha < 1$ be given. The Riemann–Liouville hybrid difference operator R^{α} is defined by

$$R^{\alpha}(u)(n) := K_1(\alpha)\Delta^{-(1-\alpha)}(u)(n+1) + K_0(\alpha)\Delta^{\alpha}(u)(n), \; n \in \mathbb{N}_0, \tag{4.1}$$

where $K_0(\alpha)$ and $K_1(\alpha)$ are functions that satisfy (2.3) and (2.4).

Remark 4.2. Due to Theorem 2.8, it is immediate that

$$R^{\alpha}u(n) = H^{\alpha}u(n) + K_0(\alpha)k^{1-\alpha}(n+1)u(0).$$

In other words, R^{α} and H^{α} coincide when $u(0) = 0$. Hence, the main difference between both operators is only the starting point.

Remark 4.3. Observe that, for all $n \in \mathbb{N}_0$, we have

$$\lim_{\alpha \to 1^-} R^{\alpha}u(n) = \lim_{\alpha \to 1^-} H^{\alpha}u(n) + K_0(\alpha)k^0(n+1)u(0) = \Delta u(n) \text{ and}$$

$$\lim_{\alpha \to 0^+} R^{\alpha}u(n) = \lim_{\alpha \to 0^+} H^{\alpha}u(n) + K_0(\alpha)k^1(n+1)u(0) = \Delta^{-1}u(n+1),$$

since $k^0(n) = \delta_0(n), n \in \mathbb{N}_0$, and $\lim_{\alpha \to 0^+} K_0(\alpha) = 0$.

The following theorem shows that the Riemann–Liouville hybrid difference operator R^{α} corresponds to a discretization of the operator $^{PRL}D_t^{\alpha}$. In this sense, the operator R^{α} appears to be appropriate and, in some sense more natural, as a discrete version of a constant proportional Riemann–Liouville hybrid operator defined as a linear combination of the Riemann–Liouville integral and derivative, instead of Caputo.

Theorem 4.4. *The operator R^α corresponds to the discretization of the operator ${}^{PRL}D_t^\alpha$ by means of the Poisson transformation.*

Proof. First, from [4, Theorem 4.5], we have the following relation between the discrete and continuous fractional derivatives in the sense of Riemann–Liouville via the Poisson transformation

$$\mathcal{P}(D_t^\alpha f)(m+1) = \Delta^\alpha(u)(m),\ m \in \mathbb{N}_0, \tag{4.2}$$

where $u = \mathcal{P}(f)$. On the other hand,

$$\mathcal{P}(J_t^\alpha f)(m) = \mathcal{P}(g_\alpha * f)(m) = (k^\alpha * u)(m) = \Delta^{-\alpha}(u)(m),\ \ m \in \mathbb{N}. \tag{4.3}$$

Finally, we get from (4.2) and (4.3) that

$$\begin{aligned}\mathcal{P}\left({}^{PRL}D_t^\alpha f\right)(m+1) &= K_1(\alpha)\mathcal{P}\left(J_t^{1-\alpha} f\right)(m+1) + K_0(\alpha)\mathcal{P}\left(D_t^\alpha f\right)(m+1)\\ &= K_1(\alpha)\Delta^{-(1-\alpha)}(u)(m+1) + K_0(\alpha)\Delta^\alpha(u)(m) = R^\alpha(u)(m),\ \ m \in \mathbb{N}_0.\end{aligned}$$

□

Remark 4.5. In case of the operator H^α, we note that we need the condition $\hat{f}(1) = \int_0^\infty e^{-s} f(s)ds = f(0)$ to obtain an analogous result. However, except for $f(t) \equiv 1$, such condition is rarely valid.

Remark 4.6. We point out that

$$R^\alpha(u)(n) = \Delta^{-(1-\alpha)}(H_b^\alpha u)(n) + (K_1(\alpha) + K_0(\alpha))k^{1-\alpha}(n+1)u(0).$$

In particular, in case $u(0) = 0$, we retrieve the analog of identity (3.6).

Theorem 4.7. *Let $0 < \alpha < 1$ be given and the operator $H^{-\alpha}$ be defined as in (3.7). Then, the following relations hold:*

$$R^\alpha(H^{-\alpha}u)(n) = u(n),\ \ n \in \mathbb{N}_0, \tag{4.4}$$

and for every $n \in \mathbb{N}$, it follows that

$$\begin{aligned}H^{-\alpha}(R^\alpha u)(n) &= u(n) - \gamma(n-1)u(1)\\ &+ \frac{K_1(\alpha)}{K_0(\alpha)+K_1(\alpha)}u(0)[(\gamma * k^\alpha)(n-1) - (\gamma * k^\alpha)(n) + \gamma(n) + (\alpha-1)\gamma(n-1)]\\ &+ \frac{K_0(\alpha)}{K_0(\alpha)+K_1(\alpha)}u(0)[(\gamma * k^\alpha)(n) - (\gamma * k^\alpha)(n+1) + \gamma(n+1) + (\alpha-1)\gamma(n)],\end{aligned} \tag{4.5}$$

and

$$H^{-\alpha}(R^\alpha u)(0) = 0.$$

Proof. From Remark 4.2, we have

$$R^\alpha(H^{-\alpha}u)(n) = H^\alpha(H^{-\alpha}u)(n) + K_0(\alpha)k^{1-\alpha}(n+1)(H^{-\alpha}u)(0) = u(n), \tag{4.6}$$

where we have used the fact that $H^\alpha(H^{-\alpha}u)(n) = u(n)$ and $H^{-\alpha}(u)(0) = 0$. Moreover, for every $n \in \mathbb{N}$, the following identity also holds:

$$
\begin{aligned}
H^{-\alpha}(R^{\alpha}u)(n) &= H^{-\alpha}(H^{\alpha}u)(n) + K_0(\alpha)H^{-\alpha}(\tau_1 k^{1-\alpha})(n)u(0) \\
&= u(n) - \gamma(n-1)u(1) + \frac{K_1(\alpha)}{K_0(\alpha)+K_1(\alpha)}u(0)[(\gamma * k^{\alpha})(n-1) \\
&\quad - (\gamma * k^{\alpha})(n) + \gamma(n) + (\alpha-1)\gamma(n-1)] \\
&\quad + \frac{K_0(\alpha)}{K_0(\alpha)+K_1(\alpha)}u(0)[(\gamma * k^{\alpha})(n) - (\gamma * k^{\alpha})(n+1) + \gamma(n+1) + (\alpha-1)\gamma(n)],
\end{aligned}
\tag{4.7}
$$

where we have used identities (3.9) and (3.21). Also, as a consequence of identity $H^{-\alpha}(H^{\alpha}u)(0) = 0$ and the fact that $H^{-\alpha}u(0) := 0$, we get

$$
H^{-\alpha}(R^{\alpha}u)(0) = H^{-\alpha}(H^{\alpha}u)(0) + K_0(\alpha)H^{-\alpha}(\tau_1 k^{1-\alpha})(0)u(0) = 0. \tag{4.8}
$$

□

Our next theorem shows that R^{α} is a Toeplitz operator for any $0 < \alpha < 1$. For more information about Toeplitz operators, we refer the reader to [34, 35].

Theorem 4.8. *For any $0 < \alpha < 1$, the operator R^{α} defines a Toeplitz operator with symbol $\Phi(z) = K_1(\alpha)(1-z)^{\alpha-1} + (K_1(\alpha)+K_0(\alpha))\frac{(1-z)^{\alpha}}{z}$.*

Proof. First, observe that by definition for every $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
R^{\alpha}u(n) &= K_1(\alpha)\Delta^{-(1-\alpha)}u(n+1) + K_0(\alpha)\Delta^{\alpha}u(n) \\
&= K_1(\alpha)\sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)u(j) + K_0(\alpha)\left[\sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)u(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j)\right].
\end{aligned}
\tag{4.9}
$$

Our first important observation is that the representation of Δ^{α} in the canonical basis $\{e_l(j)\}_{j,l\in\mathbb{N}_0}$ is a Toeplitz matrix for $0 < \alpha < 1$. In fact, we have

$$
\Delta^{\alpha}e_l(n) = \begin{cases} K_1(\alpha)k^{1-\alpha}(n-l) - \alpha(K_1(\alpha)+K_0(\alpha))\frac{k^{1-\alpha}(n-l)}{n-l+1} & \text{if } n \geq l, \\ K_1(\alpha)+K_0(\alpha) & \text{if } n = l-1, \\ 0 & \text{if } n < l-1. \end{cases} \tag{4.10}
$$

In view of (4.10), we have

$$
\Phi(z) = \frac{K_1(\alpha)+K_0(\alpha)}{z} + K_1(\alpha)\sum_{j=0}^{\infty} k^{1-\alpha}(j)z^j + (K_1(\alpha)+K_0(\alpha))\sum_{j=0}^{\infty} \frac{(-\alpha)k^{1-\alpha}(j)z^j}{(j+1)}.
$$

Let denote

$$
\varphi_1(z) = \sum_{j=0}^{\infty} k^{1-\alpha}(j)z^j \text{ and } \varphi_2(z) = \sum_{j=0}^{\infty} \frac{(-\alpha)k^{1-\alpha}(j)z^j}{(j+1)}.
$$

As a consequence of Remark 2.4, it is immediate that

$$
\varphi_1(z) = (1-z)^{\alpha-1}.
$$

On the other hand, we get

$$
z\varphi_2(z) = \sum_{j=0}^{\infty} \frac{(-\alpha)k^{1-\alpha}(j)z^{j+1}}{(j+1)},
$$

and then using again Remark 2.4, we have

$$[z\varphi_2(z)]' = -\alpha \sum_{j=0}^{\infty} k^{1-\alpha}(j)z^j = -\alpha(1-z)^{\alpha-1}.$$

After integrating once, we obtain

$$z\varphi_2(z) = (1-z)^{\alpha} + c. \tag{4.11}$$

Replacing $z = 0$ in (4.11), we get $c = -1$.
As a consequence, $\varphi_2(z) = \frac{(1-z)^{\alpha}}{z} - \frac{1}{z}$. Finally, it follows that

$$\begin{aligned}\Phi(z) &= \frac{K_1(\alpha)+K_0(\alpha)}{z} + K_1(\alpha)(1-z)^{\alpha-1} + (K_1(\alpha)+K_0(\alpha))\left[\frac{(1-z)^{\alpha}}{z} - \frac{1}{z}\right] = \frac{(1-z)^{\alpha}}{z} \\ &= K_1(\alpha)(1-z)^{\alpha-1} + (K_1(\alpha)+K_0(\alpha))\frac{(1-z)^{\alpha}}{z}\end{aligned} \tag{4.12}$$

is the symbol of the Toeplitz operator. □

The following result provides the Z-transform of R^{α}.

Theorem 4.9. *The Z-transform of the operator R^{α} is given by*

$$\widehat{R^{\alpha}u}(z) = [K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - z[K_0(\alpha)+K_1(\alpha)]u(0), \ \ |z| > 1. \tag{4.13}$$

Proof. Observe from definition (4.1), Theorem 3.23, and a straightforward computation that

$$\begin{aligned}\widehat{R^{\alpha}u}(z) &= \widehat{H^{\alpha}u}(z) + K_0(\alpha)[z\widehat{k^{1-\alpha}}(z) - zk^{1-\alpha}(0)]u(0) \\ &= [K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - \left[zK_1(\alpha) + \frac{z^{2-\alpha}}{(z-1)^{1-\alpha}}K_0(\alpha)\right]u(0) \\ &\quad + z\left[\left(\frac{z}{z-1}\right)^{1-\alpha} - 1\right]K_0(\alpha)u(0) \\ &= [K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - z[K_0(\alpha)+K_1(\alpha)]u(0).\end{aligned}$$

□

We end this article with the following illustrative examples.

Example 3. Let consider the following difference equation:

$$R^{\alpha}u(n) = 0, \ \ u(0) = C.$$

Applying Z-transform to both sides and using (4.13), we arrive to

$$[K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - z[K_0(\alpha)+K_1(\alpha)]u(0) = 0.$$

Denoting $s = \frac{z-1}{z}$ and using (3.34), this identity reduces to

$$\begin{aligned}\hat{u}(z) &= (K_1(\alpha)+K_0(\alpha))u(0)\left[K_1(\alpha)\frac{1}{s^{1-\alpha}} + K_0(\alpha)\frac{1}{s^{-\alpha}}\right]^{-1} \\ &= (K_1(\alpha)+K_0(\alpha))u(0)s^{\alpha-1}[K_1(\alpha)+K_0(\alpha)s]^{-1} \\ &= \frac{K_1(\alpha)+K_0(\alpha)}{K_0(\alpha)}u(0)s^{\alpha-1}\left[\frac{K_1(\alpha)}{K_0(\alpha)}+s\right]^{-1},\end{aligned}$$

which leads to (see 2.9):

$$u(n) = \frac{K_1(\alpha)+K_0(\alpha)}{K_0(\alpha)}(k^{1-\alpha} * \gamma)(n)u(0), \; n \in \mathbb{N}_0,$$

with $\gamma(n) := \left(\frac{1}{\frac{K_1(\alpha)}{K_0(\alpha)}+1}\right)^{n+1}$.

Example 4. Let us consider the following difference equation:

$$R^\alpha u(n) = \lambda u(n), \quad u(0) = 1. \tag{4.14}$$

Applying Z-transform to both sides and using (4.13), we arrive to

$$[K_1(\alpha)z + K_0(\alpha)(z-1)]\left(\frac{z}{z-1}\right)^{1-\alpha}\hat{u}(z) - z[K_0(\alpha)+K_1(\alpha)] = \lambda\hat{u}(z). \tag{4.15}$$

Denoting $s = \frac{z-1}{z}$ and using (3.38), identity (4.15) leads to

$$\hat{u}(z) = \frac{K_1(\alpha)+K_0(\alpha)}{\lambda}\frac{1}{s}\left[1 + \frac{K_1(\alpha)}{\lambda}\frac{1}{s^{2-\alpha}} + \frac{K_0(\alpha)}{\lambda}\frac{1}{s^{1-\alpha}} - \frac{1}{s}\right]^{-1}. \tag{4.16}$$

Hence, using (3.32), we obtain

$$u(n) = \frac{K_1(\alpha)+K_0(\alpha)}{\lambda}k^1 * E^1_{2-\alpha,1-\alpha,1,2-\alpha}\left(\frac{-K_1(\alpha)}{\lambda}, \frac{-K_0(\alpha)}{\lambda}, 1, \bullet + 1\right), \tag{4.17}$$

which is the solution of (4.14).

Remark 4.10. Note that formula (4.17) for the solution of the discrete counterpart of (3.41) is more consistent with the solution (3.42) when compared with formula (3.40). This corroborates our feeling that the operator R^α should be a better discrete counterpart for the constant proportional Caputo hybrid operator in the continuous setting than the operator H^α.

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All authors have contributed equally to this work. All authors read and approved the final manuscript.

CONFLICT OF INTEREST STATEMENT

The authors have no relevant financial or non-financial interests to disclose.

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