IMA Teaching Mathematics and its Applications (2016) Page 1 of 13
doi:10.1093/imanum/drnxxx

Emphasizing visualization and physical applications in the study of eigenvectors and eigenvalues

MARÍA JOSÉ BELTRÁN-MENEU†,
Departament de Didàctica de la Matemàtica, Universitat de València.
Avda. Tarongers, 4. 46022 Valencia, Spain.

AND

MARINA MURILLO-ARCILA‡
IUMPA, Universitat Politècnica de València.
Camino de Vera, s/n. 46022 Valencia, Spain.

AND

LLUÍS ALBARRACÍN§
Departament de Didàctica de la Matemàtica i de les Ciències Experimentals, Universitat Autònoma de Barcelona.
Catalonia, Spain.

[Received on ; revised on]

In this paper we present a teaching proposal which emphasizes on visualization and physical applications in the study of eigenvectors and eigenvalues. More concretely, we introduce these concepts using the notion of the moment of inertia of a rigid body and the GeoGebra software. The proposal was designed after observing, during a linear algebra course in the Architecture degree, that students had problems when treating eigenvectors and eigenvalues meanings from a geometrical perspective. The aim of this research is to determine if the designed teaching proposal allows students to give a geometrical meaning to the concepts of eigenvectors and eigenvalues. To this end, we analyze the responses given to a test by the students attending the teaching proposal and others attending a traditional course with no emphasis in visualization. We classify their reasoning in order to check differences between both groups. As our findings show, the students who attended the course where the teaching proposal was developed, obtained better results in questions formulated from a visual point of view than those attending the traditional course. Moreover, we observe that, whereas in the traditional group we did not find responses reasoned in the embodied world of mathematics, in the group attending the teaching proposal we found students giving responses in each of the three worlds of mathematics given by Tall: embodied, symbolic and formal.

Keywords: undergraduate mathematics education; linear algebra; eigenvectors and eigenvalues; moments of inertia; dynamic geometry software.

†Corresponding author. Email: maria.jose.beltran@uv.es
‡Email:mamuar1@posgrado.upv.es
§Email:lluis.albarracin@uab.cat

1. Introduction and Theoretical framework

Many science and engineering students are introduced to the formal presentation of mathematics through a first course in linear algebra. Unlike calculus, that often emphasizes manipulation of symbols in order to solve problems, linear algebra is based on the description of concepts, often through word definitions, and deductive derivation of further concepts from these (Stewart & Thomas, 2006a). The abstract and formal nature of linear algebra generates two sources of difficulty in its understanding identified by Dorier & Sierpinska (2001): "the nature of linear algebra itself (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)".

Various authors have investigated the learning and teaching of linear algebra from an educational perspective (see e.g. Uhlig (2002); Sierpinska *et al.* (1999)). Thomas & Stewart (2011) showed that students tend to think about the concepts of eigenvalues and eigenvectors as the application of techniques. They do not usually understand the meaning of definitions and they are unable to apply them, even in simple problems. In fact, their findings show that most students manipulate algebraically the symbols without understanding the concepts they refer to. Usually, university students are introduced to the topic of eigenvectors and eigenvalues through a formal definition, without such prior motivation, and they are soon manipulating algebraic and matrix representations. However, eigenvectors have a strong visual or embodied image in the vector space $\mathbb{R}^n$, which is sometimes hidden from students by this formal and symbolic emphasis, as shown by Stewart & Thomas (2006a,b).

We noticed this lack of geometrical understanding when analyzing the responses given by students in a traditional linear algebra course in the Architecture degree to a test with a first question focused on manipulation of symbolic expressions, and a second one stated from a visual point of view. As Thomas & Stewart (2011) showed in their findings, students seemed confident with the algebraic and matrix procedures, but most of them had no geometric view of eigenvectors or eigenvalues, and could not establish a relationship between a diagram and eigenvectors. These results suggested that the ability to visualize linear algebra concepts does not come naturally to many students and it needs to be trained, even for architecture students, who are expected to be good visualizers.

Some authors explored the benefits of visualization in linear algebra. Tall (2004) stated that it would be helpful to present university students the embodied aspects of concepts, before focusing on the formal ideas. Thomas & Stewart (2011) also found some evidence to suggest that students who receive encouragement to think in a geometric way find it useful for understanding the procedural calculations they carried out. Also Harel (2000) reinforces these assertions, arguing that "understanding an algebraic system which does not have an easily accessible concrete or visual representation may derive in cognitive obstacles for students". The influence of visualization in the teaching of other concepts in linear algebra such as the independence and span of vectors was studied by Hannah *et al*. (2013).

These facts encouraged us to design a teaching proposal in order to emphasize visualization of eigenvectors and eigenvalues with the help of GeoGebra software, which has been proved to be effective promoting different modes of thinking when studying algebraic concepts (Caglayan, 2015). One recommendation from the Linear Algebra Curriculum Study Group (LACSG) suggested in (Carlson *et al.*, 1997) was the use of technology in the first linear algebra course. Other authors such as Schonefeld (1995) and Tabaghi & Sinclair (2013) also show the benefits of using computer applets during the learning process of these mathematical concepts. In our teaching proposal, the concepts of eigenvectors and eigenvalues are not presented through an abstract definition with no meaning for the students, but motivated by problems formulated in the world of physics: the search of the maximum and minimum moment of inertia of a rigid body, which is a useful matter in architecture. Other authors such as Salgado & Trigueros (2012) report the benefits of teaching eigenvalues and eigenvectors using modeling and physical applications.

In order to check whether our teaching proposal strengthened the geometrical understanding of eigenvectors and eigenvalues, we used the test shown in Figure 3. For the analysis and classification of students' responses, we used the three worlds of mathematical thinking: embodied, symbolic and formal, given by Tall (2004, 2008) and that has been used in the analysis of students' outputs when working with linear algebra concepts (Hannah *et al.*, 2016). The *embodied world* is where we make use of visual and physical attributes of concepts, combined with enactive sensual experiences to build mental conceptions. It is based on perception, initially seen and sensed in the real world but then imagined in the mind. The *symbolic world* is where the symbolic representations of concepts are manipulated, and where it is possible to "switch effortlessly from processes to do mathematics, to concepts to think about" (Tall, 2004, p. 30). The *formal world* is where properties of objects are formalized as axioms and logical deduction is used to build and prove theorems. This suggests the existence of different ways of thinking in mathematics which are not isolated, but interact, offering advantages. The embodied world gives visual understanding that can be translated into a symbolic language. The symbolic world offers tools in order to compute and develop the embodied world. And finally, the formal world provides the logical reasoning in order to create abstract mathematical objects.

2. Purpose of the study

The aim of this research is to determine if the designed teaching proposal, focused in the notion of the moment of inertia of a rigid body, allows students to give a geometrical meaning to the concepts of eigenvectors and eigenvalues. To this end, we elaborate a test and we analyze the responses given by the students attending the teaching proposal and others attending a traditional course with no emphasis in visualization. In order to check mathematical thinking differences between both groups when using the concepts of eigenvectors and eigenvalues, we classify their reasoning using Tall's worlds.

3. Methodology

In this section we detail the methodology used in this study: the features of the experiment, a brief description of the teaching proposal and the test we used to collect the data.

3.1 *The experiment*

This study is focused on the design of a teaching proposal to strengthen the geometrical understanding of eigenvalues and eigenvectors by showing some physical applications of these concepts. To check if our goal was fulfilled, the teaching proposal was developed in a group of 35 first-year students studying an algebra course in the Architecture degree at the Universitat Politècnica de València. None of them had previously received specific teaching related to eigenvectors and visualization of vectors and linear maps. The proposal was developed by the instructor during four 90 minute sessions through participative lecture classes. The teacher guided the students through questions they had to solve, and he made use of GeoGebra software to help the students to visualize the concepts. After the development of the teaching proposal, all students were given a test to check if it had had a positive influence in the geometrical understanding of the concepts of eigenvectors and eigenvalues. This test was also given to a group of 38 first-year students attending a traditional linear algebra course with no emphasis in visualization. In this group, the algebraic concepts treated in the teaching proposal were studied from a symbolic and formal point of view, with no reference to moments of inertia. Both groups had received the same formation in mathematics and had obtained similar marks in the first term of the mathematics course. In order to clarify the identification of the different types of answers given by the students to the test, we interviewed some of them.

3.2 *Development of the teaching proposal*

In this section we explain the teaching proposal in which we base our experience, related to the use of eigenvectors and eigenvalues in order to calculate the maximum and minimum moments of inertia of a rigid body. It makes emphasis in an exploratory phase placed in a real situation which motivates the introduction of the concepts and we use visualization and physical applications in order to help the students to think in the embodied world of mathematics. It has been designed to change the way of teaching, starting with a physical understanding of the concepts in the embodied world, which hardly has any presence in most algebra course lessons.

The teaching proposal in which we base our research follows a particular sequence of activities with the schema: exploration, introduction of concepts, structuring of knowledge and application proposed by Jorba & Sanmartí (1996). Exploration consists of activities that familiarize and motivate the student to study the subject by analyzing simple and real concrete situations related to the interests of the student. These activities are very important in the process of learning as stated by Edwards & Mercer (1987), Driver (1988) and Osborne & Freyberg (1985). The introduction of concepts leads the students to identify new points of view related to the topics of study, characteristics to define the concepts and relationships between the previous and the new ones. The structuring of knowledge happens when the student assimilates the concepts previously introduced and the relationships among them. This phase offers a good opportunity to relate the symbolic and formal world with the embodied one. Finally, during the application activities phase, students use the assimilated concepts in different situations and contexts. In what follows, we describe the four phases of our proposal.

3.2.1 *Exploration phase.* The exploration phase began by showing the students pictures with different daily physical situations that can be easily reproduced experimentally. These situations were related to the resistance an object offers to be spinned around with respect to an axis. The professor asked the students to determine in which situations it was easier to make the object turn around. When these questions were solved, the professor proceeded to show the students other situations slightly different, involving metal sheets of different shapes (see Figure 1).

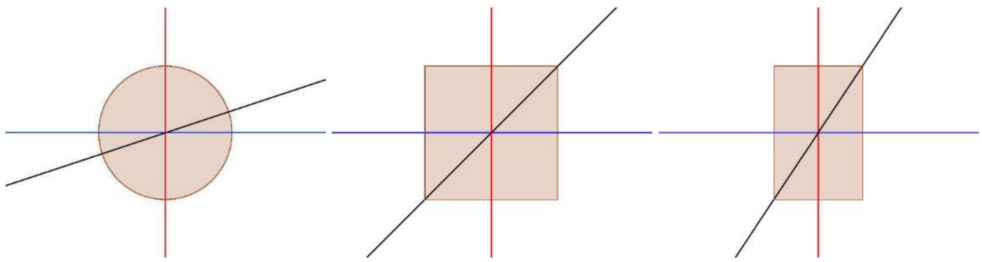


FIG. 1. Metal sheets shown to the students.

The professor formulated questions related to the effort needed to make the metal sheets spin around with respect to the different axes shown in Figure 1 and the physical property which measures this effort.

Through these questions, the previous physical knowledge of the students related to this phenomenon was revealed.

3.2.2 *Introduction of concepts.* The aim of this phase is to introduce the notion of moment of inertia. A measure of the resistance a thin flat plate offers to being spinned around respect to an axis is given by a scalar M called *area moment of inertia*. This scalar depends on the distribution of the area of the object A with respect to the axis and it is given by the multiple integral $M = \int r^2 dA$, where dA is the differential element of area and r is the distance of dA from the axis. If we fix the object in a coordinate system, its moment of inertia with respect to an axis that passes through the origin in the direction of the unitary vector v, can be calculated from its moments of inertia M_x and M_y with respect to the coordinate axes x and y, respectively. The expression that provides M is given by:

$$M = v^T \begin{pmatrix} M_x & -M_{x,y} \\ -M_{x,y} & M_y \end{pmatrix} v, \tag{3.1}$$

where $M_{x,y}$ is the so-called *product of inertia* respect to the coordinate axes. See (Meriam & Kraige, 2002, Appendix A) and (Meriam & Kraige, 2012, Appendix B) for more background and details.

In this phase, the professor explained the information above focusing on the fact that all the data about the resistance a thin flat plate offers to be spinned around with respect to an axis is given by the real and symmetric matrix $A = \begin{pmatrix} M_x & -M_{x,y} \\ -M_{x,y} & M_y \end{pmatrix}$, which is called *inertia matrix*. Moreover, he showed the students that expression (3.1) yields

$$M = v^T(Av) = \|v\|\|Av\|cos\alpha = \|Av\|cos\alpha, \tag{3.2}$$

where α is the angle between vectors v and Av. Consequently, M corresponds to the projection of vector Av onto vector v. This explanation was completed with a drawing on the board.

At this point, the teacher provided particular examples of inertia matrices of different flat surfaces to make the students find the vectors which maximized and minimized the moment of inertia M. The numerical calculations were replaced by a geometrical interpretation of the problem, with the help of the GeoGebra software. Making use of the dragging mode of GeoGebra, the students observed that unitary vectors were mapped drawing an ellipse and that the maximum and the minimum M were attained when the unitary vectors coincided with the directional vectors of the axes of the ellipse. Moreover, they could see that these vectors v maintained the same direction as Av, with $\alpha = 0$, and that the moments of inertia associated to them were equal to $\|Av\|$. See examples of the use of GeoGebra in Figure 2, where the flat surfaces under study are represented.

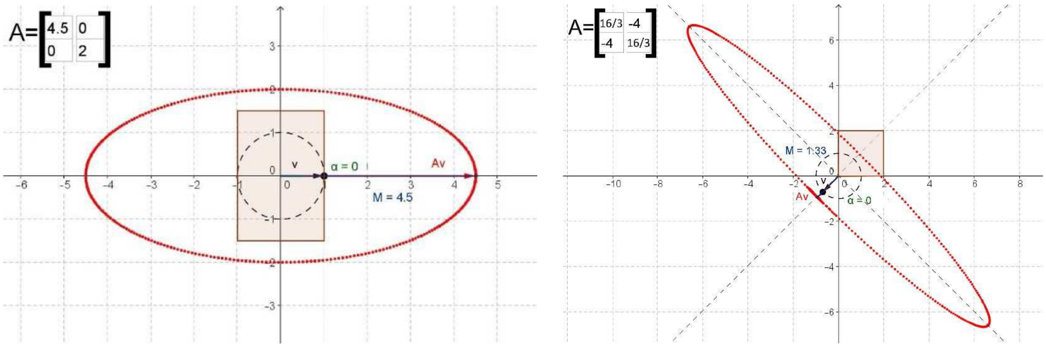


FIG. 2. GeoGebra applet for calculating moments of inertia

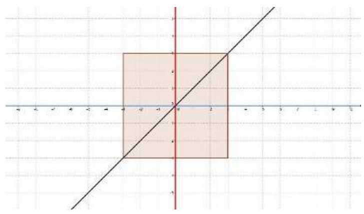
Later, the instructor gave the formal definition of an eigenvector and an eigenvalue, which can be associated to any square matrix, and he asked the students to find a relationship between these concepts and the moments of inertia. Next, the professor explained the procedure of calculating the eigenvalues and eigenvectors of a matrix justifying each step and following the suggestions given by Stewart & Thomas (2006a).

In order to visualize eigenvectors and eigenvalues of more general matrices, the professor considered the case of a non-symmetric one, which can not correspond to an inertia matrix. Making use of the GeoGebra applet, the professor showed that the unit circle is also mapped into an ellipse, although the directional vectors of the axes do not coincide with the eigenvectors of the matrix.

Throughout this phase, mathematical concepts of eigenvectors and eigenvalues are not mere abstract definitions and they are enriched through visualization and justified by physical applications.

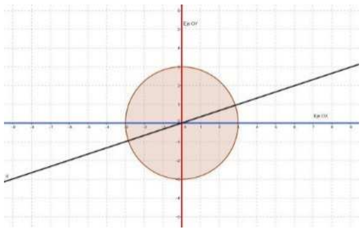
3.2.3 *Structuring phase.* In the structuring phase, in order to help the students to assimilate the concepts previously introduced and the relationships among them, the professor proposed the following activities to the students. The exercises are designed to encourage students to establish connections between the embodied, symbolic and formal worlds of mathematics.

(1) Given the matrix $A = \begin{pmatrix} 108 & 0 \\ 0 & 108 \end{pmatrix}$, which corresponds to the inertia matrix of the following surface:



Could you tell us with respect to which axis it is more difficult to spin around the figure? What would happen if we chose the axis given by $x = y$? Do these results coincide with your initial intuition?

(2) Although you do not know the inertia matrix of the following surface, could you find its eigenvectors and eigenvalues only by observing the following image? How many different eigenvalues does it have? Which form does the inertia matrix have?



3.2.4 *Application activities.* In this last phase, the professor showed the students other applications of eigenvectors and eigenvalues. One of them is the diagonalization of a matrix which is very useful in computation. He explained them how it makes easier the calculus of the nth-power of a matrix. They also studied other interesting applications in architecture such as the calculus of the tensions of a rigid solid.

3.3 *The test*

At the end of the teaching proposal, the test shown in Figure 3 was given to the students. It had also been used with the group of students attending the traditional course. Notice that no question refers to moments of inertia, since the purpose of the test was to determine if the designed teaching proposal, contextualized in the world of physics, allowed students to give a geometrical meaning to the concepts of eigenvectors and eigenvalues, and to examine possible differences in the reasoning of students attending both groups.

In question 1.a. we wanted to check if the students were able to work in the symbolic world and if they had learnt the procedure for calculating the eigenvectors of a matrix. In 1.b. we wanted to see if some students reason in the formal world of mathematics.

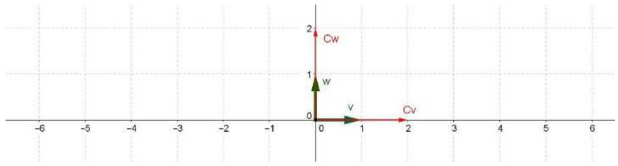
Question 2 was proposed to check if the students established connections among the three worlds of mathematical thinking. In particular, we wanted to see if the students were able to visualize eigenvectors and solve questions formulated from an embodied point of view. We also tried to examine if some students gave a non-symbolic justification of their answers.

1. Consider the following matrix and answer the following questions:

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$$

   a. Find all the eigenvalues and eigenvectors associated to A.
   b. Consider the vectors $v_1 = (2,0)$, $v_2 = (1,1)$ and $v_3 = (2,2)$. Are the vectors $v_1 + v_2$ and $v_2 + v_3$ eigenvectors of A? Justify your answer.

2. Look at the following picture:



   a. Is it possible to find a matrix C giving the previous configuration? In case it is possible, find its expression.
   b. Find the eigenvalues and eigenvectors of C. Justify your answer.
   c. Is it possible to find a matrix B giving the following configuration? Justify your answer.

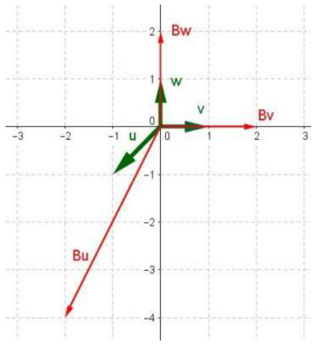


FIG. 3.Eigenvectors' geometric understanding test

4. The analysis of students' answers

In this section we detail how we have analyzed and classified the type of answers given by the students to the test, following Tall's classification of the three worlds of mathematics: symbolic, formal and embodied. For each type, we interviewed some students to clarify and confirm the kind of reasoning they used.

All students answered question 1.a. reasoning in the symbolic world of mathematics. Responses to question 1.b. were identified as **symbolic** when students:

- **Type s1:** checked that the sum of the vectors $v_1 + v_2$ and $v_2 + v_3$ verifies the eigenvectors expressions,

- **Type s2:** looked for λ such that $A(v_1 + v_2) = \lambda(v_1 + v_2)$ and $A(v_2 + v_3) = \lambda(v_2 + v_3)$.
  In Type s2, students used the formal definition of an eigenvector, but they proceeded in a symbolic way to give the answer.

The responses to this question were categorized in the **formal** world for those students who:

- **Type f1:** reasoned that the sum of eigenvectors is an eigenvector if and only if all vectors belong to the same subspace.

We interviewed student S1, who gave a Type s1 answer to question 1.b.

P: How did you check if the sum of $v_1 + v_2$ and $v_2 + v_3$ was an eigenvector?
S1: First, I summed the vectors and later I checked if the vectors had the expression (x, x) or $(x, 0)$.
P: Why did you check it with $(x, 0)$ and (x, x) and not with another expression?
S1: Because in question 1.a. I calculated all the eigenvectors and they had this form.

We observed that this student reasoned in a symbolic way by using the general expressions of the eigenvectors that he had previously obtained in question 1.a. We also interviewed student S2, who answered the question in a formal way.

P: How did you solve question 1.b.?
S2: I observed that v_1 was an eigenvector associated to $\lambda = 3$ and v_2 and v_3 were eigenvectors associated to $\lambda = 2$. Then $v_2 + v_3$ must be an eigenvector.
P: Why?
S2: Because the sum of eigenvectors is an eigenvector.
P: But in your response, you only said that $v_2 + v_3$ is an eigenvector, you didn't say anything about $v_1 + v_2$.
S2: Hmmm.... This sum is not an eigenvector.
P: Why not?
S2: Because they are associated to different eigenvalues and then they don't belong to the same subspace.

Now, we proceed to classify the answers given to question 2.a., where we observe embodied type answers among the students who attended the teaching proposal. The answers given by the students to this question were categorized in the **symbolic** world when:

- **Type s1:** students calculated the expression of the matrix using equations

in the **formal** one for those students who:

- **Type f1**: identified the column vectors as the images of the canonical basis

and in the **embodied** world for those students who:

- **Type e1:** established a connection between the vectors and the moments of inertia.
- **Type e2:** identified the canonical vectors in the picture as eigenvectors associated to the eigenvalue 2 and said that every vector is doubled.

We have considered Type e2 in the embodied world of mathematics because students are able to identify that the canonical vectors are eigenvectors, since they maintain their direction. However, students also use the formal thinking in this type of answer since they see that every vector is doubled, by linearity.

It must be stressed that one student gave the expression of the matrix after visualizing the image configuration, without any justification. After interviewing him, we could find out he had reasoned in an embodied way (Type e1):

P: How did you find A? You didn't write anything!
S3: Well, I saw the picture and I remembered the exercises about the moment of inertia we did at class. Since $(1, 0)$ and $(0, 1)$ are mapped into its double, this corresponded to the moment of inertia of a circle, and its inertia matrix was a diagonal matrix with 2 in the diagonal.

The answers given to question 2.b were classified into the **symbolic** world when:

- **Type s1:** students calculated the characteristic polynomial without realizing that C is a diagonal matrix or that the canonical vectors are eigenvectors,

we classified them as **formal** when:

- **Type f1:** they realized that C is a diagonal matrix and thus, the canonical vectors are eigenvectors and the elements in the diagonal are eigenvalues,

and they were classified into the **embodied** world when:

- **Type e1**: students related the configuration with the moments of inertia,
- **Type e2:** identified the canonical vectors in the picture as eigenvectors associated to the eigenvalue 2, since are vectors maintaining their direction.

The last question was classified following the same criteria.

5. Results

In this section we show the results of our research. In Table 1, for each question, we collect the percentages of students that answered correctly, incorrectly and did not answer, and the world of mathematical thinking they used to solve them: embodied (E), symbolic (S) or formal (F).

We observe that students attending the teaching proposal (group B) obtained better results in question 2 than students attending a traditional course (group A), especially in questions 2.a and 2.c. These questions were asked to the students in order to check if they could establish a connection between the geometric transformation shown in the diagram and its matrix form. We also observe that group B has a higher percentage of answered questions. We interpret that students have elaborated their own conception of the content in the course, although this is sometimes not enough to provide a correct answer. These combined facts provide some evidence to say that our teaching proposal, focused in the notion of moment of inertia of a surface, has had a positive effect in the geometrical understanding of the concepts of eigenvectors and eigenvalues. On the other hand, in both groups the percentage of students who succeeded in question 2 is inferior than the one in question 1, reinforcing the assertion given by Thomas & Stewart (2011) that students are more confident with the algebraic and matrix procedures than using other kind of reasoning.

Table 1. Classification of students' answers in group A and group B (% of students)

Question	Group	Correct answer			Incorrect answer			No response
		E	S	F	E	S	F	
1.a	A	0	94.7	0	0	5.3	0	0
	B	0	85.7	0	0	14.3	0	0
1.b	A	0	31.6	47.4	0	0	5.3	15.8
	B	0	22.9	62.9	0	2.9	5.7	5.7
2.a	A	0	26.3	18.4	0	21.1	0	34.2
	B	22.9	14.3	20	0	0	0	42.9
2.b	A	0	10.5	5.3	0	31.6	0	52.6
	B	8.6	5.7	5.7	28.6	2.9	5.7	42.9
2.c	A	0	18.4	13.2	0	13.2	0	55.3
	B	5.7	34.3	17.1	0	0	0	42.9

Attending to the kind of reasoning promoted, we observe that, whereas in the traditional group we did not find responses reasoned in the embodied world of mathematics, in the group attending the teaching proposal we found responses in each of the three worlds. In particular, we observe that the symbolic world's reasoning prevails among students in group A and that question 2 promotes the embodied thinking in students in group B, something which has not been observed in group A. In Table 2 it is shown that most students in both groups tend to use more than one world of mathematics to answer the test. The difference is that, whereas in group A students only combined the symbolic and formal world, in group B, 40% of students combined the three worlds of mathematics.

Table 2. Number of worlds of mathematical thinking used (% of students)

Group	1 world	2 worlds	3 worlds
A	31.6	68.4	0
B	17.1	42.9	40

We remark that, when only using one world, students used the symbolic one, and when using two of them, they combined the symbolic and formal. All students who used the embodied world to answer a question also used the formal and symbolic one in other parts of the test. When answering question 2.a., most students who reasoned in the embodied world of mathematics used the embodied Type e2 combined with the formal thinking, which was necessary to understand the behavior of linear maps.

It must be stressed that most students who reasoned in the embodied world in question 2.b. failed saying that the eigenvectors were just those represented in the picture. The students who gave a correct answer for the question 2.b. combined Type e2 answer with formal thinking, which allowed them to see the implicit information contained in the diagram. However, the higher percentage of students answering this question in group B seems to indicate that the embodied world of mathematics provides a new tool to address the problem, although sometimes it is not enough and it must be combined with others in order to solve it correctly.

In view of the results, there is some evidence to say that, in the experiment, our teaching proposal enhances the interaction between the three worlds of mathematical thinking given by Tall.

6. Discussion and conclusions

After developing our teaching proposal we have noticed that adding a geometric perspective in the teaching of linear algebra helps the students to solve problems from a new perspective. In this sense, traditional teaching limits the students' possibilities since it does not offer so many ways of structuring the knowledge. We have observed that students who attended the teaching proposal show more ability to integrate a geometric view of the concepts of eigenvectors and eigenvalues, which are usually introduced algebraically. In most traditional linear algebra courses, the students are expected to give different perspectives of the study by their own. This teaching proposal shows the benefits of providing different ways of representing algebraic concepts at class. We have noticed that this style of teaching helps students to establish connections between the different worlds of mathematical thinking given by Tall.

In contrast, our study does not allow us to analyze why some students of the group attending the teaching proposal did not make use of the geometric component for solving the questions. A possible future work could include some improvements in the experiment that allow us to analyze if their perception of the concepts is different. Also, it must be stressed that most students who solved question 2.b. geometrically gave an incorrect answer. This fact remarks the need of integrating different types of thinking to understand the implicit information contained in the diagrams.

The teaching proposal has strengthened the geometrical understanding of the concepts and it has also promoted students' thinking in the three worlds of mathematics. In this sense, we consider that our results complement Hannah *et al.* (2016)'s study, which reports the benefits of working linear algebra concepts in Tall's three worlds of mathematics. Furthermore, it contributed to relate mathematics and some concepts of physics which are important in an architecture degree. Most students, even those who did not understood completely the explanations, considered that it had helped them to improve the visualization of the concepts, and it motivated the mathematical content. Comments such as "The task design has been useful for making more visual what we study and to relate the concepts we learn with physics", reinforce this fact. For future research we consider it would be interesting to analyze the students' self-perception of the learning after the teaching proposal is developed, in order to see if they are aware of the different mathematical worlds used in their reasoning and if they have changed their way of thinking in linear algebra.

After the teaching experiment, we realized that some improvements related to our proposal could be done. We consider that it would be interesting to develop the introduction of concepts phase in a computer laboratory, where students could manipulate GeoGebra by themselves in order to familiarize with the behavior of linear maps acting on vectors. As Tabaghi & Sinclair (2013) concluded, the use of software contributes to a deeper understanding of abstract concepts. Another aspect to be considered is the visualization of eigenvectors and eigenvalues of 3×3 matrices using GeoGebra3D.

REFERENCES

CAGLAYAN, G. (2015) Making sense of eigenvalue–eigenvector relationships: Math majors linear algebra-geometry connections in a dynamic environment. *The Journal of Mathematical Behavior*, 40, 131–153.

CARLSON, D., JOHNSON, C., LAY, D. & PORTER, A. D. (1997) The Linear Algebra Curiculum Study Group recommendations for the first course in linear algebra. *Resources for Teaching Linear Algebra, MAA Notes* (D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins & W. Watkins eds), vol. 42. Mathematical Association of America: Mathematical Association of America, pp. 53–58.

DORIER, J. L. & SIERPINSKA, A. (2001) Research into the teaching and learning of linear algebra. *The teaching and learning of mathematics at university level: An ICMI study* (D. Holton ed.). Dordrecht: Kluwer Academic Publishers, pp. 255–273.

DRIVER, R. (1988)Un enfoque constructivista para el desarrollo del currículo en ciencias. *Enseñanza de la*

Ciencias, 2, 109–120.

EDWARDS, D. & MERCER, N. (1987) *Common Knowledge: The Development of Understanding in the Classroom*. London: Routledge.

HANNAH, J., STEWART, S. & THOMAS, M. (2013) Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematical Education in Science and Technology*, 44, 475–489.

HANNAH, J., STEWART, S. & THOMAS, M. (2016) Developing conceptual understanding and definitional clarity in linear algebra through the three worlds of mathematical thinking. *Teaching Mathematics and its Applications*.

HAREL, G. (2000) Three principles of learning and teaching mathematics. *The teaching of linear algebra in question* (J. L. Dorier ed.). Dordrecht: Kluwer Academic Publishers, pp. 177–189.

JORBA, J. & SANMARTÍ, N. (1996) *Enseñar, aprender y evaluar: un proceso de regulación continua. Propuestas didácticas para las áreas de Ciencias de la Naturaleza y Matemáticas*. Madrid: Ministerio de Educación y Ciencias. C.I.D.E.

MERIAM, J. & KRAIGE, L. (2002) *Engineering Mechanics: Statics*. United States of America: John Wiley and Sons Inc.

MERIAM, J. & KRAIGE, L. (2012) *Engineering Mechanics: Dynamics*. United States of America: John Wiley and Sons Inc.

OSBORNE, R. & FREYBERG, P. (1985) *Learning in Science. The implication of children's Science*. Londres: Heineman.

SALGADO, H. & TRIGUEROS, M. (2012) Teaching eigenvalues and eigenvectors with a modeling approach. *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (S. Brown, S. Larsen, K. Marrongelle & M. Oerthman eds). Portland: Portland, pp. 149–155.

SCHONEFELD, S. (1995) Eigenpictures: picturing the eigenvector problem. *The College Mathematics Journal*, 26, 316–319.

SIERPINSKA, A., DREYFUS, T. & HILLEL, J. (1999) Evaluation of a teaching design in linear algebra: The case of linear transformations. *Recherches en didactique des mathématiques*, 19, 7–40.

STEWART, S. & THOMAS, M. O. J. (2006a) Process-object difficulties in linear algebra: Eigenvalues and eigenvectors. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*. Prague, Czech Republic: Prague, Czech Republic.

STEWART, S. & THOMAS, M. O. J. (2006b) Student thinking about eigenvalues and eigenvectors: Formal, symbolic and embodied notions. *Proceedings of the 29th Mathematics Education Research Group of Australasia Conference*. Canberra, Australia: Canberra, Australia.

TABAGHI, S. G. & SINCLAIR, N. (2013) Using dynamic geometry software to explore eigenvectors: the emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning*, 18, 149–164.

TALL, D. O. (2004) Building theories: the three worlds of mathematics. *For the Learning of mathematics*, 24, 29–32.

TALL, D. O. (2008) The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20, 5–24.

THOMAS, M. O. J. & STEWART, S. (2011) Eigenvalues and eigenvectors: embodied, symbolic and formal thinking. *Mathematics Education Research Journal*, 23, 275–296.

UHLIG, F. (2002) The role of proof in comprehending and teaching elementary linear algebra. *Educational Studies in Mathematics*, 50, 335–346.