WHEN DOES CHAOS APPEAR WHILE DRIVING?: LEARNING DYNAMICAL SYSTEMS VIA CAR-FOLLOWING MODELS

J. ALBERTO CONEJERO, MARINA MURILLO-ARCILA, JESÚS M. SEOANE-SEPÚLVEDA, AND JUAN B. SEOANE-SEPÚLVEDA

To the loving memory of Dr. D. José Manuel Seoane Capote (1942–2016)

ABSTRACT. We analyze the dynamics of a system of several cars in a traffic lane, each car following the ones in front of it. The effect of small perturbations in the speed of a certain car is propagated to the cars behind of it in the lane. Nevertheless, these perturbations tend to dissipate along the lane. The results can be used as an activity taught at a third year mathematics undergraduate students to improve the knowledge of dynamical systems, modeling and physical interpretation of mathematical models.

1. Introduction

In Mathematics, a *dynamical system* is a system in which a function describes the time dependence of a point in a geometrical space. There are many examples of this, which include the very famous mathematical model describing the swinging of a clock pendulum (which started with Galileo's research in 1602) and the three dimensional famous Lorenz attractor, which provided the earliest example of chaos in a dynamical system, in the early 1960's.

Teaching dynamical systems is far from being a simple task. At the moment there are too many potential examples and models that one could use to present this notion, however not all of them are accessible to everyone due to either their technicality or their complexity and level of abstraction. The most common way to introduce this notion at an undergraduate level is by means of the famous Lotka-Volterra system. The classical Lotka-Volterra system is a two dimensional system in which the concept of stability can be easily presented by means of the typical example of a predator/prey situation. For instance, a typical example is seen in

$$\text{(LV)} \qquad \begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) \\ y'(t) = -b_1 y(t) + b_2 x(t) y(t), \end{cases}$$

with $a_1, a_2, b_1, b_2 > 0$, and Figure 1 shows the typical trajectories of such a system.

In this paper we are interested in innovating by employing a different approach on teaching dynamical systems, we propose the use of the "not that typical" car-following models.

Car-following models appeared with the intention of describing a driver's reaction to the changes in the speed to the car in front of him on a single lane. Modeling this behavior is necessary for

Key words and phrases. dynamical systems; car-following models; quick-thinking-driver model; near-nearest model.

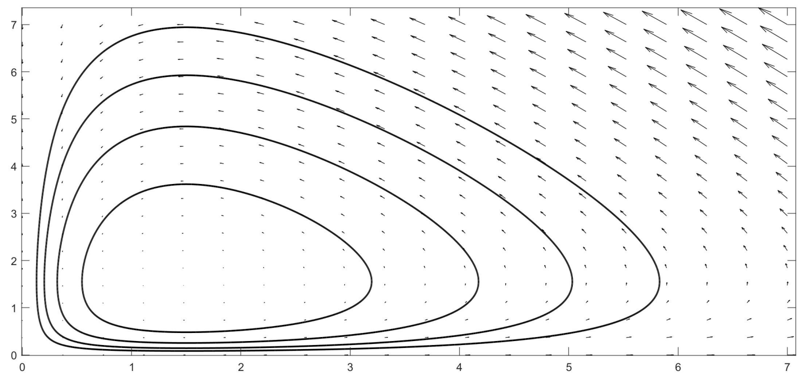

FIGURE 1. Trajectories for a the classical Lotka-Volterra system such as system (LV) for values $a_1 = 1.4$, $a_2 = 0.9$, $b_1 = 1.8$ and $b_2 = 1.2$.

the development of traffic flow theory. The first car-following models were due to Greenshields [Gre34, Gre35] in the 1930's. During 50's and 60's, car-following models were refined by taking into account considerations involved in driving a motor vehicle on a lane [Cum64] such as the difference between the velocities of a car and the car in front of it, the distance of a car respect to the preceding one, or the driver's reaction time, see for instance [For63, Pip53]. Chandler et al. [CHM58] and Herman et al. [HMPR59] proposed a mathematical model which assumes that the acceleration of the following car in each two-vehicle unit is linearly proportional to the cars relative velocities at some earlier time, with a fixed time lag of transmission of the driver-vehicle system. This model is well-known as the *Quick-Thinking Driver* QTD model.

In the practice, the acceleration of a car does not only depend on the velocity of the car in front but it also slightly depends on the velocity of a car two ahead as it is considered in the nearest and *Next-Nearest* NN model. It can be also modeled taking into account the speeds of the cars that go in front and behind of it as it is considered in the *Forward and Backward Control* FBC model. An interested reader can find a historical evolution of these models in [BM99, HB01].

Chaos is closedly linked with car-following models. Even in a simple model such as the QTD one, it is possible to find chaos relating its dynamics to certain solutions of the logistic equation [MG04, McC09]. Such a model is a particular case of a more general non-linear car-following developed by Gazis, Herman, and Rothery (GHR) for General Motors [GHR61, Rot92]. The discontinuous behavior of some of its solutions suggested the existence of chaos for a certain range of input parameters. Other authors studied the existence of chaos for this model under some assumptions: Disbro and Frame [DF89] showed chaos for the (GHR) model without taking into account signals, bottlenecks, intersections, etc. or with a coordinated signal network. In [AL98, AMLC96] chaos was observed for a platoon of vehicles described by the (GHR) model when adding a nonlinear inter-car separation dependent term.

More recently, in [BCMASS15, CMASS16] the authors studied the existence of chaos for different car-following models for an infinite number of cars driving on a road using techniques from semigroup theory. It is worth to mention that chaos can be also found when studying traffic models at macroscopic level, as it is the case of the Lighthill-Whitham-Richards [CMGPR16].

Our concern in this paper is to study the dynamics of the continuous dynamical system that represents the behavior of cars driving on a road when considering some classical car-following models such as the QTD and the NN models. More concretely, we determine their equilibria and stability in terms of the parameters involved in the models. Moreover, we illustrate the outcome with numerical solutions.

The models proposed in this paper can be considered in teaching for many different applications such as the study of dynamical systems and differential equations and the improvement of computing skills with mathematical software such as Maple, Matlab, R or Mathematica. It is also good as a proposal for developing skills in model formulation, solution and interpretation.

2. Preliminaries

We first introduce the models that we are going to study. In the basic formulation of any of the car-following models, there is a relation between the acceleration of a car and the difference between its velocity and the velocity of the car that goes in front of it. In a basic formulation for a car-following model, the driver of a car adjusts her speed according to the relative velocity between her car and the one in front, that is,

$$x_1''(t+t_1) = \lambda_1(x_2'(t) - x_1'(t)), \tag{1}$$

where $x_2(t)$ denotes the position of the car which goes in front of car 1 at time t whose position is given by $x_1(t)$, t_1 denotes the reaction time of driver 1 and the positive number λ_1 is a sensitivity coefficient that measures how strong the driver 1 responds to the acceleration of the car in front of her. Usually λ_1 lies between $0.3 - 0.4s^{-1}$ [BM99]. Under the assumption that all drivers react "very quickly", one can take $t_1 = 0$. This is known as the *Quick-Thinking-Driver* QTD model.

$$x_1''(t) = \lambda_1(x_2'(t) - x_1'(t)), \tag{2}$$

This model can be reformulated in terms of velocities $u_1(t) = x_1'(t)$ and $u_2(t) = x_2'(t)$

$$u_1'(t+t_1) = \lambda_1(u_2(t) - u_1(t)), \tag{3}$$

It can be also improved when taking into account that the reaction time depends on the speed of the car, as it is indicated in [McC09] and in [MG04, p. 92]. This leads us to formulate a modified version of it:

$$u_1'(t) = \gamma_1 u_1(t)(u_2(t) - u_1(t)), \tag{4}$$

As it is indicated in [MG04], the models in (3) and (4) should provide the same acceleration for the same relative velocity. For instance, in the case of a car moving at 45 km/h, namely about 13m/s, in order that models (3) and (4) should predict the same acceleration we take $13\gamma_1 = \lambda_1$, and typical values of γ_1 will be in the range of 0.02 to 0.03 s^{-1}. For further details on driving simulation, we refer to the excellent handbook edited by Fisher *et al* [FRCL11, Ch.5,7 & 12].

It is also interesting to investigate the effect of a control that involves the car two ahead in addition to the car in front as it usually happens in the practice. This model is known as the *nearest and next-nearest* NN model which is given by

$$u_1'(t) = \lambda_{1,1}(u_2(t) - u_1(t)) + \lambda_{1,2}(u_3(t) - u_1(t)), \tag{5}$$

in which $\lambda_{1,1}$ stands for the sensitivity coefficient in relation with the car ahead and $\lambda_{1,2}$ the one with the car two ahead, such that $\lambda_{1,1} + \lambda_{1,2}$ plays the role of λ_1.

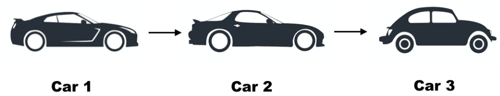

FIGURE 2. (a) On the left, the case of 3 cars, the speed of the leading car constant and equal to v, and the speed of the others behind, $u_2(t)$ and $u_1(t)$, respectively. (b) On the right, the speed of the car ahead will be denoted as $u_3(t)$, and the speed of the car behind, $u_2(t)$ and $u_1(t)$. Designed by Freepik.

Analogously, one can also improve it in the same way as (4). This yields

$$u_1'(t) = \gamma_{1,1}u_1(t)(u_2(t) - u_1(t)) + \gamma_{1,2}u_1(t)(u_3(t) - u_1(t)), \tag{6}$$

with $\gamma_{1,1} + \gamma_{1,2}$ instead of γ_1.

Some results related to the stability of dynamical systems will be needed. Let us consider a dynamical system on $\mathbb{R}^n_{+,0}$ of the form $x' = f(x)$, $x \in \mathbb{R}^n_{0,+}$ and f a differentiable function. We recall that an equilibrium point x_0 is called *hyperbolic* if all the eigenvalues of the Jacobian matrix $J(x_0)$ have nonzero real part. Such a point is called a *sink* if all the eigenvalues of $J(x_0)$ have negative real part. It is said to be a *source* if all the eigenvalues have positive real part and it is a *saddle point* if it is a hyperbolic point and it has at least one eigenvalue with positive real part and one with negative real part. In terms of stability, sinks correspond to asymptotically stable equilibria points. Hyperbolic equilibrium points are *unstable* if and only they are saddles or sources. The stability of nonhyperbolic equilibrium points is more dificult to determine and it is typically necessary to use the famous Lyapunov functions. In section 3, we study the equilibria of the Quick-Thinking-Driver model with 3 cars and we also analyze its trajectories and the ones of the perturbed model by adding an oscillation term.

In Section 4, we first study the situation of three cars with the leading car driving with a fixed speed and two cars following it and one to each other. Such a model can be perturbed in order to provide a cyclic orbit to which the speeds converge.

In Section 5, we add an additional car and we compare the QTD and NN models. We present two situations: In the first one, the cars start with different speeds, but as time goes by, their speeds tend to the one of the leading car. In the second one, we perturbate the speed of the leading car and we analyze the propagation of that perturbation along the cars on the lane.

Finally, in Section 6, we propose some possible extensions and class activities.

3. The Quick-Thinking-Driver model with 3 cars

First, we study the stability of the QTD model of 3 cars, with the leading car driving at constant speed. Here, the model describing the speed of the cars behind the leading car can be described by the following 2 equations:

$$\begin{cases} u_1'(t) = \gamma_1 u_1(t)(u_2(t) - u_1(t)) \\ u_2'(t) = \gamma_2 u_2(t)(v - u_2(t)) \end{cases} \tag{7}$$

Solving the system

$$\begin{cases} 0 = \gamma_1 u_1(u_2 - u_1) \\ 0 = \gamma_2 u_2(v - u_2) \end{cases} \tag{8}$$

we obtain the following 3 equilibrium points: $P_1 = (0,0)$, $P_2 = (0,v)$, $P_3 = (v,v)$, which can be seen by looking at the nullclines, see Figure 3. Looking at the phase plane, we can appreciate that P_3 is an attractor. To confirm this, we now calculate the eigenvalues associated to the equilibrium points P_1, P_2, and P_3. The Jacobian of the system is given by

$$J(u_1,u_2) = \begin{pmatrix} \gamma_1(u_2 - 2u_1) & \gamma_1 u_1 \\ 0 & \gamma_2(v - 2u_2) \end{pmatrix}. \tag{9}$$

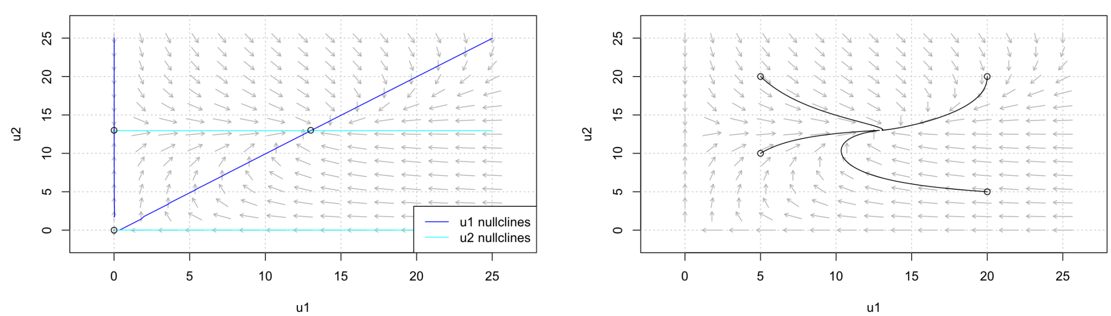

FIGURE 3. On the left, phase plane, equilibrium points, and nullclines for the solution of system (7), where $\gamma_1 = \gamma_2 = 0.03$, and $v = 13m/s$. On the right, we see the trajectories for the initial conditions $(5,10)$, $(5,20)$, $(20,5)$, and $(20,20)$ after $t = 15s$.

We next obtain the eigenvalues associated to $J(P_i)$, $i = 1,\dots,3$. $J(P_1)$ has a null and a positive eigenvalue $\gamma_1 v$, but $J(P_2)$ has a negative (- $\gamma_2 v$) and a positive ($\gamma_2 v$) eigenvalue. Anyway, both P_1 and P_2 are unstable. On the contrary, P_3 has both eigenvalues, $-\gamma_1 v$ and $-\gamma_2 v$, with negative real part and it is a stable equilibrium point for the system. This can be seen also by looking at the trajectories depicted in Figure 3.

If we perturb the speed of the leading car by a term $\sin(t)$,

$$\begin{cases} u_1'(t) = \gamma_1 u_1(t)(u_2(t) - u_1(t)) \\ u_2'(t) = \gamma_2 u_2(t)(v + \sin(t) - u_2(t)) \end{cases} \tag{10}$$

the speed falls into a cycle around the point (13,13), see Figure 4, which is a similar situation to what happens with Lotka-Volterra trajectories, but it is obtained in this case through a non-autonomous dynamical system. This can be related with the existence of pullback attractors, see for instance [HY18]. More information about the topological connections of both systems in the discrete case can be found in [Bal16]. From a physical point of view, these kind of perturbations are very typical to model dynamical systems in the presence of external perturbations as it occurs, for instance, with the pendulum [Git10]. In this specific case, this perturbation can be seen in situations in which the driver accelerates or decelerates as a consequence of a multiple collision which take place when the velocities are equal.

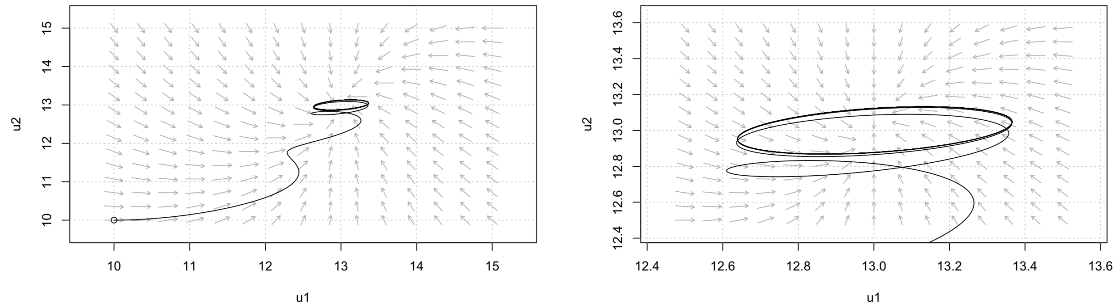

FIGURE 4. On the left, the trajectory for the initial condition $(10, 10)$ after 100 s. On the right, an amplified view of the converging cycle.

4. STABILITY OF TRAFFIC MODELS WITH 3 CARS

Let us analyze the stability of the dynamical systems that model how three cars will progress in time, following a car at fixed speed, when we consider the QTD and the NN models. We consider that the leading car goes at constant speed v, which is followed by cars 3, 2, and 1.

4.1. **Stability of the QTD model.**

First, we concentrate on the study of the dynamics of these three cars when considering they follow the leading one according to QTD model.

$$\left\{ \begin{array}{l} u_1'(t) = \gamma_1 u_1(t)(u_2(t) - u_1(t)) \\ u_2'(t) = \gamma_2 u_2(t)(u_3(t) - u_2(t)) \\ u_3'(t) = \gamma_3 u_3(t)(v - u_3(t)) \end{array} \right. \tag{11}$$

Solving the system

$$\left\{ \begin{array}{l} 0 = \gamma_1 u_1(u_2 - u_1) \\ 0 = \gamma_2 u_2(u_3 - u_2) \\ 0 = \gamma_3 u_3(v - u_3) \end{array} \right. \tag{12}$$

we obtain the equilibrium points: $P_1 = (0, 0, 0)$,$P_2 = (0, 0, v)$,$P_3 = (0, v, v)$, and $P_4 = (v, v, v)$.

We now calculate the eigenvalues associated to the equilibrium points P_1, P_2, P_3, and P_4. The Jacobian of the system is given by

$$J(u_1, u_2, u_3) = \begin{pmatrix} \gamma_1(u_2 - 2u_1) & \gamma_1 u_1 & 0 \\ 0 & \gamma_2(u_3 - 2u_2) & \gamma_2 u_2 \\ 0 & 0 & \gamma_3(v - 2u_3) \end{pmatrix}. \tag{13}$$

We next obtain the eigenvalues associated to $J(P_i)$, $i = 1, \ldots, 4$. $J(P_1)$ has $\gamma_3 v$ as a positive eigenvalue, and then P_1 is not stable. Points P_2 and P_3 are unstable equilibrium points because they both have positive and negative eigenvalues, and P_4 is a stable equilibrium point for the system since all its eigenvalues have negative real part. It is important to point out that an equilibrium for system (11) corresponds to a stationary solution, namely a solution for which each one of the three cars has constant velocity.

Since our model is nonlinear, it may present chaotic motions. For checking whether it is chaotic or not, we analyze both, the bifurcation diagrams and the Lyapunov exponent of the system in order to show the global behavior of our model for different values of the parameters.

On the one hand, *bifurcation diagrams* provide graphical information on the changes in the dynamics of the system in terms of one of its parameters [Str14]. They are very useful for checking, from a qualitative point of view, if the system is chaotic or not. For plotting it, we take an arbitrary initial condition and we compute the final state of the system versus a chosen parameter of it. If the system is periodic of period n, n points appear in a vertical line in our bifurcation diagram. Otherwise, if the system is chaotic, a continuous vertical line is depicted in the diagram.

On the other hand, the *Lyapunov exponent* is the most common tool to have a quantitative indicator to observe chaotic motions [NY12]. These tools are the most useful indicators for characterizing possible chaotic regimes in a dynamical system but, in general, they are not so known for undergraduate students in Math and other Sciences. The Lyapunov exponent, namely λ, indicates the divergence between two trajectories of the system which start their motion with very similar initial conditions. It measures this distance as a function of the time and it can be calculated as follows:

$$\lambda = \lim_{t\to\infty} = \frac{1}{t} \ln \frac{||\delta x(t_i)||}{||\delta x_0||} \tag{14}$$

where $||\delta x(t_i)||$ denotes the distance between the trajectories after the time $t = t_i$ and $||\delta x_0||$ denotes the distance between the trajectories at the initial time $t = 0$. If the system diverges, the Lyapunov exponent is positive and therefore our system presents chaotic motions. This is due to the nonlinear nature of our equations and therefore it satisfies the necessary condition for that. We can observe that these 2 nearby trajectories are sensitive to the initial conditions and their distance increases with respect to time in an exponential manner and therefore, if the equations are nonlinear, they become chaotic.

Otherwise, our system is stable and periodic motions take place. Figure 5 illustrates these last comments, showing both, the bifurcation diagram and the Lyapunov exponent of the QTD model by taking as a parameter, the sensitivity coefficient $\gamma = \gamma_i$, $i = 1, 2, 3$, representing the drivers' reactions times of a driver in a realistic situation. In Figure 5 we consider speeds around 13 m/s and values of γ around 0.3 s^{-1}. To compute both, we have taken as initial condition $(u_1(0), u_2(0), u_3(0)) = (20, 13, 10)$, a physical situation in which collisions can take place since the first and the second car are at rest and the last one has positive velocity.

Notice that, to compute numerically the Lyapunov exponents, we take high values of the integration times according to the experiment we have carried out. In our case, we have taken 500 $t.u$ which is very large in comparison with the times we use in the computations of the trajectories. In that sense, the time, in these practical situations, can be considered as infinity and therefore the estimation of the Lyapunov exponents is very accurate.

In both plots, we clearly see the final stabilization of the dynamics of our system and therefore the nonexistence of chaotic motions. This result is interesting since the equations are nonlinear but, in a practical case, there are no vehicle collisions and the cars finish in a stable situation.

4.2. **Stability of the NN model.** We now suppose that the three cars following the leading one at constant speed behave according to a NN model. Now, the acceleration of each car does not only depend on the velocity of the car just in front but also on the two cars ahead of it. For the car that only has one car ahead of them, we consider they follow the QTD model.

In this case, the system describing the model can be written as follows:

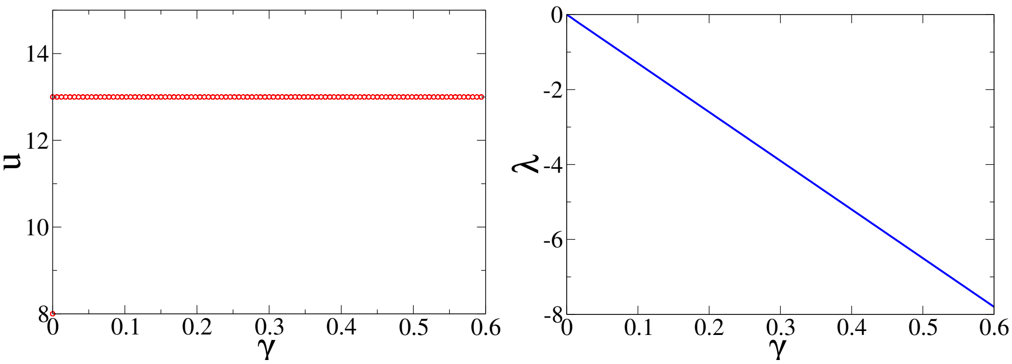

FIGURE 5. Plot of both, the bifurcation diagram, by plotting u versus γ and the Lyapunov exponent versus γ. We can observe that the motions are not chaotic for any parameter value and the system is always stable. The initial condition is given by $(u_1(0), u_2(0), u_3(0)) = (20, 13, 10)$. Observe in this case the value of velocity $v = 13$ m/s as an asymptotic fixed point and therefore there is not presence of chaos. The corresponding Lyapunov exponent distribution corroborates it properly.

$$\left\{ \begin{array}{l} u_1'(t) = \gamma_{1,1} u_1(t)(u_2(t) - u_1(t)) + \gamma_{1,2} u_1(t)(u_3(t) - u_1(t)) \\ u_2'(t) = \gamma_{2,1} u_2(t)(u_3(t) - u_2(t)) + \gamma_{2,2} u_2(t)(v - u_2(t)) \\ u_3'(t) = \gamma_3 u_3(t)(v - u_3(t)) \end{array} \right. \tag{15}$$

We now obtain the equilibrium points of the system

$$\left\{ \begin{array}{l} 0 = \gamma_{1,1} u_1(u_2 - u_1) + \gamma_{1,2} u_1(u_3 - u_1) \\ 0 = \gamma_{2,1} u_2(u_3 - u_2) + \gamma_{2,2} u_2(v - u_2) = 0 \\ 0 = \gamma_3 u_3(v - u_3) = 0 \end{array} \right. \tag{16}$$

Again, we also have $P_1 = (0,0,0)$, $P_2 = (0,0,v)$, $P_3 = (0,v,v)$, $P_4 = (v,v,v)$, as in the previous case, and $P_5 = (0, \frac{v\gamma_{2,1}}{\gamma_{2,1}+\gamma_{2,2}}, 0)$, $P_6 = (\frac{v\gamma_{1,2}}{\gamma_{1,1}+\gamma_{1,2}}, 0, v)$, and $P_7 = (\frac{v\gamma_{1,1}}{(\gamma_{1,1}+\gamma_{1,2})(\gamma_{2,1}+\gamma_{2,2})}, \frac{v\gamma_{2,1}}{\gamma_{2,1}+\gamma_{2,2}}, 0)$. We point out that points P_5, P_6 and P_7 won't be considered when analyzing the system in the car-following context since they do not represent realistic situations.

The Jacobian matrix, $J(u_1, u_2, u_3)$, in this case is given by:

$$\begin{pmatrix} \gamma_{1,1}u_2 - (2\gamma_{1,1} + 2\gamma_{1,2})u_1 + \gamma_{1,2}u_3 & \gamma_{1,1}u_1 & \gamma_{1,2}u_1 \\ 0 & \gamma_{2,2}u_3 - (2\gamma_{2,1} + 2\gamma_{2,2})u_2 + \gamma_{2,2}v & \gamma_{2,1}u_2 \\ 0 & 0 & \gamma_3 v - 2\gamma_3 u_3 \end{pmatrix}. \tag{17}$$

Substituting the equilibrium points we get that P_i, for all $1 \leq i \leq 7$, $i \neq 4$ are unstable equilibrium points. For the case of P_4, all the eigenvalues of $J(P_4)$ are negative, and then it is a stable point. So as to, all models agree in the fact that the unique stable solution is obtained when all the cars approach to the speed of the leading car.

5. STABILITY OF PERTURBED TRAFFIC MODELS

In realistic situations, when a car approaches there is a variation in the motion like the one given by a periodic force acting on the cars. When trying to maintain a certain distance, cars sometimes get a little closer and they sometimes get a little more separated. This is to prevent collisions of one car with the two others when they are too close and they circulate in the same direction. This effect can be modeled by adding in the equations a new term, namely $\alpha \sin(t)$, with $\alpha > 0$.

We discuss the stability of the previous models for the aforementioned case in which we introduce a small perturbation given by $\alpha \sin(t)$, $\alpha > 0$ in the speed of the leading car. Then, the QTD model with 3 cars following a leading one at constant speed v is given by

$$\left\{ \begin{array}{l} u_1'(t) = \gamma_1 u_1(t)(u_2(t) - u_1(t)) \\ u_2'(t) = \gamma_2 u_2(t)(u_3(t) - u_2(t)) \\ u_3'(t) = \gamma_3 u_3(t)(v + \alpha \sin(t) - u_3(t)) \end{array} \right. \tag{18}$$

In the case of the NN model the analogous perturbed model is described as follows:

$$\left\{ \begin{array}{l} u_1'(t) = \gamma_{1,1} u_1(t)(u_2(t) - u_1(t)) + \gamma_{1,2} u_1(t)(u_3(t) - u_1(t)) \\ u_2'(t) = \gamma_{2,1} u_2(t)(u_3(t) - u_2(t)) + \gamma_{2,2} u_2(t)(v - u_2(t)) \\ u_3'(t) = \gamma_3 u_3(t)(v + \alpha \sin(t) - u_3(t)) \end{array} \right. \tag{19}$$

Even in this case, chaotic motions cannot be found for any value of γ as we represent in Fig. 6, in which both, bifurcation diagrams and Lyapunov exponent are calculated for the QTD model. As in Fig. 5, we have taken as initial condition $(u_1(0), u_2(0), u_3(0)) = (20, 13, 10)$ and $v = 13$, a physical situation in which collisions can take place. If $v = 13$, the speeds of each car will converge to a cycle around (v, v, v), and the speeds can be assumed to be greater than 1, in order that all accelerations will be positive. These tools can be easily used for other systems to see the existence of irregular motions and they are very intuitive concepts for undergraduate students in Math or any degree in Sciences.

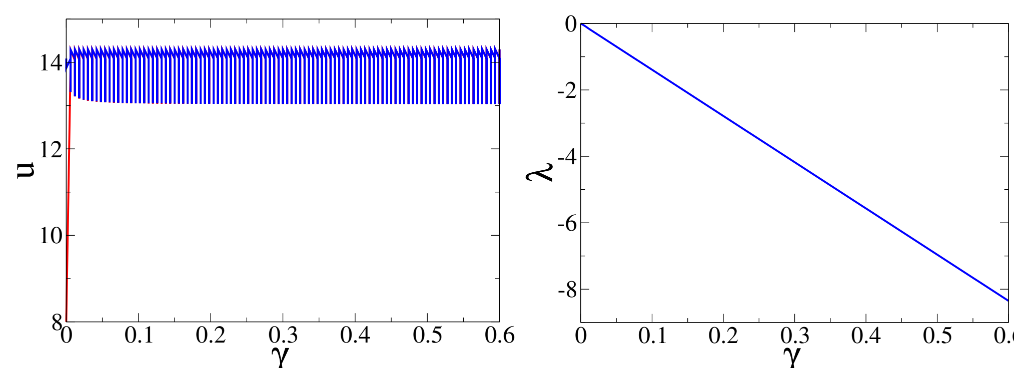

Figure 6. Plot of both, the bifurcation diagram, by plotting u versus γ and the Lyapunov exponent versus γ. We can observe that the motions are not chaotic for any parameter value and the system is always stable even under the existence of an external forcing. The initial condition is given by $(u_1(0), u_2(0), u_3(0)) = (20, 13, 10)$. We can see on the picture on the left side, that the velocity v changes periodically with period equal one. On the other hand and in the right panel, the Lyapunov exponent is zero or negative, therefore the non-chaotic behavior is corroborated.

In Figure 7, we plot the trajectories of non-perturbed and perturbed models and we see that the perturbation is transmitted to the cars following the leading one both for the QTD and the NN

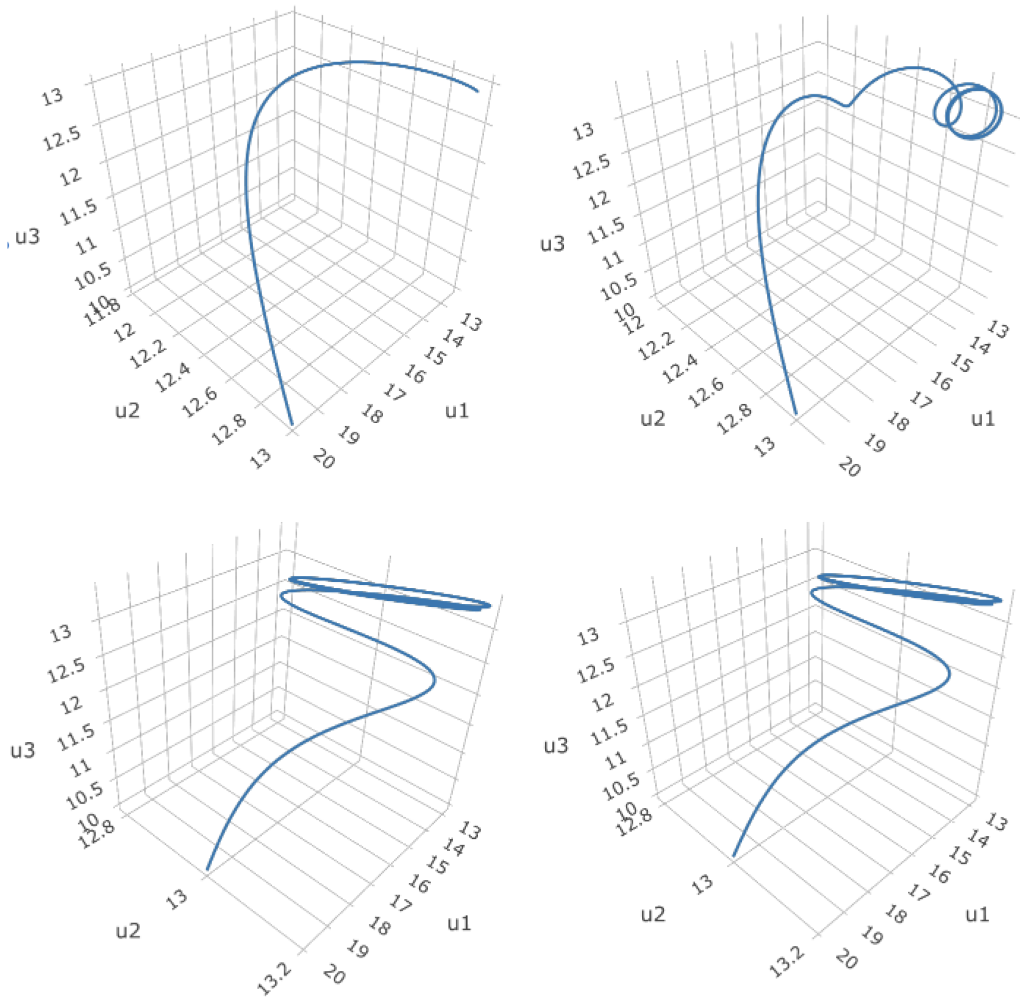

Figure 7. Top on the left, trajectory following the QTD model, see (11), where the velocity of car 1 is represented in the x-axis, the velocity of car 2 in the y-axis and the velocity of the third one in the z-axis. Top on the right, trajectory following the perturbed QTD model, see (18). Bottom on the left, trajectory following the NN model (15). Bottom on the right, trajectory following the perturbed NN model, see (19). The initial condition in all the cases is $(u_1(0), u_2(0), u_3(0)) = (20, 13, 10)$. The parameters γ_i, $i = 1, 2, 3$ are equal to 0.03. The parameters $\gamma_{i,j}$, $i, j = 1, 2$ are equal to 0.015. The parameter $\alpha = 1$.

models. Moreover, we observe that the amplitude of these perturbations tends to zero for the NN model, in contrast to what happens for the QTD.

6. Possible extensions and class activities

Once the previous models QTD and NN have been studied, and the students are familiarized with them, the most natural activities to propose are possible extensions and modifications of the models. One possible activity is to generalize and study the previous models for a finite given number of vehicles, that is, to study the system

$$\begin{cases} u_i'(t) = \gamma_i u_i(t)(u_{i+1}(t) - u_i(t)) & \text{for } 1 \leq i \leq k-1 \\ u_k'(t) = \gamma_k u_k(t)(v - u_k(t)) \end{cases}$$

and

$$\begin{cases} u_i'(t) = \gamma_{i,1} u_i(t)(u_{i+1}(t) - u_i(t)) + \gamma_{i,2} u_i(t)(u_{i+2}(t) - u_i(t)) & \text{for } 1 \leq i \leq k-2 \\ u_{k-1}'(t) = \gamma_{k-1,1} u_{k-1}(t)(u_k(t) - u_{k-1}(t)) + \gamma_{k-1,2} u_{k-1}(t)(v - u_{k-1}(t)) \\ u_k'(t) = \gamma_k u_k(t)(v - u_k(t)) \end{cases}$$

Another possible activity is to propose the same analysis but now for a new model that takes into account the speeds of the cars that drive in front and behind the main driver, which is known in its linear version as the Forward and Backward Control model developed in [HMPR59] for General Motors are considered in [BCMASS15] and to investigate if this new consideration can lead to chaotic situations in contrast to the other models:

$$\begin{aligned} u_1'(t) &= -\gamma_1 u_1(t) + \gamma_2 u_2(t)(u_2(t) - u_1(t)), \\ u_2'(t) &= \gamma_1(u_1(t) - u_2(t)) + \gamma_2 u_2(u_3(t) - u_2(t)), \end{aligned}$$

with control constants $\gamma_1, \gamma_2 > 0$, $\gamma_1 < \gamma_2$. We have considered the original model but adding a nonlinearity assuming that the control parameter γ_2 is proportional to $u_2(t)$.

6.1. **Closing remarks.** Although classical mathematical models such as population models are really useful for teaching dynamical systems, our objective in this paper is to provide a not-so-common model to emphasize the several applicactions that dynamical systems present. Moreover, the apogee of autonomous cars encourages the study of car-following models in this subject. The car-following models considered in this paper can also serve to improve students skills for modeling dynamical systems using software which contributes to a deeper understanding of abstract concepts. Moreover, it supposes an opportunity to introduce the students to a simple but important class of traffic models which are widely used in engineering and can help them to give a physical interpretation of mathematical models.

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Instituto Universitario de Matemática Pura y Aplicada,
Universitat Politècnica de València,
46022 València, Spain.
E-mail address: aconejero@upv.es

Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC)
Escuela Superior de Tecnología y Ciencias Experimentales, Universitat Jaume I
Campus de Ríu Sec, 12071 Castelló de la Plana Spain
E-mail address: murillom@uji.es

Nonlinear Dynamics, Chaos and Complex Systems Group
Departamento de Física, Universidad Rey Juan Carlos,
Tulipán s/n, 28933 Móstoles, Madrid, Spain;
E-mail address: jesus.seoane@urjc.es

Instituto de Matemática Interdisciplinar (IMI).
Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid, Plaza de las Ciencias 3, E-28040 Madrid, Spain.
E-mail address: jseoane@mat.ucm.es