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Deferred f-Statistical Convergence of Generalized Difference Sequences of Order α

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Abstract: Studies on difference sequences was introduced in the 1980s, and since then, many mathematicians have studied this kind of sequences and obtained some generalized difference sequence spaces. In this paper, using the generalized difference operator, we introduce the concept of the deferred f-statistical convergence of generalized difference sequences of the order α and give some inclusion relations between the deferred f-statistical convergence of generalized difference sequences and deferred f-statistical convergence of generalized difference sequences of the order α. Our results are more general than the corresponding results in the existing literature.

Keywords: difference sequence; deferred statistical convergence; statistical convergence of the order α

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1. Introduction, Definitions, and Preliminaries

The concept of statistical convergence was introduced by Steinhaus [1] and Fast [2] and then reintroduced independently by Schoenberg [3], and the notion was associated with summability theory by Bhardwaj et al. ([4,5]), Braha et al. [6], Çolak [7], Connor [8], Et et al. ([9]), Fridy [10], Işık et al. ([11–14]), Küçükaslan and Yılmaztürk [15], León-Saavedra et al. ([16–18]), Salat [19], Temizsu et al. ([20,21]), and many others.

The natural density of subsets of $\mathbb{N}$ plays a critical role in the definition of statistical convergence. For a subset, A, of natural numbers, if the limit

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in A\}|$$

exists, then this unique limit is called *the density of* A and mostly abbreviated by $\delta(A)$, where $|\{k \le n : k \in A\}|$ is the number of members of A not exceeding n.

A sequence, $x = (x_k)$, statistically converges to L provided that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0$$

for each $\varepsilon > 0$. It is denoted by $S - \lim x_k = L$. If $L = 0$, then x is a statistically null sequence.

The study of difference sequences reveals patterns inherent in natural growth processes. By understanding the convergence models applied to these sequences, we can make predictions and identify anomalies. In essence, summability methods, when applied to difference sequence spaces, offer a powerful tool for obtaining highly useful insights. Difference sequence spaces, a recent development in summability theory, were first introduced by Kızmaz in the 1980s and have since been extensively studied by mathematicians. The difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$, and $c_0(\Delta)$ were introduced by Kızmaz [22] as the domain of the forward difference matrix Δ^F, transforming a sequence, $x = (x_k)$, into the difference sequence $\Delta^F x = (x_k - x_{k+1})$ in the classical spaces ℓ_∞, c and c_0 of bounded, convergent, and null sequences, respectively. Quite recently, the difference space bv_p was introduced as the domain of the backward difference matrix Δ^B, transforming a sequence, $x = (x_k)$, into the difference sequence $\Delta^B x = (x_k - x_{k-1})$, in the space ℓ_p of absolutely p-summable sequences for $1 < p < \infty$ by Altay and Başar [23] and for $1 \leq p < \infty$ by Başar and Altay [24]. For more information on ℓ_p-type spaces, see [25,26]. The reader can refer to the monographs [27,28] for the background on the normed and paranormed sequence spaces and summability theory and related topics. The idea of difference sequences was generalized by Et and Çolak [29] as follows:

Given a sequence space, X, and a number, $m \in \mathbb{N}$, the space $\Delta^m(X)$ is defined as

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$, then there exists one and only one $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \tag{1}$$
$$y_{1-n} = y_{2-n} = \cdots = y_0 = 0$$

for a sufficiently large k for the instance $k > 2m$. Recently, a large amount of work has been carried out by several mathematicians regarding various generalizations of difference sequence spaces. For a detailed account of difference sequence spaces, one may refer to ([30–33]).

The deferred Cesàro mean of real valued sequences, $x = (x_k)$, was defined by Agnew [34]. Taking into account Agnew's approach, Küçükaslan and Yılmaztürk [15] introduced the concept of deferred statistical convergence as follows:

A real valued sequence, $x = (x_k)$, is called *deferred statistically convergent* to a number, L, provided, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{p_n < k \leq q_n : |x_k - L| \geq \varepsilon\}|}{q_n - p_n} = 0$$

where $p = (p_n)$ and $q = (q_n)$ are sequences of non-negative integers satisfying the conditions

$$\lim_{n\to\infty} q_n = \infty \text{ and } p_n < q_n \text{ for all } n \in \mathbb{N}. \tag{2}$$

This is a mathematical concept that offers a more nuanced and flexible approach to studying the convergence of sequences and series. Unlike traditional methods, which analyze the entire sequence or series at once, deferred convergence allows us to focus on parts of the sequence. By examining specific parts, we can identify finer convergence patterns that might be hidden when looking at the entire sequence. Throughout this paper, we assume that the sequences (p_n) and (q_n) satisfy (2) and additionally $\lim\limits_{n\to\infty}(q_n - p_n) = \infty$. We denote the set of all such (p, q) pairs by Ω. Some restrictions on (p, q) will be imposed if needed.

Modulus functions, introduced by Nakano [35], serve to bridge the gap between ordinary and statistical convergence. A modulus, f, is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$;
(ii) $f(x+y)\leq f(x)+f(y)$ for all $x,y\geq 0$;
(iii) f is increasing;
(iv) f is continuous from the right at 0.

Hence, f must be continuous everywhere in $[0,\infty)$. A modulus may be unbounded or bounded. For example, $f(x)=x^t$ $(0<t\leq 1)$ is unbounded, but $f(x)=\frac{x}{x+1}$ is bounded.

2. Δ_f^m-Deferred Statistical Convergence of Order α

Let f be an unbounded modulus, $(p,q)\in\Omega$, $\alpha\in(0,1]$, A be a subset of $\mathbb{N}$, and $A_{p,q}(n)$ denote the set $\{k: p_n<k\leq q_n, k\in A\}$. The $(D^{f,\alpha})-$density of A is defined by

$$\delta_{p,q}^{f,\alpha}(A)=\lim_{n\to\infty}\frac{1}{[f(q_n-p_n)]^{\alpha}}f\left(\left|A_{p,q}(n)\right|\right)$$

provided the limit exists.

Remark 1. *(i) If $\delta_{p,q}^{f,\alpha}(A)=0$, then A is said to be a $(D^{f,\alpha})-$null set.*
(ii) If $x=(x_k)$ is a sequence such that x_k holds the property $P(k)$ for all k except a $(D^{f,\alpha})-$null set, then we say that x_k holds $P(k)$ for "almost all k according to $D^{f,\alpha}$" and we denote this by "a.a.k $(D^{f,\alpha})$".

The proof of each of the following results is straightforward, so we choose to state these results *without* proof.

Proposition 1. *Let f be an unbounded modulus, $(p,q)\in\Omega$ and $0<\alpha\leq\beta\leq 1$. Then, $\delta_{p,q}^{f,\beta}(A)\leq\delta_{p,q}^{f,\alpha}(A)$ for any $A\subset\mathbb{N}$.*

Proposition 2. *$A\subset B$ implies that $\delta_{p,q}^{f,\alpha}(A)\leq\delta_{p,q}^{f,\alpha}(B)$ for any unbounded modulus, f, $(p,q)\in\Omega$ and $0<\alpha\leq 1$.*

Proposition 3. *$\delta_{p,q}^{f,\alpha}(A)=\delta_{p,q}^{f,\alpha}(B)=0$ implies $\delta_{p,q}^{f,\alpha}(A\cap B)=\delta_{p,q}^{f,\alpha}(A\cup B)=0$.*

Definition 1. *Let f be given as an unbounded modulus, $(p,q)\in\Omega$, $\alpha\in(0,1]$. A sequence, $x=(x_k)$, is said to be Δ_f^m-deferred statistically convergent of the order α to L if there is a real number, L, such that, for each $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{1}{[f(q_n-p_n)]^{\alpha}}f(|\{p_n<k\leq q_n:|\Delta^m x_k-L|\geq\varepsilon\}|)=0.$$

In this case, we write $S_{p,q}^{\alpha}(\Delta_f^m)-\lim x_k=L$. The set of all Δ_f^m-deferred statistically convergent sequences of the order α is denoted by $S_{p,q}^{\alpha}(\Delta_f^m)$. If $q_n=n$, $p_n=0$ for all $n\in\mathbb{N}$ and $\alpha=1$; then, $S_{p,q}^{\alpha}(\Delta_f^m)=S(\Delta_f^m)$ and $q_n=n$, $p_n=0$ for all $n\in\mathbb{N}$,. Then, $S_{p,q}^{\alpha}(\Delta_f^m)=S^{\alpha}(\Delta_f^m)$. If $f(x)=x$,, we have $S_{p,q}^{\alpha}(\Delta_f^m)=S_{p,q}^{\alpha}(\Delta^m)$. In the case of $m=0$, we have $S_{p,q}^{\alpha}(\Delta_f^m)=S_{p,q}^{f,\alpha}$.

Δ_f^m-deferred statistical convergence of the order α is not well defined for $\alpha>1$. The following example confirms this.

Example 1. *Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha > 1$, and let a sequence, $x = (x_j)$, be defined by*

$$x_j = \begin{cases} 0 & 1 \leq j \leq 3 \\ x_{j-1} + \frac{j-2}{2} & j = 2n, n \geq 2 \\ x_{j-1} + \frac{j-3}{2} & j = 2n+1, n \geq 2 \end{cases} \quad n = 1,2,3,\ldots.$$

Taking $m = 2$, we obtain

$$\Delta^2 x_j = \begin{cases} 1, & j = 2n \\ 0, & j \neq 2n \end{cases} \quad n = 1,2,3,\ldots.$$

Then, for each $\varepsilon > 0$, we have

$$\frac{f(|\{p_n < k \leq q_n : |\Delta^m x_k - 1| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} \leq \frac{f(q_n - p_n)}{[f(q_n - p_n)]^\alpha}$$

and

$$\frac{f(|\{p_n < k \leq q_n : |\Delta^m x_k - 0| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} \leq \frac{f(q_n - p_n)}{[f(q_n - p_n)]^\alpha}$$

which means that $S^\alpha_{p,q}(\Delta^2_f) - \lim x_j = 0$ and $S^\alpha_{p,q}(\Delta^2_f) - \lim x_j = 1$ for $\alpha > 1$.

We continue our work by giving some results without proof.

Theorem 1. *Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$ and $x = (x_k)$ and $y = (y_k)$ be sequences of real numbers; then, the following is true.*

(*i*) *If $S^\alpha_{p,q}(\Delta^m_f) - \lim x_k = L$ and $c \in \mathbb{R}$, then $S^\alpha_{p,q}(\Delta^m_f) - \lim cx_k = cL$.*

(*ii*) *If $S^\alpha_{p,q}(\Delta^m_f) - \lim x_k = L_1$ and $S^\alpha_{p,q}(\Delta^m_f) - \lim y_k = L_2$, then $S^\alpha_{p,q}(\Delta^m_f) - \lim(x_k + y_k) = L_1 + L_2$.*

Theorem 2. *Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$. Then, the inclusion $S^\alpha_{p,q}(\Delta^m_f) \subset S^\alpha_{p,q}(\Delta^{m+1}_f)$ strictly holds for $m \in \mathbb{N}$.*

Corollary 1. *Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$. For all $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$, the inclusion $S^\alpha_{p,q}(\Delta^{m_1}_f) \subset S^\alpha_{p,q}(\Delta^{m_2}_f)$ is strict.*

Theorem 3. *Let f be an unbounded modulus, $(p,q) \in \Omega$ and $\alpha, \beta \in (0,1]$ with $\alpha < \beta$.*

Then, the inclusion $S^\alpha_{p,q}(\Delta^m_f) \subseteq S^\beta_{p,q}(\Delta^m_f)$ is strict.

Proof. The inclusion part of the proof is straightforward. To show the strictness of the inclusion, let us consider the sequence $y = (y_j)$ by

$$y_j = \begin{cases} 1 & j = i^2 \\ 0 & j \neq i^2 \end{cases}$$

such that $\Delta^m x_j = y_j$ for some $x = (x_j)$ according to (1). Employing the modulus $f(x) = x^t$ $(0 < t \leq 1)$, $p_n = n^2$, $q_n = 4n^2$, we observe that, for each $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\left|\left\{n^2 < j \leq 4n^2 : |\Delta^m x_j| \geq \varepsilon\right\}\right| \leq n$$

and so

$$\frac{f(|n^2 < j \leq 4n^2 : |\Delta^m x_j| \geq \varepsilon|)}{[f(4n^2 - n^2)]^\beta} \leq \frac{n^t}{3^{\beta t} n^{2\beta t}}.$$

Then, taking the limit as $n \to \infty$, we have $S^{\beta}_{p,q}(\Delta^m_f) - \lim x_j = 0$ where $\frac{1}{2} < \beta \leq 1$. On the other hand, picking $\varepsilon = \frac{1}{3}$ and observing Fatih

$$\left| \left\{ n^2 < j \leq 4n^2 : \left| \Delta^m x_j \right| \geq \frac{1}{3} \right\} \right| = n$$

for each $n \in \mathbb{N}$, we have the following equality:

$$\frac{f\left(\left| n^2 < j \leq 4n^2 : \left| \Delta^m x_j \right| \geq \frac{1}{3} \right|\right)}{[f(4n^2 - n^2)]^{\alpha}} = \frac{n^t}{3^{\alpha t} n^{2\alpha t}}$$

which yields that $S^{\alpha}_{p,q}(\Delta^m_f) - \lim x_j \neq 0$ where $0 < \alpha \leq \frac{1}{2}$. □

Theorem 4. *Let f be an unbounded modulus, $(p,q) \in \Omega$ and $\alpha \in (0,1]$. Then, every Δ^m−convergent sequence is Δ^m_f−deferred statistically convergent of the order α,, but the converse does not need to hold.*

Proof. The inclusion follows from the fact that the set $\{k \in \mathbb{N} : |\Delta^m x_k - L| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$, assuming $\lim \Delta^m x_k = L$. To show that the converse does not hold for some particular cases, let us choose $p_n = n$ and $q_n = 2^n$, $f(x) = x^t$ $(0 < t \leq 1)$ and a sequence, $x = (x_k)$, such that

$$\Delta^m x_k = \begin{cases} n, & k = 2^n \\ \frac{1}{5}, & \text{else} \end{cases} \quad n = 1, 2, 3, \ldots \text{ by (1)}.$$

It is obvious that

$$\left| \left\{ n < k \leq 2^n : \left| \Delta^m x_k - \frac{1}{5} \right| \geq \varepsilon \right\} \right| \leq n$$

for each $n \in \mathbb{N}$ and $\varepsilon > 0$. Therefore, we have

$$\frac{f(\left| \left\{ n < k \leq 2^n : \left| \Delta^m x_k - \frac{1}{5} \right| \geq \varepsilon \right\} \right|)}{[f(2^n - n)]^{\alpha}} \leq \frac{f(n)}{[f(2^n - n)]^{\alpha}} = \frac{n^t}{(2^n - n)^{t\alpha}}$$

which results in $S^{\alpha}_{p,q}(\Delta^m_f) - \lim x_k = \frac{1}{5}$ for $\alpha \in \left(\frac{1}{2}, 1\right]$. However, it is clear that x is not Δ^m−convergent. □

Theorem 5. *Let f be an unbounded modulus, $(p,q) \in \Omega$ and $\alpha \in (0,1]$. Then, every Δ^m_f−deferred statistically convergent sequence of the order α is Δ^m−deferred statistically convergent, but this does not hold conversely.*

Proof. Let $x = (x_k)$ be Δ^m_f−deferred statistically convergent to L of the order α. That is, for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^{\alpha}} = 0.$$

Then, for each $r \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies that

$$f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|) \leq \frac{1}{r}[f(q_n - p_n)]^{\alpha} \leq \frac{1}{r} f(q_n - p_n)$$

Moreover, due to the subadditiveness of f, we obtain

$$\frac{1}{r} f(q_n - p_n) = \frac{1}{r} f\left(r \frac{q_n - p_n}{r}\right) \leq f\left(\frac{q_n - p_n}{r}\right).$$

It follows that
$$\frac{|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|}{q_n - p_n} \leq \frac{1}{r}$$
since f is increasing. Thus, x is Δ^m-deferred statistically convergent.

The sequence used in Theorem 4 can be reissued to see that the converse of this result need not hold. The aforementioned sequence $x = (x_k)$ is Δ^m-deferred statistically convergent to $\frac{1}{5}$ where $p_n = n$ and $q_n = 2^n$. However, we observe the inequality

$$\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right| \geq n - [\sqrt{n}] - 1 \text{ for each } n \in \mathbb{N},$$

where [.] denotes the integral part of the enclosed number. Considering the modulus $g(x) = \ln(x+1)$ and $\alpha \in (0,1]$, we have

$$\begin{aligned}\frac{g(\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right|)}{[g(2^n - n)]^\alpha} &\geq \frac{g(n - [\sqrt{n}] - 1)}{[g(2^n - n)]^\alpha} = \frac{\ln(n - [\sqrt{n}])}{[\ln(2^n - n + 1)]^\alpha} \\ &\geq \frac{\ln(n - [\sqrt{n}])}{[\ln(2^n + 1)]^\alpha} \geq \frac{\ln(n - [\sqrt{n}])}{\ln(2^n + 1)} \\ &> \frac{\ln(n - [\sqrt{n}])}{\ln(n^3 + 1)} = b_n.\end{aligned}$$

This implies that
$$\lim_{n\to\infty} \frac{g(\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right|)}{[g(2^n - n)]^\alpha} \neq 0$$
since $\lim_{n\to\infty} b_{n^2} = \lim_{n\to\infty} \frac{\ln(n^2-n)}{\ln(n^6+1)} = \frac{1}{3}$. Thus, $S^\alpha_{p,q}(\Delta^m_g) - \lim x_k \neq \frac{1}{5}$. □

Theorem 6. *Let f be given as an unbounded modulus, $(p,q) \in \Omega$, and let α be a fixed real number such that $\alpha \in (0,1]$. If the sequence $\left\{\frac{[f(q_n)]^\alpha}{[f(q_n - p_n)]^\alpha}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m_f-statistically convergent sequence of the order α is Δ^m_f-deferred statistically convergent of the order α.*

Proof. If (x_k) is a Δ^m_f-statistically convergent sequence of the order α, there exists $L > 0$ such that
$$\lim_{n\to\infty} \frac{f(|\{k \leq n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(n)]^\alpha} = 0$$
Then, due to $\lim_{n\to\infty} q_n = \infty$, the sequence
$$\left\{\frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha}\right\}_{n\in\mathbb{N}}$$
is a null sequence. Furthermore, the inclusion $\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\} \subseteq \{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}$ implies that

$$\begin{aligned}\frac{f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} &\leq \frac{[f(q_n)]^\alpha}{[f(q_n - p_n)]^\alpha} \frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha} \\ &\leq M \frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha}\end{aligned}$$

for some $M > 0$. Taking the limit as $n \to \infty$ yields that x is Δ^m_f-deferred statistically convergent to L of the order α. □

From Theorem 6, we obtain the following results.

Corollary 2. *Let f be given as an unbounded modulus, $(p,q) \in \Omega$, and let α be a fixed real number such that $\alpha \in (0,1]$. If $q_n < n$ for all $n \in \mathbb{N}$ and the sequence $\left\{ \frac{[f(n)]^\alpha}{[f(q_n - p_n)]^\alpha} \right\}_{n\in\mathbb{N}}$ is bounded, then every Δ_f^m-statistically convergent sequence of the order α is Δ_f^m-deferred statistically convergent of the order α.*

Corollary 3. *Let f be given as an unbounded modulus, $(p,q) \in \Omega$, and let α be a fixed real number such that $\alpha \in (0,1]$. If $\lim_n \frac{[f(q_n - p_n)]^\alpha}{[f(n)]^\alpha} = a > 0$ $(a \in \mathbb{R})$ and $q_n < n$,, then every Δ_f^m-statistically convergent sequence of the order α is Δ_f^m-deferred statistically convergent of the order α.*

Corollary 4. *(i) Let f be an unbounded modulus, $(p,q) \in \Omega$. If the sequence $\left\{ \frac{[f(q_n)]}{[f(q_n - p_n)]} \right\}_{n\in\mathbb{N}}$ is bounded, then every Δ_f^m-statistically convergent sequence is Δ_f^m-deferred statistically convergent.*

(ii) Let $(p,q) \in \Omega$ be given and α be a fixed real number such that $\alpha \in (0,1]$. If the sequence $\left\{ \frac{(q_n)^\alpha}{(q_n - p_n)^\alpha} \right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m-statistically convergent sequence of the order α is Δ^m-deferred statistically convergent of the order α.

(iii) Let $(p,q) \in \Omega$ be given. If the sequence $\left\{ \frac{q_n}{q_n - p_n} \right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m-statistically convergent sequence is Δ^m-deferred statistically convergent.

In the following theorem, by changing the conditions on the sequences (p_n) and (q_n) we give the same relations as in Corollary 4 (ii).

Theorem 7. *Let $m \in \mathbb{N}, (p,q) \in \Omega$ and α be a fixed real number such that $0 < \alpha \leq 1$, and $\liminf_n \frac{q_n}{p_n} > 1$. Then, every Δ^m-statistically convergent sequence of the order α is Δ^m-deferred statistically convergent of the order α.*

Proof. Since $\liminf_n \frac{q_n}{p_n} > 1$, we can find a number, $s > 0$, such that $\frac{q_n}{p_n} > 1 + s$ for sufficiently large n, which implies that

$$\frac{q_n - p_n}{q_n} \geq \frac{s}{1+s} \Longrightarrow \left(\frac{q_n - p_n}{q_n}\right)^\alpha \geq \left(\frac{s}{1+s}\right)^\alpha \Longrightarrow \frac{1}{{q_n}^\alpha} \geq \frac{s^\alpha}{(1+s)^\alpha} \frac{1}{(q_n - p_n)^\alpha}.$$

Since

$$\begin{aligned} \frac{|\{1 < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{{q_n}^\alpha} &\geq \frac{|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{q_n^\alpha} \\ &\geq \frac{s^\alpha}{(1+s)^\alpha} \frac{|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{(q_n - p_n)^\alpha} \end{aligned}$$

we have that $x = (x_k)$ is deferred Δ^m-statistically convergent of the order α

In the following, the results $S_{p,q}^\beta(\Delta_f^m)$ and $S_{r,s}^\alpha(\Delta_f^m)$ will be compared under the following conditions for $(p,q),(r,s) \in \Omega$ and

$$p_n < r_n < s_n < q_n \text{ for all } n \in \mathbb{N}. \tag{3}$$

□

Theorem 8. *Let $m \in \mathbb{N}$, (p,q), $(r,s) \in \Omega$, and α, β be two fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.*

(*i*) *If*

$$\lim_{n\to\infty} \frac{[f(s_n - r_n)]^\alpha}{[f(q_n - p_n)]^\beta} = s > 0 \quad (4)$$

then $S^\beta_{p,q}(\Delta^m_f) \subset S^\alpha_{r,s}(\Delta^m_f)$.

(*ii*) *If*

$$\lim_{n\to\infty} \frac{q_n - p_n}{[f(s_n - r_n)]^\beta} = \frac{1}{f(1)} \quad (5)$$

then $S^\alpha_{r,s}(\Delta^m_f) \subseteq S^\beta_{p,q}(\Delta^m_f)$.

Proof. (*i*) Let $x = (x_k) \in S^\beta_{p,q}(\Delta^m_f)$. Since (3) is provided, for a given $\varepsilon > 0$, we have

$$\{p_n < k \le q_n : |\Delta^m x_k - L| \ge \varepsilon\} \supseteq \{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}$$

and we also have the following inequality:

$$\begin{aligned} \frac{f(|\{p_n < k \le q_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} &\ge \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} \\ &= \frac{[f(s_n - r_n)]^\alpha}{[f(q_n - p_n)]^\beta} \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(s_n - r_n)]^\alpha}. \end{aligned}$$

So we have $x \in S^\alpha_{r,s}(\Delta^m_f)$ provided (4) holds.

(*ii*) Let (5) be satisfied and $x \in S^\alpha_{r,s}(\Delta^m_f)$. Then, for every $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{f(|\{p_n < k \le q_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} \\ &= \frac{f(|\{p_n < k \le r_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} \\ &+ \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} + \frac{f(|\{s_n < k \le q_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\beta} \\ &\le \frac{(q_n - p_n) - (s_n - r_n)}{[f(s_n - r_n)]^\beta} + \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(s_n - r_n)]^\alpha} \\ &\le \frac{(q_n - p_n) - (s_n - r_n)^\beta}{[f(s_n - r_n)]^\beta} + \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(s_n - r_n)]^\alpha} \\ &\le \frac{q_n - p_n - \frac{[f(s_n - r_n)]^\beta}{f(1)}}{[f(s_n - r_n)]^\beta} + \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(s_n - r_n)]^\alpha} \\ &= \left(\frac{q_n - p_n}{[f(s_n - r_n)]^\beta} - \frac{1}{f(1)} \right) + \frac{f(|\{r_n < k \le s_n : |\Delta^m x_k - L| \ge \varepsilon\}|)}{[f(s_n - r_n)]^\alpha} \end{aligned}$$

Therefore, $x \in S^\beta_{p,q}(\Delta^m_f)$. □

From Theorem 8, we obtain the following results.

Corollary 5. (*i*) *Let* $m \in \mathbb{N}$, $(p,q), (r,s) \in \Omega$ *and* $0 < \alpha \le 1$. *If*

$$\lim_{n\to\infty} \left[\frac{f(s_n - r_n)}{f(q_n - p_n)} \right]^\alpha = s > 0$$

then $S^\alpha_{p,q}(\Delta^m_f) \subset S^\alpha_{r,s}(\Delta^m_f)$.

(*ii*) *Let* $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$ *and* $0 < \alpha \leq 1$. *If*

$$\lim_{n\to\infty} \frac{[f(s_n - r_n)]^\alpha}{f(q_n - p_n)} = s > 0$$

then $S_{p,q}(\Delta_f^m) \subset S_{r,s}^\alpha(\Delta_f^m)$.

(*iii*) *Let* $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$. *If*

$$\lim_{i\to\infty} \frac{f(s_n - r_n)}{f(q_n - p_n)} = s > 0$$

then $S_{p,q}(\Delta_f^m) \subset S_{r,s}(\Delta_f^m)$.

Corollary 6. *Let* $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$. *If*

$$\lim_{n\to\infty} \frac{f(q_n - p_n)}{f(s_n - r_n)} = 1$$

then $S_{r,s}(\Delta_f^m) \subseteq S_{p,q}(\Delta_f^m)$.

Proof. Omitted. □

3. Strong Δ_f^m−Deferred Cesàro Summability of Order α

Now, we introduce strong Δ_f^m−deferred Cesàro summability of the order α and give some relations between strong Δ_f^m−deferred Cesàro summability of the order α and strong Δ_f^m−deferred Cesàro summability of the order β, where α and β are fixed real numbers such that $\beta \geq \alpha > 0$.

Definition 2. *Let* f *be a modulus and* α *be a positive real number. We define*

$$w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) = \left\{x \in w : \lim_{n\to\infty} \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) = 0\right\},$$

$$w_{p,q}^{\alpha}\left(\Delta_f^m\right) = \left\{x \in w : \lim_{n\to\infty} \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) = 0 \text{ for some number } L\right\},$$

$$w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right) = \left\{x \in w : \sup_n \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) < \infty\right\}.$$

If $x \in w_{p,q}^{\alpha}\left(\Delta_f^m\right)$, *we shall say that the sequence* $x = (x_k)$ *is strongly* Δ_f^m*−deferred Cesàro summable of the order* α *to L (or strongly* $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$*−Cesàro summable to L).*

Some spaces are obtained by specializing f, α *and a pair of* (p,q).

(*i*) *In the case* $f(x) = x$, *we write* $w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}(\Delta^m)$ *and* $w_{p,q}^{\alpha,\infty}(\Delta^m)$ *instead of* $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ *and* $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, *respectively.*

(*ii*) *In the case* $\alpha = 1$, *we write* $w_{p,q}^{0}\left(\Delta_f^m\right)$, $w_{p,q}\left(\Delta_f^m\right)$ *and* $w_{p,q}^{\infty}\left(\Delta_f^m\right)$ *instead of* $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ *and* $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, *respectively.*

(*iii*) *In the special cases* $f(x) = x$ *and* $\alpha = 1$, *we write* $w_{p,q}^{0}(\Delta^m)$, $w_{p,q}(\Delta^m)$ *and* $w_{p,q}^{\infty}(\Delta^m)$ *instead of* $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ *and* $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, *respectively.*

(iv) *If $q_n = n$ and $p_n = 0$ (for all $n \in \mathbb{N}$), then we write we write $w^{\alpha,0}\left(\Delta_f^m\right)$, $w^{\alpha}\left(\Delta_f^m\right)$ and $w^{\alpha,\infty}\left(\Delta_f^m\right)$ instead of $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, respectively.*

Theorem 9. *(i) For any modulus, f, and positive α, $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.*

(ii) For any modulus f and $\alpha \geq 1$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.

Proof. (ii) Let $x \in w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $\alpha \geq 1$. Since f is subadditive and increasing, we have

$$\begin{aligned}\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) &\leq \frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} 1 \\ &= \frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)(q_n-p_n)}{(q_n-p_n)^{\alpha}} \\ &= \frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)}{(q_n-p_n)^{\alpha-1}}\end{aligned}$$

and since $\alpha \geq 1$, we have $x \in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.
□

Theorem 10. *For any modulus, f, and $\alpha \geq 1$, we have*

(i) $w_{p,q}^{\alpha,0}(\Delta^m) \subset w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$.

(ii) $w_{p,q}^{\alpha}(\Delta^m) \subset w_{p,q}^{\alpha}\left(\Delta_f^m\right)$.

(iii) $w_{p,q}^{\alpha,\infty}(\Delta^m) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.

Proof. We consider only the last inclusion; the others can be proved in the same way. Let $x \in w_{p,q}^{\alpha,\infty}(\Delta^m)$; then, there exists a number, $M > 0$, such that

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} |\Delta^m x_k| < M, \text{ for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 < t \leq \delta$. We can write

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) = \frac{1}{(q_n-p_n)^{\alpha}}\sum_{\substack{p_n+1 \\ |\Delta^m x_k| \leq \delta}}^{q_n} f(|\Delta^m x_k|) + \frac{1}{(q_n-p_n)^{\alpha}}\sum_{\substack{p_n+1 \\ |\Delta^m x_k| > \delta}}^{q_n} f(|\Delta^m x_k|).$$

For $|\Delta^m x_k| \leq \delta$, we have

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{\substack{p_n+1 \\ |\Delta^m x_k| \leq \delta}}^{q_n} f(|\Delta^m x_k|) \leq \frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} \varepsilon = \frac{\varepsilon}{(q_n-p_n)^{\alpha-1}}. \tag{6}$$

For $|\Delta^m x_k| > \delta$, we first use the inequality $|\Delta^m x_k| < \frac{|\Delta^m x_k|}{\delta} < 1 + \left[\frac{|\Delta^m x_k|}{\delta}\right]$ where [.] denotes the integral part of the enclosed number; then, by the definition of the modulus function, we can write

$$f(|\Delta^m x_k|) \leq \left(1 + \left[\frac{|\Delta^m x_k|}{\delta}\right]\right) f(1) \leq 2f(1)\frac{|\Delta^m x_k|}{\delta}$$

and so

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{\substack{p_n+1\\|\Delta^m x_k|>\delta}}^{q_n} f(|\Delta^m x_k|)\leq 2f(1)\delta^{-1}\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n}|\Delta^m x_k| \tag{7}$$

From (6) and (7), we have

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k|)\leq \frac{\varepsilon}{(q_n-p_n)^{\alpha-1}}+2f(1)\delta^{-1}\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n}|\Delta^m x_k|.$$

Since $\alpha\geq 1$ and $x\in w_{p,q}^{\alpha,\infty}(\Delta^m)$, we have $x\in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, and the proof is complete.

We pause to recall that Maddox [36] proved that for any modulus, f, $\lim_{t\to\infty}\frac{f(t)}{t}$ exists and equals $\eta=\inf\{f(t)/t;t>0\}$ such that $0\leq\eta\leq f(1)$. In the next theorem, we show that the reciprocals of the inclusions in Theorem 10 also hold under a restriction on the modulus f. □

Theorem 11. *Let f be a modulus and α be a positive real number. If $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)\subset w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)\subset w_{p,q}^{\alpha}(\Delta^m)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)\subset w_{p,q}^{\alpha,\infty}(\Delta^m)$.*

Proof. Suppose that $x\in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$ and $\lim_{t\to\infty}\frac{f(t)}{t}=\eta=\inf\{f(t)/t;t>0\}>0$. Then, we have $f(t)\geq\eta t$ which yields $t\leq\eta^{-1}f(t)$ for all $t\geq 0$. This gives rise to the inequality

$$\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n}|\Delta^m x_k|\leq\eta^{-1}\frac{1}{(q_n-p_n)^{\alpha}}\sum_{p_n+1}^{q_n} f(|\Delta^m x_k|).$$

Thus, $x\in w_{p,q}^{\alpha,\infty}(\Delta^m)$. The proofs of the other inclusions are analogous, so we omit them. □

Theorems 10 and 11 yield the next result.

Theorem 12. *Let f be any modulus such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $\alpha\geq 1$. Then, we have $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)=w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)=w_{p,q}^{\alpha}(\Delta^m)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)=w_{p,q}^{\alpha,\infty}(\Delta^m)$.*

In the next result, we compare the sequence spaces $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\beta}\left(\Delta_f^m\right)$ without any restriction on the modulus f and $(p,q)\in\Omega$.

Theorem 13. *Let f be a modulus, $(p,q)\in\Omega$ and $\beta\geq\alpha>0$. Then, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)\subset w_{p,q}^{\beta}\left(\Delta_f^m\right)$, and the inclusion may be particularly strict for certain specific choices of α and β.*

Proof. The inclusion part of the proof is straightforward. To show that the inclusion may be strict, let f be a modulus, $q_n=2n^2$ and $p_n=n^2$ (for all $n\in\mathbb{N}$), and consider the sequence $x=(x_k)$ defined by

$$x_k=\begin{cases}1-n, & (n-1)^2+1\leq k\leq n^2\\ 0, & k=1\end{cases}\quad n=2,3,\ldots$$

Observe that Δx_k equals 1 when k is a square and 0 when k is a non-square. Therefore, using the fact that $f(0)=0$, for every $n\in\mathbb{N}$, we have

$$\frac{1}{(2n^2-n^2)^{\beta}}\sum_{p_n+1}^{q_n} f(|\Delta x_k-0|)\leq\frac{nf(1)}{n^{2\beta}}\to 0,\text{ as } n\to\infty$$

so $x \in w_{p,q}^{\beta}\left(\Delta_f\right)$ for $\beta > \frac{1}{2}$. On the other hand,

$$\frac{1}{(2n^2 - n^2)^{\alpha}} \sum_{p_n+1}^{q_n} f(|\Delta x_k - 0|) \geq \frac{\sqrt{n} - 1}{n^{2\alpha}} f(1) \to \infty, \text{ as } n \to \infty$$

which implies that $x \notin w_{p,q}^{\alpha}\left(\Delta_f\right)$ for $0 < \alpha < \frac{1}{4}$.

Finally, we give a fairly general relation between strong Δ_f^m-deferred Cesàro summability of the order α and Δ_f^m-deferred statistical convergence of the order α. □

Theorem 14. *Let $0 < \alpha \leq \beta \leq 1$, f be an unbounded modulus such that there exists a positive constant, c, such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable of the order α to L, then it is Δ_f^m-deferred statistically convergent of the order β to L.*

Proof. Let $x = (x_k) \in w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $\varepsilon > 0$; using the definition of modulus function, we have

$$\begin{aligned} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) &\geq f\left(\sum_{p_n+1}^{q_n} |\Delta^m x_k - L|\right) \\ &\geq f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|\varepsilon) \\ &\geq cf(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)f(\varepsilon) \end{aligned}$$

and since $\alpha \leq \beta$,

$$\begin{aligned} \frac{1}{(q_n - p_n)^{\alpha}} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) &\geq \frac{cf(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)f(\varepsilon)}{(q_n - p_n)^{\alpha}} \\ &\geq \frac{cf(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)f(\varepsilon)}{(q_n - p_n)^{\beta}} \\ &= \frac{cf(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)f(\varepsilon)[f(q_n - p_n)]^{\beta}}{[f(q_n - p_n)]^{\beta}(q_n - p_n)^{\beta}}. \end{aligned}$$

This completes the proof. □

The following results are derivable from Theorem 14.

Corollary 7. *Let $0 < \alpha \leq 1$, f be an unbounded modulus such that there exists a positive constant, c, such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable of the order α to L, then it is Δ_f^m-deferred statistically convergent of the order α to L.*

Corollary 8. *Let f be an unbounded modulus such that there exists a positive constant, c, such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable to L, then it is Δ_f^m-deferred statistically convergent to L.*

By combining Theorem 11 of this article and Theorem 2.10 of Temizsu et al. [20] for the cases $\alpha = \beta$ and $r = 1$, we immediately obtain the next theorem.

Theorem 15. *Let f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\alpha \in (0, 1]$. If a sequence is strongly Δ_f^m-deferred Cesàro summable of the order α to L, then it is Δ^m-deferred statistically convergent of the order α to L.*

Specializing f and α in Theorem 15, we derive the following results.

Corollary 9. *Let f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable to L, then it is Δ^m-deferred statistically convergent to L.*

Corollary 10. *If a sequence is strongly Δ^m-deferred Cesàro summable to L, then it is Δ^m-deferred statistically convergent to L.*

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