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Numerical Semigroups with a Given Frobenius Number and Some Fixed Gaps

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Abstract: If P is a nonempty finite subset of positive integers, then $\mathscr{A}(P) = \{S \mid S \text{ is a numerical semigroup}, S \cap P = \emptyset \text{ and } \max(P) \text{ is the Frobenius number of } S\}$. In this work, we prove that $\mathscr{A}(P)$ is a covariety; therefore, we can arrange the elements of $\mathscr{A}(P)$ in the form of a tree. This fact allows us to present several algorithms, including one that calculates all the elements of $\mathscr{A}(P)$, another that obtains its maximal elements (with respect to the set inclusion order) and one more that computes the elements of $\mathscr{A}(P)$ that cannot be expressed as an intersection of two elements of $\mathscr{A}(P)$, that properly contain it.

Keywords: Frobenius number; gap; multiplicity; algorithm; covariety; irreducible element; R variety

MSC: 20M14; 11D07; 13H10

1. Introduction

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A submonoid of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$, which is closed for the sum and contains zero. A numerical semigroup submonoid (S) of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S = \{x \in \mathbb{N} \mid x \notin S\}$ is finite.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S \backslash \{0\})$, $\mathrm{F}(S) = \max\{z \in \mathbb{Z} \mid z \notin S\}$ and $\mathrm{g}(S) = \sharp(\mathbb{N} \backslash S)$ (where $\sharp X$ denotes the cardinality of a set (X)) are three important invariants of S, which we shall refer to as the multiplicity, Frobenius number and genus of S, respectively.

If X is a nonempty subset of $\mathbb{N}$, then $\langle X \rangle$ denotes the submonoid of $(\mathbb{N}, +)$ generated by X, that is,

$$\langle X \rangle = \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid n \in \mathbb{N} \backslash \{0\}, \{x_1, \ldots, x_n\} \subseteq X \text{ and } \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{N}\}.$$

In [1] (Lemma 2.1), it is shown that $\langle X \rangle$ is a numerical semigroup if and only if $\gcd(X) = 1$.

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle X \rangle$, then we say that X is a system of generators of M. Moreover, if $M \neq \langle Y \rangle$ for all $Y \subsetneq X$, then we say that X is a minimal system of generators of M. In [1] (Corollary 2.8), it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which is finite. $\mathrm{msg}(M)$ denotes the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the embedding dimension of M and is denoted by $\mathrm{e}(M)$.

The Frobenius problem for numerical semigroups (see [2]) consists of looking for formulas that calculate the Frobenius number and the genus from its minimal system

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of generators. This problem was solved in [3] for numerical semigroups of embedding dimension two. At present, the problem remains unsolved for numerical semigroups of embedding dimensions greater than or equal to three. However, this problem has been extensively dealt with in the numerical semigroup literature (see, for instance, [4–9]).

The non-habitual reader studying numerical semigroups may find the nomenclature, i.e., multiplicity, genus, embedding dimension, etc., somewhat surprising. With reference to this, we will say that in the bibliography of the topic, there is a long list of publications devoted to the study of one-dimensional, analytically irreducible local Noetherian domains via their value semigroup, which is a numerical semigroup (see, for instance, [10–16]). All the introduced invariants have their interpretation in this context; hence, their names.

The study of the gaps and the genus of a numerical semigroup has also been extensively discussed (see, for example, [17–20]). Much of these works is motivated by the conjecture of Bras-Amoros presented in [21], where she predicted that there are more numerical semigroups of genus $g+1$ than of genus g.

Let P be a nonempty, finite subset of $\mathbb{N}\backslash\{0\}$.

$$\mathscr{A}(P) = \{S \mid S \text{ is a numerical semigroup, } \mathrm{F}(S) = \max(P) \text{ and } P \subseteq \mathbb{N}\backslash S\}.$$

If $S \in \mathscr{A}(P)$, then we say that S is a $\mathscr{A}(P)$*-semigroup*. The main aim of this work is to study set $\mathscr{A}(P)$.

We begin Section 2 by seeing that $\mathscr{A}(P)$ is a covariety. This fact, together with the results of [22], allow us to arrange the elements of $\mathscr{A}(P)$ in the form of a tree. As a consequence, we obtain an algorithm that calculates all the elements of $\mathscr{A}(P)$.

If X is a subset of an element from $\mathscr{A}(P)$, then we say that X is an $\mathscr{A}(P)$*-set*. In Section 3, we prove that if X is an $\mathscr{A}(P)$ set, then there exists the least element from $\mathscr{A}(P)$ containing X. This numerical semigroup is denoted by $\mathscr{A}(P)[X]$, and we say that it is the $\mathscr{A}(P)$*-semigroup generated* by X. If $S = \mathscr{A}(P)[X]$ and $S \neq \mathscr{A}(P)[Y]$ for all $Y \subsetneq X$, then we say that X is an $\mathscr{A}(P)$*-minimal system of generators* of S. In Section 3, we prove that every $\mathscr{A}(P)$ semigroup admits a unique $\mathscr{A}(P)$-minimal system of generators. $\mathscr{A}(P)\mathrm{msg}(S)$ denotes the minimal system of generators of S. The cardinality of $\mathscr{A}(P)\mathrm{msg}(S)$ is called the $\mathscr{A}(P)$ *rank* of S. We finish Section 3 by showing an algorithmic process that computes all the elements of $\mathscr{A}(P)$ with a given $\mathscr{A}(P)$-rank .

$\mathscr{M}(P)$ denotes the set of maximal elements of $\mathscr{A}(P)$ with respect to the set inclusion order. We start Section 4 by proposing an algorithm for calculating $\mathscr{M}(P)$.

An element of $\mathscr{A}(P)$ is $\mathscr{A}(P)$*-irreducible* if it cannot be written as the intersection of two elements of $\mathscr{A}(P)$ containing it properly. $\mathscr{I}(\mathscr{A}(P)) = \{S \in \mathscr{A}(P) \mid S \text{ is } \mathscr{A}(P)\text{-irreducible}\}$. In Section 4, we prove that $\mathscr{M}(P) \subseteq \mathscr{I}(\mathscr{A}(P))$, and we proposed an algorithm that computes $\mathscr{I}(\mathscr{A}(P))$.

A numerical semigroup is *irreducible* if it cannot be written as the intersection of two numerical semigroups containing it properly. If F is a positive integer, $\mathscr{I}(F) = \{S \mid S \text{ is an irreducible numerical semigroup and } \mathrm{F}(S) = F\}$.

If S is a numerical semigroup, then $\theta(S) = \{s \in S\backslash\{0\} \mid s < \frac{\mathrm{F}(S)}{2}\}$. $\mathscr{L}(F) = \{S \mid S \text{ is a numerical semigroup, and } \mathrm{zF}(S) = F\}$ the following equivalence binary relation: $S\mathscr{R}T$ if and only if $\theta(S) = \theta(T)$. Then, the quotient set expressed as $\dfrac{\mathscr{L}(F)}{\mathscr{R}}$ is $\{[S]_{\mathscr{R}} \mid S \in \mathscr{L}(F)\}$. If $S \in \mathscr{I}(\max(P))$, then $\mathrm{A}(S) = \{T \in [S]_{\mathscr{R}} \mid T \cap P = \emptyset\}$. In Section 5, we show that $\{\mathrm{A}(S) \mid S \in \mathscr{I}(\max(P)) \text{ and } \mathrm{A}(S) \neq \emptyset\}$ is a partition of $\mathscr{A}(P)$. As a consequence and applying the results of [23], we propose new algorithm to calculate $\mathscr{A}(P)$.

If A and B are numerical semigroups such that $A \subseteq B$, then $[A, B] = \{S \mid S \text{ is a numerical semigroup and } A \subseteq S \subseteq B\}$. In Section 6, we prove that if $\mathscr{M}(P) =$

$\{A_1, A_2, \ldots, A_r\}$, then $\mathscr{A}(P) = \bigcup_{i=1}^{r} [\{0, \max(P)+1, \rightarrow\}, A_i]$, where the $\rightarrow$ symbol means that every integer greater than $\max(P)+1$, belongs to the set. Therefore, $\mathscr{A}(P)$ is the union of R varieties. Finally, using the results of [24], we present a new algorithm for calculating $\mathscr{A}(P)$.

This paper contains a large number of examples to illustrate how our proposed algorithms work. The computation of these examples was performed using the `numericalsgps` package in the GAP System (see [25,26]).

2. The Tree Associated with $\mathscr{A}(P)$

A *covariety* is a nonempty family ($\mathscr{C}$) of numerical semigroups that fulfills the following conditions:

(1) There is a minimum of $\mathscr{C}$ with respect to set inclusion, denoted by $\Delta(\mathscr{C}) = \min(\mathscr{C})$.
(2) If $\{S, T\} \subseteq \mathscr{C}$, then $S \cap T \in \mathscr{C}$.
(3) If $S \in \mathscr{C}$ and $S \neq \Delta(\mathscr{C})$, then $S \backslash \{\mathrm{m}(S)\} \in \mathscr{C}$.

The next result appears in [22] (Lemma 2.2).

Lemma 1. *Let S and T be numerical semigroups and $x \in S$. Then, the following conditions hold:*

1. $S \cap T$ *is a numerical semigroup and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.
2. $S \backslash \{x\}$ *is a numerical semigroup if and only if* $x \in \mathrm{msg}(S)$.
3. $\mathrm{m}(S) = \min(\mathrm{msg}(S))$.

In the rest of this section, P denotes a nonempty, finite set of $\mathbb{N} \backslash \{0\}$.

The following result has an easy proof.

Proposition 1. *$\mathscr{A}(P)$ is a covariety and $\Delta(\mathscr{A}(P)) = \{0, \max(P)+1, \rightarrow\}$.*

The next result is a consequence from [22] (Proposition 2.1).

Corollary 1. *$\mathscr{A}(P)$ is a set with finite cardinality.*

A *graph* (G) is a pair (V, E), where V is a nonempty set and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of V and E are called *vertices* and *edges* of G, respectively.

A *path (of length n)* connecting vertices a and b of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = a$ and $v_n = b$.

A graph (G) is *a tree* if there exists a vertex (r, known as *the root* of G) such that for any other vertex (x) of G, there exists a unique path connecting x and r. If (u, v) is an edge of tree G, we say that u is a *child* of v.

Graph $\mathrm{G}(\mathscr{A}(P))$ is defined in the following way: $\mathscr{A}(P)$ is its set of vertices, and $(S, T) \in \mathscr{A}(P) \times \mathscr{A}(P)$ is an edge if and only if $T = S \backslash \{\mathrm{m}(S)\}$.

By applying Proposition 1 and [22] (Proposition 2.3), we obtain the following result.

Proposition 2. *$\mathrm{G}(\mathscr{A}(P))$ is a tree, and $\Delta(\mathscr{A}(P))$ is its root.*

A tree can be built recurrently by starting from the root and connecting each vertex already built with its children by means of an edge. Hence, it is very interesting to characterize the children of an arbitrary vertex of $\mathrm{G}(\mathscr{A}(P))$. This is our next aim. For this reason, we introduce the following concept.

We say that an integer (x) is a *special gap* of a numerical semigroup (S) if $x \notin S$ and $S \cup \{x\}$ is a numerical semigroup. $\mathrm{SG}(S)$ denotes the set formed by the special gaps of S.

The above result is obtained by applying Proposition 1 and [22] (Proposition 2.4).

Proposition 3. *If $S \in \mathscr{A}(P)$, then the set formed by the children of S in tree $\mathrm{G}(\mathscr{A}(P))$ is $\{S \cup \{x\} \mid x \in \mathrm{SG}(S), x < \mathrm{m}(S)$ and $S \cup \{x\} \in \mathscr{A}(P)\}$.*

As an immediate consequence of the previous proposition, we obtain the next result.

Corollary 2. *If $S \in \mathscr{A}(P)$, then the set formed by the children of S in tree $\mathrm{G}(\mathscr{A}(P))$ is $\{S \cup \{x\} \mid x \in \mathrm{SG}(S), x < \mathrm{m}(S)$ and $x \notin P\}$.*

Our next objective is to propose an algorithm that computes $\mathscr{A}(P)$ using the above results.

If S is a numerical semigroup and $n \in S \backslash \{0\}$, then the Apéry set (see [27]) is defined as $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$.

From [1] (Lemma 2.4), we can deduce the following.

Lemma 2. *Let S be a numerical semigroup and $n \in S \backslash \{0\}$. Then, $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n for all $i \in \{0, \ldots, n-1\}$.*

Remark 1. *If S is a numerical semigroup and we know that $\mathrm{Ap}(S, x) = \{0 = w(0), w(1), \ldots, w(x-1)\}$ for some $x \in S \backslash \{0\}$, then*

1. *The problem of membership of S is solved, since an integer (n) belongs to S if and only if $n \geq w\,(n \bmod x)$, where $n \bmod x$ denotes the remainder after dividing n by x.*
2. *Using Remark 1 from [22], we can compute $\mathrm{SG}(S)$.*
3. *Applying Remark 2 from [22], we can calculate $\mathrm{Ap}(S \cup \{b\}, x)$ for all $b \in \mathrm{SG}(S)$.*

Thus, we have all the ingredients to propose Algorithm 1, that computes $\mathscr{A}(P)$

Algorithm 1 Computation of $\mathscr{A}(P)$

INPUT : A nonempty finite subset P of $\mathbb{N} \backslash \{0\}$.
OUTPUT: $\mathscr{A}(P)$.

(1) $\Delta = \{0, \max(P)+1, \rightarrow\}$ and $\mathrm{Ap}(\Delta, \max(P)+1) = \{0, \max(P)+2, \max(P)+3, \cdots, 2\max(P)+1\}$.
(2) $A = B = \{\Delta\}$.
(3) For every $S \in B$, by using Remark 1, compute

$$\theta(S) = \{x \in \mathrm{SG}(S) \mid x \notin P \text{ and } x < \mathrm{m}(S)\}.$$

(4) If $\bigcup_{S \in B} \theta(S) = \emptyset$, then return A and stop.
(5) $B := \bigcup_{S \in B} \{S \cup \{x\} \mid x \in \theta(S)\}$.
(6) $A := A \cup B$.
(7) For every $S \in B$, by using Remark 1, compute $\mathrm{Ap}(S, \max(P)+1)$ and go to Step (3).

In the following example we show how the previous algorithm works.

Example 1. *Let $P = \{4, 7, 9\}$. Our aim is to compute $\mathscr{A}(P)$.*

- $\Delta = \{0, 10, \rightarrow\}$ *and* $\mathrm{Ap}(\Delta, 10) = \{0, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.
- $A = B = \{\Delta\}$.
- $\theta(\Delta) = \{5, 6, 8\}$.
- $B = \{\Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}\}$.
- $A = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}\}$,
  $\mathrm{Ap}(\Delta \cup \{5\}, 10) = \{0, 11, 12, 13, 14, 5, 16, 17, 18, 19\}$,

$\mathrm{Ap}(\Delta \cup \{6\}, 10) = \{0, 11, 12, 13, 14, 15, 6, 17, 18, 19\}$,
$\mathrm{Ap}(\Delta \cup \{8\}, 10) = \{0, 11, 12, 13, 14, 15, 16, 17, 8, 19\}$.

- $\theta(\Delta \cup \{5\}) = \emptyset$, $\theta(\Delta \cup \{6\}) = \{5\}$, $\theta(\Delta \cup \{8\}) = \{5, 6\}$.
- $B = \{\Delta \cup \{6, 5\}, \Delta \cup \{8, 5\}, \Delta \cup \{8, 6\}\}$.
- $A = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}, \Delta \cup \{6, 5\}, \Delta \cup \{8, 5\}, \Delta \cup \{8, 6\}\}$.
- $\mathrm{Ap}(\Delta \cup \{6, 5\}, 10) = \{0, 11, 12, 13, 14, 5, 6, 17, 18, 19\}$.
  $\mathrm{Ap}(\Delta \cup \{8, 5\}, 10) = \{0, 11, 12, 13, 14, 5, 16, 17, 8, 19\}$,
  $\mathrm{Ap}(\Delta \cup \{8, 6\}, 10) = \{0, 11, 12, 13, 14, 15, 6, 17, 8, 19\}$.
- $\theta(\Delta \cup \{5, 6\}) = \emptyset = \theta(\Delta \cup \{8, 5\})$, $\theta(\Delta \cup \{8, 6\}) = \{5\}$.
- $B = \{\Delta \cup \{8, 6, 5\}\}$.
- $A = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}, \Delta \cup \{6, 5\}, \Delta \cup \{8, 5\}, \Delta \cup \{8, 6\}, \Delta \cup \{8, 6, 5\}\}$.
- $\mathrm{Ap}(\Delta \cup \{8, 6, 5\}, 10) = \{0, 11, 12, 13, 14, 5, 6, 17, 8, 19\}$.
- $\theta(\Delta \cup \{8, 6, 5\}) = \emptyset$.

Therefore the algorithm returns

$$\mathscr{A}(P) = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}, \Delta \cup \{6, 5\}, \Delta \cup \{8, 5\}, \Delta \cup \{8, 6\}, \Delta \cup \{8, 6, 5\}\}.$$

3. The Least Element of $\mathscr{A}(P)$ That Contains an $\mathscr{A}$ Set

Throughout this section, P denotes a nonempty, finite set of $\mathbb{N} \backslash \{0\}$.

Recall that a subset (X) of $\mathbb{N}$ is an $\mathscr{A}(P)$ *set* if there is $S \in \mathscr{A}(P)$ such that $X \subseteq S$.

Proposition 4. *Let $X \subseteq \mathbb{N}$. Then, X is an $\mathscr{A}(P)$ set if and only if $\langle X \rangle \cap P = \emptyset$.*

Proof. *(Necessity).* If X is an $\mathscr{A}(P)$ set, then there is $S \in \mathscr{A}(P)$ such that $X \subseteq S$. If $\langle X \rangle \subseteq S$ and $P \cap S = \emptyset$, then $\langle X \rangle \cap P = \emptyset$.
(Sufficiency). It is clear that $S = \langle X \rangle \cup \{\max(P) + 1, \rightarrow\}$ is an element of $\mathscr{A}(P)$ that contains X. Therefore, X is an $\mathscr{A}(P)$ set. □

By applying Proposition 1 and Corollary 1, we deduce the following.

Lemma 3. *The intersection of elements belonging to $\mathscr{A}(P)$ is, again, an element of $\mathscr{A}(P)$.*

If X is an $\mathscr{A}(P)$ set, $\mathscr{A}(P)[X]$ denotes the intersection of all elements of $\mathscr{A}(P)$ containing X.

By applying Lemma 3, we can easily obtain the following result.

Proposition 5. *If X is an $\mathscr{A}(P)$ set, then $\mathscr{A}(P)[X]$ is the smallest element of $\mathscr{A}(P)$ (with respect to set inclusion order) containing X.*

Lemma 4. *If $S \in \mathscr{A}(P)$, then $X = \{x \in \mathrm{msg}(S) \mid x \notin \Delta(\mathscr{A}(P))\}$ is an $\mathscr{A}(P)$ set and $\mathscr{A}(P)[X] = S$.*

Proof. According to Proposition 1, $\mathscr{A}(P)$ is a covariety, and according to [22] (Proposition 4.2), we have the result. □

Remember that if X is an $\mathscr{A}(P)$ set and $S = \mathscr{A}(P)[X]$, then we say that X is a $\mathscr{A}(P)$ *system of generators* of S. Moreover, if $S \neq \mathscr{A}(P)[Y]$ for all $Y \subsetneq X$, then X is called a *minimal* $\mathscr{A}(P)$ *system of generators* of S.

Lemma 5. *If X is an $\mathscr{A}(P)$ set, then $\mathscr{A}(P)[X] = \langle X \rangle \cup \{\max(P) + 1, \rightarrow\}$.*

Proof. From the proof of Proposition 4, we know that $\langle X\rangle \cup \{\max(P)+1, \rightarrow\} \in \mathscr{A}(P)$. Moreover, it is clear that if $S \in \mathscr{A}(P)$ and $X \subseteq S$, then $\langle X\rangle \cup \{\max(P)+1, \rightarrow\} \subseteq S$. By applying Proposition 5, we find that $\mathscr{A}(P)[X] = \langle X\rangle \cup \{\max(P)+1, \rightarrow\}$. □

Theorem 1. *If $S \in \mathscr{A}(P)$, then $X = \{x \in \mathrm{msg}(S) \mid x < \max(P)\}$ is the unique minimal $\mathscr{A}(P)$ system of generators of S.*

Proof. According to Lemma 4, we know that X is an $\mathscr{A}(P)$-set and $\mathscr{A}(P)[X] = S$. To conclude the proof, we see that if Y is an $\mathscr{A}(P)$ set and $\mathscr{A}(P)[Y] = S$, then $X \subseteq Y$. According to Lemma 5, $S = \mathscr{A}(P)[Y] = \langle Y\rangle \cup \{\max(P)+1, \rightarrow\}$. Therefore, we can easily deduce that $X \subseteq Y$. □

If $S \in \mathscr{A}(P)$, then $\mathscr{A}(P)\mathrm{msg}(S)$ denotes the unique minimal $\mathscr{A}(P)$ system of generators of S. Recall that the cardinality of $\mathscr{A}(P)\mathrm{msg}(S)$ is called the $\mathscr{A}(P)$ *rank* of S, denoted by $\mathscr{A}(P)_{\mathrm{rank}}(S)$.

The proof of the next result is easy.

Lemma 6. *If $S \in \mathscr{A}(P)$, then we have the following conditions.*

(1) $\mathscr{A}(P)_{\mathrm{rank}}(S) \leq \mathrm{e}(S)$.
(2) $\mathscr{A}(P)_{\mathrm{rank}}(S) = 0$ *if and only if* $S = \Delta(\mathscr{A}(P))$.

It is straightforward to prove the following result.

Proposition 6. *Let $r \in \mathbb{N}\backslash\{0\}$. If $X \subseteq \{1, 2, \cdots, \max(P)-1\}$, $P \cap \langle X\rangle = \emptyset$, $\mathrm{msg}(\langle X\rangle) = X$ and $\sharp X = r$, then $S = \langle X\rangle \cup \{\max(P)+1, \rightarrow\}$ is an element of $\mathscr{A}(P)$ and $\mathscr{A}(P)_{\mathrm{rank}}(S) = r$. Moreover, every element of $\mathscr{A}(P)$ with an $\mathscr{A}(P)$* rank *equal to r has this form.*

Example 2. *Let $P = \{5, 7\}$. Using Proposition 6, we can build all the elements of $\mathscr{A}(P)$ with an $\mathscr{A}(P)$* rank *equal to* 2*. Based on Proposition 6, we know that these numerical semigroups have the form of $\langle X\rangle \cup \{8, \rightarrow\}$, where $X \subseteq \{1, 2, 3, 4, 5, 6\}$, $\langle X\rangle \cap \{5, 7\} = \emptyset$, $\mathrm{msg}(\langle X\rangle) = X$ and $\sharp X = 2$. Therefore, $S = \langle 4, 6\rangle \cup \{8, \rightarrow\}$ is the unique numerical semigroup with these conditions.*

4. The Maximal Elements in $\mathscr{A}(P)$

Throughout this section, P denotes a nonempty, finite subset of $\mathbb{N}\backslash\{0\}$. $\mathscr{M}(P)$ denotes the maximal elements (with respect to inclusion order) of $\mathscr{A}(P)$.

If C is a subset of $\mathbb{N}\backslash\{0\}$, then

$$\mathscr{L}(C) = \{S \mid S \text{ is a numerical semigroup and } S \cap C = \emptyset\}.$$

Observe that $\mathscr{A}(P) \subseteq \mathscr{L}(P)$ and $\mathscr{A}(P) = \{S \in \mathscr{L}(P) \mid \mathrm{F}(S) = \max(P)\}$. However, both sets have the same set of maximal elements, that is, S is a maximal element of $\mathscr{A}(P)$ if and only if S is a maximal element of $\mathscr{L}(P)$. This is the content of the following result.

Proposition 7. *With the above notation, $\mathscr{M}(P) = \mathrm{Maximals}(\mathscr{L}(P))$.*

Proof. It is clear that if S is an element of $\mathscr{L}(P)$, then $S \cup \{\max(P)+1, \rightarrow\}$ is, again, an element of $\mathscr{L}(P)$. Therefore, if S is a maximal element of $\mathscr{L}(P)$, then $S \in \mathscr{A}(P)$.

If $S \in \mathscr{M}(P)$, then there is a maximal element of $\mathscr{L}(P)$ (T) such that $S \subseteq T$. As T is a maximal element of $\mathscr{L}(P)$, $T \in \mathscr{A}(P)$. Hence, $S = T$ is a maximal element of $\mathscr{L}(P)$. □

An algorithm to compute the maximal elements of $\mathscr{L}(P)$ appears in [28]. Therefore, we can assert that we have an algorithm to calculate $\mathscr{M}(P)$. This algorithm will be Algorithm 2, Our next aim in this section is to show another algorithm to compute $\mathscr{M}(P)$. For this, we need to introduce some results. The following result is derived from [1] (Lemma 4.35).

Lemma 7. *Let S and T be numerical semigroups such that $S \subsetneq T$ and $x = \max(T \backslash S)$. Then, $S \cup \{x\}$ is also a numerical semigroup.*

Theorem 2. *Let S be a numerical semigroup. Then, $S \in \mathscr{M}(P)$ if and only if $S \cap P = \emptyset$ and $\mathrm{SG}(S) \subseteq P$.*

Proof. *(Necessity).* If $S \in \mathscr{M}(P)$, then $S \in \mathscr{A}(P)$, so $S \cap P = \emptyset$. If $\mathrm{SG}(S) \nsubseteq P$, then there is $x \in \mathrm{SG}(S)$ such that $x \notin P$. Then, $T = S \cup \{x\} \in \mathscr{A}(P)$ and $S \subsetneq T$. Therefore, $S \notin \mathscr{M}(P)$. *(Sufficiency).* If $\mathrm{SG}(S) \subseteq P$, then $\mathrm{F}(S) \in P$. As $S \cap P = \emptyset$ and $P \subseteq \mathbb{N} \backslash S$, $\max(P) = \mathrm{F}(S)$. Therefore, $S \in \mathscr{A}(P)$. If $S \notin \mathscr{M}(P)$, then there is $T \in \mathscr{A}(P)$ such that $S \subsetneq T$. Let $b = \max(T \backslash S)$. Then, according to Lemma 7, we know that $b \in \mathrm{SG}(S)$. As $\mathrm{SG}(S) \subseteq P$, $b \in P$. Thus, $b \in P \cap T$, contradicting $T \in \mathscr{A}(P)$. □

Algorithm 2 Computation of $\mathscr{M}(P)$

INPUT: A nonempty finite subset (P) of $\mathbb{N} \backslash \{0\}$.
OUTPUT: $\mathscr{M}(P)$.

(1) By using Algorithm 1, compute

$$A = \{(S, \mathrm{Ap}(S, \max(P) + 1)) \mid S \in \mathscr{A}(P)\}.$$

(2) By using Remark 1, compute $\mathrm{SG}(S)$ for all $S \in \mathscr{A}(P)$.
(3) Return $\{S \in \mathscr{A}(P) \mid \mathrm{SG}(S) \subseteq P\}$.

In the following example we see how the above algorithm works.

Example 3. *Let $P = \{4, 7, 9\}$. Now, we compute $\mathscr{M}(P)$ by applying the previous algorithm.*

As in Example 1, we have

$$\mathscr{A}(P) = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}, \Delta \cup \{6,5\}, \Delta \cup \{8,5\}, \Delta \cup \{8,6\}, \Delta \cup \{8,6,5\}\}.$$

Then,

- $A = \{(\Delta, \mathrm{Ap}(\Delta, 10)), (\Delta \cup \{5\}, \mathrm{Ap}(\Delta \cup \{5\}, 10)), (\Delta \cup \{6\}, \mathrm{Ap}(\Delta \cup \{6\}, 10)), (\Delta \cup \{8\}, \mathrm{Ap}(\Delta \cup \{8\}, 10)), (\Delta \cup \{5,6\}, \mathrm{Ap}(\Delta \cup \{5,6\}, 10)), (\Delta \cup \{5,8\}, \mathrm{Ap}(\Delta \cup \{5,8\}, 10)), (\Delta \cup \{6,8\}, \mathrm{Ap}(\Delta \cup \{6,8\}, 10)), (\Delta \cup \{5,6,8\}, \mathrm{Ap}(\Delta \cup \{5,6,8\}, 10))\} = \{\{0, 11, 12, 13, 14, 15, 16, 17, 18, 19\}, \{0, 11, 12, 13, 14, 5, 16, 17, 18, 19\}, \{0, 11, 12, 13, 14, 15, 6, 17, 18, 19\}, \{0, 11, 12, 13, 14, 15, 16, 17, 8, 19\}, \{0, 11, 12, 13, 14, 5, 6, 17, 18, 19\}, \{0, 11, 12, 13, 14, 5, 16, 17, 8, 19\}, \{0, 11, 12, 13, 14, 15, 6, 17, 8, 19\}, \{0, 11, 12, 13, 14, 5, 6, 17, 8, 19\}\}$.
- $\mathrm{SG}(\Delta) = \{5, 6, 7, 8, 9\}$, $\mathrm{SG}(\Delta \cup \{5\}) = \{6, 7, 8, 9\}$, $\mathrm{SG}(\Delta \cup \{6\}) = \{5, 7, 8, 9\}$, $\mathrm{SG}(\Delta \cup \{8\}) = \{4, 5, 6, 7, 9\}$, $\mathrm{SG}(\Delta \cup \{5,6\}) = \{7, 8, 9\}$, $\mathrm{SG}(\Delta \cup \{5,8\}) = \{6, 7, 9\}$, $\mathrm{SG}(\Delta \cup \{6,8\}) = \{4, 5, 7, 9\}$, $\mathrm{SG}(\Delta \cup \{5,6,8\}) = \{7, 9\}$.
- $\mathscr{M}(P) = \{\Delta \cup \{5, 6, 8\}\}$.

Recall that an element of $\mathscr{A}(P)$ is $\mathscr{A}(P)$-*irreducible* if it cannot be written as the intersection of two elements of $\mathscr{A}(P)$ containing it properly. $\mathscr{J}(\mathscr{A}(P)) = \{S \in \mathscr{A}(P) \mid S \text{ is } \mathscr{A}(P)\text{-irreducible}\}$. The following proposition has an immediate proof.

Proposition 8. *With the previous notation, $\mathscr{M}(P) \subseteq \mathscr{J}(\mathscr{A}(P))$.*

Our next objective is to propose Algorithm 3, that computes $\mathscr{J}(\mathscr{A}(P))$. For this, the following result is essential.

Theorem 3. *Let $S \in \mathscr{A}(P)$. Then, S is $\mathscr{A}(P)$-irreducible if and only if $\sharp(\mathrm{SG}(S)\backslash P) \le 1$.*

Proof. *(Necessity).* If $\sharp(\mathrm{SG}(S)\backslash P) \ge 2$, then there is $\{u,v\} \subseteq \mathrm{SG}(S)$ such that $u \ne v$ and $P \cap \{u,v\} = \emptyset$. Then, $\{S \cup \{u\}, S \cup \{v\}\} \subseteq \mathscr{A}(P)$, $S \subsetneq S \cup \{u\}$, $S \subsetneq S \cup \{v\}$ and $(S \cup \{u\}) \cap (S \cup \{v\}) = S$. Therefore, S is not $\mathscr{A}(P)$-irreducible.

(Sufficiency). If S is not $\mathscr{A}(P)$-irreducible, then there is $\{A,B\} \subseteq \mathscr{A}(P)$ such that $S \subsetneq A$, $S \subsetneq B$ and $S = A \cap B$. Let $u = \max(A\backslash S)$ and $v = \max(B\backslash S)$. The, according to Lemma 7, we know that $\{u,v\} \subseteq \mathrm{SG}(S)$. Moreover, it is clear that $u \ne v$ and $\{u,v\} \cap P = \emptyset$. Hence, $\sharp(\mathrm{SG}(S)\backslash P) \ge 2$. □

Algorithm 3 Computation of $\mathscr{J}(\mathscr{A}(P))$

INPUT: A nonempty finite subset P of $\mathbb{N}\backslash\{0\}$.
OUTPUT: $\mathscr{J}(\mathscr{A}(P))$.

(1) Compute $A = \{(S, \mathrm{Ap}(S, \max(P)+1)) \mid S \in \mathscr{A}(P)\}$.
(2) Return $\{S \mid S \in \mathscr{A}(P) \text{ and } \sharp(\mathrm{SG}(S)\backslash P) \le 1\}$.

Next, we illustrate the previous algorithm.

Example 4. *Following Example 3, let $P = \{4,7,9\}$ be the nonempty, finite set. Now, we compute $\mathscr{J}(\mathscr{A}(P))$ by applying Algorithm 3.*

- *In Example 3, by using Algorithm 1 we obtained*

$$\mathscr{A}(P) = \{\Delta, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{8\}, \Delta \cup \{6,5\}, \Delta \cup \{8,5\}, \Delta \cup \{8,6\}, \Delta \cup \{8,6,5\}\}.$$

  and
  $\mathrm{SG}(\Delta) = \{5,6,7,8,9\}$, $\mathrm{SG}(\Delta \cup \{5\}) = \{6,7,8,9\}$, $\mathrm{SG}(\Delta \cup \{6\}) = \{5,7,8,9\}$, $\mathrm{SG}(\Delta \cup \{8\}) = \{4,5,6,7,9\}$, $\mathrm{SG}(\Delta \cup \{5,6\}) = \{7,8,9\}$, $\mathrm{SG}(\Delta \cup \{5,8\}) = \{6,7,9\}$, $\mathrm{SG}(\Delta \cup \{6,8\}) = \{4,5,7,9\}$, $\mathrm{SG}(\Delta \cup \{5,6\}) = \{7,8,9\}$, $\mathrm{SG}(\Delta \cup \{5,6,8\}) = \{7,9\}$.
- *Therefore,* $\mathscr{J}(\mathscr{A}(P)) = \{\Delta \cup \{5,6\}, \Delta \cup \{5,8\}, \Delta \cup \{6,8\}, \Delta \cup \{5,6,8\}\}$.

As a consequence of Theorems 2 and 3, we have the following.

Corollary 3. $\mathscr{M}(P) = \{S \in \mathscr{A}(P) \mid \sharp(\mathrm{SG}(S)\backslash P) = 0\}$.

5. A Partition of $\mathscr{A}(P)$

Throughout this section, P denotes a nonempty, finite subset of $\mathbb{N}\backslash\{0\}$.

Recall that a numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

If F is a positive integer,

$$\mathscr{L}(F) = \{S \mid S \text{ is a numerical semigroup and } \mathrm{F}(S) = F\},$$

and

$$\mathscr{J}(F) = \{S \mid S \text{ is an irreducible numerical semigroup and } \mathrm{F}(S) = F\}.$$

The following result is deduced from [29] (Theorem 1).

Lemma 8. *If $F \in \mathbb{N}\backslash\{0\}$, then $\mathscr{J}(F) = \mathrm{Maximals}(\mathscr{L}(F))$ (with respect to inclusion order).*

If S is a numerical semigroup, then $\theta(S) = \{s \in S\backslash\{0\} \mid s < \frac{\mathrm{F}(S)}{2}\}$.

Over $\mathscr{L}(F)$, we define the following binary equivalence relation: $S\mathscr{R}T$ if and only if $\theta(S) = \theta(T)$. If $S \in \mathscr{L}(F)$, $[S]_{\mathscr{R}} = \{T \in \mathscr{L}(F) \mid S\mathscr{R}T\}$. The quotient set of $\mathscr{L}(F)$, according to relation $\mathscr{R}$, is $\dfrac{\mathscr{L}(F)}{\mathscr{R}} = \{[S]_{\mathscr{R}} \mid S \in \mathscr{L}(F)\}$.

The following result is deduced from [23] (Theorem 3).

Proposition 9. *If $F \in \mathbb{N}\backslash\{0\}$, then $\{[S]_{\mathscr{R}} \mid S \in \mathscr{J}(F)\}$ is a partition of $\mathscr{L}(F)$.*

If $S \in \mathscr{J}(\max(P))$, then $\mathrm{A}(S) = \{T \in [S]_{\mathscr{R}} \mid T \cap P = \emptyset\}$.

The next result is an immediate consequence of previous proposition.

Corollary 4. *The set expressed as $\{\mathrm{A}(S) \mid S \in \mathscr{J}(\max(P))$ and $\mathrm{A}(S) \neq \emptyset\}$ is a partition of $\mathscr{A}(P)$.*

As a consequence of Corollary 4, to build all the elements of $\mathscr{A}(P)$, the following is enough:

1. Compute the set of $\beta(P) = \{S \mid S \in \mathscr{J}(\max(P))$ and $\mathrm{A}(S) \neq \emptyset\}$.
2. For every $S \in \beta(P)$, compute $\mathrm{A}(P)$.

From [23] (Lemma 12), we deduce the following.

Lemma 9. *Let $F \in \mathbb{N}\backslash\{0\}$, $S \in \mathscr{J}(F)$ and T be a numerical semigroup. Then, $T \in [S]_{\mathscr{R}}$ if and only if $\langle\theta(S)\rangle \cup \{F+1, \rightarrow\} \subseteq T \subseteq S$.*

$\Delta(S)$ denotes the numerical semigroup $(\langle\theta(S)\rangle \cup \{F+1, \rightarrow\})$ considered in Lemma 9. The following result is a consequence of it.

Lemma 10. *Let $S \in \mathscr{J}(\max(P))$. Then, $\mathrm{A}(S) \neq \emptyset$ if and only if $\Delta(S) \cap P = \emptyset$.*

In [20], an algorithmic procedure is shown to compute $\mathscr{J}(F)$. Therefore, we can propose Algorithm 4, to calculate $\beta(P)$.

Algorithm 4 Computation of $\beta(P)$

INPUT: A nonempty finite subset P of $\mathbb{N}\backslash\{0\}$.
OUTPUT: $\beta(P)$.

(1) Compute $\mathscr{J}(\max(P))$.
(2) Return $\{S \in \mathscr{J}(\max(P)) \mid \Delta(S) \cap P = \emptyset\}$.

In the following example, we obtain $\beta(\{4,7,9\})$ by using the previous algorithm.

Example 5. *Let $P = \{4,7,9\}$ be a nonempty, finite set.*

- $\mathscr{J}(\max(P)) = \{\langle 5,6,7,8\rangle, \langle 4,6,7\rangle, \langle 2,11\rangle\}$.
- $\theta(\langle 5,6,7,8\rangle) = \emptyset$, $\Delta(\langle 5,6,7,8\rangle) = \{0, 10, \rightarrow\}$, $\Delta(\langle 5,6,7,8\rangle) \cap P = \emptyset$.
  $\theta(\langle 4,6,7\rangle) = \{4\}$, $\Delta(\langle 4,6,7\rangle) = \langle 4,10,11,13\rangle$, $\Delta(\langle 4,6,7\rangle) \cap P = \{4\}$.
  $\theta(\langle 2,11\rangle) = \{2,4\}$, $\Delta(\langle 2,11\rangle) = \langle 2,11\rangle$, $\Delta(\langle 2,11\rangle) \cap P = \{4\}$.
  Then,
- $\beta(\{4,7,9\}) = \{\langle 5,6,7,8\rangle\}$.

Our aim in this section is to present an algorithm that calculates $\mathrm{A}(S)$ given $S \in \beta(S)$. For this reason, we introduce the following notation.

If A and B are nonempty subsets of $\mathbb{Z}$, we write $A + B = \{a + b \mid a \in A, b \in B\}$. If $F \in \mathbb{N}\backslash\{0\}$ and $S \in \mathscr{J}(F)$, $\mathrm{D}(S) = S\backslash\Delta(S)$. If $d \in \mathrm{D}(S)$, then $T(d) = (\{d\} + \Delta(S)) \cap \mathrm{D}(S)$. If $B \subseteq \mathrm{D}(S)$, then $\mathrm{T}(B) = \bigcup_{b\in B} \mathrm{T}(b)$.

The following result appears in [23] (Proposition 14).

Lemma 11. *If* $F \in \mathbb{N}\backslash\{0\}$ *and* $S \in \mathscr{J}(F)$, *then* $[S]_{\mathscr{R}} = \{\Delta(S) \cup \mathrm{T}(B) \mid B \subseteq \mathrm{D}(S)\}$.

By applying this last lemma, we easily deduce the next result.

Proposition 10. *If* $S \in \beta(P)$, *then* $\mathrm{A}(S) = \{\Delta(S) \cup \mathrm{T}(B) \mid B \subseteq \mathrm{D}(S)$ *and* $\mathrm{T}(B) \cap P = \emptyset\}$.

We are now in a position to present the proposed algorithm, that is, Algorithm 5

Algorithm 5 Computation of $\mathrm{A}(S)$

INPUT: An element S of $\beta(P)$.
OUTPUT: $\mathrm{A}(S)$.

(1) Compute $\mathrm{C} = \{b \in \mathrm{D}(S) \mid \mathrm{T}(b) \cap P = \emptyset\}$.
(2) Return $\{\Delta(S) \cup \mathrm{T}(B) \mid B \subseteq C\}$.

In the next example, we show how set $\mathrm{A}(S)$ can be calculated by using the previous algorithm.

Example 6. *Let* $\beta(\{4,7,9\}) = \{\langle 5,6,7,8\rangle\}$ *obtained in Example 5. We now compute* $\mathrm{A}(\langle 5,6,7,8\rangle)$.

- $\mathrm{D}(\langle 5,6,7,8\rangle) = \langle 5,6,7,8\rangle\backslash\Delta(\langle 5,6,7,8\rangle) = \langle 5,6,7,8\rangle\backslash\{0,10,\rightarrow\} = \{5,6,7,8\}$.
- $\mathrm{T}(5) = (\{5\} + \Delta(\langle 5,6,7,8\rangle)) \cap \{5,6,7,8\} = (\{5\} + \{0,10,\rightarrow\}) \cap \{5,6,7,8\} = \{5,15,\rightarrow\} \cap \{5,6,7,8\} = \{5\}$.
  $\mathrm{T}(5) \cap P = \emptyset$.
- $\mathrm{T}(6) = (\{6\} + \Delta(\langle 5,6,7,8\rangle)) \cap \{5,6,7,8\} = (\{6\} + \{0,10,\rightarrow\}) \cap \{5,6,7,8\} = \{6,16,\rightarrow\} \cap \{5,6,7,8\} = \{6\}$.
  $\mathrm{T}(6) \cap P = \emptyset$.
- $\mathrm{T}(7) = (\{7\} + \Delta(\langle 5,6,7,8\rangle)) \cap \{5,6,7,8\} = (\{7\} + \{0,10,\rightarrow\}) \cap \{5,6,7,8\} = \{7,17,\rightarrow\} \cap \{5,6,7,8\} = \{7\}$.
  $\mathrm{T}(7) \cap P = \{7\}$.
- $\mathrm{T}(8) = (\{8\} + \Delta(\langle 5,6,7,8\rangle)) \cap \{5,6,7,8\} = (\{8\} + \{0,10,\rightarrow\}) \cap \{5,6,7,8\} = \{8,18,\rightarrow\} \cap \{5,6,7,8\} = \{8\}$.
  $\mathrm{T}(8) \cap P = \emptyset$.

Then, $\mathrm{C} = \{5,6,8\}$.

- $T(\{\emptyset\}) = \emptyset$,
- $T(\{5\}) = \{5\}$, $T(\{6\}) = \{6\}$, $T(\{8\}) = \{8\}$.
- $T(\{5,6\}) = \{5,6\}$, $T(\{5,8\}) = \{5,8\}$, $T(\{6,8\}) = \{6,8\}$.
- $T(\{5,6,8\}) = \{5,6,8\}$.

Therefore,

$$\mathrm{A}(\langle 5,6,7,8\rangle) = \{\Delta, \Delta\cup\{5\}, \Delta\cup\{6\}, \Delta\cup\{8\}, \Delta\cup\{5,6\}, \Delta\cup\{5,8\}, \Delta\cup\{6,8\}, \Delta\cup\{5,6,8\}\}.$$

As a consequence of Algorithm 5, we present Algorithm 6, that provides an alternative method to Algorithm 1 for computing $\mathscr{A}(P)$

Algorithm 6 Computation of $\mathscr{A}(P)$

INPUT: A nonempty finite subset P from $\mathbb{N}\backslash\{0\}$.
OUTPUT: $\mathscr{A}(P)$.

(1) Compute $\beta(P)$ by using Algorithm 4.
(2) For every $S \in \beta(P)$ compute $\mathrm{A}(S)$ by using Algorithm 5.
(3) Return $\bigcup_{S\in\beta(P)} \mathrm{A}(S)$.

In the next example, by using Algorithms 4 and 5, we show how we can compute set $\mathscr{A}(P)$ in an alternative way to that provided by Algorithm 1.

Example 7. *Let $P = \{4,5,7\}$ be a finite, nonempty subset of $\mathbb{N}\backslash\{0\}$.*

- *Using Algorithm 4, we have seen in Example 5 that $\beta(\{4,7,9\}) = \{\langle 5,6,7,8\rangle\}$.*
- *For $\langle 5,6,7,8\rangle \in \beta(\{4,7,9\})$, using Algorithm 5, we have seen in Example 6 that*

$$\mathrm{A}(\langle 5,6,7,8\rangle)$$
$$= \{\Delta, \Delta\cup\{5\}, \Delta\cup\{6\}, \Delta\cup\{8\}, \Delta\cup\{5,6\}, \Delta\cup\{5,8\}, \Delta\cup\{6,8\}, \Delta\cup\{5,6,8\}\}.$$

- *The algorithm returns*

$$\mathscr{A}(P) = \mathrm{A}(\langle 5,6,7,8\rangle)$$
$$= \{\Delta, \Delta\cup\{5\}, \Delta\cup\{6\}, \Delta\cup\{8\}, \Delta\cup\{5,6\}, \Delta\cup\{5,8\}, \Delta\cup\{6,8\}, \Delta\cup\{5,6,8\}\}.$$

6. Decomposition of $\mathscr{A}(P)$ into R Varieties

Throughout this section, P denotes a nonempty, finite subset of $\mathbb{N}\backslash\{0\}$. Moreover, we suppose that $\mathscr{M}(P) = \{A_1, A_2, \cdots, A_r\}$.

If S is a numerical semigroup, then the set formed by the *small elements* of S is $\mathrm{N}(S) = \{x \in S \mid x < \mathrm{F}(S)\}$.

By applying Lemma 5, we easily deduce the next result.

Proposition 11. *With the above notation, $\mathscr{A}(P) = \{\langle X\rangle \cup \{\max(P)+1, \rightarrow\}\}$ such that $X \subseteq \mathrm{N}(A_i)$ for some $i \in \{1,\cdots,r\}$.*

As a consequence of this proposition, we have the following.

Corollary 5. $\mathscr{A}(P) = \{S \mid$ *S is a numerical semigroup and* $\{0, \max(P)+1, \rightarrow\} \subseteq S \subseteq A_i$ *for some* $i \in \{1,\cdots,r\}\}$.

Let A and B be numerical semigroups such that $A \subseteq B$. $[A,B] = \{S \mid S$ is a numerical semigroup and $A \subseteq S \subseteq B\}$. With this new notation, we can rewrite Corollary 5 as follows.

Corollary 6. $\mathscr{A}(P) = \bigcup_{i=1}^{r} [\{0, \max(P)+1, \rightarrow\}, A_i]$.

If S and T are numerical semigroups such that $S \subsetneq T$, then the $\max(T\backslash S)$ is called the *Frobenius number of S restricted to T*, denoted by $\mathrm{F}_T(S)$.

An *R variety* is a nonempty family ($\mathscr{R}$) of numerical semigroups that verifies the following:

1. $\mathscr{R}$ has a maximum (with respect to inclusion order), denoted by $\Delta(\mathscr{R})$.
2. If $\{S,T\} \subseteq \mathscr{R}$, then $S \cap T \in \mathscr{R}$.

3. If $S \in \mathscr{R}$ and $S \neq \Delta(\mathscr{R})$, then $S \cup \{\mathrm{F}_{\Delta(\mathscr{R})}(S)\} \in \mathscr{R}$.

From [24] (Example 2.3), the following can be deduced.

Proposition 12. *If A and B are numerical semigroups such that $A \subseteq B$, then $[A, B]$ is an R variety.*

Given an R variety ($\mathscr{R}$) Algorithm 4.7 from [24] allows us obtain the set of all elements of $\mathscr{R}$ with a fixed genus. This is Algorithm 7.

The proof of the following lemma is straightforward.

Lemma 12. *If A and B are numerical semigroups and $A \subseteq B$, then $\{\mathrm{g}(S) \mid S \in [A, B]\} = \{\mathrm{g}(B), \mathrm{g}(B)+1, \cdots, \mathrm{g}(A)\}$.*

Algorithm 7 Computation of $[A, B]$

INPUT: A and B numerical semigroups such that $A \subseteq B$.
OUTPUT: $[A, B]$.

(1) For every $g \in \{\mathrm{g}(B), \mathrm{g}(B)+1, \cdots, \mathrm{g}(A)\}$, compute

$$\mathrm{C}(g) = \{S \in [A, B] \mid \mathrm{g}(S) = g\}.$$

(2) Return $\bigcup_{g=\mathrm{g}(B)}^{\mathrm{g}(A)} \mathrm{C}(g)$.

Now, we illustrate how this algorithm works.

Example 8. *Let $A = \{0, 10, \rightarrow\}$ and $B = \langle 5, 6, 8 \rangle$ be numerical semigroups. We now compute the R variety $[A, B]$.*

- *$g(A) = 9$ and $g(B) = 6$. Then, $g \in \{6, 7, 8, 9\}$.*
  $\mathrm{C}(6) = \{\langle 5, 6, 8 \rangle\}$.
  $\mathrm{C}(7) = \{\langle 6, 8, 10, 11, 13, 15 \rangle, \langle 5, 8, 11, 12, 14 \rangle, \langle 5, 6, 13, 14 \rangle\}$.
  $\mathrm{C}(8) = \{\langle 8, 10, 11, 12, 13, 14, 15, 17 \rangle, \langle 6, 10, 11, 13, 14, 15 \rangle, \langle 5, 11, 12, 13, 14 \rangle\}$. $\mathrm{C}(9) = \{\{0, 10, \rightarrow\}\}$.
- *The algorithm returns*

$$\{\langle 5, 6, 8 \rangle, \langle 6, 8, 10, 11, 13, 15 \rangle, \langle 5, 8, 11, 12, 14 \rangle, \langle 5, 6, 13, 14 \rangle, \langle 8, 10, 11, 12, 13, 14, 15, 17 \rangle,$$
$$\langle 6, 10, 11, 13, 14, 15 \rangle, \langle 5, 11, 12, 13, 14 \rangle, \{0, 10, \rightarrow\}\}.$$

Finally, we will present Algorithm 8, which is an alternative to Algorithms 1 and 6 for calculating $\mathscr{A}(P)$.

Algorithm 8 Computation of $\mathscr{A}(P)$

INPUT: A nonempty finite subset P of $\mathbb{N} \backslash \{0\}$.
OUTPUT: $\mathscr{A}(P)$.

(1) Compute $\mathscr{M}(P)$ by using Algorithm 2.
(2) For every $B \in \mathscr{M}(P)$, compute $[\{0, \max(P)+1, \rightarrow\}, B]$ by using Algorithm 7.
(3) Return $\bigcup_{B \in \mathscr{M}(P)} [\{0, \max(P)+1, \rightarrow\}, B]$.

We conclude this paper by presenting an example that illustrates how this algorithm works.

Example 9. *By using the previous algorithm, we calculate set $\mathscr{A}(P)$ as $P = \{4,7,9\}$.*

- *In Example 3 and using Algorithm 2, we find that $\mathscr{M}(P) = \{\Delta \cup \{5,6,8\}\} = \{\langle 5,6,8\rangle\}$.*
- *For $\langle 5,6,8\rangle \in \mathscr{M}(P)$, in Example 8 and using Algorithm 7 we compute the R variety:*

$$[\{0,10,\rightarrow\},\langle 5,6,8\rangle] =$$

$$\{\langle 5,6,8\rangle,\langle 6,8,10,11,13,15\rangle,\langle 5,8,11,12,14\rangle,\langle 5,6,13,14\rangle,\langle 8,10,11,12,13,14,15,17\rangle,$$
$$\langle 6,10,11,13,14,15\rangle,\langle 5,11,12,13,14\rangle,\{0,10,\rightarrow\}\}.$$

- *The algorithm returns*

$$\mathscr{A}(P) =$$

$$\{\langle 5,6,8\rangle,\langle 6,8,10,11,13,15\rangle,\langle 5,8,11,12,14\rangle,\langle 5,6,13,14\rangle,\langle 8,10,11,12,13,14,15,17\rangle,$$
$$\langle 6,10,11,13,14,15\rangle,\langle 5,11,12,13,14\rangle,\{0,10,\rightarrow\}\}.$$

7. Conclusions

This work shows a new application of the concept of covariety to the study of numerical semigroups. Indeed, if P is a nonempty, finite subset of positive integers, we have proven that $\mathscr{A}(P) = \{S \mid S \text{ is a numerical semigroup}, S \cap P = \emptyset \text{ and } \max(P) \text{ is the Frobenius number of } S\}$ is a covariety. This fact has allowed us to do the following:

1. To arrange the elements of $\mathscr{A}(P)$ in the form of a tree and, as a consequence, to propose an algorithm that calculates all its elements;
2. To study the maximal elements from $\mathscr{A}(P)$ with respect to the set inclusion order; in particular, we propose an algorithm for calculating them;
3. To investigate the elements of $\mathscr{A}(P)$ that cannot be expressed as an intersection of two elements of $\mathscr{A}(P)$ that properly contain it; we also propose an algorithm for this purpose.

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