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Independent subcontexts and blocks of concept lattices. Definitions and relationships to decompose fuzzy contexts ☆

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ABSTRACT

The decomposition of datasets is a useful mechanism in the processing of large datasets and it is required in many cases. In formal concept analysis (FCA), the dataset is interpreted as a context and the notion of independent context is relevant in the decomposition of a context. In this paper, we have introduced a formal definition of independent context within the multi-adjoint concept lattice framework, which can be translated to other fuzzy approaches. Furthermore, we have analyzed the decomposition of a general bounded lattice in pieces, that we have called blocks. This decomposition of a lattice has been related to the existence of a decomposition of a context into independent subcontexts. This study will allow to develop algorithms to decompose datasets with imperfect information.

1. Introduction

The processing of large amounts of data is a challenge that continues to be a leading research topic in recent times. One of the strategies to tackle knowledge extraction from relational databases is its factorization/decomposition [5,21,22,24,31]. The ability to decompose a dataset enables the reduction of the complexity of information processing, thereby facilitating the more efficient solution of the problem at hand [7,12]. In addition, two further fundamental aspects can be identified. Firstly, the extracted factors reveal valuable knowledge regarding the entirety of the dataset, and secondly, these factors can be regarded as new variables, which had initially been obscured within the data and have now been exposed by the decomposition.

Formal Concept Analysis (FCA, for short) is a mathematical theory, introduced in the eighties [20,35], in which different tools are developed in order to gather information from relational datasets, and enabling the representation of the extracted knowledge in terms of the algebraic structure of a complete lattice [30,32,33,36]. Different approaches can be found in order to extend FCA to a fuzzy environment [2,6,10,26]. Amongst these approaches, the multi-adjoint framework is one of the most flexible [1,14,17,27,28], offering a set of features that facilitate the modeling of complex real-world problems. Recently, following the same philosophy presented in [19], in [3,4] the authors introduced a study on the properties that these independent subcontexts satisfy in the concept lattice associated with the original context. Moreover, they analyze the extension of these properties to a fuzzy environment, providing a first step to finding a relationship between independent subcontexts and blocks (or intervals) of concepts of the concept lattice. Although the notions of either independent subcontexts and blocks of concepts are intuitive, a formal definition is still required.

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The introduction of these definitions is a key milestone since it will lay the foundations to develop procedures to decompose formal contexts within a fuzzy framework.

One of the most relevant lines of research within FCA, where several methods have already been proposed, is the decomposition of contexts [8,9,11,19,25,34]. In [19], the authors proposed a mechanism based on modal operators to extract subtables, denominated as independent subcontexts, from Boolean tables. Therefore, in this paper, we formally introduce the notion of block of elements of a general bounded lattice and study different properties concerning to this notion, with the purpose of applying all the obtained results to the theory of FCA. In addition, we introduce the formal definition of independent subcontext of a formal context within the fuzzy setting provided by the multi-adjoint paradigm. We also analyze some properties that relate independent subcontexts of a given context to blocks of concepts of the corresponding multi-adjoint concept lattice, and vice versa. As a consequence of this study, we provide a characterization of the contexts that contain independent subcontexts by means of blocks of the associated concept lattice. This final result relates “independent” parts of the original dataset with an algebraic substructure inside the associated concept lattice. This algebraic point of view will facilitate the design of automatic algorithms to decompose datasets with imperfect information. We also include several examples to illustrate the notions and results introduced in this paper.

The organization of this paper is as follows: Section 2 recalls some preliminary notions, several results and fixes the algebraic structure that will be used in this study. In Section 3, the notion of block of elements of a lattice is introduced together with different properties that it satisfies. The notion of independent subcontext in a multi-adjoint framework is presented in Section 4. Furthermore, the connections between independent subcontexts and blocks of concepts are analyzed in-depth in Section 5. Lastly, we summarize our conclusions and present our prospects for future research in Section 6.

2. Preliminaries

In this section, several necessary notions and results related to the fuzzy generalization of FCA given by the multi-adjoint framework [28] are recalled. The basic operators considered in this environment are the adjoint triples [13], whose definition is included below.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

where $x \in P_1$, $y \in P_2$ and $z \in P_3$. Condition (1) is called *adjoint property*.

The following result states some properties of adjoint triples that will be used in this paper.

Proposition 2 *([16]). Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to three posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, then the following properties are satisfied:*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*
3. *$\bot_1 \,\&\, y = \bot_3$, $\top_3 \swarrow y = \top_1$, for all $y \in P_2$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
4. *$x \,\&\, \bot_2 = \bot_3$ and $\top_3 \nwarrow x = \top_2$, for all $x \in P_1$, when $(P_2, \leq_2, \bot_2, \top_2)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
5. *$z \nwarrow \bot_1 = \top_2$ and $z \swarrow \bot_2 = \top_1$, for all $z \in P_3$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded posets.*
6. *$z \swarrow y = \max\{x \in P_1 \mid x \,\&\, y \leq_3 z\}$, for all $y \in P_2$ and $z \in P_3$.*
7. *$z \nwarrow x = \max\{y \in P_2 \mid x \,\&\, y \leq_3 z\}$, for all $x \in P_1$ and $z \in P_3$.*

Gödel, product and Łukasiewicz t-norms together with their residuated implications are some examples of adjoint triples that we will use in this work. It is convenient to note that, since these t-norms are commutative, their residuated implications coincide, that is, $\swarrow^G = \nwarrow_G$, $\swarrow^P = \nwarrow_P$ and $\swarrow^Ł = \nwarrow_Ł$ [15]. Another important property we will use in this paper is the possibility of a conjunctor has zero-divisors.

Definition 3. Given three lower bounded posets, $(P_1, \leq_1, \bot_1)$, $(P_2, \leq_2, \bot_2)$, $(P_3, \leq_3, \bot_3)$, an operator $\&\colon P_1 \times P_2 \to P_3$ has *zero-divisors*, if there exist at least two elements $x \in P_1 \setminus \{\bot_1\}$ and $y \in P_2 \setminus \{\bot_2\}$, such that $x \,\&\, y = \bot_3$.

On the other hand, in order to consider a formal context within this multi-adjoint framework, it is necessary to define an algebraic structure called multi-adjoint frame.

Definition 4. A *multi-adjoint frame* is a tuple $(L_1, L_2, P, \&_1, \dots, \&_n)$,where $(L_1, \preceq_1, \bot_1, \top_1)$ and $(L_2, \preceq_2, \bot_2, \top_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P, for all $i \in \{1, \dots, n\}$.

When $L_1 = L_2 = P$, we will simply write $(L, \&_1, \dots, \&_n)$. From a fixed multi-adjoint frame, a context is defined as follows.

Definition 5. Given a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$, a *context* is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon A \times B \to \{1, \ldots, n\}$ is a mapping which associates any element in $A \times B$ to some particular adjoint triple of the frame.

When P is bounded, that is, $(P, \preceq, \bot_P)$, a context (A, B, R, σ) will be called *normalized* if for every attribute $a \in A$ there exist $b_1, b_2 \in B$ such that $R(a, b_1) \neq \bot_P$ and $R(a, b_2) = \bot_P$ and for every object $b \in B$ there exist $a_1, a_2 \in B$ such that $R(a_1, b) \neq \bot_P$ and $R(a_2, b) = \bot_P$.

The fuzzy generalization of derivation operators $^{\uparrow}\colon L_2^B \to L_1^A$ and $^{\downarrow}\colon L_1^A \to L_2^B$ are given below:

$$g^{\uparrow}(a) = \inf\{R(a, b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\}$$

$$f^{\downarrow}(b) = \inf\{R(a, b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\}$$

for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, where L_2^B and L_1^A denote the set of mappings $g\colon B \to L_2$ and $f\colon A \to L_1$, respectively. In this environment, a *multi-adjoint concept* is a pair $\langle g, f\rangle$, where $g \in L_2^B$ is a fuzzy subset of objects and $f \in L_1^A$ is a fuzzy subset of attributes, satisfying that $g^{\uparrow} = f$ and $f^{\downarrow} = g$. Furthermore, the set of multi-adjoint concepts together with the usual ordering forms a complete lattice.

Definition 6. The *multi-adjoint concept lattice* associated with a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ and a context (A, B, R, σ) given, is the set

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow} = f, f^{\downarrow} = g\}$$

where the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$), for all $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathcal{M}$.

In addition, the fuzzy sets $g \in L_2^B$ and $f \in L_1^A$ such that $g(b) = \top_2$, for all $b \in B$, and $f(a) = \top_1$, for all $a \in A$, will be denoted as $g_{\top}$ and $f_{\top}$, respectively. Similarly, when $g(b) = \bot_2$, for all $b \in B$, and $f(a) = \bot_1$, for all $a \in A$, they will be denoted as $g_{\bot}$ and $f_{\bot}$, respectively.

In the following definition, we recall the notion of meet-irreducible element of a lattice, which plays a key role in several results developed in this paper.

Definition 7. Given a lattice $(L, \preceq)$, such that $\wedge$ is the meet operator, and an element $x \in L$ verifying

1. If L has a top element $\top$, then $x \neq \top$.
2. If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$.

x is called *meet-irreducible ($\wedge$-irreducible) element* of L.

These elements can be seen as a generator system of the rest of elements of the lattice, when the ascending chain condition holds [18].

Proposition 8 *([18]). Given a lattice $(L, \preceq)$, satisfying the ascending chain condition, and the set of meet-irreducible elements $M(L)$, we have for each $x \in L$ that*

$$x = \bigwedge\{m \in L \mid m \in M(L), x \preceq m\}$$

The characterization of the meet-irreducible concepts of a multi-adjoint concept lattice given in [14], will be also used in this work. The following definition is necessary in order to recall the characterization.

Definition 9. For each $a \in A$, the fuzzy subsets of attributes $\phi_{a,x} \in L_1^A$ defined, for all $x \in L_1$, as

$$\phi_{a,x}(a') = \begin{cases} x & \text{if } a' = a \\ \bot_1 & \text{if } a' \neq a \end{cases}$$

will be called *fuzzy-attributes*. The set of all fuzzy-attributes will be denoted as $\Phi = \{\phi_{a,x} \mid a \in A, x \in L_1\}$.

Analogously, the fuzzy-objects are defined in the same way.

The characterization of the meet-irreducible concepts in the multi-adjoint framework is showed below.

Theorem 10 *([14]). The set of $\wedge$-irreducible elements of $\mathcal{M}$, $M_F(A, B, R, \sigma)$, is:*

$$\left\{\langle \phi_{a,x}^{\downarrow}, \phi_{a,x}^{\downarrow\uparrow}\rangle \mid \phi_{a,x}^{\downarrow} \neq \bigwedge\{\phi_{a_i,x_i}^{\downarrow} \mid \phi_{a_i,x_i} \in \Phi, \phi_{a,x}^{\downarrow} \prec_2 \phi_{a_i,x_i}^{\downarrow}\} \text{ and } \phi_{a,x}^{\downarrow} \neq g_{\top}\right\}$$

The following technical result will be used in the proof of several results introduced in this paper.

Lemma 11 *([28]). Let $(L_1, L_2, P, \&_i, \dots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) a context. Given $a \in A, b \in B, x \in L_1$ and $y \in L_2$, the following equalities hold:*

- $\phi^{\downarrow}_{a,x}(b') = R(a, b') \nwarrow_{\sigma(a,b')} x$, *for all* $b' \in B$,
- $\phi^{\uparrow}_{b,y}(a') = R(a', b) \swarrow^{\sigma(a',b)} y$, *for all* $a' \in A$.

Another important result which has been used in this paper is the fundamental theorem for multi-adjoint concept lattices. In order to recall this result, it is necessary to introduce the following definition.

Definition 12. Let $(L_1, L_2, P, \&_i, \dots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) a context. The multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ is *represented* by a complete lattice $(V, \sqsubseteq)$ if there exists a pair of mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ such that:

1a) $\alpha[A \times L_1]$ is infimum-dense;
1b) $\beta[B \times L_2]$ is supremum-dense; and
2) For each $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$:

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \text{if and only if} \quad x \,\&_{\sigma(a,b)}\, y \le R(a, b)$$

Once the previous notion has been included, the fundamental theorem for multi-adjoint concept lattices is recalled.

Theorem 13 *([28]). A complete lattice $(V, \sqsubseteq)$ represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ if and only if $(V, \sqsubseteq)$ is isomorphic to $(\mathcal{M}, \preceq)$.*

The following corollary is derived from the previous theorem.

Corollary 14. *Let $(L_1, L_2, P, \&_i, \dots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) a context. Then, for each $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, the following equivalence holds:*

$$\langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y} \rangle \preceq \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle \quad \textit{if and only if} \quad x \,\&_{\sigma(a,b)}\, y \le R(a, b)$$

Proof. The proof straightforwardly follows from Theorem 13 considering the lattice $(V, \sqsubseteq) = (\mathcal{M}, \preceq)$ and the mappings $\alpha\colon A \times L_1 \to \mathcal{M}$, $\beta\colon B \times L_2 \to \mathcal{M}$, defined as $\alpha(a, x) = \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle$, $\beta(b, y) = \langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y} \rangle$, for all $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$. □

The following section will be devoted to study blocks of elements of bounded lattices.

3. Block of elements of a lattice

In this section, we are going to formalize the notion of block of elements of a non-trivial bounded lattice as well as different properties that this notion satisfies in order to apply them to concept lattices in FCA. Hence, in this paper a bounded lattice $(L, \preceq, \bot, \top)$ with at least three elements will be fixed. First of all, we introduce the central notion of this section.

Definition 15. A sublattice $K \subset L$ is called a *block of elements* of L if $K \setminus \{\bot, \top\} \neq \varnothing$ and $(\uparrow k \cup \downarrow k) \setminus \{\bot, \top\} \subseteq K$, for all $k \in K \setminus \{\bot, \top\}$, where $\uparrow k = \{x \in L \mid k \preceq x\}$ and $\downarrow k = \{x \in L \mid x \preceq k\}$.

A block of elements of a lattice L will be called a block of L, for short. Notice that, given a block K of L, we have that $K \cup \{\bot, \top\}$ is both a proper ideal and a proper filter of L in ordered set theory [18]. Analogously, a subset of L being a proper ideal and a proper filter is a block of L, due to L having at least three elements. One important feature of the notion of block of a lattice is that it could not include the top and/or the bottom elements of the bounded lattice. Next, we present two special kinds of blocks with a significant role in this paper.

Definition 16. Let $K \subset L$ be a block of L. Then,

- K is called a *minimal block* of elements of L if there is no block K' of L such that $K' \subset K$.
- K is called a *complete block* of elements of L if $\bot, \top \in K$.

Notice that the notion of complete block does not imply the maximal notion with respect to the inclusion, we will illustrate this fact in the following example.

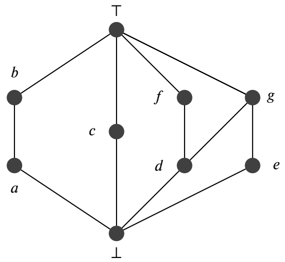

Fig. 1. Hasse diagram of the lattice L of Example 17.

Example 17. Let us consider a bounded lattice $(L, \preceq, \bot, \top)$ represented in Fig. 1. We can define different blocks, for example, the sets $K_1 = \{a, b\}$ and $K_2 = \{c\}$ are minimal blocks, and $K_3 = \{\bot, d, e, f, g, \top\}$ is a complete block. It is easy to check that the complete block K_3 is not maximal with the inclusion, we can find other blocks which contain the complete block K_3. For instance, the set $K_4 = K_3 \cup K_1$ is also a complete block as well as the set $K_5 = K_3 \cup K_2$ and both contain the complete block K_3. Therefore, having a complete block does not imply maximality with respect to inclusion.

In addition, note that $K_1 \cup K_2$ is not a block since it is not a sublattice of L and neither is $K_1 \cup K_2 \cup K_3$, since a block must not be the whole lattice.

In addition, as we commented above, the union of blocks is not a block in general, except when the considered blocks are complete and the union is not the whole lattice.

Proposition 18. *Let $\{K_i\}_{i\in I}$ be a family of blocks of L, with I a non-empty index set. If there exists $j \in I$ such that K_j is a complete block and $\bigcup_{i\in I} K_i \subset L$, then $\bigcup_{i\in I} K_i$ is a complete block.*

Proof. Let us consider a family of blocks $\{K_i\}_{i\in I}$ of L, with I an index set, such that $\bigcup_{i\in I} K_i \subset L$ and there exists $j \in I$ where K_j is a complete block. Hence, this last hypothesis implies that $\top, \bot \in \bigcup_{i\in I} K_i$.

Now, given $k \in \bigcup_{i\in I} K_i \setminus \{\bot, \top\}$, there exists $j \in I$, such that $k \in K_j$. Since K_j is a block, then by Definition 15 we have that

$$(\uparrow k \cup \downarrow k) \setminus \{\bot, \top\} \subset K_j$$

Therefore, we have that

$$(\uparrow k \cup \downarrow k) \setminus \{\bot, \top\} \subseteq \bigcup_{i\in I} K_i$$

In addition, due to $\bigcup_{i\in I} K_i \neq L$ by hypothesis, it only remains to prove that $\bigcup_{i\in I} K_i$ is a sublattice. Given $k_1, k_2 \in \bigcup_{i\in I} K_i$, we clearly have that

$$k_1 \wedge k_2 \in \downarrow k_1 \subseteq \bigcup_{i\in I} K_i \text{ and } k_1 \vee k_2 \in \uparrow k_1 \subseteq \bigcup_{i\in I} K_i$$

because of the properties of infimum and supremum, and that $\bot$ and $\top$ are already in $\bigcup_{i\in I} K_i$. Thus, $\bigcup_{i\in I} K_i$ is also a complete block. □

We can find several blocks within a lattice. In particular, we are interested in finding families of blocks that have no elements in common except for the top and bottom elements of the bounded lattice.

Definition 19. We say that

- Two blocks K_1 and K_2 of L are *independent* if $(K_1 \cap K_2) \subseteq \{\bot, \top\}$.
- A family of blocks $\{K_i\}_{i\in I}$ of L, with I an index set, is called *a family of independent blocks* of L, if the elements in the family are independent pairwise.
- $(L, \preceq, \bot, \top)$ is *decomposed into independent blocks* if there exists a family of independent blocks $\{K_i\}_{i\in I}$ of L, such that $\bigcup_{i\in I} K_i = L$.

Note that the inclusion $(K_1 \cap K_2) \subseteq \{\bot, \top\}$ is equivalent to the equality $(K_1 \cap K_2) \setminus \{\bot, \top\} = \varnothing$, which will also be used in the proofs of some results.

An interesting property about blocks is that the intersection of blocks is also a block, whenever the blocks considered are not independent blocks.

Proposition 20. *Given two blocks K_1 and K_2 of L, we have that either $K_1 \cap K_2$ is a block of L or they are independent blocks of L.*

Proof. Let us consider K_1 and K_2 two blocks of L, and we will assume that they are not independent. Hence, $(K_1 \cap K_2) \nsubseteq \{\bot, \top\}$, and we will prove that $P = K_1 \cap K_2$ is a block of L. Due to the previous expression we only need to prove that P is a sublattice of L and that $(\uparrow p \cup \downarrow p) \setminus \{\bot, \top\} \subseteq P$, for all $p \in P \setminus \{\bot, \top\}$. We will begin with this last property.

Given $p \in P$, since $P \subseteq K_i$ and $P \subseteq K_j$, by Definition 15, it is satisfied that $(\uparrow p \cup \downarrow p) \setminus \{\bot, \top\} \subseteq K_i$ and $(\uparrow p \cup \downarrow p) \setminus \{\bot, \top\} \subseteq K_j$. Therefore, $(\uparrow p \cup \downarrow p) \setminus \{\bot, \top\} \subseteq K_i \cap K_j = P$.

It remains to prove that P is a sublattice of L. Indeed, for every $p_1, p_2 \in P$, we have that $p_1, p_2 \in K_1$ and $p_1, p_2 \in K_2$, and therefore, since K_1 and K_2 are sublattices, we have $p_1 \vee p_2 \in K_1$ and $p_1 \vee p_2 \in K_2$. Thus $p_1 \vee p_2 \in K_1 \cap K_2 = P$. Analogously, we have that $p_1 \wedge p_2 \in P$ and therefore, P is a sublattice of L. Consequently, $K_1 \cap K_2 = P$ is a block of L.

Thus, either K_1 and K_2 are independent blocks of L or $K_1 \cap K_2$ is a block of L. □

Proposition 20 together with the notion of minimal block leads us to determine the relationship between minimal blocks and independence.

Proposition 21. *Every non-empty family of minimal blocks of L with non-repeated blocks is a family of independent blocks.*

Proof. We consider a family $\{K_i\}_{i \in I}$ of minimal blocks of L with non-repeated blocks. We proceed by reductio ad absurdum, let us suppose that there exist two minimal blocks of the family, K_i and K_j with $i \neq j$ and $i, j \in I$, such that they are not independent. Hence, by Proposition 20, $P = K_i \cap K_j$ is a block of L. Since $P \subseteq K_i$ and $P \subseteq K_j$, by the minimality of K_i and K_j, we obtain that $P = K_i$ and $P = K_j$, that is, $K_i = K_j$, which is a contradiction since the family of minimal blocks does not contains repeated blocks.

Therefore, it is satisfied that $(K_i \cap K_j) \subseteq \{\bot, \top\}$ for any two minimal blocks K_i and K_j of the family, that is, the family of minimal blocks is a family of independent blocks. □

Notice that the independence among blocks does not imply the minimality as the following example shows.

Example 22. Let us consider again Example 17 in which the bounded lattice is represented in Fig. 1. We consider again the block $K_3 = \{\bot, d, e, f, g, \top\}$ and we can define a new block $K_6 = \{\bot, a, b, \top\}$. It is easy to verify that K_3 and K_6 are independent blocks, since $K_3 \cap K_6 \subseteq \{\bot, \top\}$, and K_6 is not a minimal block since $K_1 = \{a, b\} \subset K_6$.

The following result asserts that a bounded lattice can be decomposed into independent blocks from a single block of the lattice.

Proposition 23. *Given a block K of L, if $(L \setminus K) \nsubseteq \{\bot, \top\}$, then the set $P = (L \setminus K) \cup \{\bot, \top\}$ is a complete block of L. Moreover, L can be decomposed into the independent blocks K and P.*

Proof. We consider a bounded lattice L such that K is a block of L and $(L \setminus K) \nsubseteq \{\bot, \top\}$. Hence, if we denote $P = (L \setminus K) \cup \{\bot, \top\}$, we have that $P \setminus \{\bot, \top\} \neq \varnothing$. Let us prove that P is a block of L. It is clear that $L = K \cup P$ and $K \cap P \subseteq \{\bot, \top\}$, hence, by Definition 15 and that K is a block of L, we have that any element $p \in P \setminus \{\bot, \top\}$ and any element $k \in K \setminus \{\bot, \top\}$ are incomparable. Now, given $p \in P \setminus \{\bot, \top\}$ we will prove that $\uparrow p \subseteq P$. By reductio ad absurdum, if $\uparrow p \nsubseteq P$, there would exist $p' \in \uparrow p$ satisfying that $p' \in K$, which lead us to $p \in \downarrow p'$ and, since K is a block of L and $p \neq \bot$, we have that $p \in K$, which is a contradiction because $p \neq \top$. With an analogous reasoning, we have that $\downarrow p \subseteq P$. Therefore, $(\uparrow p \ \cup \downarrow p) \subseteq P$, for all $p \in P \setminus \{\bot, \top\}$. Moreover, since L is a bounded lattice, we have that $p_1 \wedge p_2$ and $p_1 \vee p_2$ exist for any two elements $p_1, p_2 \in P$. Now, if $p_1 \in \{\bot, \top\}$, then clearly $p_1 \wedge p_2, p_1 \vee p_2 \in P$. Otherwise, the chain of inclusions $p_1 \wedge p_2 \preceq p_1 \preceq p_1 \vee p_2$ implies that $p_1 \wedge p_2, p_1 \vee p_2 \in (\uparrow p_1 \ \cup \downarrow p_1) \subseteq P$. Thus, P is a sublattice of L. Consequently, P is a block of L and it is clear that is a complete block of L by its definition.

On the other hand, K and P are independent blocks since, by the definition of P, we have straightforwardly that $K \cap P \subseteq \{\bot, \top\}$. Moreover, it is clear that $L = K \cup P$, by the definition of P. Thus, L can be decomposed into K and P. □

As a consequence, we straightforwardly obtain the following corollary.

Corollary 24. *If there exists at least a block in L, then there exists a family of complete blocks of L, $\{K_i\}_{i \in \Lambda}$, with Λ a index set, satisfying that $\bigcup_{i \in \Lambda} K_i = L$.*

Proof. The proof follows from Proposition 23. □

As a consequence, if the bounded lattice L has independent blocks, then, by Proposition 23, L can be decomposed into independent blocks.

Corollary 25. *If the bounded lattice $(L, \preceq, \bot, \top)$ has independent blocks, then it can be decomposed into independent blocks.*

Once all the results related to the notion of block have been introduced, in the following section we will address the notion of subcontext of a formal context within the theory of FCA.

4. Independent subcontexts in the multi-adjoint framework

This section is devoted to provide the formal definition of independent subcontext, as well as the definition of decomposition of a context into independent subcontexts. The formalization of both notions is essential to be able to develop mechanisms of decomposition of contexts within the fuzzy framework provided by the multi-adjoint paradigm.

Throughout this section, a multi-adjoint frame $\mathcal{L} = (L, \&_1, \dots, \&_n)$, where $(L, \preceq, \bot, \top)$ is a complete lattice, and the boundary condition is satisfied in both arguments, that is, $x \&_i \top = \top \&_i x = x$, for all $x \in L$, will be fixed. With this purpose, the first definition we need to introduce is the notion of separable subcontext.

Definition 26. Given the multi-adjoint frame $\mathcal{L}$ and a context (A, B, R, σ), a *separable subcontext* is a tuple[1] $(Y, X, R_{Y\times X}, \sigma_{Y\times X})$ such that

- $Y \subset A$ and $X \subset B$ are non-empty sets.
- There exist $a \in Y$ and $b \in X$ such that $R(a, b) \neq \bot$.
- $R(a, b') = \bot$, for all $(a, b') \in Y \times X^c$.
- $R(a', b) = \bot$, for all $(a', b) \in Y^c \times X$.

Remark 27. Note that we do not allow that the whole context be a separable subcontext of itself. Moreover, when the context considered is normalized, if $(Y, X, R_{Y\times X}, \sigma_{Y\times X})$ is a separable subcontext, then other separable subcontext is the tuple $(Y^c, X^c, R_{Y^c\times X^c}, \sigma_{Y^c\times X^c})$, since for every $a' \in Y^c$, there exists $b' \in X^c$ such that $R(a', b') \neq \bot$ and it is straightforward that this tuple satisfies the last two conditions in Definition 26.

In addition, we will denote a normalized context (A, B, R, σ) by C_n. Moreover, its associated multi-adjoint concept lattice will be denoted as $\mathcal{M}_\text{n}$. From now on, we will assume that the concept lattice satisfies the ascending chain condition, which straightforwardly holds in a finite setting. One important feature of the frames and the contexts considered in this work is that every attribute and every object generate concepts different from the bottom and the top element of the concept lattices, as the following result states. This proposition will be important to proof several of the main results introduced in this work.

Proposition 28. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n, then the following hold:*

- $\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle \in \mathcal{M}$.
- $\langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x}\rangle, \langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y}\rangle \notin \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$, *for all $a \in A$, $b \in B$ and $x, y \in L \setminus \{\bot\}$.*

Proof. Let us consider any attribute $a \in A$. Given an object $b \in B$, by Lemma 11, Proposition 2 and the fact that $f_\bot = \phi_{a,\bot}$, we have that

$$f^{\downarrow}_\bot(b) = R(a, b) \nwarrow_{\sigma(a,b)} \bot = \top$$

Therefore, $f^{\downarrow}_\bot = g_\top$. In addition, we have that

$$f^{\downarrow\uparrow}_\bot(a') = g_\top{}^{\uparrow}(a') = \inf\{R(a', b) \swarrow^{\sigma(a',b)} g_\top(b) \mid b \in B\} = \inf\{R(a', b) \swarrow^{\sigma(a',b)} \top \mid b \in B\}$$

Since the context is normalized, for every $a \in A$ there exists $b_a \in B$ such that $R(a, b_a) = \bot$. Thus,

$$R(a, b_a) \swarrow^{\sigma(a,b_a)} \top = \bot \swarrow^{\sigma(a,b_a)} \top = \max\{x \in L_1 \mid x \&_{\sigma(a,b_a)} \top \preceq \bot\} = \bot$$

Due to the fact that every $\&_i$ satisfies the boundary condition in the first argument. Therefore, we obtain $f^{\downarrow\uparrow}_\bot = f_\bot$, and so $\langle g_\top, f_\bot\rangle \in \mathcal{M}$. Analogously, we can obtain that $\langle g_\bot, f_\top\rangle \in \mathcal{M}$ considering the fuzzy-objects.

Finally, in order to prove the second item, it is sufficient to show that $\langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x}\rangle \notin \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$, for all $a \in A$ and $x \in L \setminus \{\bot\}$, since the proof for the objects follows analogously. Notice that, due to $f_\bot = \phi_{a,\bot}$, for all $a \in A$, it is necessary to remove the bottom element from the lattice. Let us consider any attribute $a \in A$ of the context C_n and $x \in L \setminus \{\bot\}$. Hence, for every object $b \in B$, by Lemma 11 and Proposition 2, we have the following chain of equalities:

$$\phi^{\downarrow}_{a,x}(b) = R(a, b) \nwarrow_{\sigma(a,b)} x = \max\{y \in L \mid x \&_{\sigma(a,b)} y \preceq R(a, b)\}$$

Since the context is normalized, there exist $b_0, b_1 \in B$ such that $R(a, b_0) = \bot$ and $R(a, b_1) \neq \bot$. Therefore,

- Considering b_1 to compute $\phi^{\downarrow}_{a,x}(b_1)$, by the monotonicity of the operator and the boundary condition, we have that

$$x \&_{\sigma(a,b_1)} R(a, b_1) \preceq \top \&_{\sigma(a,b_1)} R(a, b_1) = R(a, b_1)$$

[1] Notice that $R_{Y\times X}$ and $\sigma_{Y\times X}$ denote the restriction of the relation R and the mapping σ to the Cartesian product $Y \times X$.

Therefore, $R(a,b_1) \in \{y \in L \mid x \,\&_{\sigma(a,b_1)}\, y \le R(a,b_1)\}$, that is, $\phi^{\downarrow}_{a,x}(b_1) \ne \bot$ and hence, $\phi^{\downarrow}_{a,x} \ne g_{\bot}$.

- Considering b_0, we have that $x \,\&_{\sigma(a,b_0)} \top \not\le R(a,b_0) = \bot$ due to the boundary condition and $x \in L \setminus \{\bot\}$. Therefore, $\top \notin \{y \in L \mid x \,\&_{\sigma(a,b_0)}\, y \le R(a,b_0)\}$, i.e., $\phi^{\downarrow}_{a,x}(b_0) \ne \top$ and hence, $\phi^{\downarrow}_{a,x} \ne g_{\top}$.

In conclusion, we can assert that $\langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle \notin \{\langle g_{\top}, f_{\bot}\rangle, \langle g_{\bot}, f_{\top}\rangle\}$. □

Next, the formal definition of decomposition of a fuzzy context into subcontexts is introduced, which is one if the main notions of the paper.

Definition 29. The context C_{n} has a *decomposition into independent subcontexts*, if there exists a non-empty index set Λ such that:

- $(A_\lambda, B_\lambda, R_{A_\lambda \times B_\lambda}, \sigma_{A_\lambda \times B_\lambda})$ is a separable subcontext of C_{n}, for all $\lambda \in \Lambda$.
- $\bigcup_{\lambda\in\Lambda} A_\lambda = A$, $\bigcup_{\lambda\in\Lambda} B_\lambda = B$, and $A_\lambda \cap A_\mu = \varnothing$, $B_\lambda \cap B_\mu = \varnothing$, for all $\lambda, \mu \in \Lambda$ with $\lambda \ne \mu$.
- The mapping σ associates conjunctors with no zero-divisor for the subsets $A^c_\lambda \times B_\lambda$ and $A_\lambda \times B^c_\lambda$ of $A \times B$, for all $\lambda \in \Lambda$.

Every tuple $(A_\lambda, B_\lambda, R_{A_\lambda \times B_\lambda}, \sigma_{A_\lambda \times B_\lambda})$ is called *independent subcontext of C_n*.

In order to simplify the notation when we consider a decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_{A_\lambda \times B_\lambda}, \sigma_{A_\lambda \times B_\lambda}) \mid \lambda \in \Lambda\}$, we will denote each of them by $(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda)$. In order to have a better understanding about the existing differences between the notions of separable subcontext and independent subcontext, we will introduce the following example.

Example 30. Let us consider the multi-adjoint frame $(L, \le, \&^*_{\mathrm{G}}, \&^*_{\mathrm{Ł}})$ where $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ represents the partition of the unit interval in five pieces, and $\&^*_{\mathrm{G}}$ and $\&^*_{\mathrm{Ł}}$ are the discretization of the Gödel and Łukasiewicz t-norms, respectively [16,29]. Operators $\&^*_{\mathrm{G}}, \&^*_{\mathrm{Ł}} \colon L \times L \to$ are defined as:

$$x \,\&^*_{\mathrm{G}}\, y = \frac{\lceil 5 \cdot \min\{x, y\} \rceil}{5} \qquad x \,\&^*_{\mathrm{Ł}}\, y = \frac{\lceil 5 \cdot \max\{0, x + y - 1\} \rceil}{5}$$

for all $x, y \in L$, where $\lceil\_\rceil$ is the ceiling function. In this case, the residuated implications $\swarrow^*_{\mathrm{G}}, \nwarrow^*_{\mathrm{G}}, \swarrow^*_{\mathrm{Ł}}, \nwarrow^*_{\mathrm{Ł}} \colon L \times L \to L$ are defined, for all $x, y, z \in L$, as:

$$z \swarrow^*_{\mathrm{G}} y = \frac{\lfloor 5 \cdot (z \leftarrow_{\mathrm{G}} y) \rfloor}{5} \qquad z \swarrow^*_{\mathrm{Ł}} y = \frac{\lfloor 5 \cdot \min\{1, 1 - y + z\} \rfloor}{5}$$

$$z \nwarrow^*_{\mathrm{G}} x = \frac{\lfloor 5 \cdot (z \leftarrow_{\mathrm{G}} x) \rfloor}{5} \qquad z \nwarrow^*_{\mathrm{Ł}} x = \frac{\lfloor 5 \cdot \min\{1, 1 - x + z\} \rfloor}{5}$$

where $\lfloor\_\rfloor$ is the floor function and $\leftarrow \colon [0,1] \times [0,1] \to [0,1]$ is the residuated implication of the Gödel t-norm, defined for all $y, z \in [0,1]$ as:

$$z \leftarrow_{\mathrm{G}} y = \begin{cases} 1 & \text{if } y \le z \\ z & \text{otherwise} \end{cases}$$

Now, we consider two normalized contexts (A, B, R, σ) and (A, B, R, σ') given by the set of attributes $A = \{a_1, a_2, a_3\}$, the set of objects $B = \{b_1, b_2, b_3, b_4\}$, the relation $R \colon A \times B \to L$ defined in Table 1 and where σ and σ' are defined as:

$$\sigma(a,b) = \begin{cases} \&^*_{\mathrm{Ł}} & \text{if } (a,b) = (a_1, b_2) \\ \&^*_{\mathrm{G}} & \text{otherwise} \end{cases} \qquad \sigma'(a,b) = \begin{cases} \&^*_{\mathrm{Ł}} & \text{if } (a,b) \in \{(a_1, b_2), (a_3, b_3)\} \\ \&^*_{\mathrm{G}} & \text{otherwise} \end{cases}$$

Table 1
Fuzzy relation of Example 30.

R	b_1	b_2	b_3	b_4
a_1	0.6	0.8	0	0
a_2	0	0	0.4	0
a_3	0	0	0	1

On the one hand, considering the context (A, B, R, σ) we obtain the following six different separable subcontexts:

- $(A_1, B_1, R_{A_1 \times B_1}, \sigma_{A_1 \times B_1})$, where $A_1 = \{a_1\}$ and $B_1 = \{b_1, b_2\}$.
- $(A_2, B_2, R_{A_2 \times B_2}, \sigma_{A_2 \times B_2})$, where $A_2 = \{a_2\}$ and $B_2 = \{b_3\}$.
- $(A_3, B_3, R_{A_3 \times B_3}, \sigma_{A_3 \times B_3})$, where $A_3 = \{a_3\}$ and $B_3 = \{b_4\}$.
- $(A_4, B_4, R_{A_4 \times B_4}, \sigma_{A_4 \times B_4})$, where $A_4 = A_1 \cup A_2$ and $B_3 = B_1 \cup B_2$.
- $(A_5, B_5, R_{A_5 \times B_5}, \sigma_{A_5 \times B_5})$, where $A_5 = A_1 \cup A_3$ and $B_5 = B_1 \cup B_3$.
- $(A_6, B_6, R_{A_6 \times B_6}, \sigma_{A_6 \times B_6})$, where $A_6 = A_2 \cup A_3$ and $B_6 = B_2 \cup B_3$.

On the other hand, if the context (A, B, R, σ') is considered, the separable subcontexts are the same except for the mapping σ, since the notion of separable subcontext does not depend on the considered mapping σ but on the fuzzy relation R, which is the same in both contexts. Therefore, we can see that there exists a one-to-one correspondence between the separable subcontexts in both contexts, that is, $(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda})$ is a separable subcontext in (A, B, R, σ) if and only if $(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma'_{A_\lambda\times B_\lambda})$ is a separable subcontext in (A, B, R, σ').

However, being a separable subcontext does not always imply being an independent subcontext. This is due to the fact that the mapping of the context plays a key role in Definition 29. In this case, we can build four different decompositions into independent subcontexts from the context (A, B, R, σ), which are the following:

- $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda}) \mid \lambda \in \{1, 2, 3\}\}$.
- $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda}) \mid \lambda \in \{1, 6\}\}$.
- $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda}) \mid \lambda \in \{2, 5\}\}$.
- $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda}) \mid \lambda \in \{3, 4\}\}$.

If we consider the context (A, B, R, σ'), we only have one possible decomposition into independent subcontexts, that is:

- $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma'_{A_\lambda\times B_\lambda}) \mid \lambda \in \{1, 6\}\}$.

For instance, $(A_2, B_2, R_{A_2\times B_2}, \sigma_{A_2\times B_2})$ is an independent subcontext in the decomposition $\{(A_\lambda, B_\lambda, R_{A_\lambda\times B_\lambda}, \sigma_{A_\lambda\times B_\lambda}) \mid \lambda \in \{2, 5\}\}$ of (A, B, R, σ), but the subcontext $(A_2, B_2, R_{A_2\times B_2}, \sigma'_{A_2\times B_2})$ is not an independent subcontext of any decomposition of (A, B, R, σ'), since a conjunctor with zero-divisors is assigned by σ' to a pair which does not belong to the separable subcontext $(A_2, B_2, R_{A_2\times B_2}, \sigma'_{A_2\times B_2})$, in particular $\sigma'(a_3, b_3) = \&^*_{\text{Ł}}$ where $(a_3, b_3) \in A_2^c \times B_2$.

Therefore, the different assignments carried out by the mappings σ and σ' cause that the context (A, B, R, σ) has 4 different decompositions into independent subcontexts and (A, B, R, σ') only one. □

Remark 31. Notice that considering a frame $\mathcal{L}$ and a context C_n, if there exists a separable subcontext $(Y, X, R_{Y\times X}, \sigma_{Y\times X})$, then the index set Λ in Definition 29 has at least two elements, since $(Y^c, X^c, R_{Y^c\times X^c}, \sigma_{Y^c\times X^c})$ is also a separable subcontext (see Remark 27) and both subcontexts form a decomposition into independent subcontexts of $\mathcal{L}$, when the mapping σ associates conjunctors with no zero-divisors for the subsets $Y^c \times X$ and $Y \times X^c$ of $A \times B$.

The following lemma is a technical result which will be useful to demonstrate the main results of this paper.

Lemma 32. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n such that it has a decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$, an attribute $a \in A_\lambda$ and $x \in L\backslash\{\bot\}$, then the extents of the fuzzy attributes, that is $\phi^\downarrow_{a,x} \colon B \to L$, specifically are*

$$\phi^\downarrow_{a,x}(b) = \begin{cases} R(a,b) \nwarrow_{\sigma(a,b)} x & \text{if } b \in B_\lambda \\ \bot & \text{otherwise} \end{cases}$$

for all $b \in B$.

Proof. Given $a \in A_\lambda$ and $x \in L\backslash\{\bot\}$, by Lemma 11, we have that $\phi^\downarrow_{a,x}(b) = R(a,b) \nwarrow_{\sigma(a,b)} x$, for every $b \in B$. Now, if $b \in {B_\lambda}^c$, then we have that $R(a', b) = \bot$, for all $a' \in A_\lambda$, by Definition 26. Therefore, due to $a \in A_\lambda$, we obtain that

$$\phi^\downarrow_{a,x}(b) = \bot \nwarrow_{\sigma(a,b)} x$$

Since the conjunctors associated with $A_\lambda \times B^c_\lambda$ have no zero-divisors, we can assert that $\phi^\downarrow_{a,x}(b) = \bot$. Thus, we obtain the result. □

An essential part to achieve the goal of this paper is to study the relationship between the concepts of the concept lattice and the independent subcontexts of a decomposition of the corresponding context. To this end, we will consider an index set I such that $M_F(A) = \{\langle \phi^\downarrow_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i}\rangle \mid i \in I\}$ is the set of $\wedge$-irreducible elements of $\mathcal{M}_\text{n}$. By the definition of $M_F(A)$, two different indices $i, j \in I$ (with $i \neq j$) may exist such that $a_i = a_j$ and $x_i \neq x_j$. Moreover, if $A' \subseteq A$ and $\langle g, f\rangle \in \mathcal{M}_\text{n}$, we will denote the sets

$$M_F^{A'} = \{\langle \phi^\downarrow_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i}\rangle \in M_F(A) \mid a_i \in A'\}$$

$$M_g^{A'} = \{\langle \phi^\downarrow_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i}\rangle \in M_F^{A'} \mid \langle g, f\rangle \preceq \langle \phi^\downarrow_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i}\rangle\}$$

The following result shows that, if a context C_n has a decomposition into independent subcontexts, then every concept different from $\langle g_\top, f_\bot\rangle$ and $\langle g_\bot, f_\top\rangle$[2] is decomposed into $\wedge$-irreducible elements associated with attributes in only one of the separable subcontexts.

Proposition 33. *Given the multi-adjoint frame $\mathcal{L}$, the context C_n such that $\{(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$ is a decomposition into independent subcontexts, and a concept $\langle g, f\rangle \in \mathcal{M}_n$, with $g \neq g_\top$ and $g \neq g_\bot$, then there exists $\lambda \in \Lambda$ such that*

$$\langle g, f\rangle = \bigwedge M_g^{A_\lambda} \quad \text{and} \quad M_g^{A_\lambda^c} = \varnothing$$

Proof. Given a concept $\langle g, f\rangle \in \mathcal{M}_\mathrm{n}$, being $g \neq g_\top$ and $g \neq g_\bot$, since $\mathcal{M}_\mathrm{n}$ satisfied the ascending chain condition, by Proposition 8, we can ensure that there exists $\lambda \in \Lambda$ and $\langle \phi_{a,x}^{\downarrow}, \phi_{a,x}^{\downarrow\uparrow}\rangle \in M_F^{A_\lambda}$, such that $\langle g, f\rangle \preceq \langle \phi_{a,x}^{\downarrow}, \phi_{a,x}^{\downarrow\uparrow}\rangle$, that means $M_g^{A_\lambda} \neq \varnothing$. Moreover, by means of its expression by $\wedge$-irreducible elements and the fact that $\bigcup_{\lambda\in\Lambda} A_\lambda = A$, we can divide it into the subsets $I_1 = \{i \in I \mid a_i \in A_\lambda, g \preceq_2 \phi_{a_i,x_i}^{\downarrow}\}$ and $I_2 = \{j \in I \mid a_j \in A_\lambda^c, g \preceq_2 \phi_{a_j,x_j}^{\downarrow}\}$, which implies that

$$\langle g, f\rangle = \left(\bigwedge_{i\in I_1} \langle \phi_{a_i,x_i}^{\downarrow}, \phi_{a_i,x_i}^{\downarrow\uparrow}\rangle\right) \wedge \left(\bigwedge_{j\in I_2} \langle \phi_{a_j,x_j}^{\downarrow}, \phi_{a_j,x_j}^{\downarrow\uparrow}\rangle\right)$$

Let us see that the concept $\langle g, f\rangle$ can only be expressed with elements with $i \in I_1$, but not with both. Clearly, by the selection of λ, we have that $I_1 \neq \varnothing$. Now, if we assume that $I_2 \neq \varnothing$, then we consider $j \in I_2$ and obtain, by Lemma 32 and that $\bigcup_{\lambda\in\Lambda} B_\lambda = B$, the following statements.

- If $b \in B_\lambda$, then $\bigwedge_{j\in I_2} \phi_{a_j,x_j}^{\downarrow}(b) = \bot$. Therefore, $g(b) = \bot$, for all $b \in B_\lambda$.
- Furthermore, due to $I_1 \neq \varnothing$, if $b \in B_\lambda^c$, then $\bigwedge_{i\in I_1} \phi_{a_i,x_i}^{\downarrow}(b) = \bot$. Consequently, $g(b) = \bot$, for all $b \in B_\lambda^c$.

Resulting in $g = g_\bot$ which contradicts the hypothesis that the concept is not the bottom concept of the concept lattice. Thus, $I_2 = \varnothing$ and we obtain the results. □

The following example illustrates the previous result.

Example 34. Returning to Example 30 and considering the context(A, B, R, σ'), the list of multi-adjoint concepts is given on the left side of Fig. 2. In this case, we only have a decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_\lambda, \sigma'_\lambda) \mid \lambda \in \{1, 6\}\}$. Let us consider

$C_0 = \langle \{\}, \{a_1/1, a_2/1, a_3/1\}\rangle$

$C_1 = \langle \{b_1/0.6, b_2/0.8\}, \{a_1/1\}\rangle$

$C_2 = \langle \{b_3/0.2\}, \{a_2/1, a_3/0.8\}\rangle$

$C_3 = \langle \{b_4/1\}, \{a_3/1\}\rangle$

$C_4 = \langle \{b_1/0.6, b_2/1\}, \{a_1/0.8\}\rangle$

$C_5 = \langle \{b_1/1, b_2/1\}, \{a_1/0.6\}\rangle$

$C_6 = \langle \{b_1/1, b_2/1, b_3/1, b_4/1\}, \{\}\rangle$

$C_7 = \langle \{b_3/0.4\}, \{a_2/1, a_3/0.6\}\rangle$

$C_8 = \langle \{b_3/0.2, b_4/1\}, \{a_3/0.8\}\rangle$

$C_9 = \langle \{b_3/0.6\}, \{a_2/0.4, a_3/0.4\}\rangle$

$C_{10} = \langle \{b_3/0.4, b_4/1\}, \{a_3/0.6\}\rangle$

$C_{11} = \langle \{b_3/0.8\}, \{a_2/0.4, a_3/0.2\}\rangle$

$C_{12} = \langle \{b_3/0.6, b_4/1\}, \{a_3/0.4\}\rangle$

$C_{13} = \langle \{b_3/1\}, \{a_2/0.4\}\rangle$

$C_{14} = \langle \{b_3/0.8, b_4/1\}, \{a_3/0.2\}\rangle$

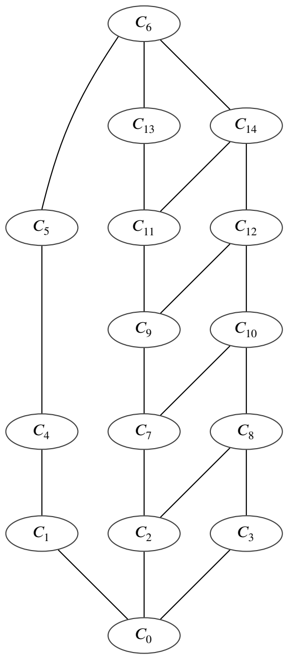

Fig. 2. List of multi-adjoint concepts of the context (A, B, R, σ') and its associated multi-adjoint concept lattice.

[2] Recall that, by Proposition 28, the top and bottom concepts of $\mathcal{M}_\mathrm{n}$ are $\langle g_\top, f_\bot\rangle$ and $\langle g_\bot, f_\top\rangle$, respectively.

the concept $C_2 = \langle\{b_3/0.2\}, \{a_2/1, a_3/0.8\}\rangle$. This concept can be expressed as infimum of two $\wedge$-irreducible concepts, in particular, $C_2 = C_{13} \wedge C_8$, as Fig. 2 shows. Moreover, it can be verified that $C_8 = \langle\phi^{\downarrow}_{a_3,0.8}, \phi^{\downarrow\uparrow}_{a_3,0.8}\rangle$ and $C_{13} = \langle\phi^{\downarrow}_{a_2,0.4}, \phi^{\downarrow\uparrow}_{a_2,0.4}\rangle = \langle\phi^{\downarrow}_{a_2,0.2}, \phi^{\downarrow\uparrow}_{a_2,0.2}\rangle$. Therefore, the attributes generating these concepts belong to the same independent subcontext $(A_6, B_6, R_6, \sigma_6')$. □

The definition and results above have formally fixed the notion of independent subcontexts in the multi-adjoint framework, in which the mapping σ plays a fundamental role. The following section will relate this notion with the blocks of the concept lattice associated with the original context.

5. Relationship between blocks of concepts and independent subcontexts

The goal of this section is to analyze the existing relationships between the notions presented in the two previous sections. Specifically, we are interested in discovering when a fuzzy normalized context contains subcontexts associated with blocks of concepts of the multi-adjoint concept lattice. From now on, the same algebraic structure as in the previous section will be considered.

The two following results relate the independent subcontexts of a decomposition of a context to the blocks of the associated multi-adjoint concept lattice. In particular, the following one takes into account the previous result to determine complete blocks of concepts from independent subcontexts.

Proposition 35. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n that has a decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$, then the set*

$$K_\lambda = \{\langle g, f\rangle \in \mathcal{M}_n \mid \langle g, f\rangle = \bigwedge M_g^{A_\lambda}\} \cup \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$$

is a complete block of $\mathcal{M}_n$, for all $\lambda \in \Lambda$.

Proof. First of all, it is clear that $K_\lambda \setminus \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$ is not empty for any $\lambda \in \Lambda$ by Proposition 28 and Proposition 33.

Then, we will verify that K_λ is a sublattice of $\mathcal{M}_\text{n}$. Given $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in K_\lambda$, it is clear that $\langle g_1, f_1\rangle \wedge \langle g_2, f_2\rangle \in K_\lambda$. Moreover, $\langle g_1, f_1\rangle \vee \langle g_2, f_2\rangle \in \mathcal{M}_\text{n}$ is a concept which will be denoted as $\langle g_3, f_3\rangle$, that is, $\langle g_1, f_1\rangle \vee \langle g_2, f_2\rangle = \langle g_3, f_3\rangle$. If $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ or $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ or $\langle g_3, f_3\rangle = \langle g_\top, f_\bot\rangle$, then we trivially have that $g_3 \in K_\lambda$. Otherwise, by Proposition 33, we have that $M_{g_1}^{A_\lambda^c} = \varnothing$ and $M_{g_2}^{A_\lambda^c} = \varnothing$ which implies that $M_{g_3}^{A_\lambda^c} = \varnothing$. Therefore, $\langle g_3, f_3\rangle = \bigwedge M_{g_3}^{A_\lambda}$, that is, $\langle g_3, f_3\rangle \in K_\lambda$. Consequently, K_λ is a sublattice of $\mathcal{M}_\text{n}$. Finally, it only remains to be verified that

$$(\uparrow\langle g, f\rangle \cup \downarrow\langle g, f\rangle) \setminus \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\} \subseteq K_\lambda$$

for all $\langle g, f\rangle \in K_\lambda$, but this fact straightforwardly holds by the definition of K_λ and Proposition 33. □

As a consequence of the previous result, when a decomposition into independent subcontexts exists, the associated concept lattice can be decomposed into independent blocks of concepts, as the following result states.

Theorem 36. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n which has a decomposition into independent subcontexts, then $\mathcal{M}_n$ has a decomposition into independent blocks. Specifically, if $\{(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$ is a decomposition into independent subcontexts of C_n, then the family $\{K_\lambda\}_{\lambda\in\Lambda}$, where*

$$K_\lambda = \{\langle g, f\rangle \in \mathcal{M}_n \mid \langle g, f\rangle = \bigwedge M_g^{A_\lambda}\} \cup \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$$

for all $\lambda \in \Lambda$, is a decomposition into independent blocks of $\mathcal{M}_n$.

Proof. Let us consider a decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$ of C_n where Λ is an index set with at least two elements ($|\Lambda| \geq 2$, see Remark 31) and $\bigcup_{\lambda\in\Lambda} A_\lambda = A$, $\bigcup_{\lambda\in\Lambda} B_\lambda = B$. Hence, by Proposition 35, we have that

$$K_\lambda = \{\langle g, f\rangle \in \mathcal{M}_\text{n} \mid \langle g, f\rangle = \bigwedge M_g^{A_\lambda}\} \cup \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$$

is a block, for all $\lambda \in \Lambda$.

Now, we will prove that they are independent. Given $\alpha, \beta \in \Lambda$, if there exists a concept $\langle g, f\rangle \in \mathcal{M}_\text{n}$ such that $\langle g, f\rangle \in (K_\alpha \cap K_\beta) \setminus \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$, then $\langle g, f\rangle$ has a non-trivial decomposition of $\wedge$-irreducible elements in $M_F^{A_\alpha}$ and in $M_F^{A_\beta}$, which contradicts Proposition 33. Thus,

$$(K_\alpha \cap K_\beta) \setminus \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\} = \varnothing$$

As a consequence, $\{K_\lambda \subseteq \mathcal{M}_\text{n} \mid \lambda \in \Lambda\}$ is a set of independent blocks and, by Corollary 25, we have that $\mathcal{M}_\text{n}$ can be decomposed into independent blocks. Specifically, the family $\{K_\lambda\}_{\lambda\in\Lambda}$ is a decomposition into independent blocks of $\mathcal{M}_\text{n}$ since for any concept $\langle g, f\rangle \in \mathcal{M}_\text{n}$, by Proposition 33, there exists $\lambda \in \Lambda$, such that $\langle g, f\rangle = \bigwedge M_g^{A_\lambda}$ which straightforwardly implies that $\langle g, f\rangle \in K_\lambda$ and thus $\bigcup_{\lambda\in\Lambda} K_\lambda = \mathcal{M}_\text{n}$. □

Let us come back to Example 34 to illustrate the previous results.

Example 37. We can easily check that the sets K_λ (described in Proposition 35) that we can obtain from the decomposition into independent subcontexts $\{(A_\lambda, B_\lambda, R_\lambda, \sigma'_\lambda) \mid \lambda \in \{1,6\}\}$ are the following:

$$K_1 = \{C_0, C_1, C_4, C_5, C_6\}$$
$$K_6 = \{C_0, C_2, C_3, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_6\}$$

As we can observe in Fig. 2, both sets K_1 and K_6 are complete blocks. Indeed, these blocks are independent and form a decomposition into independent blocks of the multi-adjoint concept lattice. □

Now, we are interested in the other implication, that is, determining a decomposition of independent subcontexts from a given multi-adjoint concept lattice containing independent blocks. Given a family of blocks $\{K_\mu\}_{\mu\in\Lambda}$ of the multi-adjoint concept lattice $\mathcal{M}_{\mathrm{n}}$, then we can define for every $\mu \in \Lambda$ the sets

$$A_\mu = \{a \in A \mid \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x}\rangle \in K^*_\mu, \text{ with } x \in L\}$$
$$B_\mu = \{b \in B \mid \langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y}\rangle \in K^*_\mu, \text{ with } y \in L\}$$

where $K^*_\mu = K_\mu \setminus \{\langle g_\top, f_\bot\rangle, \langle g_\bot, f_\top\rangle\}$.

The following result provides a sufficient condition in order to ensure that these sets form a partition of the corresponding sets.

Proposition 38. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n, if $\mathcal{M}_n$ has a decomposition on independent blocks $\{K_\mu\}_{\mu\in\Lambda}$, then the sets $\{A_\mu \mid \mu \in \Lambda\}$ and $\{B_\mu \mid \mu \in \Lambda\}$ form a partition of the set of attributes A and the set of objects B, respectively.*

Proof. Since $\mathcal{M}_{\mathrm{n}}$ has a decomposition on independent blocks $\{K_\mu\}_{\mu\in\Lambda}$, we have that $\bigcup_{\mu\in\Lambda} K_\mu = \mathcal{M}_{\mathrm{n}}$. Therefore, by Proposition 28, every $a \in A$ belong to a subset A_μ.

Let us prove that $A_\lambda \cap A_\mu = \varnothing$, for all $\lambda, \mu \in \Lambda$, with $\lambda \neq \mu$. Given $a \in A$, if there exists λ and μ, such that $a \in A_\lambda \cap A_\mu$, then there exist $x_i, x_j \in L$ such that $\langle \phi^{\downarrow}_{a,x_i}, \phi^{\downarrow\uparrow}_{a,x_i}\rangle \in K^*_\lambda$ and $\langle \phi^{\downarrow}_{a,x_j}, \phi^{\downarrow\uparrow}_{a,x_j}\rangle \in K^*_\mu$. Therefore, we have that $x_i \vee x_j \in L$, $\phi_{a,x_i} \preceq \phi_{a,x_i\vee x_j}$ and $\phi_{a,x_j} \preceq \phi_{a,x_i\vee x_j}$. Hence, by the monotonicity of the operator $^{\downarrow}$, we obtain that $\phi^{\downarrow}_{a,x_i\vee x_j} \preceq \phi^{\downarrow}_{a,x_i}$ and $\phi^{\downarrow}_{a,x_i\vee x_j} \preceq \phi^{\downarrow}_{a,x_j}$ and this implies, by the definition of block, that $\langle \phi^{\downarrow}_{a,x_i\vee x_j}, \phi^{\downarrow\uparrow}_{a,x_i\vee x_j}\rangle \in K_\lambda$ and $\langle \phi^{\downarrow}_{a,x_i\vee x_j}, \phi^{\downarrow\uparrow}_{a,x_i\vee x_j}\rangle \in K_\mu$. Now, we consider the following cases:

- If $\phi^{\downarrow}_{a,x_i\vee x_j} = g_\top$, then $\phi^{\downarrow}_{a,x_i} = g_\top = \phi^{\downarrow}_{a,x_j}$, which contradicts the assumption on $\langle \phi^{\downarrow}_{a,x_i}, \phi^{\downarrow\uparrow}_{a,x_i}\rangle \in K^*_\lambda$ and $\langle \phi^{\downarrow}_{a,x_j}, \phi^{\downarrow\uparrow}_{a,x_j}\rangle \in K^*_\mu$.
- If $\phi^{\downarrow}_{a,x_i\vee x_j} = g_\bot$, then $\phi^{\downarrow}_{a,\top} \preceq \phi^{\downarrow}_{a,x_i\vee x_j} = g_\bot$. Hence, $\phi^{\downarrow}_{a,\top} = g_\bot$ which is a contradiction since, by Proposition 28, we have that $\phi^{\downarrow}_{a,\top} \neq g_\bot$.
- Otherwise, it contradicts the fact of being independent blocks.

Analogously, we obtain a partition of the set of objects by means of the sets B_μ. □

The following lemma shows a relationship between the partitions of attributes and objects given in the previous result.

Lemma 39. *Given the multi-adjoint frame $\mathcal{L}$, the context C_n and the partitions $\{A_\mu \mid \mu \in \Lambda\}$ and $\{B_\mu \mid \mu \in \Lambda\}$ obtained from a decomposition on independent blocks $\{K_\mu\}_{\mu\in\Lambda}$ of $\mathcal{M}_n$, if $R(a,b) \neq \bot$, with $a \in A$ and $b \in B$, then there exists $\mu \in \Lambda$ such that $a \in A_\mu$ and $b \in B_\mu$.*

Proof. Let us consider an attribute $a \in A$ and an object $b \in B$ such that $R(a,b) \neq \bot$. Therefore, by Proposition 38, there exists $\mu \in \Lambda$ such that $a \in A_\mu$. Now, we consider $y = R(a,b)$ and it is clear that $\top \,\&_{\sigma(a,b)}\, y \leq R(a,b)$, by the boundary condition. Therefore, applying Corollary 14, the inequality $\phi^{\uparrow\downarrow}_{b,y} \preceq \phi^{\downarrow}_{a,\top}$ holds. In addition, by Proposition 28, we know that $g_\top \neq \phi^{\downarrow}_{a,\top} \neq g_\bot$. Then, since $a \in A_\mu$ and by Definition 15, we have that $\langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y}\rangle \in K^*_\mu$, and therefore, $b \in B_\mu$. □

Now, in order to illustrate the previous results, we will come back to Example 30 to build partitions of the sets of attributes and objects from the independent blocks of the concept lattice.

Example 40. Let us consider the multi-adjoint concept lattice, which is depicted in Fig. 3, associated with the context (A, B, R, σ) of Example 30.

We can find several decompositions into independent blocks of the multi-adjoint concept lattice. Let us consider the one given by $\{K_\mu \mid \mu \in \{1,2,3\}\}$ where the independent blocks are the following:

$$K_1 = \{C_0, C_1, C_4, C_5, C_6\}, \; K_2 = \{C_0, C_2, C_6, C_7\}, \text{ and } \; K_3 = \{C_0, C_3, C_6\}$$

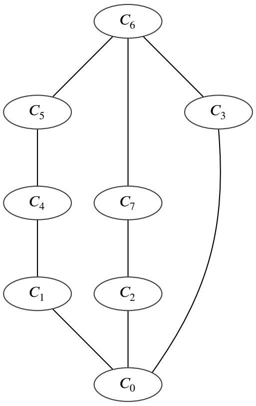

Fig. 3. The multi-adjoint concept lattice associated with the context of Example 30.

In addition, we need the list given in Table 2 which includes the multi-adjoint concepts of the multi-adjoint concept lattice (except for the bottom and the top elements) together with the fuzzy-attributes and fuzzy-objects from which these concepts are obtained. In

Table 2
List of fuzzy-attributes and fuzzy-objects which generate the multi-adjoint concepts in Example 40.

$$
\begin{aligned}
C_1 &= \langle \phi^{\downarrow}_{a_1,1}, \phi^{\downarrow\uparrow}_{a_1,1} \rangle = \langle \phi^{\uparrow\downarrow}_{b_1,0.8}, \phi^{\uparrow}_{b_1,0.8} \rangle = \langle \phi^{\uparrow\downarrow}_{b_1,1}, \phi^{\uparrow}_{b_1,1} \rangle \\
C_2 &= \langle \phi^{\downarrow}_{a_2,0.6}, \phi^{\downarrow\uparrow}_{a_2,0.6} \rangle = \langle \phi^{\downarrow}_{a_2,0.8}, \phi^{\downarrow\uparrow}_{a_2,0.8} \rangle = \langle \phi^{\downarrow}_{a_2,1}, \phi^{\downarrow\uparrow}_{a_2,1} \rangle = \langle \phi^{\uparrow\downarrow}_{b_3,0.6}, \phi^{\uparrow}_{b_3,0.6} \rangle \\
&= \langle \phi^{\uparrow\downarrow}_{b_3,0.8}, \phi^{\uparrow}_{b_3,0.8} \rangle = \langle \phi^{\uparrow\downarrow}_{b_3,1}, \phi^{\uparrow}_{b_3,1} \rangle \\
C_3 &= \langle \phi^{\downarrow}_{a_3,0.2}, \phi^{\downarrow\uparrow}_{a_3,0.2} \rangle = \langle \phi^{\downarrow}_{a_3,0.4}, \phi^{\downarrow\uparrow}_{a_3,0.4} \rangle = \langle \phi^{\downarrow}_{a_3,0.6}, \phi^{\downarrow\uparrow}_{a_3,0.6} \rangle = \langle \phi^{\downarrow}_{a_3,0.8}, \phi^{\downarrow\uparrow}_{a_3,0.8} \rangle \\
&= \langle \phi^{\downarrow}_{a_3,1}, \phi^{\downarrow\uparrow}_{a_3,1} \rangle = \langle \phi^{\uparrow\downarrow}_{b_4,0.2}, \phi^{\uparrow}_{b_4,0.2} \rangle = \langle \phi^{\uparrow\downarrow}_{b_4,0.4}, \phi^{\uparrow}_{b_4,0.4} \rangle = \langle \phi^{\uparrow\downarrow}_{b_4,0.6}, \phi^{\uparrow}_{b_4,0.6} \rangle \\
&= \langle \phi^{\uparrow\downarrow}_{b_4,0.8}, \phi^{\uparrow}_{b_4,0.8} \rangle = \langle \phi^{\uparrow\downarrow}_{b_4,1}, \phi^{\uparrow}_{b_4,1} \rangle \\
C_4 &= \langle \phi^{\downarrow}_{a_1,0.8}, \phi^{\downarrow\uparrow}_{a_1,0.8} \rangle = \langle \phi^{\uparrow\downarrow}_{b_1,1}, \phi^{\uparrow}_{b_1,1} \rangle \\
C_5 &= \langle \phi^{\downarrow}_{a_1,0.2}, \phi^{\downarrow\uparrow}_{a_1,0.2} \rangle = \langle \phi^{\downarrow}_{a_1,0.4}, \phi^{\downarrow\uparrow}_{a_1,0.4} \rangle = \langle \phi^{\downarrow}_{a_1,0.6}, \phi^{\downarrow\uparrow}_{a_1,0.6} \rangle = \langle \phi^{\uparrow\downarrow}_{b_1,0.2}, \phi^{\uparrow}_{b_1,0.2} \rangle \\
&= \langle \phi^{\uparrow\downarrow}_{b_1,0.4}, \phi^{\uparrow}_{b_1,0.4} \rangle = \langle \phi^{\uparrow\downarrow}_{b_1,0.6}, \phi^{\uparrow}_{b_1,0.6} \rangle = \langle \phi^{\uparrow\downarrow}_{b_2,0.2}, \phi^{\uparrow}_{b_2,0.2} \rangle = \langle \phi^{\uparrow\downarrow}_{b_2,0.4}, \phi^{\uparrow}_{b_2,0.4} \rangle \\
&= \langle \phi^{\uparrow\downarrow}_{b_2,0.6}, \phi^{\uparrow}_{b_2,0.6} \rangle = \langle \phi^{\uparrow\downarrow}_{b_2,0.8}, \phi^{\uparrow}_{b_2,0.8} \rangle \\
C_7 &= \langle \phi^{\downarrow}_{a_2,0.2}, \phi^{\downarrow\uparrow}_{a_2,0.2} \rangle = \langle \phi^{\downarrow}_{a_2,0.4}, \phi^{\downarrow\uparrow}_{a_2,0.4} \rangle = \langle \phi^{\uparrow\downarrow}_{b_3,0.2}, \phi^{\uparrow}_{b_3,0.2} \rangle = \langle \phi^{\uparrow\downarrow}_{b_3,0.4}, \phi^{\uparrow}_{b_3,0.4} \rangle
\end{aligned}
$$

this case, the subsets of attributes and objects defined from the block K_1 according to Proposition 38, are the following:

$$A_1 = \{a \in A \mid \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle \in K_1^*, \text{ with } x \in L\} = \{a_1\}$$

$$B_1 = \{b \in B \mid \langle \phi^{\uparrow\downarrow}_{b,y}, \phi^{\uparrow}_{b,y} \rangle \in K_1^*, \text{ with } y \in L\} = \{b_1, b_2\}$$

Equivalently, the subsets of attributes and objects defined from the blocks K_2 and K_3 are given below:

$$A_2 = \{a_2\} \qquad B_2 = \{b_3\}$$

$$A_3 = \{a_3\} \qquad B_3 = \{b_4\}$$

It is clear that $\{A_\mu \mid \mu \in \{1,2,3\}\}$ and $\{B_\mu \mid \mu \in \{1,2,3\}\}$ form a partition of the set of attributes and objects, respectively, as Proposition 38 states.

Furthermore, we have that if $R(a,b) \neq \bot$, with $a \in A$ and $b \in B$, then there exists $\mu \in \{1,2,3\}$ such that $a \in A_\mu$ and $b \in B_\mu$, as Lemma 39 showed. □

Conversely to Proposition 35 and Theorem 36, the following proposition determines separable subcontexts from independent blocks of concepts of a decomposition of the concept lattice.

Proposition 41. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n whose associated multi-adjoint concept lattice has a decomposition into independent blocks $\{K_\mu\}_{\mu\in\Lambda}$, then the tuple $(A_\mu, B_\mu, R_\mu, \sigma_\mu)$ is a separable subcontext of C_n, for all $\mu \in \Lambda$.*

Proof. Let us consider the partitions given by Proposition 38 associated with an index set Λ. Therefore, given any attribute $a \in A$, there exists $\mu \in \Lambda$ such that $a \in A_\mu$. Moreover, by Proposition 28, there exists $b_a \in B$ such that $\phi^{\downarrow}_{a,\top}(b_a) \neq \bot$. In particular, the following chain of equalities holds:

$$\phi^{\downarrow}_{a,\top}(b_a) = R(a, b_a) \nwarrow_{\sigma(a,b_a)} \top = \max\{x \in L \mid \top \,\&_{\sigma(a,b_a)}\, x \leq R(a, b_a)\} = R(a, b_a) \neq \bot$$

where the first equality is satisfied by Lemma 11, the second one by Proposition 2 and the last one holds because $\&_{\sigma(a,b_a)}$ satisfies the boundary condition on the left argument. Moreover, by Lemma 39, $b_a \in B_\mu$, since $R(a, b_a) \neq \bot$. Thus, there exist $a \in A_\mu$ and $b_a \in B_\mu$ such that $R(a, b_a) \neq \bot$. In order to prove that the tuple $(A_\mu, B_\mu, R_\mu, \sigma_\mu)$ is a separable subcontext, it only remains to show that $R(a, b') = \bot$, for all $(a, b') \in A_\mu \times B^c_\mu$; the proof of $R(a', b) = \bot$, for all $(a', b) \in A^c_\mu \times B_\mu$, is analogous. We will proceed by reductio ad absurdum, we suppose that there exists $b' \in B^c_\mu$ such that $R(a, b') \neq \bot$. Hence, by Lemma 39, since $a \in A_\mu$ we have that $b' \in B_\mu$ which is a contradiction. Thus, $R(a, b') = \bot$, for all $(a, b') \in A_\mu \times B^c_\mu$. Consequently, the tuple $(A_\mu, B_\mu, R_\mu, \sigma_\mu)$ is a separable subcontext of C_{n}. □

The following result is an extension of the previous one. It shows that when a multi-adjoint concept lattice has a decomposition into independent blocks, it is also possible to obtain a decomposition into independent subcontexts.

Theorem 42. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n, if $\mathcal{M}_n$ has a decomposition into independent blocks, then C_n can be decomposed into independent subcontexts.*

Proof. First of all, by Proposition 38, we have that if $\mathcal{M}_{\mathrm{n}}$ has a decomposition into independent blocks there exists a partition of the set of attributes and the set of objects associated with an index set Λ. In addition, by Proposition 41, we know that $(A_\mu, B_\mu, R_\mu, \sigma_\mu)$ is a separable subcontext of C_{n}, for all $\mu \in \Lambda$. Consequently, according to Definition 29, it only remains to prove that σ associates conjunctors with no zero-divisors in $A_\mu \times B^c_\mu$ (the proof for $A^c_\mu \times B_\mu$ follows analogously), for all $\mu \in \Lambda$. Let us proceed by reductio ad absurdum. Suppose that there exists $\mu \in \Lambda$ such that σ associates a conjunctor with zero-divisors to a pair $(a, b_0) \in A_\mu \times B^c_\mu$. Hence, there exist $x, y \in L \setminus \{\bot\}$ such that $x \,\&_{\sigma(a,b_0)}\, y = \bot$. Notice that x and y cannot be $\top$, since we get a contradiction with the boundary condition. In addition, by Corollary 14, the inequality $\phi^{\downarrow\uparrow}_{a,x} \leq \phi^{\uparrow}_{b_0,y}$ holds and, by Proposition 28, $f_\top \neq \phi^{\uparrow}_{b_0,y} \neq f_\bot$. Then, by Definition 15 and since $a \in A_\mu$, we have that $\langle \phi^{\uparrow\downarrow}_{b_0,y}, \phi^{\uparrow}_{b_0,y} \rangle \in K^*_\mu$ which contradicts the fact of independent blocks, since $b_0 \in B^c_\mu$ and so b_0 belongs to another block different from K_μ. Therefore, σ cannot associate a conjunctor of $\mathcal{L}$ with zero-divisors in $A_\mu \times B^c_\mu$.

Consequently, $\{(A_\mu, B_\mu, R_\mu, \sigma_\mu) \mid \mu \in \Lambda\}$ is a decomposition into independent subcontexts of the context C_{n}. □

The last result of the paper is a direct consequence of Theorem 36 and Theorem 42.

Corollary 43. *Given the multi-adjoint frame $\mathcal{L}$ and the context C_n, then the following statements are equivalent:*

- *C_n has a decomposition into independent subcontexts.*
- *$\mathcal{M}_n$ has a decomposition into independent blocks.*

Finally, we come back and continue with Example 40 in order to illustrate these last results.

Example 44. We have that $\{A_\mu \mid \mu \in \{1, 2, 3\}\}$ and $\{B_\mu \mid \mu \in \{1, 2, 3\}\}$ are the partitions of the set of attributes and objects obtained in Example 40. Let us consider the subsets A_1 and B_1. Observing the fuzzy relation R in Table 3, we can verify that the tuple $(A_1, B_1, R_{A_1 \times B_1}, \sigma_{A_1 \times B_1})$ is a separable subcontext of (A, B, R, σ) as Proposition 41 states. Moreover, it is easy to check that the tuples $(A_2, B_2, R_{A_2 \times B_2}, \sigma_{A_2 \times B_2})$ and $(A_3, B_3, R_{A_3 \times B_3}, \sigma_{A_3 \times B_3})$ are also separable subcontexts.

Table 3
Fuzzy relation R and the mapping σ of context (A, B, R, σ) of Example 44.

R		B_1		B_2	B_3
		b_1	b_2	b_3	b_4
A_1	a_1	0.6	0.8	0	0
A_2	a_2	0	0	0.4	0
A_3	a_3	0	0	0	1

σ		B_1		B_2	B_3
		b_1	b_2	b_3	b_4
A_1	a_1	$\&^*_G$	$\&^*_Ł$	$\&^*_G$	$\&^*_G$
A_2	a_2	$\&^*_G$	$\&^*_G$	$\&^*_G$	$\&^*_G$
A_3	a_3	$\&^*_G$	$\&^*_G$	$\&^*_G$	$\&^*_G$

In addition, we can see in Table 3 that the mapping σ does not assign conjunctors with zero-divisors to the pairs $A_\mu \times B^c_\mu$ and $A^c_\mu \times B_\mu$, for all $\mu \in \{1, 2, 3\}$. Therefore, as Theorem 42 states, the set $\{(A_\mu, B_\mu, R_\mu, \sigma_\mu) \mid \mu \in \{1, 2, 3\}\}$ is a decomposition into independent subcontexts of (A, B, R, σ).

Finally, as Corollary 43 claims $\{(A_\mu, B_\mu, R_\mu, \sigma_\mu) \mid \mu \in \{1, 2, 3\}\}$ is a decomposition into independent subcontexts of (A, B, R, σ) if and only if $\{K_\mu \mid \mu \in \{1, 2, 3\}\}$ is a decomposition into independent blocks of the multi-adjoint concept lattice. □

6. Conclusions and future work

This paper has started with the notion of block of elements of a general bounded lattice. Different properties have been studied, such as they can decompose the given lattice. In particular, we have proved that minimal blocks are independent blocks and the existence of one block implies the existence of a decomposition of the lattice. These properties are key to study the existence of independent subcontexts of a given context. Before that, this last notion has been formally introduced together with some properties in a particular multi-adjoint framework. Based on this definition we have analyzed the close existing relationship between independent subcontexts and blocks in the multi-adjoint concept lattice. As a consequence of this study, we have provided a characterization of the contexts that contain independent subcontexts by means of blocks of the associated multi-adjoint concept lattice. This fact will allow to lay the foundations to the decomposition of contexts in the multi-adjoint paradigm.

In [23], "block relations" in formal fuzzy concept analysis was introduced with a clear different meaning from the notion of "block of concepts" introduced in this paper. A detail relationship will be given in the future. Furthermore, we will extend these results to more general multi-adjoint frameworks. In addition, we will develop a decomposition mechanism to compute either a decomposition into independent subcontexts of a given context or a decomposition into independent blocks of a given multi-adjoint concept lattice. We are also interested in applying the obtained results to decompose real databases.

CRediT authorship contribution statement

Roberto G. Aragón: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Jesús Medina:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Eloísa Ramírez-Poussa:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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