Research Article

Francisco Javier García-Pacheco*

On sup- and inf-attaining functionals

https://doi.org/10.1515/math-2024-0090
received January 12, 2024; accepted October 15, 2024

Abstract: Reflexivity is characterized by the weak-compactness of the unit ball. The weak-compactness of bounded, closed, and convex sets is characterized through sup-attaining functionals in view of the famous James' theorem. The aim of this mathematical note is to provide the construction of bounded, closed, and convex subsets for which there exists a functional attaining its supremum on such a set but not its infimum. This construction leads to a characterization of reflexivity in the category of normed spaces and a characterization of full norm-attainment also in the category of normed spaces.

Keywords: reflexivity, norm-attaining operator, Bishop-Phelps theorem, supporting vector

MSC 2020: 46A03, 46A35

Dedicated to Prof. Javier Pérez-Fernández

1 Introduction

There are three main algebraic structures involved in Quantum Mechanics: Effect algebras, C^*-algebras, and Clifford algebras. The representation of measurements in a quantum mechanical system via self-adjoint operators led to the development of the study of the previous algebras. The norm of the self-adjoint operator represents the magnitude of the measurement. If such self-adjoint operator is norm-attaining on the unit ball of the Hilbert space, then it means that the measurement achieves its magnitude at a given state. This is why it is important to study norm-attaining functionals and operators, or more generally, sup- and inf-attaining functionals and operators on general bounded, closed, and convex subsets.

Regarding the geometric structure of bounded, closed, and convex sets, it is worth recalling the remarkable Krein-Milman theorem [1], which states that in a Hausdorff locally convex real or complex topological vector space, each compact convex subset coincides with the closed convex hull of its extreme points. The Krein-Milman theorem gave birth to the Krein-Milman property [2–4], which deeply connects with Measure Theory via the remarkable Radon-Nikodym theorem and its subsequent Radon-Nikodym property [5–8]. Actually, a longstanding open problem asks for the equivalence between the Krein-Milman property and the Radon-Nikodym property, although several partial solutions have been already provided [9–11]. There are two forms of the Krein-Milman property (weak and strong), which have been recently proved to be equivalent for Hausdorff locally convex real topological vector spaces enjoying the Bishop-Phelps property [12]. As expected, the Bishop-Phelps property originated from the well-known Bishop-Phelps theorem [13,14] and is defined for general real topological vector spaces.

The aim of this mathematical note is to provide the construction of bounded, closed, and convex subsets for which there can be found functionals attaining their supremum on that sets but not their infimum (Theorem 3.3). This construction yields a characterization of reflexivity in the category of normed spaces (Corollary 3.4). This characterization, when stated in the scope of Banach spaces, can actually be derived from

*** Corresponding author: Francisco Javier García-Pacheco,** Department of Mathematics, College of Engineering, University of Cadiz, Avenida de la Universidad 10, Puerto Real, Spain, e-mail: garcia.pacheco@uca.es

the classic literature [15]; however, our approach is different and originates from the ambience of topological vector spaces. Another characterization is provided, which is about full norm-attainment (every functional is norm-attaining), also in the category of normed spaces, involving supporting vectors of dual operators [16]. A supporting vector is simply a vector at which a bounded linear operator attains its norm. It is shown in Lemma 3.5 that certain supporting vectors of a dual operator can be computed as functionals which are norm-attaining at certain supporting vectors of the predual operator. This result allows a characterization of full norm-attainment in terms of supporting vectors of predual operators (Theorem 3.7 and Corollary 3.8).

2 Preliminaries

Let X be a real topological vector space and consider a non-empty subset C of X. The set of functionals attaining their sup on C is by definition $\mathsf{SA}(C) \coloneqq \{f \in X^* : \sup f(C) \text{ is attained on } C\}$. The set of functionals attaining its inf on C is denoted by $\mathsf{IA}(C)$. Obviously, both $\mathsf{SA}(C)$ and $\mathsf{IA}(C)$ can be empty. In fact, they are interesting when C is bounded, closed, and convex.

The notion of supporting vector appears implicitly and scattered throughout the literature of Banach space theory and operator theory [17–22]. However, it was formerly introduced for the first time in [16]. Let X, Y be real or complex normed spaces and consider a continuous linear operator $T : X \to Y$ between them. The set of supporting vectors of T is by definition

$$\operatorname{suppv}(T) = \{x \in X : \|T(x)\| = \|T\|\|x\|\}.$$

The geometric and topological structure of $\operatorname{suppv}(T)$ has been studied differently in [23,24]. Supporting vectors also have strong applications to applied disciplines, such as Bioengineering, Physics, and Statistics [25,26], in the sense that supporting vectors serve to solve multiobjective optimization problems that commonly arise in the previous disciplines, improving considerably previous results obtained by means of weaker techniques, like for instance Heuristic methods [27–29].

The existence of non-zero supporting vectors of a bounded linear operator implies the existence of non-zero supporting vectors of the dual operator. Indeed, let X, Y be real or complex normed spaces and $T : X \to Y$ a continuous linear operator such that $\operatorname{suppv}(T) \neq \{0\}$, then we will show that $\operatorname{suppv}(T^*) \neq \{0\}$. If $x \neq 0$ satisfies that $\|T(x)\| = \|T\|\|x\|$, then Hahn-Banach theorem assures the existence of a non-zero $y^* \in X^*$ in such a way that $|y^*(T(x))| = \|y^*\|\|T(x)\| = \|y^*\|\|T\|\|x\|$, resulting in $\|T^*(y^*)\| = \|y^* \circ T\| = \|y^*\|\|T\| = \|T^*\|\|y^*\|$. As a consequence, $y^* \in \operatorname{suppv}(T^*)\backslash\{0\}$.

3 Results

Recall that in a Hausdorff locally convex real or complex topological vector space, the closed convex hull of any bounded subset is bounded as well. This fact will be used in the next lemmas.

Lemma 3.1. *Let X be a real topological vector space. Let Y be a proper subspace of X. Let $C \subseteq X$, $x \in X\backslash(Y \cup C)$, and $D \coloneqq \operatorname{co}(C \cup \{x\}) \cap Y$. If D is dense in $\operatorname{co}(C \cup \{x\})$ and $f \in X^*$ satisfies that $f(x) > \sup f(C)$ and attains its infimum on C at some $c_0 \in C \cap Y$, then $f|_Y \in \mathsf{IA}(D)\backslash\mathsf{SA}(D)$.*

Proof. By assumption, $f(c_0) \leq f(c)$ for all $c \in C$. Fix an arbitrary $d \in D$. There are $c \in C$ and $t \in [0, 1]$ such that $d = tc + (1 - t)x$. Then, $f(c_0) \leq f(c) = tf(c) + (1 - t)f(c) \leq tf(c) + (1 - t)f(x) = f(tc + (1 - t)x) = f(d)$, meaning that $f(c_0) = \inf f|_Y(D)$, hence $f|_Y \in \mathsf{IA}(D)$. Next, we will show that $f \notin \mathsf{SA}(D)$. Fix again any arbitrary $d \in D$ and consider $c \in C$ and $t \in [0, 1]$ such that $d = tx + (1 - t)c$. Observe that $t < 1$, since otherwise $d = x \notin Y$, which is impossible. Then, $f(d) = tf(x) + (1 - t)f(c) < tf(x) + (1 - t)f(x) = f(x)$. By density of D in $\operatorname{co}(C \cup \{x\})$, we can find $d' \in D$ in such a way that $f(d) < f(d') < f(x)$. As a consequence, $f \notin \mathsf{SA}(D)$. □

Lemma 3.2. *Let X be a real topological vector space. Let $A \subseteq X$ and $f \in X^*$ such that $f(A)$ is bounded above. Then, $B \coloneqq \overline{\mathrm{co}}(A \cup \{x\})$ is a closed and convex subset of X satisfying that $A \subseteq B$ and $f \in \mathsf{SA}(B)$. If X is Hausdorff and locally convex and A is bounded, then B is bounded as well.*

Proof. Fix an arbitrary $x \in X$ such that $f(x) \geq \sup f(A)$ (note that such an x exists because, by linearity, either f is null or unbounded above). Observe that $B \coloneqq \overline{\mathrm{co}}(A \cup \{x\})$ is clearly closed and convex. Note that $f(x) = tf(x) + (1 - t)f(x) \geq tf(x) + (1 - t)f(a) = f(tx + (1 - t)a)$ for every $a \in A$, meaning that $f(x) = \sup f(\mathrm{co}(A \cup \{x\})) = \sup f(\overline{\mathrm{co}}(A \cup \{x\})) = \sup f(B)$. As a consequence, $f \in \mathsf{SA}(B)$. Finally, if A is bounded and X is Hausdorff and locally convex, then B is bounded as well. □

Note that if C is a bounded, closed, and convex subset of a Hausdorff locally convex real topological vector space, then $\mathrm{co}(C \cup \{x\})$ is bounded, closed, and convex as well.

Theorem 3.3. *Let X be a Hausdorff locally convex real topological vector space. If there exist a bounded, closed, and convex subset C of X and $f \in X^* \backslash \mathsf{IA}(C)$, then there exists another bounded, closed, and convex subset D of X such that $C \subseteq D$ and $f \in \mathsf{SA}(D) \backslash \mathsf{IA}(D)$.*

Proof. In first place, note that $f(C)$ is bounded. Fix an arbitrary $x \in X$ such that $f(x) \geq \sup f(C)$. We already know from Lemma 3.2 that $D \coloneqq \mathrm{co}(C \cup \{x\})$ is a bounded, closed, and convex subset of X such that $f(x) = \sup f(D)$, i.e., $f \in \mathsf{SA}(D)$. Let us show that $f \notin \mathsf{IA}(D)$. Indeed, fix an arbitrary element $tx + (1 - t)c \in D$ with $t \in [0, 1]$ and $c \in C$. Since $f \notin \mathsf{IA}(C)$, there exists $c' \in C$ in such a way that $f(c') < f(c)$. Next, assume $t = 1$. Then, $f(c') < f(c) \leq f(x)$ since $f(x) = \sup f(D)$ with $c' \in D$. Now, assume $0 \leq t < 1$. Then, $f(tx + (1 - t)c') = tf(x) + (1 - t)f(c') < tf(x) + (1 - t)f(c) = f(tx + (1 - t)c)$ with $tx + (1 - t)c' \in D$. As a consequence, $f \notin \mathsf{IA}(D)$. □

It is well known that a normed space is reflexive if and only if it is complete and satisfies full norm-attainment (every functional is norm-attaining) in view of James' theorem [20,21]. In [30], James constructed an example of a non-complete normed space over which every functional is norm-attaining. Refer [31,32] for other weakenings of reflexivity.

Corollary 3.4. *A real normed space X is reflexive if and only if $\mathsf{IA}(C) = \mathsf{SA}(C)$ for each bounded, closed, and convex subset C of X.*

Proof. If X is reflexive, then weak compactness of the unit ball assures that $\mathsf{IA}(C) = \mathsf{SA}(C) = X^*$ for each bounded, closed, and convex subset C of X. Conversely, assume that $\mathsf{IA}(C) = \mathsf{SA}(C)$ for each bounded, closed, and convex subset C of X. We will prove first that $\mathsf{NA}(X) = X^*$, where $\mathsf{NA}(X)$ stands for the set of norm-attaining functionals on X. Suppose on the contrary, there exists $f \in \mathsf{S}_{X^*}$, which is not norm-attaining. Then, $f \notin \mathsf{IA}(\mathsf{B}_X)$. In view of Theorem 3.3, there exists a bounded, closed, and convex subset D of X such that $\mathsf{B}_X \subseteq D$ and $f \in \mathsf{SA}(D) \backslash \mathsf{IA}(D)$, contradicting our initial assumption. As a consequence, $\mathsf{NA}(X) = X^*$. Finally, let us prove that X is complete. Assume X is not complete and denote by Z to the completion of X. Consider $z \in Z \backslash X$ with $\|z\| = 2$. Let $D \coloneqq \mathrm{co}(\mathsf{B}_Z \cup \{z\}) \cap X$, which is clearly bounded, closed, and convex in X and dense in $\mathrm{co}(\mathsf{B}_Z \cup \{z\})$. Choose $z^* \in \mathsf{S}_{Z^*}$ such that $z^*(z) = 2$ and denote $x^* \coloneqq z^*|_X$. Note that x^* attains its infimum on B_X because $\mathsf{NA}(X) = X^*$; therefore, z^* also attains its infimum on B_Z at the same point. In accordance with Lemma 3.1, $x^* \in \mathsf{IA}(D) \backslash \mathsf{SA}(D)$, contradicting our initial assumption. As a consequence, X is complete. □

In Section 2, it was mentioned that the existence of non-zero supporting vectors of an operator implies the existence of non-zero supporting vectors of the dual operator. The converse to the previous sentence is only possible under full norm-attainment. Next lemma provides a sufficient condition to obtain supporting vectors of a dual operator.

Lemma 3.5. *Let X, Y be real or complex normed spaces and $T : X \to Y$ a continuous linear operator. Then,*

$$\mathrm{suppv}(T^*) \supseteq \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

If $\mathsf{NA}(X) = X^*$, *then*

$$\mathrm{suppv}(T^*)\backslash\{0\} \subseteq \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

Proof. Fix an arbitrary $y^* \in Y^*$ and take a non-zero $x \in T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T)$. Then, $|y^*(T(x))| = \|y^*\|\|T(x)\| = \|y^*\|\|T\|\|x\|$. This chain of equalities, together with the fact that $\|y^* \circ T\| \leq \|y^*\|\|T\|$, implies that $\|y^* \circ T\| = \|y^*\|\|T\|$. As a consequence, $\|T^*(y^*)\| = \|y^* \circ T\| = \|y^*\|\|T\| = \|y^*\|\|T^*\|$, meaning that $y^* \in \mathrm{suppv}(T^*)$. Let us assume now that $\mathsf{NA}(X) = X^*$. Take any supporting vector $y^* \in \mathrm{suppv}(T^*)\backslash\{0\}$. By assumption, there exists $x \in \mathsf{S}_X$ satisfying that $y^* \circ T \in X^*$ attains its norm at x, i.e., $|(y^* \circ T)(x)| = \|y^* \circ T\|$. We will prove next that $x \in \mathrm{suppv}(T)$. Indeed,

$$\|y^*\|\|T(x)\| \geq |y^*(T(x))| = |(y^* \circ T)(x)| = \|y^* \circ T\| = \|T^*(y^*)\| = \|T^*\|\|y^*\| = \|y^*\|\|T\|,$$

reaching the conclusion that $\|T(x)\| \geq \|T\|$, i.e., $\|T(x)\| = \|T\|$. It only remains to show that $x \in T^{-1}(\mathrm{suppv}(y^*))$, which holds because $|y^*(T(x))| = \|y^*\|\|T\| = \|y^*\|\|T(x)\|$. □

From Lemma 3.5, we can derive an immediate corollary, whose proof we omit for simplicity.

Corollary 3.6. *Let X, Y be real or complex normed spaces and $T : X \to Y$ a continuous linear operator. Suppose that* $\mathsf{NA}(X) = X^*$. *If* $\mathrm{suppv}(T^*) \neq \{0\}$, *then* $\mathrm{suppv}(T) \neq \{0\}$.

We will finalize this note with a characterization of full norm-attainment in terms of supporting vectors of dual operators.

Theorem 3.7. *A real or complex normed space X satisfies that* $\mathsf{NA}(X) = X^*$ *if and only if for every normed space Y and for every continuous linear operator $T : X \to Y$ such that* $\mathrm{suppv}(T) \neq \{0\}$,

$$\mathrm{suppv}(T^*) = \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

Proof. Assume first that $\mathsf{NA}(X) = X^*$. Take any arbitrary normed space Y and any arbitrary continuous linear operator $T : X \to Y$ such that $\mathrm{suppv}(T) \neq \{0\}$. According to Lemma 3.5,

$$\mathrm{suppv}(T^*) \supseteq \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}$$

and

$$\mathrm{suppv}(T^*)\backslash\{0\} \subseteq \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

It only remains to show that if $y^* = 0$, then $T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}$. Indeed, if $y^* = 0$, then $\mathrm{suppv}(y^*) = Y$, so $T^{-1}(\mathrm{suppv}(y^*)) = X$, meaning that $T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) = \mathrm{suppv}(T) \neq \{0\}$. Conversely, assume that for every normed space Y and for every continuous linear operator $T : X \to Y$ such that $\mathrm{suppv}(T) \neq \{0\}$,

$$\mathrm{suppv}(T^*) = \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

Suppose on the contrary that $\mathsf{NA}(X) \neq X^*$. There exists $x^* \in X^*$ which is not norm-attaining. Then, take $Y := X$ and $T := I_X$. Observe that $\mathrm{suppv}(x^*) = \{0\}$, and $T^{-1}(\mathrm{suppv}(x^*)) = \{0\}$, meaning that $T^{-1}(\mathrm{suppv}(x^*)) \cap \mathrm{suppv}(T) = \{0\}$, hence

$$x^* \notin \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$

Finally, $\mathrm{suppv}(T^*) = X^*$; therefore,

$$\mathrm{suppv}(T^*) \neq \{y^* \in Y^* : T^{-1}(\mathrm{suppv}(y^*)) \cap \mathrm{suppv}(T) \neq \{0\}\}.$$ □

Theorem 3.7 can in fact be rephrased as follows.

Corollary 3.8. *A real or complex normed space X satisfies that* $\mathsf{NA}(X) = X^*$ *if and only if for every normed space* Y *and for every continuous linear operator* $T : X \to Y$, $\operatorname{suppv}(T^*) \neq \{0\}$ *implies* $\operatorname{suppv}(T) \neq \{0\}$.

Proof. If $\mathsf{NA}(X) = X^*$, then we simply call on Corollary 3.6. If $\mathsf{NA}(X) \neq X^*$, then we simply take $Y \coloneqq \mathbb{R}$ or $\mathbb{C}$ and $T \coloneqq x^*$ a non-norm-attaining functional on X (note that $T^* : Y^* \to X^*$ satisfies that $\operatorname{suppv}(T^*) \neq \{0\}$ because Y^* is finite dimensional). □

The proof of Corollary 3.8 can be readapted for completeness purposes.

Scholium 3.9. A real or complex Banach space X is reflexive if and only if for every Banach space Y and for every continuous linear operator $T : X \to Y$, $\operatorname{suppv}(T^*) \neq \{0\}$ implies $\operatorname{suppv}(T) \neq \{0\}$.

Acknowledgements: The author would like to thank the Reviewer for valuable comments and suggestions.

Funding information: This research was funded by Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía: ProyExcel00780 (Operator theory: An interdisciplinary approach), and ProyExcel01036 (Multifísica y optimización multiobjetivo de estimulación magnética transcraneal).

Author contributions: The author confirms the sole responsibility for the conception of the study, presented results, and manuscript preparation.

Conflict of interest: The author states no conflict of interest.

References

[1] M. Krein and D. Milman, *On extreme points of regular convex sets*, Studia Math. **9** (1940), 133–138, DOI: https://doi.org/10.4064/sm-9-1-133-138.
[2] R. Armario, F. J. García-Pacheco, and F. J. Pérez-Fernández, *On the Krein-Milman property and the Bade property*, Linear Algebra Appl. **436** (2012), no. 5, 1489–1502, DOI: https://doi.org/10.1016/j.laa.2011.08.018.
[3] F. J. García-Pacheco, *Convex components and multi-slices in real topological vector spaces*, Ann. Funct. Anal. **6** (2015), no. 3, 73–86, DOI: https://doi.org/10.15352/afa/06-3-7.
[4] R. E. Huff and P. D. Morris, *Dual spaces with the Krein-Milman property have the Radon-Nikodým property*, Proc. Amer. Math. Soc. **49** (1975), 104–108, DOI: https://doi.org/10.2307/2039800.
[5] J. Bourgain, *La propriété de Radon-Nikodym: cours de 3ème cycle, année 1979*, Université Pierre et Marie Curie, Paris VI, 1979.
[6] K. R. Etse, A. K. Lakmon, and Y. Mensah, *The Radon-Nikodým property for the Fourier space of some hypergroups*, Creat. Math. Inform. **32** (2023), no. 2, 153–164.
[7] P. Mikusiński and J. P. Ward, *On the Radon-Nikodym property for vector measures and extensions of transfunctions*, Ann. Math. Sil. **35** (2021), no. 1, 77–89, DOI: https://doi.org/10.2478/amsil-2020-0022.
[8] M. Passenbrunner, *Spline characterizations of the Radon-Nikodým property*, Proc. Amer. Math. Soc. **148** (2020), no. 2, 811–824, DOI: https://doi.org/10.1090/proc/14711.
[9] S. A. Argyros and I. Deliyanni, *Nondentable sets in Banach spaces with separable dual*, Israel J. Math. **81** (1993), no. 1–2, 53–64, DOI: https://doi.org/10.1007/BF02761297.
[10] R. C. James, *The Radon-Nikodým and Krein-Milman properties for convex sets*, in: Geometry of Normed Linear Spaces (Urbana-Champaign, Ill., 1983), Contemporary Mathematics, vol. 52, American Mathematical Society, Providence, RI, 1986, pp. 43–54.
[11] W. Schachermayer, *The Radon-Nikodým property and the Krein-Milman property are equivalent for strongly regular sets*, Trans. Amer. Math. Soc. **303** (1987), no. 2, 673–687, DOI: https://doi.org/10.2307/2000690.
[12] F. J. García-Pacheco, *On Bishop-Phelps and Krein-Milman properties*, Mathematics **11** (2023), no. 21, 4473, DOI: https://doi.org/10.3390/math11214473.
[13] E. Bishop and R. R. Phelps, *The support functionals of a convex set*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, R.I., 1963, pp. 27–35.

[14] R. R. Phelps, *The Bishop-Phelps theorem*, in: Ten Mathematical Essays on Approximation in Analysis and Topology, Elsevier B. V., Amsterdam, 2005, pp. 235–244.
[15] B. Cascales, J. Orihuela, and A. Pérez, *One-sided James' compactness theorem*, J. Math. Anal. Appl. **445** (2017), no. 2, 1267–1283, DOI: https://doi.org/10.1016/j.jmaa.2016.03.080.
[16] C. Cobos-Sánchez, F. J. García-Pacheco, S. Moreno-Pulido, and S. Sáez-Martínez, *Supporting vectors of continuous linear operators*, Ann. Funct. Anal. **8** (2017), no. 4, 520–530, DOI: https://doi.org/10.1215/20088752-2017-0016.
[17] A. Aizpuru and F. J. García-Pacheco, *A short note about exposed points in real Banach spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) **28** (2008), no. 4, 797–800, DOI: https://doi.org/10.1016/S0252-9602(08)60080-6.
[18] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98, DOI: https://doi.org/10.1090/S0002-9904-1961-10514-4.
[19] F. J. García-Pacheco and D. Puglisi, *Lineability of functionals and operators*, Studia Math. **201** (2010), no. 1, 37–47, DOI: https://doi.org/10.4064/sm201-1-3.
[20] R. C. James, *Characterizations of reflexive Banach spaces*, Studia Math. **1** (1963), no. 1, 55–56, DOI: http://eudml.org/doc/215850.
[21] R. C. James, *Characterizations of reflexivity*, Studia Math. **23** (1964), no. 3, 205–216, http://eudml.org/doc/217072.
[22] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148, DOI: https://doi.org/10.1007/BF02759700.
[23] F. J. García-Pacheco, *Lineability of the set of supporting vectors*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 2, 41, DOI: https://doi.org/10.1007/s13398-020-00981-6.
[24] F. J. García-Pacheco and E. Naranjo-Guerra, *Supporting vectors of continuous linear projections*, Int. J. Funct. Anal. Oper. Theory Appl. **9** (2017), no. 3, 85–95, DOI: https://doi.org/10.17654/FA009030085.
[25] C. Cobos-Sanchez, F. J. Garcia-Pacheco, J. M. Guerrero-Rodriguez, and J. R. Hill, *An inverse boundary element method computational framework for designing optimal TMS coils*, Eng. Anal. Bound. Elem. **88** (2018), 156–169, DOI: https://doi.org/10.1016/j.enganabound.2017.11.002.
[26] C. Cobos-Sánchez, F. J. Garcia-Pacheco, J. M. Guerrero-Rodriguez, and L. Garcia-Barrachina, *Solving an IBEM with supporting vector analysis to design quiet TMS coils*, Eng. Anal. Bound. Elem. **117** (2020), 1–12, DOI: https://doi.org/10.1016/j.enganabound.2020.04.013.
[27] L. Marin, H. Power, R. W. Bowtell, C. Cobos-Sanchez, A. A. Becker, P. Glover, et al., *Numerical solution of an inverse problem in magnetic resonance imaging using a regularized higher-order boundary element method*, in: Boundary Elements and Other Mesh Reduction Methods XXIX, WIT Transactions on Modelling and Simulation, vol. 44, WIT Press, Southampton, 2007, pp. 323–332.
[28] L. Marin, H. Power, R. W. Bowtell, C. Cobos-Sanchez, A. A. Becker, P. Glover, et al., *Boundary element method for an inverse problem in magnetic resonance imaging gradient coils*, CMES Comput. Model. Eng. Sci. **23** (2008), no. 3, 149–173, DOI: https://doi.org/10.3970/cmes.2008.023.149.
[29] C. Cobos-Sánchez, J. M. Guerrero-Rodriguez, A. Quirós-Olozábal, and D. Blanco-Navarro, *Novel TMS coils designed using an inverse boundary element method*, Phys. Med. Biol. **62** (2016), no. 1, 73–90, DOI: https://doi.org/10.1088/1361-6560/62/1/73.
[30] R. C. James, *A counterexample for a sup theorem in normed spaces*, Israel J. Math. **9** (1971), 511–512, DOI: https://doi.org/10.1007/BF02771466.
[31] R. W. Cross, *A characterisation of almost reflexive normed spaces*, Proc. Roy. Irish Acad. Sect. A **92** (1992), no. 2, 225–228, DOI: https://www.jstor.org/stable/20489420.
[32] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998.