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MULTIPLE CONVEX BODY SEMIGROUPS AND BUCHSBAUM RINGS

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In this work, using the concept of multiple convex body semigroup, we present new families of Buchsbaum semigroups. We characterize Buchsbaum circle and convex polygonal semigroups and we describe algorithmic methods to check such characterizations.

Introduction

Buchsbaum rings were introduced in the last half of the twentieth century and they have been treated from different points of view. Two good introductions to Buchsbaum rings are [8] and [15]. Given a field $\Bbbk$, r indeterminates over it, $t_1, \dots, t_r$, and an affine cancellative commutative semigroup $S \subset \mathbb{N}^r$, the semigroup ring $\Bbbk[S]$ is defined as the subring of $\Bbbk[t_1, \dots, t_r]$ generated by $t^s = t_1^{s_1} \cdots t_r^{s_r}$ with $s = (s_1, \dots, s_r) \in S$. The semigroup S is Buchsbaum if its associated semigroup ring $\Bbbk[S]$ is a Buchsbaum ring. There are many works devoted to the study of Buchsbaum affine semigroup rings (see for example [1; 2; 7; 12; 15; 16]). A recurrent problem proposed in many of them is to find a criteria, expressed in terms of the affine semigroup S, to know if $\Bbbk[S]$ is Buchsbaum (see [2]).

In this work we focus on the study of Buchsbaum multiple convex body semigroup rings. Given $F \subset \mathbb{R}^r_{\geq}$ a nonempty convex body, we consider the so-called multiple convex body semigroup $\mathcal{F} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^r$, where $F_i = i \cdot F$ with $i \in \mathbb{N}$ (see [4] for further details). These semigroups are a generalization of the semigroups generated by intervals appearing in [14], where F is a real interval $[\alpha, \beta]$. It is well known that affine semigroups are finitely generated if and only if the convex cone spanned by the semigroup is a rational polyhedral cone (see [11]). Thus, in general, multiple convex body semigroups are not finitely generated; in this work, we only consider the case they are finitely generated.

The importance of multiple convex body semigroups is because they are a useful tool to obtain examples of different kinds of rings. For instance, in [3], the authors use these semigroups to characterize some families of Cohen–Macaulay and Gorenstein rings and they give computational methods to get examples. All these characterizations are based on the easy method to check whether an element belongs or not to a multiple convex body semigroup. In this work, taking advantage of this fact, we give a way to check the Buchsbaumness of multiple convex body semigroups by using basic tools of Linear Algebra and Basic Geometry. These tools are also used in the construction of Buchsbaum semigroup rings. Besides,

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in Corollaries 8 and 9, we give explicitly families of Buchsbaum semigroups. We also introduce the `Mathematica` package `PolySGTools` [6]. This package is used to compute the minimal generating set of a convex polygonal semigroup given by a rational polygon and to check Buchsbaumness of an affine convex polygonal semigroup.

The contents of this paper are organized as follows. In Section 1, we provide some basic tools and definitions that are used in the rest of the work. In Section 2, Buchsbaum affine circle semigroups are characterized. Finally, Section 3 is devoted to the study of properties that characterize Buchsbaum affine convex polygonal semigroups (Theorem 6) and to give explicit families of Buchsbaum affine convex polygonal semigroups.

1. Preliminaries

For any A subset of $\mathbb{R}^r$, denote by $A_\geq$ the set $\{(x_1, \dots, x_r) \in A \mid x_i \geq 0,\ i = 1, \dots, r\}$. Let G be a nonempty subset of $\mathbb{R}^r_\geq$, denote by $C(G)$ the cone $\{\sum_{i=1}^p q_i f_i \mid p \in \mathbb{N}, q_i \in \mathbb{Q}_\geq, f_i \in G\}$, and by $\mathrm{int}(G)$ the part of G lying in the interior of $C(G)$. We use $\mathrm{d}(P)$ to represent the Euclidean distance from a point P to the origin O.

Let $S \subset \mathbb{N}^r$ be the affine semigroup generated by the finite set $\{n_1, \dots, n_r, n_{r+1}, \dots, n_{r+m}\}$. A semigroup S is called simplicial if $C(S) = C(\{n_1, \dots, n_r\})$. All semigroups appearing in this work are simplicial, so in the sequel we will assume such property. In the case the semigroup ring $\Bbbk[S]$ is called a Cohen–Macaulay ring, S is a Cohen–Macaulay semigroup. The set $C(S) \cap \mathbb{N}^r$ is an affine semigroup which is denoted by $\mathcal{C}$.

Let $\hat{S}$ be the semigroup $\{a \in \mathbb{N}^r \mid a + n_i \in S,\ \forall i = 1, \dots, r+m\}$. It is straightforward to prove that $\hat{S} \subset \mathcal{C}$ and $\hat{S} + (S \setminus \{0\}) \subseteq S$. The following results are based on the general criteria for Cohen–Macaulayness and Buchsbaumness properties appearing in [10; 16] which laid the ground for the research on this topic.

Theorem 1 [7, Theorem 5]. *The following conditions are equivalent*:

(1) *S is Buchsbaum.*

(2) *$\hat{S}$ is Cohen–Macaulay.*

In [7, Theorem 9], it is given a method to check if a simplicial semigroup is Buchsbaum. To apply such method it is necessary to compute the intersection of the Apéry sets of the generators of the rational cone of S (the elements $n_1, \dots, n_r$). Such intersection is computed using the method presented in [13] that uses some bounds to describe a region where the elements of the Apéry set are; the high value of the bound obtained makes the algorithm impractical in many cases. Thus, to determine if S is Buchsbaum, it is necessary to check if the semigroup $\hat{S}$ is Cohen–Macaulay. In this work we focus on solving algorithmically this problem for some kinds of subsemigroups of $\mathbb{N}^2$.

Given $S \subseteq \mathbb{N}^2$ an affine semigroup, denote by R_1 and R_2 the extremal rays of $\mathcal{C} = C(S) \cap \mathbb{N}^2$ with R_1 the ray with greater slope and by $n_1 \in R_1$ the element of $S \cap R_1$ with less module, similarly define $n_2 \in R_2$. One can verify that $\hat{S} \cap R_i = S \cap R_i$ if S contains only a minimal generator in R_i. Note that $\mathcal{C} \cap R_j = \mathbb{N}^2 \cap R_j$ (with $j = 1, 2$) is a subsemigroup of $\mathbb{N}^2$ and that it is generated only by an element.

Corollary 2 [3, Corollary 2]. *Let $S \subseteq \mathbb{N}^2$, the following conditions are equivalent*:

(1) *S is Cohen–Macaulay.*

(2) *For all $a \in \mathcal{C} \setminus S$, $a + n_1$ or $a + n_2$ does not belong to S.*

Lemma 3 [3, Lemma 3]. *Let $S \subseteq \mathbb{N}^2$ be a simplicial affine semigroup such that* $\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(S)$ *is a nonempty finite set. Then S is not Cohen–Macaulay.*

From now on, we consider only semigroups associated to convex bodies. Let $F \subset \mathbb{R}^r_{\geq}$ be a nonempty convex body, the multiple convex body semigroup generated by F is the semigroup $\mathcal{F} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^r$. In general, these semigroups are not finitely generated. An interesting property of them is that it is easy to check if an element P belongs to a given semigroup. Just proceed as follows: take R the ray defined by P and the segment $R \cap F = \overline{AB}$ with $\mathrm{d}(A) \leq \mathrm{d}(B)$; the element P belongs to $\mathcal{F}$ if and only if the set $\{k \in \mathbb{N} \mid \mathrm{d}(P)/\mathrm{d}(B) \leq k \leq \mathrm{d}(P)/\mathrm{d}(A)\}$ is nonempty.

In [4], multiple convex body semigroups are characterized when the initial convex body is a circle or a convex polygon. In both cases, a multiple convex body semigroup is affine if and only if the intersection of the initial convex body with each extremal ray of its associated positive integer cone contains at least a rational point. Besides, the minimal system of generators of these multiple convex body semigroups can be computed algorithmically (see [4, Theorems 14 and 18] for further details). Let $F \subset \mathbb{R}^2$ be a convex body, in this case the positive integer cone $\mathcal{C}$ is equal to $C(F \cap \mathbb{R}^2_{\geq}) \cap \mathbb{N}^2$ and $\mathrm{int}(\mathcal{C}) = \mathcal{C} \setminus \{R_1, R_2\}$.

2. Buchsbaum affine circle semigroups

Let $C \subset \mathbb{R}^2$ be the circle with center (a, b) and radius $r > 0$ with $a, b, r \in \mathbb{R}$; define C_i the circle with center (ia, ib) and radius ir and $\mathcal{S} = \bigcup_{i=0}^{\infty} C_i \cap \mathbb{N}^2$ the so-called circle semigroup associated to C. These semigroups satisfy that $\mathrm{int}(\mathcal{C}) \setminus \mathcal{S}$ is finite (see [4, Lemma 17]). Note that when $C \cap \mathbb{R}^2_{\geq}$ has at least two points the circle semigroup is simplicial, so in this section we consider that $\mathcal{S}$ is always a simplicial affine circle semigroup. Let $\hat{\mathcal{S}}$ be the semigroup $\{a \in \mathbb{N}^2 \mid a + n_i \in \mathcal{S},\ \forall i = 1, \dots, m\}$ with $\{n_1, n_2, \dots, n_m\}$ the minimal system of generators of $\mathcal{S}$.

Proposition 4. *Let $\mathcal{S} \subset \mathbb{N}^2$ be an affine circle semigroup. The semigroup $\mathcal{S}$ is Buchsbaum if and only if* $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{S}})$ *and $\hat{\mathcal{S}} \cap R_j$ is generated only by one element for $j = 1, 2$.*

Proof. By results of [9; 17], a simplicial semigroup T is Buchsbaum if and only if $T' + (T \setminus \{0\}) \subseteq T$, where T' is the extended semigroup of T given by $T' = T_1 \cap T_2$ with $T_j = \{P - Q \mid P \in T \text{ and } Q \in T \cap R_j\}$ (see the definition of T' for any dimension in the above papers). It is straightforward to prove that $\hat{T} \subset T'$.

By the assumption of $\mathcal{S}$, one can verify that $\mathrm{int}(\mathcal{S}') = \mathcal{S}' \setminus \bigcup_{j=1}^{2} R_j = \mathcal{C} \setminus \bigcup_{j=1}^{2} R_j = \mathrm{int}(\mathcal{C})$. Moreover, $\mathcal{S}' \cap R_j = \mathcal{S} \cap R_j$ if $\mathcal{S}$ contains only a minimal generator in R_j.

If $\mathcal{S}$ is Buchsbaum, $\mathcal{S}' + (\mathcal{S} \setminus \{0\}) \subseteq \mathcal{S}$; thus, $\mathcal{S}' = \hat{\mathcal{S}}$, $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{S}})$, and $\hat{\mathcal{S}} \cap R_j$ is generated by one element for $j = 1, 2$.

In order to finish the proof, we show that if $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{S}})$ and $\hat{\mathcal{S}} \cap R_j$ is generated only by one element for $j = 1, 2$, then $\hat{\mathcal{S}} = \mathcal{S}'$. It is straightforward to prove that $\mathcal{S}' \cap R_j = \hat{\mathcal{S}} \cap R_j$ if $\hat{\mathcal{S}} \cap R_j$ is generated only by one element. Since $\mathrm{int}(\hat{\mathcal{S}}) = \mathrm{int}(\mathcal{C}) = \mathrm{int}(\mathcal{S}')$, $\hat{\mathcal{S}} = \mathcal{S}'$. So $\mathcal{S}' + (\mathcal{S} \setminus \{0\}) \subseteq \mathcal{S}$, and $\mathcal{S}$ is Buchsbaum. □

The conditions of Proposition 4 can be determined from the initial circle. To check whether $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{S}})$, we only have to compute the finite set $\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\hat{\mathcal{S}})$ by using the bound provided by [4, Lemma 17]. The second condition is satisfied whether $C \cap R_j$ is a point or, in the case $C \cap R_j$ is a segment, if the generator of $\mathcal{C} \cap R_j$ belongs to $\hat{\mathcal{S}}$. Both conditions can be checked algorithmically.

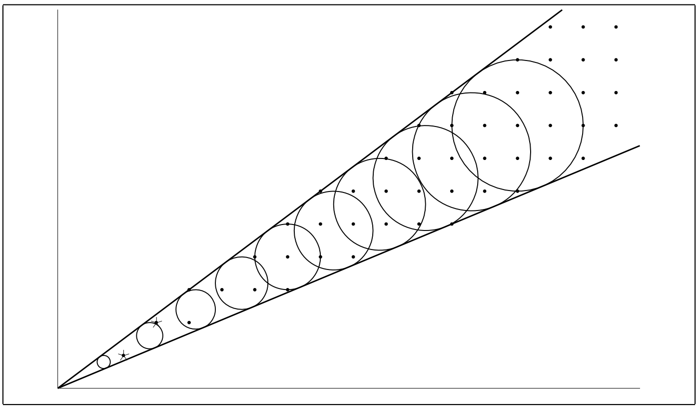

Figure 1. The affine circle semigroup associated to the circle with center $\left(\frac{7}{5}, \frac{4}{5}\right)$ and radius $\frac{1}{5}$.

Example 5. Let C be the circle with center $\left(\frac{7}{5}, \frac{4}{5}\right)$ and radius $\frac{1}{5}$. Using the program `CircleSG` (see [5]), we obtain that the affine circle semigroup[1] $\mathcal{S}$ associated to C is minimally generated by the set

$$\Big\{(4, 2), (5, 3), (6, 3), (6, 4), (7, 3), (7, 4), (7, 5), (8, 5), (9, 4), (9, 6), (10, 7), (11, 8), (15, 11), (19, 8), (19, 14), (23, 17), (27, 20), (31, 13), (31, 23), (32, 24), (35, 26), (43, 18), (55, 23), (67, 28), (79, 33), (91, 38), (96, 40)\Big\}$$

and $\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\mathcal{S})$ is $\{(2, 1), (3, 2)\}$ (see Figure 1). It is easy to check that the points $(2, 1)$ and $(3, 2)$ belong to $\hat{\mathcal{S}}$. Thus, $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{S}})$. Besides, $\hat{\mathcal{S}} \cap R_1 = \langle(32, 24)\rangle$ and $\hat{\mathcal{S}} \cap R_2 = \langle(96, 40)\rangle$. By Proposition 4, the affine circle semigroup $\mathcal{S}$ is Buchsbaum.

For any affine semigroup, a problem of a high computational complexity is the problem of determining whether an element belongs to it. As explained above, in the particular case of circle semigroups this problem is simple. This fact simplifies the computation of the above example and allows us to obtain the result very quickly.

3. Buchsbaum affine convex polygonal semigroups

Denote by $F \subset \mathbb{R}^2_{\geq}$ a compact convex polygon (not equal to a segment) with vertex set $\boldsymbol{P} = \{P_1, \ldots, P_t\}$ arranged in counterclockwise direction and let $\mathcal{P} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^2$ be its associated semigroup. Note that since the fixed convex polygon F is not a segment, $\mathcal{P}$ is a simplicial affine convex polygonal semigroup. As in previous sections, R_1 and R_2 are the extremal rays of $\mathcal{C}$ assuming R_1 with a slope greater than the slope of R_2. Let $\hat{\mathcal{P}}$ be the semigroup $\{a \in \mathbb{N}^2 \mid a + n_i \in \mathcal{P},\ \forall i = 1, \ldots, m\}$ with $\{n_1, n_2, \ldots, n_m\}$ the minimal system of generators of $\mathcal{P}$ and let n'_j be a minimal generator of $\hat{\mathcal{P}} \cap R_j$ with $j = 1, 2$.

In order to prove the results of this section, we consider different special subsets of the cone $\mathcal{C}$ and some points and lines in $C(F)$. We distinguish two cases, $F \cap R_i$ is a point or a segment.

[1] Note that $C \cap R_1 = \left(\frac{32}{25}, \frac{24}{25}\right)$ and $C \cap R_2 = \left(\frac{96}{65}, \frac{8}{13}\right)$.

Assume $F \cap R_1 = \{P_1\} \subset \boldsymbol{P}$, let j be the least positive integer such that $j\overline{P_1P_t} \cap (j+1)\overline{P_1P_2}$ is not empty. Since $\overline{P_1P_t}$ and $\overline{P_1P_2}$ are not parallel, there exists a point $\{V_1\} = j\overline{P_1P_t} \cap (j+1)\overline{P_1P_2}$ (using [4, Lemma 11], V_1 can be easily computed). Denote by T_1 the triangle with vertex set $\{O, P_1, V_1 - jP_1\}$, and by $\mathring{T}_1$ its topological interior. By [4, Lemma 11], for every $h \in \mathbb{N}$ with $h \geq j$ the points $h\overline{P_1P_t} \cap (h+1)\overline{P_1P_2}$ are in the same straight line, which we denote by L_1. Note that $\big((\mathring{T}_1 \cup (\overline{OP_1} \setminus \{O, P_1\})) + \mu P_1\big) \cap \mathcal{P} = \varnothing$ for all $\mu \in \mathbb{Z}_{\geq}$. This construction allows us to define the set

$$\mathcal{B}_1 = \big\{D + \lambda n_1 \mid D \in \overline{(jP_1)V_1} \text{ and } \lambda \in \mathbb{Q}_{\geq}\big\} \cap \mathcal{C}$$

whose elements are in $\mathcal{P}$ or they are in $\bigcup_{\mu \in \mathbb{N},\, \mu \geq j}\big((\mathring{T}_1 \cup (\overline{OP_1} \setminus \{O, P_1\})) + \mu P_1\big)$. The elements of $\mathcal{B}_1$ verify that if $P \in \mathcal{B}_1 \setminus \mathcal{P}$ then $P + n_1 \notin \mathcal{P}$ and thus $P \notin \hat{\mathcal{P}}$; this implies that $\mathcal{P} \cap \mathcal{B}_1 = \hat{\mathcal{P}} \cap \mathcal{B}_1$. Denote by Υ_1 the finite set ConvexHull$(\{O, jP_1, V_1, L_1 \cap R_2\}) \cap \mathbb{N}^2$. Analogously, if the set $F \cap R_2 = \{P_1\} \subset \boldsymbol{P}$ (for the sake of simplicity, we call again this point P_1) there exists the least integer j such that $j\overline{P_1P_2} \cap (j+1)\overline{P_1P_t}$ is equal to $\{V_2\}$. Let T_2 be the triangle with vertex set $\{O, P_1, V_2 - jP_1\}$, and denote by L_2 the line containing the points $\{h\overline{P_1P_2} \cap (h+1)\overline{P_1P_t} \mid h \geq j,\ h \in \mathbb{N}\}$ and by $\mathcal{B}_2$ the set $\{D + \lambda n_2 \mid D \in \overline{(jP_1)V_2} \text{ and } \lambda \in \mathbb{Q}_{\geq}\} \cap \mathcal{C}$. All of the properties of these sets are analogous to the properties of the sets defined previously for R_1. Denote by Υ_2 the finite set ConvexHull$(\{O, jP_1, V_2, L_2 \cap R_1\}) \cap \mathbb{N}^2$. In the case $F \cap R_i$ is a segment for some i, we take $L_i = R_i$ and $\Upsilon_i = \{O\}$.

We define the set $\Upsilon = (Q + C(F)) \cap \mathbb{N}^2 \subset \mathcal{C}$ with $\{Q\} = L_1 \cap L_2 \subset C(F)$. Note that the boundary of the set Υ intersects with two different sides of the polygon i_0F when $i_0 \gg 0$ and therefore the sets $\Upsilon \setminus \mathcal{P}$ and $\Upsilon \setminus \hat{\mathcal{P}}$ are finite. The last set we define is the finite set $\Upsilon' = \{a \in (\Upsilon_1 \cup \Upsilon_2) \setminus \hat{\mathcal{P}} \mid a + n_1',\ a + n_2' \in \hat{\mathcal{P}}\}$. It is straightforward to prove that the cone $\mathcal{C}$ is the union of $\mathcal{B}_1, \mathcal{B}_2, \Upsilon_1, \Upsilon_2$ and Υ.

Theorem 6. *Let $\mathcal{P}$ be a simplicial affine convex polygonal semigroup. Then*

(1) *if* $\operatorname{int}(\mathcal{C}) = \operatorname{int}(\hat{\mathcal{P}})$, *the semigroup $\mathcal{P}$ is Buchsbaum if and only if $\hat{\mathcal{P}} \cap R_j$ is generated by only one element for $j = 1, 2$,*

(2) *if* $\operatorname{int}(\mathcal{C}) \neq \operatorname{int}(\hat{\mathcal{P}})$, *the semigroup $\mathcal{P}$ is Buchsbaum if and only if $\Upsilon' = \varnothing$ and $\Upsilon \subset \hat{\mathcal{P}}$.*

Proof. The proof of the first case is similar to the proof of Proposition 4, so it is left to the reader.

Assume that $\operatorname{int}(\mathcal{C}) \neq \operatorname{int}(\hat{\mathcal{P}})$ and that $\hat{\mathcal{P}}$ is Cohen–Macaulay. By Corollary 2, the set Υ' has to be empty. If $\Upsilon \not\subset \hat{\mathcal{P}}$, choose $a \in \Upsilon \setminus \hat{\mathcal{P}}$ such that $\mathrm{d}(a) = \max\{\mathrm{d}(a') \mid a' \in \Upsilon \setminus \hat{\mathcal{P}}\}$. Then $a + n_1'$ and $a + n_2'$ belong to $\hat{\mathcal{P}}$ which implies that $\hat{\mathcal{P}}$ is not Cohen–Macaulay. Thus $\Upsilon \subset \hat{\mathcal{P}}$.

Conversely, let a be an element of $\mathcal{C} \setminus \hat{\mathcal{P}}$ (note that $a \notin \Upsilon \subset \hat{\mathcal{P}}$). We discuss the possibilities we have. If $F \cap R_1$ is a point and a belongs to the strip bounded by the parallel lines R_1 and L_1, we have that $\mathcal{P} \cap R_1 = \hat{\mathcal{P}} \cap R_1$ and $n_1 = n_1'$. Besides, the element a belongs to $\Upsilon_1 \setminus \hat{\mathcal{P}}$ or it belongs to $\mathcal{B}_1 \setminus \hat{\mathcal{P}}$. Since $\Upsilon' = \varnothing$, if $a \in \Upsilon_1 \setminus \hat{\mathcal{P}}$, the element $a + n_1'$ or $a + n_2'$ does not belong to $\hat{\mathcal{P}}$ and if $a \in \mathcal{B}_1 \setminus \hat{\mathcal{P}}$ then $a + n_1' \notin \hat{\mathcal{P}}$. We proceed similarly in the case of $F \cap R_2$ is a point and a belongs to the strip bounded by the parallel lines R_2 and L_2, obtaining that the element $a + n_1'$ or $a + n_2'$ does not belong to $\hat{\mathcal{P}}$. If $F \cap R_2$ is a segment, since $\Upsilon \subset \hat{\mathcal{P}}$, we have that $\mathcal{C} \setminus \hat{\mathcal{P}} \subset \Upsilon_1 \cup \mathcal{B}_1$, and thus every $a \in \mathcal{C} \setminus \hat{\mathcal{P}}$ verifies $a + n_1'$ or $a + n_2'$ is not in $\hat{\mathcal{P}}$. Likewise, when $F \cap R_1$ is a segment and $F \cap R_2$ is a single point, for every $a \in \mathcal{C} \setminus \hat{\mathcal{P}}$ we obtain again that $a + n_1'$ or $a + n_2'$ is not in $\hat{\mathcal{P}}$. In any of the above cases, every element $a \in \mathcal{C} \setminus \hat{\mathcal{P}}$ fulfills that at least one element of the set $\{a + n_1', a + n_2'\}$ does not belong to $\hat{\mathcal{P}}$, and hence $\hat{\mathcal{P}}$ is Cohen–Macaulay (Corollary 2). Finally, if $F \cap R_1$ and $F \cap R_2$ are both segments, then $\mathcal{C} = \Upsilon \subset \hat{\mathcal{P}}$; this implies $\operatorname{int}(\mathcal{C}) = \operatorname{int}(\hat{\mathcal{P}})$, which is a contradiction. □

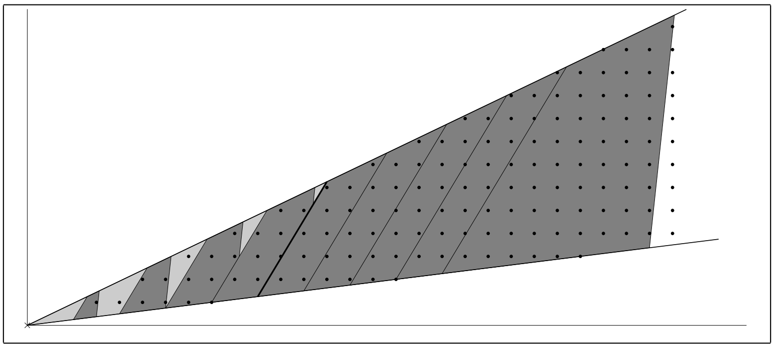

Figure 2. The affine polygonal semigroup $\mathcal{P}$ associated to the polygon

$$\{(2, 0.25), (3, 0.375), (2.6, 1.25), (3.12, 1.5)\}.$$

Since $\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\mathcal{P}) = \{(4, 1), (7, 3)\}$, $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{P}})$. Besides, $\hat{\mathcal{P}} \cap R_j$ is generated by only one element for $j = 1, 2$, and therefore $\mathcal{P}$ is Buchsbaum.

As in the circle semigroup case, to apply the above result it is necessary to check whether $\mathrm{int}(\mathcal{C}) = \mathrm{int}(\hat{\mathcal{P}})$. The different situations are the following:

(1) If $F \cap R_1$ is a segment $\overline{P_1 P_t}$ and $F \cap R_2$ is a segment $\overline{P_{d-1} P_d}$, the set Υ is equal to the positive integer cone $\mathcal{C}$ and the sets $\mathcal{C} \setminus \mathcal{P}$ and $\mathcal{C} \setminus \hat{\mathcal{P}}$ are finite. Let $j \in \mathbb{N}$ be the least integer such that

$$j\,\overline{P_1 P_t} \cap (j+1)\,\overline{P_1 P_t} \neq \varnothing \quad \text{and} \quad j\,\overline{P_{d-1} P_d} \cap (j+1)\,\overline{P_{d-1} P_d} \neq \varnothing,$$

and let T be the triangle with vertex set $\{O, jP_1, jP_{d-1}\}$. Clearly, $T \cap \mathbb{N}^2$ is finite and $\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\hat{\mathcal{P}}) \subseteq T \cap \mathbb{N}^2$. This is illustrated in Figure 2.

(2) Let us suppose that $F \cap R_1$ is a point P_1 and $F \cap R_2$ is a point P_d. If $P \in (\mathrm{int}(\mathcal{C}) \cap (\mathcal{B}_1 \cup \mathcal{B}_2)) \setminus \mathrm{int}(\mathcal{P})$, the element $P + n_1$ or $P + n_2$ does not belong to $\mathcal{P}$ and thus $P \notin \hat{\mathcal{P}}$. This implies that $\mathcal{P} \cap (\mathcal{B}_1 \cup \mathcal{B}_2) = \hat{\mathcal{P}} \cap (\mathcal{B}_1 \cup \mathcal{B}_2)$. Let $j \in \mathbb{N}$ be such that $j\,\overline{P_1 P_t} \cap (j+1)\,\overline{P_1 P_2} = \{V_1\}$ and let $t \in \mathbb{N}$ satisfying $tP_1 = n_1$. For every $r, k \in \mathbb{Z}_{\geq}$ there exists $h \in \{0, \ldots, t-1\}$ such that $(\mathring{T}_1 + (r+j)P_1) \cap \mathbb{N}^2 = (\mathring{T}_1 + (h+j)P_1) \cap \mathbb{N}^2 + kn_1$. A similar construction must be done for $\mathcal{B}_2$ proceeding similarly with the triangle T_2. So to compare $\mathrm{int}(\mathcal{C}) \cap (\mathcal{B}_1 \cup \mathcal{B}_2)$ with $\mathrm{int}(\hat{\mathcal{P}}) \cap (\mathcal{B}_1 \cup \mathcal{B}_2)$ it is only necessary to check if there are nonnegative integer points in the sets $\mathring{T}_1 + (h+j)P_1$ (with $h \in \{0, \ldots, t-1\}$) and, analogously, in some translations of $\mathring{T}_2$ in the direction of P_d. Since Υ_1 and Υ_2 are included in two parallelograms, $\Upsilon_1 \cup \Upsilon_2$ is a finite set and therefore $(\mathrm{int}(\mathcal{C}) \cap (\Upsilon_1 \cup \Upsilon_2)) \setminus \mathrm{int}(\hat{\mathcal{P}})$ can be computed.

In order to compute $(\mathrm{int}(\mathcal{C}) \cap \Upsilon) \setminus \mathrm{int}(\hat{\mathcal{P}})$, just take $j \in \mathbb{N}$ the least integer such that both sets

$$j\,\overline{P_1 P_t} \cap (j+1)\,\overline{P_1 P_2} = \{V\} \quad \text{and} \quad j\,\overline{P_d P_{d+1}} \cap (j+1)\,\overline{P_d P_{d-1}} = \{V'\}$$

are formed by only one point and let T be the triangle with vertex set $\{Q, V, V'\}$. By construction, the sets $(\mathrm{int}(\mathcal{C}) \cap \Upsilon) \setminus T$, $(\mathrm{int}(\mathcal{P}) \cap \Upsilon) \setminus T$ and $(\mathrm{int}(\hat{\mathcal{P}}) \cap \Upsilon) \setminus T$ are equal. Therefore $\mathrm{int}(\mathcal{C}) \cap \Upsilon = \mathrm{int}(\hat{\mathcal{P}}) \cap \Upsilon$ if and only if the finite sets $\mathrm{int}(\mathcal{C}) \cap T$ and $\{a \in \mathrm{int}(\mathcal{P}) \cap T \mid a + n_i \in \mathcal{P},\ \forall i = 1, \ldots, m\}$ are equal. This case is illustrated in Example 7 (see Figure 3).

(3) If $F \cap R_1 = \{P_1\}$ and $F \cap R_2$ is a segment $\overline{P_{d-1}P_d}$, to compare the sets $\mathrm{int}(\mathcal{C}) \setminus \Upsilon$ and $\mathrm{int}(\hat{\mathcal{P}}) \setminus \Upsilon$, just proceed as in the second case with the sets $\mathcal{B}_1$ and Υ_1. Let now $j \in \mathbb{N}$ be the least integer such that $j\,\overline{P_1P_t} \cap (j+1)\,\overline{P_1P_2}$ is a point V and $j\,\overline{P_{d-1}P_d} \cap (j+1)\,\overline{P_{d-1}P_d} \neq \varnothing$, and let T be the triangle with vertex set $\{Q, V, jP_{d-1}\}$. Then $\mathrm{int}(\mathcal{C}) \cap \Upsilon = \mathrm{int}(\hat{\mathcal{P}}) \cap \Upsilon$ if and only if the finite sets $\mathrm{int}(\mathcal{C}) \cap T$ and $\mathrm{int}(\hat{\mathcal{P}}) \cap T$ are equal.

(4) Finally, the case $F \cap R_2$ is a point and $F \cap R_1$ is a segment is analogous to the above case.

In any case, all the necessary sets to compare $\mathrm{int}(\mathcal{C})$ with $\mathrm{int}(\hat{\mathcal{P}})$ are finite and they can be obtained algorithmically. Besides, the conditions $\Upsilon' = \varnothing$ and $\Upsilon \subset \hat{\mathcal{P}}$ can be checked algorithmically and "$\hat{\mathcal{P}} \cap R_j$ is generated by only one element" can be tested in a similar way to the case of circle semigroup.

Example 7. Let F be the polygon determined by the rational points

$$\{(3.6, 1.8), (3.6, 0.6), (3.3, 1.05), (4.2, 1.5), (4.14, 0.99)\}$$

and $\mathcal{P}$ its associated affine convex polygonal semigroup (the dark gray region in Figure 3). The minimal system of generators of $\mathcal{P}$ can be computed with the function `PolygonalSG` (see [6]),

```
In[1]:= PolygonalSG[{{3.6,1.8},{3.6,0.6},{3.3,1.05},{4.2,1.5},{4.14,0.99}}]
Out[1]= {{4,1},{7,2},{7,3},{8,3},{10,3},{11,2},{11,5},{14,3},{18,3},{18,9},
        {20,8},{23,10}}
```

We obtain that $\mathcal{P}$ is minimally generated by

$$G = \{(18,9), (18,3), (4,1), (20,8), (23,10), (8,3), (11,5), (11,2), (10,3), (14,3), (7,2), (7,3)\},$$

following the notation of the above sections $n_1 = (18, 9)$ and $n_2 = (18, 3)$.

Using basic tools of Linear Algebra we compute the sets Υ_1, Υ_2, the triangle T and the necessary translations of T_1 and T_2 (the above sets are needed to check the conditions of Theorem 6). Those translations are the lighter gray triangles in Figure 3, $(\Upsilon_1 \cup \Upsilon_2) \setminus \mathcal{P}$ is the region in middle light gray and $(\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\mathcal{P})) \cap \Upsilon = (T \cap \mathbb{N}^2) \setminus \mathcal{P} = \{(13, 4)\}$. Since $(13, 4)$ does not belong to $\mathcal{P}$, by Corollary 2, the semigroup $\mathcal{P}$ is not Cohen–Macaulay. We also have $(13, 4) + n \in \mathcal{P}$ for all $n \in G$. This can be checked with the function `BelongToSG` in [6]. For $n_1 = (18, 9)$, we obtain

```
In[2]:= BelongToSG[{13,4}+{18,9},{{3.6,1.8},{3.6,0.6},{3.3,1.05},{4.2,1.5},
                   {4.14,0.99}}]
Out[2]= True
```

Thus $(13, 4) \in \hat{\mathcal{P}}$ and therefore $\Upsilon \subset \hat{\mathcal{P}}$.

The set $(\mathrm{int}(\mathcal{C}) \setminus \mathrm{int}(\mathcal{P})) \cap (\Upsilon_1 \cup \Upsilon_2)$ is equal to

$$D = \{(3,1), (5,1), (5,2), (6,2), (9,2), (10,2), (9,3), (13,3), (16,3), (17,3), (9,4), (10,4), (17,4), (12,5), (13,5), (13,6)\},$$

but none of these points are in $\hat{\mathcal{P}}$. Besides, for all $a \in D$, $a + n_1'$ or $a + n_2'$ does not belong to $\hat{\mathcal{P}}$. Therefore Υ' is the empty set. By Theorem 6, we conclude that $\mathcal{P}$ is a non-Cohen–Macaulay Buchsbaum affine semigroup.

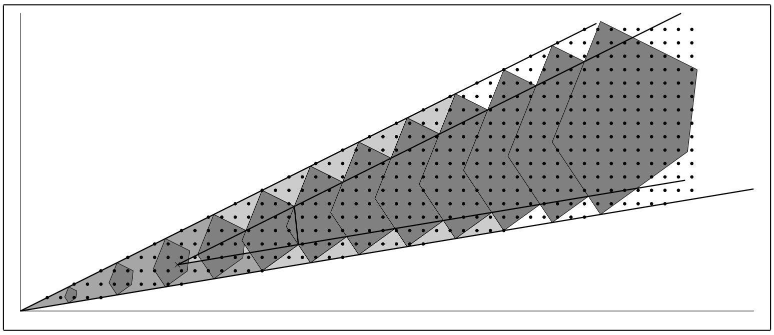

Figure 3. The affine polygonal semigroup $\mathcal{P}$ associated to the polygon

$$\{(3.6, 1.8), (3.6, 0.6), (3.3, 1.05), (4.2, 1.5), (4.14, 0.99)\}.$$

Using the function `PSGIsBuchsbaumQ` of [6], the above Buchsbaumness problem can be solved in less than a second (all the examples of this work have been done in an Intel Core i7 with 16 GB of main memory),

```
In[3]:=AbsoluteTiming[PSGIsBuchsbaumQ[{{3.6,1.8},{3.6,0.6},{3.3,1.05},
                                        {4.2,1.5},{4.14,0.99}}]]
Out[3]= {0.734412, True}
```

The return value is `{0.734412, True}`, where `0.734412` are the seconds required for this computation and `True` is the answer to the Buchsbaumness question.

If we use the method of Theorem 9 in [7], it is necessary to compute the intersection of the Apéry set of n_1 and the Apéry set of n_2 by checking if $2 \times 7\,771\,556\,800\,000$ elements belong to $\mathcal{P}$. This is clearly inefficient.

As indicated before, the problem of determining whether an element belongs to a convex polygonal semigroup is straightforward; this implies a reduction of the time of computation.

In Example 7, it used only elementary algebra, but Buchsbaum semigroups can be generated using an even simpler approach. The following results provide two user-friendly properties which allow us to obtain easily Buchsbaum rings.

Corollary 8. *Every affine convex polygonal semigroup associated to a triangle with rational vertices is Buchsbaum.*

Proof. Note that if F is a triangle, $\mathcal{P}$ and $\hat{\mathcal{P}}$ are equal. Corollary 12 in [3] proves that every affine convex polygonal semigroup associated to a triangle with rational vertices is Cohen–Macaulay. Thus, $\hat{\mathcal{P}} = \mathcal{P}$ is Cohen–Macaulay and therefore $\mathcal{P}$ is Buchsbaum (Theorem 1). □

Corollary 9. *Let F be a convex polygon with vertices $P_1, \dots, P_4 \in \mathbb{Q}^2_{\geq}$ and let $\mathcal{P}$ be its associated affine convex polygonal semigroup. If $P_1 \in \mathcal{P} \cap R_1$, $P_4 \in \mathcal{P} \cap R_2$ and the points O, P_2 and P_3 are aligned, $\mathcal{P}$ is Buchsbaum.*

Proof. Let $\mathcal{C}_1$ be the positive integer cone delimited by the ray R_1 and the line OP_2, and let $\mathcal{C}_2$ be the cone delimited by the ray R_2 and the line OP_2. Trivially $\mathcal{C} = C(F) \cap \mathbb{N}^2$ is the union of $\mathcal{C}_1$ and $\mathcal{C}_2$,

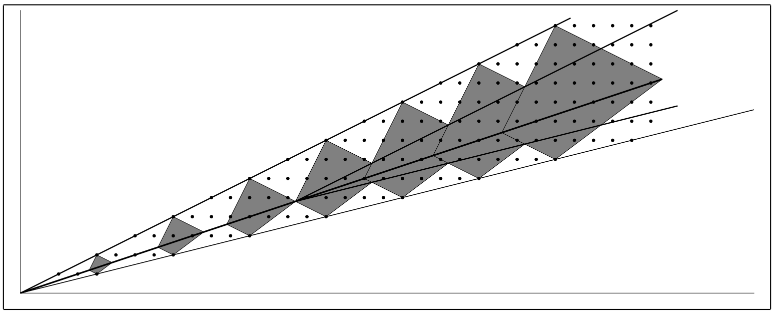

Figure 4. The affine polygonal semigroup $\mathcal{P}$ associated to the polygon

$$\{(3.6, 1.2), (4.8, 1.6), (4, 2), (4, 1)\}.$$

and the semigroup $\mathcal{P}$ is the union of the affine convex polygonal semigroups, $\mathcal{P}_1$ and $\mathcal{P}_2$, associated to the triangles with vertex sets $\{P_1, P_2, P_3\}$ and $\{P_2, P_3, P_4\}$, respectively. With that decomposition of the affine convex polygonal semigroup $\mathcal{P}$ and from the hypothesis, we can assert that $\mathcal{P}$ is equal to $\hat{\mathcal{P}}$, that $\Upsilon \subset \mathcal{P}$ and that $\mathcal{P} \cap R_1$ and $\mathcal{P} \cap R_2$ are semigroups generated by only one element (Figure 4 illustrates this situation). Under such conditions, let a be an element belonging to $\mathcal{C} \setminus \mathcal{P}$. Note that if $a \in \mathcal{C}_1 \setminus \mathcal{P}_1$ then $a + n_1 \notin \mathcal{P}$, otherwise, if $a \in \mathcal{C}_2 \setminus \mathcal{P}_2$ then $a + n_2 \notin \mathcal{P}$. In any case, $a + n_1$ or $a + n_2$ does not belong to $\mathcal{P}$. Thus $\hat{\mathcal{P}}$ $(= \mathcal{P})$ is Cohen–Macaulay (Corollary 2) and then $\mathcal{P}$ is Buchsbaum. □

References

[1] H. Bresinsky, "Monomial Buchsbaum ideals in $\mathbb{P}^r$", *Manuscripta Math.* **47**:1-3 (1984), 105–132.

[2] W. Bruns, J. Gubeladze, and N. V. Trung, "Problems and algorithms for affine semigroups", *Semigroup Forum* **64**:2 (2002), 180–212.

[3] J. I. García-García and A. Vigneron-Tenorio, "Computing families of Cohen–Macaulay and Gorenstein rings", *Semigroup Forum* **88**:3 (2014), 610–620.

[4] J. I. García-García, M. A. Moreno-Frías, A. Sánchez-R.-Navarro, and A. Vigneron-Tenorio, "Affine convex body semigroups", *Semigroup Forum* **87**:2 (2013), 331–350.

[5] J. I. García-García, M. A. Moreno-Frías, A. Sánchez-R.-Navarro, and A. Vigneron-Tenorio, "Computing affine convex body semigroups", 2013, Available at http://hdl.handle.net/10498/15832.

[6] J. I. García-García, A. Sánchez-R.-Navarro, and A. Vigneron-Tenorio, "`PolySGTools`, a `Mathematica` package for convex polygonal semigroups", 2017, Available at http://hdl.handle.net/10498/19438.

[7] P. A. García-Sánchez and J. C. Rosales, "On Buchsbaum simplicial affine semigroups", *Pacific J. Math.* **202**:2 (2002), 329–339.

[8] S. Goto, "On Buchsbaum rings", *J. Algebra* **67**:2 (1980), 272–279.

[9] S. Goto and K. Watanabe, "On graded rings, II ($\mathbb{Z}^n$-graded rings)", *Tokyo J. Math.* **1**:2 (1978), 237–261.

[10] S. Goto, N. Suzuki, and K. Watanabe, "On affine semigroup rings", *Japan. J. Math.* (*N.S.*) **2**:1 (1976), 1–12.

[11] J. Herzog, T. Hibi, and N. V. Trung, "Symbolic powers of monomial ideals and vertex cover algebras", *Adv. Math.* **210**:1 (2007), 304–322.

[12] Y. Kamoi, "Defining ideals of Buchsbaum semigroup rings", *Nagoya Math. J.* **136** (1994), 115–131.

[13] J. C. Rosales and P. A. García-Sánchez, "On Cohen–Macaulay and Gorenstein simplicial affine semigroups", *Proc. Edinburgh Math. Soc.* (2) **41**:3 (1998), 517–537.

[14] J. C. Rosales, P. A. García-Sánchez, J. I. García-García, and J. M. Urbano-Blanco, "Proportionally modular Diophantine inequalities", *J. Number Theory* **103**:2 (2003), 281–294.

[15] J. Stückrad and W. Vogel, *Buchsbaum rings and applications: an interaction between algebra, geometry and topology*, Springer, 1986.

[16] N. V. Trung, "Classification of the double projections of Veronese varieties", *J. Math. Kyoto Univ.* **22**:4 (1982/83), 567–581.

[17] N. V. Trung and L. T. Hoa, "Affine semigroups and Cohen–Macaulay rings generated by monomials", *Trans. Amer. Math. Soc.* **298**:1 (1986), 145–167.

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