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Coefficients of some distinguished Drinfeld modular forms

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ABSTRACT

We determine some coefficients of the Drinfeld modular forms h , the discriminant Δ and the Eisenstein series E_{q^l-1} . For the function h the coefficients are obtained from a formula of C. Armana; for Δ and E_{q^l-1} the coefficients are obtained from extensions of that formula.

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1. Introduction

Distinguished Drinfeld modular forms like the discriminant function Δ or the Eisenstein series E_k admit expansions with rational coefficients (i.e. elements of $\mathbb{F}_q(T)$). The study of these coefficients was initiated by D. Goss (see for example [5]); among other interesting results, he got a bound for the denominators of Eisenstein series.

Another direction in the study of the coefficients of modular forms was proposed by E.-U. Gekeler in [4]: he used the action of Hecke operators on the function h and obtained some congruences (modulo primes) which determine the coefficients with subscripts

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$(q - 1)q^d + 1$, $d \geq 1$ (for the t -expansion of h). Similar results were obtained for other modular forms using the same idea: the coefficients of Δ with subscripts $(q - 1)(q^{d+1} + 1)$, $d \geq 1$, were determined in [6], and the coefficients of the Eisenstein series E_{q^l-1} with subscripts $(q - 1)(q^{d+l} + q^{l-1} + \dots + 1)$, $d \geq 1$, were determined (up to a factor of relatively low degree) in [2].

Empirical formulas (without proof) were also found for other coefficients of h , with subscripts $(q - 1)Jq^d + 1$, $2 \leq J \leq q - 1$, $d \geq 1$, and Δ , the coefficients with subscripts $(q - 1)(Jq^{d+1} + 1)$, $2 \leq J \leq q - 1$, $d \geq 1$ (see [7, p. 74]). For the function h these coefficients could be determined if some congruences (modulo powers of primes) were proven, but this becomes more complicated than in the case of the coefficients with subscripts $(q - 1)q^d + 1$. Armana’s formula determines the coefficients of h with subscripts $(q - 1)Jq^d + 1$, $1 \leq J \leq q - 1$, $d \geq 1$, as power sums of coefficients of the Carlitz module (Eq. (8)). These power sums can be explicitly computed to obtain a simple formula (Theorem 3.1) from which the empirical formulas for h are easily derived (Corollary 3.1); these two results are proven in Section 3 of the present work.

In Section 4 we extend the formula for h to the functions Δ and E_{q^l-1} (Theorem 4.1 and Corollary 4.2). From these formulas and Theorem 3.1, it is possible to find the expression of the coefficients of $\bar{\pi}^{1-q^2} \Delta$ with subscripts $(q - 1)(Jq^{d+1} + 1)$, $1 \leq J \leq q - 1$, $d \geq 1$ (Theorem 4.2), and the coefficients of $\bar{\pi}^{1-q^l} E_{q^l-1}$ with subscripts $(q - 1)Jq^{d+l} + q^l - 1$, $1 \leq J \leq q - 1$, $d \geq 1$ (Theorem 4.3).

2. Preliminaries

Let \mathbb{F}_q be the field of q elements; let $A = \mathbb{F}_q[T]$ be the ring of polynomials over \mathbb{F}_q , and $K = \mathbb{F}_q(T)$ its quotient field. We consider the field $K_\infty = \mathbb{F}_q((1/T))$, and its completed algebraic closure $\mathbb{C}_\infty = \widehat{K}_\infty$.

Let $\mathbb{C}_\infty\{\tau\}$ be the ring of non-commutative polynomials whose product is given by $\tau\alpha = \alpha^q\tau$, $\alpha \in \mathbb{C}_\infty$. The ring $\mathbb{C}_\infty\{\tau\}$ can be identified with the ring of q -additive polynomials $\sum_{i=0}^m c_i X^{q^i}$ where the product is given by substitution.

A *Drinfeld module of rank r over \mathbb{C}_∞* is a ring \mathbb{F}_q -homomorphism $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$ determined by $\phi_T = T\tau^0 + \sum_{i=1}^r c_i \tau^i$, with $c_r \neq 0$.

An *A -lattice in \mathbb{C}_∞ of rank r* is a discrete free A -module $\Lambda \subset \mathbb{C}_\infty$ of rank r . The exponential function associated to a lattice Λ is

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

Through the function e_Λ it is possible to construct a Drinfeld module of rank r . This construction establishes a bijection between the set of lattices in \mathbb{C}_∞ of rank r and the set of Drinfeld modules of rank r over \mathbb{C}_∞ .

The *Carlitz module* is the rank one Drinfeld module $C : A \rightarrow \mathbb{C}_\infty\{\tau\}$ determined by

$$C_T = T\tau^0 + \tau = TX + X^q.$$

Let $L = \bar{\pi}A$ be the lattice in \mathbb{C}_∞ which corresponds to the Carlitz module C . We consider the function $t(z) = e_L(\bar{\pi}z)^{-1} = \bar{\pi}^{-1} \sum_{a \in A} \frac{1}{z+a}$; this function will be used as parameter of the expansions of the modular forms considered here (it is an analogue of the classical $e^{2\pi iz}$).

Let $a \in A$, $\deg a = d$; let C_a denote the image of a by C , and we write $C_a = \sum_{i=0}^d C_{a,i} \tau^i$ (we adopt the notation used in [1]). The coefficients $C_{a,i}$ satisfy the following identities. Let $P \in A$, $\deg P = d$; then $C_{TP} = TC_P + C_P^q$; therefore

$$\begin{aligned} C_{TP,0} &= TC_{P,0}, \\ C_{TP,i} &= TC_{P,i} + C_{P,i-1}^q, \quad i = 1, \dots, d. \end{aligned} \tag{1}$$

Furthermore, $C_{T^{i+1}} = C_{T^i} C_T$; it follows that

$$C_{T^{i+1},k} = C_{T^i,k-1} + C_{T^i,k} T^q, \quad k = 1, \dots, i + 1. \tag{2}$$

Now, if $P \in A$, then $C_{TP} = C_T C_P = C_P C_T$; we get that

$$(T^{q^{i+1}} - T) C_{P,i+1} = C_{P,i}^q - C_{P,i}. \tag{3}$$

Let $\Omega = \mathbb{C}_\infty - K_\infty$. A function $f : \Omega \rightarrow \mathbb{C}_\infty$ is a *modular form of weight k and type m for the group $GL(2, A)$* if f is holomorphic on Ω , satisfies

$$f\left(\frac{az + b}{cz + d}\right) = (ad - bc)^{-m} (cz + d)^k f(z) \tag{4}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, A)$, and admits an expansion of the type $f = \sum_{n \geq 0} c_n t(z)^n$. A *cuspidal form* is a modular form whose expansion is $\sum_{n \geq 1} c_n t(z)^n$.

The Eisenstein series of weight k , defined as

$$E_k(z) = \sum_{\substack{c,d \in A \\ (c,d) \neq (0,0)}} \frac{1}{(cz + d)^k},$$

is a Drinfeld modular form of weight k and type 0. The discriminant function, defined as

$$\Delta(z) = (T^{q^2} - T) E_{q^2-1} + (T^q - T)^q (E_{q-1})^{q+1}$$

is a cuspidal form of weight $q^2 - 1$ and type 0.

The set $M_{k,m}$ of modular forms of weight k and type m is a \mathbb{C}_∞ -vector space of finite dimension; the subspace of cuspidal forms of $M_{k,m}$ is denoted by $M_{k,m}^0$. The spaces $M_{q^2-1,0}^0$

and $M_{q+1,1}^0$ are one-dimensional spaces; the first one is generated by Δ , and the second one is generated by one of its $(q - 1)$ -st roots h (cf. [3, p. 681]). We determine h as the modular form of weight $q + 1$ and type 1 whose expansion is $-t(z) + \sum_{n \geq 2} a_n t(z)^n$.

Let $P \in A$ be a monic polynomial of degree d . The P -th Hecke operator T_P acts on the spaces $M_{k,m}$ and $M_{k,m}^0$; this action can be given explicitly on the t -expansions of the modular forms: if $f \in M_{k,m}$, $f = \sum_{n \geq 0} c_n t(z)^n$, then

$$T_P f = P^{k-1} \sum_{i \geq 0} \sum_{\delta \in A_+, \delta | P} \delta^{-k} c_i G_{i,\delta} (\delta t_P / \delta), \tag{5}$$

where A_+ is the set of monic polynomials in A , $t_a = t(az) = t^{q^{\deg a}} / f_a(t)$, $f_a(X) = X^{q^{\deg a}} C_a(X^{-1})$, and $G_{i,\delta}$ is the i -th Goss polynomial with respect to the lattice $\ker C_\delta$ (cf. Proposition 5.2 of [1, p. 1446]). Goss polynomials can be recursively defined as follows (cf. [3, p. 675]): let $a \in A$, $\alpha_l = C_{a,l}/a$; then $G_{i,a} = 0$ if $i \leq 0$, $G_{1,a} = X$, and for $i > 1$,

$$G_{i,a} = X(G_{i-1,a} + \alpha_1 G_{i-q,a} + \alpha_2 G_{i-q^2,a} + \dots).$$

There also exists the following explicit formula for $G_{i,a}$ (cf. [3, p. 676]): if $\deg a = d$, then

$$G_{i+1,a}(X) = \sum_{j \leq i} \sum_{\underline{n}} \binom{j}{\underline{n}} \alpha^{\underline{n}} X^{j+1}, \tag{6}$$

where $\underline{n} = (n_0, n_1, \dots, n_d)$ runs over the set of $(d + 1)$ -tuples satisfying $n_0 + n_1 + \dots + n_d = j$ and $n_0 + n_1 q + \dots + n_d q^d = i$, $\alpha^{\underline{n}} = \alpha_0^{n_0} \alpha_1^{n_1} \dots \alpha_d^{n_d}$, and $\binom{j}{\underline{n}} = j! / (n_0! n_1! \dots n_d!)$. We adopt again the notation of [1], and write

$$C(a)^{\underline{n}} = C_{a,0}^{n_0} C_{a,1}^{n_1} \dots C_{a,d}^{n_d}.$$

Then the formula of Eq. (6) is written as

$$G_{i+1,a}(X) = \sum_{j \leq i} a^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(a)^{\underline{n}} X^{j+1}, \tag{7}$$

where $\underline{n} = (n_0, n_1, \dots, n_d)$ runs over the set of $(d + 1)$ -tuples satisfying $n_0 + n_1 + \dots + n_d = j$ and $n_0 + n_1 q + \dots + n_d q^d = i$.

We conclude this section with Armana’s formula, which gives the expression of some coefficients of modular forms with fixed weight and types 0 and 1 for the group $\Gamma_0(\mathfrak{n})$.

Proposition 2.1. *Let $P \in A$ be a monic polynomial of degree d . Let $h = \sum_{i \geq 0} a_i t^i$ be the expansion of the function h . The t^q -coefficient of $T_P(h)$ is*

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_d) \\ n_0+n_1+\dots+n_d=q-1}} \binom{q-1}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1 q+\dots+n_d q^d} - \sum_{Q \in A_{1+}} Q^q.$$

Proof. Cf. [1, Proposition 5.5, p. 1447]. □

Notation 2.1. For $d \geq 1$, $\underline{i} = (i_0, i_1, \dots, i_{d-1})$, $i_k \geq 0$, let

$$S_d(i_0, i_1, \dots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} = \sum_{P \in A_{d+}} C_{P,0}^{i_0} C_{P,1}^{i_1} \cdots C_{P,d-1}^{i_{d-1}},$$

where A_{d+} denotes the set of monic polynomials of degree d in A .

Corollary 2.1. The t^q -coefficient of $\sum_{P \in A_{d+}} C(P)^{\underline{i}} T_P(h)$ is

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_d) \\ n_0+n_1+\dots+n_d=q-1}} \binom{q-1}{\underline{n}} S_d(n_0 + i_0, \dots, n_{d-1} + i_{d-1}) a_{1+n_0+n_1q+\dots+n_dq^d} \\ - \sum_{P \in A_{d+}} C_{P,0}^{i_0} C_{P,1}^{i_1} \cdots C_{P,d-1}^{i_{d-1}} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right).$$

The following proposition, that we will use along the work, is crucial for proving the formula for the coefficients of h which is given in Corollary 2.2.

Proposition 2.2. Let $d > 0$ and $i_j \in \{0, 1, \dots, 2(q-1)\}$ for all $j \in \{0, 1, \dots, d-1\}$. Then

$$S_d(i_0, \dots, i_{d-1}) = \begin{cases} (-1)^d & \text{if, for all } j, i_j \in \{q-1, 2(q-1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Cf. [1, Proposition 3.5, pp. 1440–1441]. □

Theorem 2.1. Let $c = \sum_{j=0}^d c_j q^j$ with $0 \leq c_j < q$, $\sum_{j=0}^d c_j = q-1$ and $c_d \neq q-1$, and let $i_k = q-1 - c_k$ for $0 \leq k \leq d-1$. Then the t^q -coefficient of $\sum_{P \in A_{d+}} C(P)^{\underline{i}} T_P(h)$ is

$$(-1)^d \binom{q-1}{c_0, \dots, c_d} a_{c+1} - \sum_{P \in A_{d+}} C_{P,0}^{i_0} C_{P,1}^{i_1} \cdots C_{P,d-1}^{i_{d-1}} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right).$$

Proof. Cf. [1, proof of part 2 of Theorem 7.2, p. 1455]. □

Corollary 2.2. Let $c = \sum_{j=0}^d c_j q^j$ with $0 \leq c_j < q$, $\sum_{j=0}^d c_j = q-1$ and $c_d \neq q-1$. Assume $\binom{q-1}{c_0, \dots, c_d} \neq 0$. Then

$$a_{c+1} = (-1)^d \binom{q-1}{c_0, \dots, c_d}^{-1} \sum_{P \in A_{d+}} C_{P,0}^{i_0} C_{P,1}^{i_1} \cdots C_{P,d-1}^{i_{d-1}} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right).$$

As it was indicated in Section 1, we are interested in the coefficients $a_{J(q-1)q^d+1}$, $1 \leq J \leq q-1$, $d \geq 1$. For $c+1 = 1 + \sum_{j=0}^{d+1} c_j q^j = J(q-1)q^d + 1$, we get the sequence $c_j = 0$ for $j = 0, \dots, d-1$, $c_d = q - J$ and $c_{d+1} = J - 1$; now from the formula of Corollary 2.2,

$$a_{J(q-1)q^d+1} = (-1)^{d+1} \left(\frac{(q-1)!}{(J-1)!(q-J)!} \right)^{-1} \times \sum_{P \in A_{(d+1)+}} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right). \tag{8}$$

We will determine these coefficients in Corollary 3.1. The key for proving this corollary is given in Theorem 3.1: the power sums of the coefficients of the Carlitz module are explicitly computed to obtain a simple formula from which the result is derived. Formulas similar to that of Eq. (8) are obtained for coefficients of the Drinfeld modular forms Δ and E_{q^t-1} (Corollary 4.2), and these coefficients are determined in Theorems 4.2 and 4.3; as in the case of Corollary 3.1, these theorems are proven from the formula of Theorem 3.1.

Remark 2.1. (a) The multinomial coefficients that we consider in this work are always viewed as elements of \mathbb{F}_q .

(b) Let $m \geq 1$, $1 \leq r \leq q-1$; then $\frac{mq+r}{r} = 1$ and $\frac{mq-r}{r} = -1$ in \mathbb{F}_q . It follows that $\binom{q-1}{k} = (-1)^k$ if $0 \leq k \leq q-1$.

(c) In Eq. (8) the coefficient $\frac{(q-1)!}{(J-1)!(q-J)!}$ is nonzero for any q : by part (b), if $k \leq q-1$, then $\binom{q-1}{k} = \frac{(q-1)!}{k!(q-1-k)!} = (-1)^k$ (compare with [1, Theorem 7.2, p. 1453, and Proposition 8.1, p. 1458]).

3. Coefficients of the function h

We will determine in this section the coefficients of the expansion of $h = \sum_{i \geq 0} a_i t^i$ with subscripts $(q-1)Jq^d + 1$, $1 \leq J \leq q-1$.

By Remark 2.1, Eq. (8) can be rewritten as

$$a_{J(q-1)q^d+1} = (-1)^{d+J} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right). \tag{9}$$

Observe now that

$$\begin{aligned} & \sum_{P \in A_{(d+1)+}} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \left(\sum_{Q \in A_{1+}, Q|P} Q^q \right) \\ &= \sum_{Q \in A_{1+}} Q^q \left(\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \right). \end{aligned} \tag{10}$$

For fixed $Q \in A_{1+}$, we are computing

$$\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q-1} C_{P,1}^{q-1} \cdots C_{P,d-1}^{q-1} C_{P,d}^{J-1}.$$

We first do it for $Q = T$ and then we extend the result to any $Q \in A_{1+}$. We need two auxiliary results.

Lemma 3.1. *Let $P, Q \in A$, and $0 \leq r \leq q - 1$. Then*

$$\sum_{a \in \mathbb{F}_q} (P + a)^{q-1} (Q + a)^r = \epsilon - (Q - P)^r,$$

where $\epsilon = -1$ if $r = q - 1$, and $\epsilon = 0$ if $r < q - 1$.

Proof. Observe that $\sum_{a \in \mathbb{F}_q} a^k = -1$ if $k (> 0)$ is a multiple of $q - 1$, and $\sum_{a \in \mathbb{F}_q} a^k = 0$ otherwise; we deduce that

$$\begin{aligned} \sum_{a \in \mathbb{F}_q} (P + a)^{q-1} (Q + a)^r &= \sum_{a \in \mathbb{F}_q} a^{r+(q-1)} + \sum_{a \in \mathbb{F}_q} a^{q-1} \sum_{k=0}^r \binom{q-1}{k} \binom{r}{r-k} P^k Q^{r-k} \\ &= \sum_{a \in \mathbb{F}_q} a^{r+(q-1)} + \left(\sum_{a \in \mathbb{F}_q} a^{q-1} \right) \left(\sum_{k=0}^r \binom{q-1}{k} \binom{r}{r-k} P^k Q^{r-k} \right) \\ &= \epsilon - (Q - P)^r, \end{aligned}$$

where $\epsilon = -1$ if $r = q - 1$, and $\epsilon = 0$ if $r < q - 1$. \square

Lemma 3.2. *Let $d \geq 1$, $0 \leq i < d$, $1 \leq r \leq q - 1$. Let $(a_d, a_{d-1}, \dots, a_{i+1})$ be a $(d - i)$ -tuple with $a_j \in \mathbb{F}_q$; we consider the polynomial*

$$P_{i+1}(a_d, a_{d-1}, \dots, a_{i+1}) = T^{d-i} + a_d T^{d-i-1} + \cdots + a_{i+1}.$$

Then, for each $(d - i - 1)$ -tuple $(a_d, a_{d-1}, \dots, a_{i+2})$, it holds that

$$\begin{aligned} \sum_{a_{i+1} \in \mathbb{F}_q} \left(\sum_{k=0}^i C_{T^i, k} C_{P_{i+1}, i-k}^{q^k} \right)^{q-1} \left(\sum_{k=0}^{i+1} C_{T^{i+1}, k} C_{P_{i+1}, i+1-k}^{q^k} \right)^r \\ = \epsilon - \left(T^{q^{i+1}} \sum_{k=0}^{i+1} C_{T^{i+1}, k} C_{P_{i+2}, i+1-k}^{q^k} \right)^r, \end{aligned}$$

where $\epsilon = -1$ if $r = q - 1$, and $\epsilon = 0$ if $r < q - 1$.

Proof. We have that $C_{P_{i+1},0} = TP_{i+2} + a_{i+1}$, and so

$$\sum_{k=0}^i C_{T^i,k} C_{P_{i+1},i-k}^{q^k} = \sum_{k=0}^{i-1} C_{T^i,k} C_{P_{i+1},i-k}^{q^k} + (TP_{i+2})^{q^i} + a_{i+1},$$

$$\sum_{k=0}^{i+1} C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} = \sum_{k=0}^i C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} + (TP_{i+2})^{q^{i+1}} + a_{i+1}.$$

Now, since $\sum_{k=0}^{i-1} C_{T^i,k} C_{P_{i+1},i-k}^{q^k} + (TP_{i+2})^{q^i}$, and $\sum_{k=0}^i C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} + (TP_{i+2})^{q^{i+1}}$ do not depend on a_{i+1} , we deduce from [Lemma 3.1](#) that

$$\sum_{a_{i+1} \in \mathbb{F}_q} \left(\sum_{k=0}^i C_{T^i,k} C_{P_{i+1},i-k}^{q^k} \right)^{q-1} \left(\sum_{k=0}^{i+1} C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} \right)^r$$

$$= \epsilon - \left(\left(\sum_{k=0}^i C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} + (TP_{i+2})^{q^{i+1}} \right) - \left(\sum_{k=0}^{i-1} C_{T^i,k} C_{P_{i+1},i-k}^{q^k} + (TP_{i+2})^{q^i} \right) \right)^r, \tag{11}$$

where $\epsilon = -1$ if $r = q - 1$, and $\epsilon = 0$ if $r < q - 1$. From the identity

$$C_{P_{i+1},k} = TC_{P_{i+2},k} + C_{P_{i+2},k-1}^q, \quad k = 1, \dots, i + 1,$$

it follows that

$$\left(\sum_{k=0}^i C_{T^{i+1},k} C_{P_{i+1},i+1-k}^{q^k} + (TP_{i+2})^{q^{i+1}} \right) - \left(\sum_{k=0}^{i-1} C_{T^i,k} C_{P_{i+1},i-k}^{q^k} + (TP_{i+2})^{q^i} \right)$$

$$= \left(\sum_{k=0}^i C_{T^{i+1},k} (TC_{P_{i+2},i+1-k} + C_{P_{i+2},i-k}^q)^{q^k} + (TP_{i+2})^{q^{i+1}} \right)$$

$$- \left(\sum_{k=0}^{i-1} C_{T^i,k} (TC_{P_{i+2},i-k} + C_{P_{i+1},i-k-1}^q)^{q^k} + (TP_{i+2})^{q^i} \right). \tag{12}$$

We rearrange the terms of the right-hand side of Eq. (12) in order to obtain the recursive formula established in the lemma. We know that the polynomials $C_{T^{i+1},k}$ satisfy

$$C_{T^{i+1},k} = C_{T^i,k-1} + C_{T^i,k} T^{q^k}, \quad k = 1, \dots, i + 1. \tag{13}$$

Furthermore, if $P \in A$, then

$$(T^{q^{i+1}} - T)C_{P,i+1} = C_{P,i}^q - C_{P,i}. \tag{14}$$

From Eqs. (13) and (14) it follows that

$$\begin{aligned}
 & C_{T^{i+1},k}(TC_{P_{i+2},i+1-k} + C_{P_{i+2},i-k}^q)^{q^k} - C_{T^i,k-1}C_{P_{i+2},i-k}^{q^k} - C_{T^i,k}(TC_{P_{i+2},i-k})^{q^k} \\
 &= C_{T^{i+1},k}(TC_{P_{i+2},i+1-k} + C_{P_{i+2},i-k}^q)^{q^k} - C_{T^{i+1},k}C_{P_{i+2},i-k}^{q^k} \\
 &= T^{q^k}C_{T^{i+1},k}C_{P_{i+2},i+1-k}^{q^k} + C_{T^{i+1},k}(C_{P_{i+2},i-k}^{q^{k+1}} - C_{P_{i+2},i-k}^{q^k}) \\
 &= T^{q^{i+1}}(C_{T^{i+1},k}C_{P_{i+2},i+1-k}^{q^k}),
 \end{aligned}$$

for $1 \leq k \leq i$. On the other hand,

$$\begin{aligned}
 & C_{T^{i+1},0}(TC_{P_{i+2},i+1} + C_{P_{i+2},i}^q) - TC_{T^i,0}C_{P_{i+2},i} \\
 &= T^{i+2}C_{P_{i+2},i+1} + T^{i+1}(C_{P_{i+2},i}^q - C_{P_{i+2},i}) \\
 &= T^{i+2}C_{P_{i+2},i+1} + T^{i+1}((T^{q^{i+1}} - T)C_{P_{i+2},i+1}) \\
 &= T^{q^{i+1}}(C_{T^{i+1},0}C_{P_{i+2},i+1}).
 \end{aligned}$$

We finally observe that $(TP_{i+2})^{q^{i+1}} = T^{q^{i+1}}(C_{T^{i+1},i+1}C_{P_{i+2},i+1}^{q^{i+1}})$. \square

Theorem 3.1. *Let $1 \leq J \leq q - 1$, and $a \in \mathbb{F}_q$. Then*

$$\sum_{P \in A_{(d+1)+}, (T+a)|P} C_{P,0}^{q-1} \cdots C_{P,d-1}^{q-1} C_{P,d}^{J-1} = (-1)^d (T+a)^{Jq^d-1}.$$

Proof. Let us first consider $a = 0$. We have

$$\begin{aligned}
 & \sum_{P \in A_{(d+1)+}, T|P} C_{P,0}^{q-1} \cdots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \\
 &= \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T,0}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T,d-1}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T,d}^{J-1}.
 \end{aligned} \tag{15}$$

Now, if $P \in A$, then

$$C_{T^i P,i} = \sum_{k=0}^i C_{T^i,k} C_{P,i-k}^{q^k}. \tag{16}$$

Using this identity for $C_{T^{d+1}+a_d T^d+\dots+a_1 T,0}$ and $C_{T^{d+1}+a_d T^d+\dots+a_1 T,1}$ in Eq. (15) we obtain that

$$\begin{aligned} & \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 0}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 1}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1} \\ &= \sum_{a_1, \dots, a_d \in \mathbb{F}_q} (C_{T,0} C_{P_1,0})^{q-1} (C_{T,0} C_{P_1,1} + C_{T,1} C_{P_1,1}^q)^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1}, \end{aligned}$$

where P_1 is the polynomial defined in Lemma 3.2. Now, since the polynomials $C_{T^{d+1}+a_d T^d+\dots+a_1 T, r}$, $2 \leq r \leq d$, do not depend on a_1 , it follows that

$$\begin{aligned} & \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 0}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 1}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1} \\ &= \sum_{a_2, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1} S_1, \end{aligned} \tag{17}$$

where

$$S_1 = \sum_{a_1 \in \mathbb{F}_q} (C_{T,0} C_{P_1,0})^{q-1} (C_{T,0} C_{P_1,1} + C_{T,1} C_{P_1,1}^q)^{q-1}.$$

By Lemma 3.2,

$$S_1 = -T^{q-1} - T^{q^2-1} (C_{T,0} C_{P_2,1} + C_{T,1} C_{P_2,0}^q)^{q-1}. \tag{18}$$

Coming back to Eq. (17),

$$\begin{aligned} & \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 0}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 1}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1} \\ &= \sum_{a_2, \dots, a_d \in \mathbb{F}_q} (-T^{q-1} - T^{q^2-1} (C_{T,0} C_{P_2,1} + C_{T,1} C_{P_2,0}^q)^{q-1}) \\ & \quad \times C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1} \\ &= -T^{q-1} \sum_{a_2, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1} \\ & \quad - \sum_{a_2, \dots, a_d \in \mathbb{F}_q} T^{q^2-1} (C_{T,0} C_{P_2,1} + C_{T,1} C_{P_2,0}^q)^{q-1} \\ & \quad \times C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1}. \end{aligned} \tag{19}$$

Since $0 \leq J - 1 < q - 1$, the term

$$-T^{q-1} \sum_{a_2, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1} = 0$$

(cf. Proposition 2.2). Hence

$$\begin{aligned}
 & \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 0}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 1}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1} \\
 &= - \sum_{a_2, \dots, a_d \in \mathbb{F}_q} T^{q^2-1} (C_{T, 0} C_{P_2, 1} + C_{T, 1} C_{P_2, 0}^q)^{q-1} \\
 & \quad \times C_{T^{d+1}+\dots+a_2 T^2, 2}^{q-1} \cdots C_{T^{d+1}+\dots+a_2 T^2, d}^{J-1}.
 \end{aligned} \tag{20}$$

Repeating this argument we obtain that

$$\begin{aligned}
 & \sum_{a_1, \dots, a_d \in \mathbb{F}_q} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 0}^{q-1} C_{T^{d+1}+a_d T^d+\dots+a_1 T, 1}^{q-1} \cdots C_{T^{d+1}+a_d T^d+\dots+a_1 T, d}^{J-1} \\
 &= (-1)^{d-1} T^{q^d-1} \sum_{a_d \in \mathbb{F}_q} \left(\sum_{k=0}^{d-1} C_{T^{d-1}, k} C_{P_d, d-1-k}^{q^k} \right)^{q-1} C_{T^{d+1}+a_d T^d, d}^{J-1} \\
 &= (-1)^{d-1} T^{q^d-1} \sum_{a_d \in \mathbb{F}_q} \left(\sum_{k=0}^{d-1} C_{T^{d-1}, k} C_{P_d, d-1-k}^{q^k} \right)^{q-1} \left(\sum_{k=0}^d C_{T^d, k} C_{P_d, d-k}^{q^k} \right)^{J-1} \\
 &= (-1)^d T^{q^d-1} \left(T^{q^d} \sum_{k=0}^d C_{T^d, k} C_{P_{d+1}, d-k}^{q^k} \right)^{J-1} = (-1)^d T^{q^d-1} T^{(J-1)q^d} = (-1)^d T^{Jq^d-1}.
 \end{aligned}$$

Let now $a \in \mathbb{F}_q$; if $P(T) \in \mathbb{F}_q[T]$, then

$$C_{P(T+a), i} = C_{P(T), i}(T + a).$$

It follows that

$$\sum_{P \in A_{(d+1)+}, (T+a)|P} C_{P, 0}^{q-1} \cdots C_{P, d-1}^{q-1} C_{P, d}^{J-1} = (-1)^d (T + a)^{Jq^d-1},$$

for any $a \in \mathbb{F}_q$. \square

Corollary 3.1. *Let $d \geq 1$, $1 \leq J \leq q - 1$. Let $h = \sum_{i \geq 0} a_i t^i$. Then*

$$a_{J(q-1)q^d+1} = (-1)^{J+1} (T^{q^d} - T)^J + \epsilon,$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ otherwise.

Proof. From Eqs. (9), (10) and Theorem 3.1, we conclude that

$$a_{J(q-1)q^d+1} = (-1)^{d+J} \sum_{a \in \mathbb{F}_q} (-1)^d (T + a)^q (T + a)^{Jq^d-1} = (-1)^J \sum_{a \in \mathbb{F}_q} (T + a)^{Jq^d+q-1}.$$

Now, by [Lemma 3.1](#),

$$\sum_{a \in \mathbb{F}_q} (T + a)^{Jq^d + q - 1} = \sum_{a \in \mathbb{F}_q} (T^{q^d} + a)^J (T + a)^{q - 1} = \epsilon - (T^{q^d} - T)^J,$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ if $J < q - 1$. \square

4. Coefficients of Δ and E_{q^l-1}

Let us consider the discriminant Δ and the Eisenstein series E_{q^l-1} of weight $q^l - 1$, $l \geq 1$. The functions Δ and E_{q^l-1} are modular forms of type 0 for $GL(2, A)$, so they admit expansions in the parameter $s = t^{q-1}$; nevertheless, we will use their t -expansions since the action of Hecke operators is given with respect to t -expansions. These modular forms satisfy the following property. Let $f = \sum_{i \geq 0} b_i t^i$; if $f = \Delta$, then $b_i \neq 0 \Rightarrow i \equiv 0, q - 1 \pmod{q(q - 1)}$ (cf. [\[3, Proposition 6.10, p. 684\]](#)), and if $f = E_{q^l-1}$, then $b_i \neq 0 \Rightarrow i \equiv 0, q^{l-1}(q - 1), q^{l-2}(q^2 - 1), \dots, q^l - 1 \pmod{q^l(q - 1)}$ (cf. [\[2, Proposition 1, p. 111\]](#)).

As we indicated previously, another property of these modular forms is that the coefficients of the expansions of $\pi^{1-q^2} \Delta$ and $\pi^{1-q^l} E_{q^l-1}$ are elements of $\mathbb{F}_q(T)$. The goal of this section is to determine the coefficient $b_{(q-1)(Jq^{d+1}+1)}$ of the t -expansion of $\pi^{1-q^2} \Delta$, and the coefficient $c_{(q-1)Jq^{d+l}+q^l-1}$ of the t -expansion of $\pi^{1-q^l} E_{q^l-1}$, where $1 \leq J \leq q - 1$, $d \geq 1$. The first step is to extend the formula of [Eq. \(8\)](#) to Δ and to the Eisenstein series E_{q^l-1} .

Remark 4.1. We will use the following result (Lucas' formula) in the proofs of [Lemmas 4.1 and 4.3](#). Let p be a prime, let $q = p^r$ and $m = \sum_{i=0}^s m_i q^i$, $n = \sum_{i=0}^s n_i q^i$, $0 \leq m_i, n_i \leq q - 1$. Then

$$\binom{m}{n} \equiv \prod_{i=0}^s \binom{m_i}{n_i} \pmod{p}.$$

Lemma 4.1. *Let $a \in A$, $\deg a = d$, $l \geq 1$, and $i + 1 \equiv q^l - 1 \pmod{q^l(q - 1)}$. If the X^k -coefficient of the Goss polynomial $G_{i+1,a}(X)$ is nonzero for some $k \leq q^l - 1$ such that $k \not\equiv 0 \pmod{q}$, then $i + 1 = q^l - 1$ and $k = q^l - 1$.*

Proof. We have from [Eq. \(7\)](#) that

$$G_{i+1,a}(X) = \sum_{j \leq i} a^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(a)^{\underline{n}} X^{j+1},$$

where $\underline{n} = (n_0, n_1, \dots, n_d)$ runs over the set of $(d + 1)$ -tuples satisfying $n_0 + n_1 + \dots + n_d = j$ and $n_0 + n_1 q + \dots + n_d q^d = i$.

From the assumption on the polynomial $G_{i+1,a}(X)$, there exists a $(d + 1)$ -tuple (n_0, n_1, \dots, n_d) such that $1 + n_0 + n_1 + \dots + n_d = k = j + 1$, $k \leq q^l - 1$, $k \not\equiv 0 \pmod q$, $n_0 + n_1q + \dots + n_dq^d = i$, and $\binom{j}{\underline{n}} \neq 0$. We divide $1 + n_0 + n_1q + \dots + n_dq^d$ by $q^l(q - 1)$ (considering q as an indeterminate). The remainder of this division is

$$(n_d + \dots + n_l)q^l + n_{l-1}q^{l-1} + \dots + n_0 + 1.$$

Since $i + 1 \equiv q^l - 1 \pmod{q^l(q - 1)}$, it follows that

$$n_{l-1}q^{l-1} + \dots + n_1q + n_0 + 2 \equiv 0 \pmod{q^l}.$$

Hence

$$n_sq^s + \dots + n_0 + 2 \equiv 0 \pmod{q^{s+1}}, \quad s = 0, 1, \dots, l - 1. \tag{21}$$

We now have that

$$\begin{aligned} \binom{j}{\underline{n}} &= \frac{(n_0 + n_1 + \dots + n_d)!}{n_0!n_1! \dots n_d!} \\ &= \binom{n_0 + \dots + n_d}{n_0} \cdot \binom{n_1 + \dots + n_d}{n_1} \dots \binom{n_{d-1} + n_d}{n_{d-1}}. \end{aligned} \tag{22}$$

Let $n_s = \sum_{r=0}^{l-1} n_{s,r}q^r$, $s = 0, 1, \dots, d$, $0 \leq n_{s,r} \leq q - 1$. Since $\binom{j}{\underline{n}} \neq 0$, applying Lucas' formula to Eq. (22), we conclude that

$$n_{0,r} + n_{1,r} + \dots + n_{d,r} \leq q - 1, \quad r = 0, 1, \dots, l - 1. \tag{23}$$

From Eqs. (21) and (23), we deduce (using induction) that

$$\begin{aligned} n_{0,0} &= q - 2, \\ n_{s,0} + n_{s-1,1} + \dots + n_{0,s} + 1 &= q, \quad s = 1, \dots, l - 1. \end{aligned}$$

By Lemma 4.2, $n_{0,r_0} + n_{1,r_0} + \dots + n_{l-1-r_0,r_0} = q - 2$ for exactly one $r_0 \in \{0, 1, \dots, l - 1\}$, and $n_{0,r} + n_{1,r} + \dots + n_{l-1-r,r} = q - 1$ for $r \neq r_0$. Now, since $k = 1 + n_0 + n_1 + \dots + n_d \leq q^l - 1$, we get that

$$k = 1 + n_0 + n_1 + \dots + n_d = 1 + (q - 2)q^{r_0} + \sum_{r \neq r_0} (q - 1)q^r = q^l - q^{r_0}.$$

The condition $k \not\equiv 0 \pmod q$ implies $r_0 = 0$; by Lemma 4.2, $n_0 = q^l - 2$. \square

Lemma 4.2. Let $k \in \mathbb{N}$. Let $\{n_{s,r}: s, r \in \{0, 1, \dots, k\}, s + r \leq k\}$ be a set of natural numbers. Assume that $n_{0,r} + n_{1,r} + \dots + n_{k-r,r} \leq q - 1$ for $r = 0, 1, \dots, k$, and

$$n_{0,0} = q - 2,$$

$$n_{s,0} + n_{s-1,1} + \dots + n_{0,s} + 1 = q, \quad s = 1, \dots, k.$$

Then $n_{0,r_0} + n_{1,r_0} + \dots + n_{k-r_0,r_0} = q - 2$ for exactly one $r_0 \in \{0, 1, \dots, k\}$, and $n_{0,r} + n_{1,r} + \dots + n_{k-r,r} = q - 1$ for $r \neq r_0$. Furthermore, if $r_0 = 0$, then $n_{0,r} = q - 1$ for $r = 1, 2, \dots, k$.

Proof. Induction on k . \square

Proposition 4.1. Let $P \in A$ be a monic polynomial of degree d .

(a) Let $\bar{\pi}^{1-q^2} \Delta = \sum_{i \geq 0} b_i t^i$ be the expansion of the discriminant function. The t^{q^2-1} -coefficient of $T_P(\bar{\pi}^{1-q^2} \Delta)$ is

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_d) \\ n_0+n_1+\dots+n_d=q^2-2}} \binom{q^2-2}{\underline{n}} C(P)^{\underline{n}} b_{1+n_0+n_1q+\dots+n_dq^d} - P^{q-2} \sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1},$$

where $1 + n_0 + n_1q + \dots + n_dq^d \equiv q - 1 \pmod{q(q-1)}$.

(b) Let $\bar{\pi}^{1-q^l} E_{q^l-1} = \sum_{i \geq 0} c_i t^i$ be the expansion of the Eisenstein series. The $t^{q^{l+1}-1}$ -coefficient of $T_P(\bar{\pi}^{1-q^l} E_{q^l-1})$ is

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_d) \\ n_0+n_1+\dots+n_d=q^{l+1}-2}} \binom{q^{l+1}-2}{\underline{n}} C(P)^{\underline{n}} c_{1+n_0+n_1q+\dots+n_dq^d} - P^{q^l-2} \sum_{Q \in A_{1+}, Q|P} Q,$$

where $1 + n_0 + n_1q + \dots + n_dq^d \equiv q^l - 1 \pmod{q^l(q-1)}$.

Proof. (a) We make $f = \bar{\pi}^{1-q^2} \Delta$ in Eq. (5); then

$$T_P(\bar{\pi}^{1-q^2} \Delta) = P^{q^2-2} \sum_{i \geq 0} \sum_{\delta \in A_+, \delta|P} \delta^{1-q^2} b_i G_{i,\delta}(\delta t_{P/\delta}). \tag{24}$$

Now $t_a = t(az) = t^{q^{\deg a}}/f_a(t)$, $f_a(X) = C_a(X^{-1})$; so $t^{q^{\deg a}}$ divides t_a . Therefore, in order to obtain the coefficient of t^{q^2-1} in the expansion of $T_P(\bar{\pi}^{1-q^2} \Delta)$, we only have to consider in Eq. (24) the polynomials δ such that $\deg(P/\delta) \leq 1$. Furthermore, $b_i \neq 0 \Rightarrow i \equiv 0, q-1 \pmod{q(q-1)}$; if $i \equiv 0 \pmod{q(q-1)}$, then $G_{i,\delta}(X)$ is a q -th power of some polynomial (cf. Proposition 3.4 of [3, p. 675]); hence, if $i \equiv 0 \pmod{q(q-1)}$, then the t^{q^2-1} -coefficient of $G_{i,\delta}(\delta t_{P/\delta})$ is zero. Thus, we only consider the case $i \equiv q-1 \pmod{q(q-1)}$.

Let $\deg(P/\delta) = 0$; then $\delta = P$. The sum corresponding to P in Eq. (24) is

$$P^{-1} \sum_{i \geq 0} b_i G_{i,P}(Pt).$$

The coefficient of t^{q^2-1} in this expansion is

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_d) \\ n_0+n_1+\dots+n_d=q^2-2}} \binom{q^2-2}{\underline{n}} C(P)^{\underline{n}} b_{1+n_0+n_1q+\dots+n_dq^d}$$

(cf. Eq. (7)); by the previous argument, the coefficients b_i that occur in the sum satisfy $i \equiv q-1 \pmod{q(q-1)}$.

Let now $\deg(P/\delta) = 1$. The sum corresponding to δ in Eq. (24) is

$$(P/\delta)^{q^2-2} \sum_{i \geq 0} \delta^{-1} b_i G_{i,\delta}(\delta t_{P/\delta}). \tag{25}$$

Let $i \equiv q-1 \pmod{q(q-1)}$; the t^{q^2-1} -coefficient of $G_{i,\delta}(\delta t_{P/\delta})$ is nonzero only if the X^k -coefficient of the Goss polynomial $G_{i,\delta}(X)$ is nonzero for some $k \leq q-1$. By Lemma 4.1, $i = q-1$. Thus, we just determine the t^{q^2-1} -coefficient of

$$(P/\delta)^{q^2-2} \delta^{-1} b_{q-1} G_{q-1,\delta}(\delta t_{P/\delta}) = -(P/\delta)^{q^2-2} \delta^{-1} (\delta t_{P/\delta})^{q-1}.$$

We have that

$$t_{P/\delta} = \frac{t^q}{1 + (P/\delta)t^{q-1}} = t^q \sum_{n \geq 0} (-1)^n (P/\delta)^n t^{n(q-1)}.$$

Then the t^{q^2-1} -coefficient of $(t_{P/\delta})^{q-1}$ is P/δ . Therefore the t^{q^2-1} -coefficient of the expansion of Eq. (25) is $-(P/\delta)^{q^2-1} \delta^{q-2}$.

(b) We have that

$$T_P(\bar{\pi}^{1-q^l} E_{q^l-1}) = P^{q^l-2} \sum_{i \geq 0} \sum_{\delta \in A_+, \delta|P} \delta^{1-q^l} c_i G_{i,\delta}(\delta t_{P/\delta}). \tag{26}$$

The expansion $\bar{\pi}^{1-q^l} E_{q^l-1} = \sum_{i \geq 0} c_i t^i$ satisfies that $c_i \neq 0 \Rightarrow i \equiv 0, q^{l-1}(q-1), q^{l-2}(q^2-1), \dots, q^l-1 \pmod{q^l(q-1)}$; as in part (a), due to the property of Goss polynomials, in order to obtain the coefficient of $t^{q^{l+1}-1}$ in the expansion of $T_P(\bar{\pi}^{1-q^l} E_{q^l-1})$, we only have to consider in Eq. (26) the case $i \equiv q^l-1 \pmod{q^l(q-1)}$. Let δ be such that $\deg(P/\delta) \geq 1$; if the term $t^{q^{l+1}-1}$ occurs in $t_{P/\delta}^k$, then $k \leq q^l-1$ and $k \not\equiv 0 \pmod{q}$. By Lemma 4.1, only the terms corresponding to $\deg(P/\delta) \leq 1$ contribute to the term $t^{q^{l+1}-1}$ of the expansion of $T_P(\bar{\pi}^{1-q^l} E_{q^l-1})$ (Eq. (26)).

Now the proof for E_{q^l-1} is similar to that of part (a). We just indicate that the coefficient c_{q^l-1} has to be determined for the case $\deg(P/\delta) = 1$. This coefficient is -1 ; this follows from the formula

$$\bar{\pi}^{1-q^l} E_{q^l-1} = (-1)^{l+1} L_l^{-1} - \left(\sum_{i < l} (-1)^i L_i^{-1} \sum_{a \in A_+} t_a^{q^l - q^i} \right),$$

where $L_0 = 1$ and $L_i = (T^{q^i} - T)(T^{q^{i-1}} - T) \cdots (T^q - T)$ (cf. [2, Eq. (6), p. 111]). \square

Corollary 4.1. *Let $\underline{i} = (i_0, i_1, \dots, i_d)$, $i_k \geq 0$, and let $S_{d+1}(i_0, i_1, \dots, i_d)$ be as in Notation 2.1.*

(a) *The t^{q^2-1} -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^i T_P(\bar{\pi}^{1-q^2} \Delta)$ is*

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_{d+1}) \\ n_0+n_1+\dots+n_{d+1}=q^2-2}} \binom{q^2-2}{\underline{n}} S_{d+1}(n_0+i_0, \dots, n_d+i_d) b_{1+n_0+n_1q+\dots+n_{d+1}q^{d+1}} \\ - \sum_{P \in A_{(d+1)+}} C_{P,0}^{q-2+i_0} C_{P,1}^{i_1} \cdots C_{P,d}^{i_d} \left(\sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1} \right),$$

where $1 + n_0 + n_1q + \dots + n_{d+1}q^{d+1} \equiv q - 1 \pmod{q(q-1)}$.

(b) *The $t^{q^{l+1}-1}$ -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^i T_P(\bar{\pi}^{1-q^l} E_{q^l-1})$ is*

$$\sum_{\substack{\underline{n}=(n_0, n_1, \dots, n_{d+1}) \\ n_0+n_1+\dots+n_{d+1}=q^{l+1}-2}} \binom{q^{l+1}-2}{\underline{n}} S_{d+1}(n_0+i_0, \dots, n_d+i_d) c_{1+n_0+n_1q+\dots+n_{d+1}q^{d+1}} \\ - \sum_{P \in A_{(d+1)+}} C_{P,0}^{q^l-2+i_0} C_{P,1}^{i_1} \cdots C_{P,d}^{i_d} \left(\sum_{Q \in A_{1+}, Q|P} Q \right),$$

where $1 + n_0 + n_1q + \dots + n_{d+1}q^{d+1} \equiv q^l - 1 \pmod{q^l(q-1)}$.

Lemma 4.3. (a) *Let $l \geq 1$, and $i + 1 \equiv q^l - 1 \pmod{q^l(q-1)}$. Let $d \geq 1$; we consider a $(d+2)$ -tuple $\underline{n} = (n_0, n_1, \dots, n_{d+1})$ satisfying $n_k \geq 0$ for $k = 0, 1, \dots, d+1$, $n_0 + n_1 + \dots + n_{d+1} = q^{l+1} - 2$, $n_0 + n_1q + \dots + n_{d+1}q^{d+1} = i$, and*

$$\binom{q^{l+1}-2}{\underline{n}} \neq 0.$$

Then $n_0 + 2 \equiv 0 \pmod{q^l}$, and $n_k \equiv 0 \pmod{q^l}$ for $k = 1, 2, \dots, d+1$.

(b) *Let $1 \leq J \leq q - 1$. Then*

$$\frac{(q^{l+1}-2)!}{(q^l-2)!(q^l(q-J))!(q^l(J-1))!} = (-1)^{J-1}.$$

Proof. (a) This can be done using the arguments of the proof of Lemma 5 of [2, p. 114]; nevertheless, since the claims of this reference are slightly different from our claim, we briefly indicate the proof.

As in Lemma 4.1, we get the congruences

$$n_s q^s + n_{s-1} q^{s-1} + \dots + n_0 + 2 \equiv 0 \pmod{q^{s+1}}, \quad s = 0, 1, \dots, l - 1. \tag{27}$$

Let v_p be the p -adic valuation on \mathbb{Q} (here $p = \text{char}(\mathbb{F}_q)$); then either $n_0 = q^{l+1} - 2$ or $v_p(q^{l+1} - 2 - n_0) = v_p(n_0 + 2)$. If $n_0 = q^{l+1} - 2$, then $n_k = 0$ for $k = 1, 2, \dots, d + 1$. Assume now that $v_p(q^{l+1} - 2 - n_0) = v_p(n_0 + 2)$; since $\binom{q^{l+1} - 2}{n_k} \neq 0$, it follows that $\binom{q^{l+1} - 2 - n_0}{n_k} \neq 0$ for each $k = 1, 2, \dots, d + 1$. By Lucas' formula $v_p\binom{q^{l+1} - 2 - n_0}{n_k} \geq v_p(q^{l+1} - 2 - n_0) = v_p(n_0 + 2)$ for $k = 1, 2, \dots, d + 1$; we obtain from the congruences of Eq. (27) that $v_p(n_0 + 2) \geq v_p(q^l)$, and so $v_p(n_k) \geq v_p(q^l)$ for $k = 1, 2, \dots, d + 1$.

(b) We write

$$\frac{(q^{l+1} - 2)!}{(q^l - 2)!(q^l(q - J))!(q^l(J - 1))!} = \binom{q^{l+1} - 2}{q^l - 2} \binom{q^{l+1} - q^l}{q^l - Jq^l}. \tag{28}$$

By Lucas' formula, $\binom{q^{l+1} - 2}{q^l - 2} = 1$, and $\binom{q^{l+1} - q^l}{q^l - Jq^l} = \binom{q-1}{q-J}$. By Remark 2.1, $\binom{q-1}{q-J} = (-1)^{q-J} = (-1)^{J-1}$. \square

Theorem 4.1. Let $1 \leq J \leq q - 1$, $d \geq 1$.

(a) Let $i_0 = q(q - 1) - (q - 2)$, $i_k = q(q - 1)$ for $1 \leq k \leq d - 1$, $i_d = q(q - 1) - q(q - J)$. Then the t^{q^2-1} -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^i T_P(\bar{\pi}^{1-q^2} \Delta)$ is

$$(-1)^{d+J} b_{(q-1)(Jq^{d+1}+1)} - \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1} \right).$$

(b) Let $i_0 = q^l(q - 1) - (q^l - 2)$, $i_k = q^l(q - 1)$ for $2 \leq k \leq d - 1$, $i_d = q^l(q - 1) - q^l(q - J)$. Then the $t^{q^{l+1}-1}$ -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^i T_P(\bar{\pi}^{1-q^l} E_{q^l-1})$ is

$$(-1)^{d+J} c_{(q-1)Jq^{d+1}+q^l-1} - \sum_{P \in A_{(d+1)+}} C_{P,0}^{q^l(q-1)} C_{P,1}^{q^l(q-1)} \dots C_{P,d}^{q^l(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q \right).$$

Proof. (a) We prove in the formula of Corollary 4.1 (a) that, if $n_0 + n_1 + \dots + n_{d+1} = q^2 - 2$, $1 + n_0 + n_1 q + \dots + n_{d+1} q^{d+1} \equiv q - 1 \pmod{q(q - 1)}$ and

$$\underline{n} \neq (q - 2, 0, \dots, 0, q(q - J), q(J - 1)),$$

then

$$\binom{q^2 - 2}{\underline{n}} S_{d+1}(n_0 + i_0, \dots, n_d + i_d) = 0.$$

By [Lemma 4.3](#) (a), $n_0 + 2 \equiv 0 \pmod q$ and $n_k \equiv 0 \pmod q$, for $k = 1, 2, \dots, d + 1$. Let $n_0 = mq - 2$, $1 \leq m \leq q$. If $m = q$, then $n_0 = q^2 - 2$ and $n_1 = n_2 = \dots = n_{d+1} = 0$. Hence

$$\begin{aligned} S_{d+1}(n_0 + i_0, \dots, n_d + i_d) &= \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(2q-2)} C_{P,1}^{q(q-1)} \dots C_{P,d-1}^{q(q-1)} C_{P,d}^{q(J-1)} \\ &= \left(\sum_{P \in A_{(d+1)+}} C_{P,0}^{2q-2} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} \right)^q. \end{aligned}$$

Since $0 \leq J - 1 \leq q - 2$,

$$\sum_{P \in A_{(d+1)+}} C_{P,0}^{2q-2} C_{P,1}^{q-1} \dots C_{P,d-1}^{q-1} C_{P,d}^{J-1} = 0$$

(cf. [Proposition 2.2](#)).

Assume now that $n_0 = mq - 2$, $1 \leq m \leq q - 1$. Let $n_k = qm_k$, $1 \leq k \leq d$; then $m_1 + \dots + m_d \leq q - m$. We have that

$$\begin{aligned} S_{d+1}(n_0 + i_0, \dots, n_d + i_d) &= \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-2+m)} C_{P,1}^{q(q-1+m_1)} \dots C_{P,d-1}^{q(q-1+m_{d-1})} C_{P,d}^{q(J-1+m_d)} \\ &= \left(\sum_{P \in A_{(d+1)+}} C_{P,0}^{q-2+m} C_{P,1}^{q-1+m_1} \dots C_{P,d-1}^{q-1+m_{d-1}} C_{P,d}^{J-1+m_d} \right)^q; \end{aligned}$$

if $m > 1$, then $S_{d+1}(n_0 + i_0, \dots, n_d + i_d) = 0$. Assume now that $n_0 = q - 2$; then $m_1 + \dots + m_d \leq q - 1$. If $m_k \geq 1$ for some $1 \leq k \leq d - 1$, then $S_{d+1}(n_0 + i_0, \dots, n_d + i_d) = 0$. In summary, the only case for which

$$\binom{q^2 - 2}{\underline{n}} S_{d+1}(n_0 + i_0, \dots, n_d + i_d)$$

may be nonzero is $n_0 = q - 2$, $n_k = 0$, $1 \leq k \leq d - 1$, $n_d = q(q - J)$, $n_{d+1} = q(J - 1)$; in this case $S_{d+1}(n_0 + i_0, \dots, n_d + i_d) = (-1)^{d+1}$. Now, by [Lemma 4.3](#) (b),

$$\frac{(q^2 - 2)!}{(q - 2)!(q(q - J))!(q(J - 1))!} = (-1)^{J-1}.$$

(b) Similar to part (a): using [Lemma 4.3](#), it is proven in the formula of [Corollary 4.1](#) (b) that

$$\binom{q^{l+1} - 2}{\underline{n}} S_{d+1}(n_0 + i_0, \dots, n_d + i_d) \neq 0$$

only for $n_0 = q^l - 2$, $n_k = 0$, $1 \leq k \leq d - 1$, $n_d = q^l(q - J)$, $n_{d+1} = q^l(J - 1)$, and in this case $\binom{q^{l+1}-2}{\underline{n}} S_{d+1}(n_0 + i_0, \dots, n_d + i_d) = (-1)^{d+J}$. \square

Corollary 4.2. *Let $1 \leq J \leq q - 1$, $d \geq 1$.*

(a) *We have that*

$$b_{(q-1)(Jq^{d+1}+1)} = (-1)^{d+J} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1} \right).$$

(b) *We have that*

$$c_{(q-1)Jq^{d+1}+q^l-1} = (-1)^{d+J} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q^l(q-1)} C_{P,1}^{q^l(q-1)} \dots C_{P,d}^{q^l(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q \right).$$

Proof. (a) Δ is an eigenfunction for all the Hecke operators T_P and the corresponding eigenvalues are P^{q-2} (see Eq. (5)). Let \underline{i} be as in Theorem 4.1, i.e., $i_0 = q(q - 1) - (q - 2)$, $i_k = q(q - 1)$ for $1 \leq k \leq d - 1$, $i_d = q(q - 1) - q(q - J)$; it follows that the t^{q^2-1} -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^{\underline{i}} T_P \Delta$ is

$$b_{q^2-1} \sum_{P \in A_{(d+1)+}} P^{q-2} C(P)^{\underline{i}} = b_{q^2-1} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)}.$$

Since $J \leq q - 1$, we conclude that $\sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} = 0$ (cf. Proposition 2.2). The corollary now follows from Theorem 4.1, since

$$(-1)^{d+J} b_{(q-1)(Jq^{d+1}+1)} - \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1} \right)$$

is also the t^{q^2-1} -coefficient of $\sum_{P \in A_{(d+1)+}} C(P)^{\underline{i}} T_P \Delta$.

(b) Similar to part (a). \square

We can already obtain the coefficients $b_{(q-1)(Jq^{d+1}+1)}$ and $c_{(q-1)Jq^{d+1}+q^l-1}$. We first determine $b_{(q-1)(Jq^{d+1}+1)}$.

Lemma 4.4. *Let $P, Q \in A$, $1 \leq J \leq q - 1$, and $\epsilon' = 0, 1$. Then*

$$\sum_{a \in \mathbb{F}_q} a^{\epsilon'} (P + a)^J (Q + a)^{q-2} = \epsilon - \sum_{r=0}^{J-1+\epsilon'} \binom{J}{r} \binom{q-2}{J-1+\epsilon'-r} P^r Q^{J-1+\epsilon'-r},$$

where $\epsilon = -1$ if $J = q - 1$ and $\epsilon' = 1$, and $\epsilon = 0$ if $J < q - 1$ or $\epsilon' = 0$.

Proof. The lemma follows from the fact that $\sum_{a \in \mathbb{F}_q} a^k = -1$ if $k(> 0)$ is a multiple of $q - 1$, and $\sum_{a \in \mathbb{F}_q} a^k = 0$ otherwise (cf. [Lemma 3.1](#)). \square

Theorem 4.2. *Let $d \geq 1$, $1 \leq J \leq q - 1$. Let $\bar{\pi}^{1-q^2} \Delta = \sum_{i \geq 0} b_i t^i$. Then*

$$b_{(q-1)(Jq^{d+1}+1)} = (-1)^{J+1} (T^{q^{d+1}} - T^q)^{J-1} (JT - (J + 1)T^q + T^{q^{d+1}}) + \epsilon,$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ otherwise.

Proof. We first observe that

$$\begin{aligned} b_{(q-1)(Jq^{d+1}+1)} &= (-1)^{d+J} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q^{q^2-q+1} \right) \\ &= (-1)^{d+J} \sum_{Q \in A_{1+}} Q^{q^2-q+1} \left(\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q(q-1)} C_{P,1}^{q(q-1)} \dots C_{P,d}^{q(J-1)} \right) \\ &= (-1)^{d+J} \sum_{Q \in A_{1+}} Q^{q^2-q+1} \left(\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d}^{J-1} \right)^q. \end{aligned}$$

Now it follows from [Theorem 3.1](#) that

$$\begin{aligned} b_{(q-1)(Jq^{d+1}+1)} &= (-1)^{d+J} \sum_{Q \in A_{1+}} Q^{q^2-q+1} \left(\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d}^{J-1} \right)^q \\ &= (-1)^{d+J} \sum_{a \in \mathbb{F}_q} (T + a)^{q^2-q+1} ((-1)^d (T + a)^{Jq^d-1})^q \\ &= (-1)^J \sum_{a \in \mathbb{F}_q} (T + a)^{Jq^{d+1}+q^2-2q+1} \\ &= (-1)^J \sum_{a \in \mathbb{F}_q} (T^{q^{d+1}} + a)^J (T^q + a)^{q-2} (T + a). \end{aligned}$$

Let us calculate this last sum. From [Lemma 4.4](#) we deduce that

$$\begin{aligned} &(-1)^J \sum_{a \in \mathbb{F}_q} (T^{q^{d+1}} + a)^J (T^q + a)^{q-2} (T + a) \\ &= (-1)^{J+1} \sum_{r=0}^{J-1} \binom{J}{r} \binom{q-2}{J-1-r} T^{rq^{d+1}} T^{(J-1-r)q+1} \\ &\quad + (-1)^{J+1} \sum_{r=0}^J \binom{J}{r} \binom{q-2}{J-r} T^{rq^{d+1}} T^{(J-r)q} + (-1)^J \epsilon, \end{aligned}$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ otherwise. From the identity

$$\binom{q-2}{r} = (-1)^r(r+1),$$

where $0 \leq r \leq q - 1$, we derive that

$$\begin{aligned} & (-1)^{J+1} \sum_{r=0}^{J-1} \binom{J}{r} \binom{q-2}{J-1-r} T^{rq^{d+1}} T^{(J-1-r)q+1} \\ &= (-1)^{J+1} \sum_{r=0}^{J-1} (-1)^{J-1-r} J \binom{J-1}{r} T^{(J-1-r)q+1} T^{rq^{d+1}}. \end{aligned}$$

From the identities

$$(J-r) \binom{J}{r} = J \binom{J-1}{r}, \quad \binom{J}{r} = \binom{J-1}{r} + \binom{J-1}{r-1},$$

we derive that

$$\begin{aligned} & (-1)^{J+1} \sum_{r=0}^J \binom{J}{r} \binom{q-2}{J-r} T^{rq^{d+1}} T^{(J-r)q} \\ &= (-1)^{J+1} \sum_{r=0}^J (-1)^{J-r} (J-r+1) \binom{J}{r} T^{rq^{d+1}} T^{(J-r)q} \\ &= (-1)^{J+1} \left(\sum_{r=0}^J (-1)^{J-r} J \binom{J-1}{r} T^{rq^{d+1}} T^{(J-r)q} + \sum_{r=0}^J (-1)^{J-r} \binom{J}{r} T^{rq^{d+1}} T^{(J-r)q} \right) \\ &= (-1)^{J+2} \sum_{r=0}^{J-1} (-1)^{J-1-r} (J+1) \binom{J-1}{r} T^{rq^{d+1}} T^{(J-r)q} \\ &\quad + (-1)^{J+1} \sum_{r=0}^{J-1} (-1)^{J-1-r} \binom{J-1}{r} T^{(r+1)q^{d+1}} T^{(J-1-r)q}. \end{aligned}$$

In summary,

$$\begin{aligned} b_{(q-1)(Jq^{d+1}+1)} - \epsilon &= (-1)^{J+1} \sum_{r=0}^{J-1} (-1)^{J-1-r} J \binom{J-1}{r} T^{(J-1-r)q+1} T^{rq^{d+1}} \\ &\quad + (-1)^{J+2} \sum_{r=0}^{J-1} (-1)^{J-1-r} (J+1) \binom{J-1}{r} T^{rq^{d+1}} T^{(J-r)q} \\ &\quad + (-1)^{J+1} \sum_{r=0}^{J-1} (-1)^{J-1-r} \binom{J-1}{r} T^{(r+1)q^{d+1}} T^{(J-1-r)q}, \quad (29) \end{aligned}$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ otherwise. We finally observe that

$$(-1)^{J+1}(T^{q^{d+1}} - T^q)^{J-1}(JT - (J + 1)T^q + T^{q^{d+1}})$$

equals the right-hand side of Eq. (29). \square

Let us conclude this work with the coefficient $c_{(q-1)Jq^{d+l}+q^l-1}$.

Theorem 4.3. *Let $d \geq 1$, $1 \leq J \leq q - 1$. Let $\bar{\pi}^{1-q^l} E_{q^l-1} = \sum_{i \geq 0} c_i t^i$. Then*

$$c_{(q-1)Jq^{d+l}+q^l-1} = (-1)^J \frac{(T^{q^d} - T)^{Jq^l} (T^{q^l} - T)}{(T^q - T)^{q^l}} + \epsilon$$

where $\epsilon = -1$ if $J = q - 1$, and $\epsilon = 0$ otherwise.

Proof. From Corollary 4.2 (b) and Theorem 3.1, we obtain that

$$\begin{aligned} c_{(q-1)Jq^{d+l}+q^l-1} &= (-1)^{d+J} \sum_{P \in A_{(d+1)+}} C_{P,0}^{q^l(q-1)} C_{P,1}^{q^l(q-1)} \dots C_{P,d}^{q^l(J-1)} \left(\sum_{Q \in A_{1+}, Q|P} Q \right) \\ &= (-1)^{d+J} \sum_{Q \in A_{1+}} Q \left(\sum_{P \in A_{(d+1)+}, Q|P} C_{P,0}^{q-1} C_{P,1}^{q-1} \dots C_{P,d}^{J-1} \right)^{q^l} \\ &= (-1)^{d+J} \sum_{a \in \mathbb{F}_q} (T+a) ((-1)^d (T+a)^{Jq^d-1})^{q^l} \\ &= (-1)^J \sum_{a \in \mathbb{F}_q} (T+a)^{Jq^{d+l}-q^l+1}. \end{aligned}$$

We write

$$\begin{aligned} &(-1)^J \sum_{a \in \mathbb{F}_q} (T+a)^{Jq^{d+l}-q^l+1} \\ &= (-1)^J T^{Jq^{d+l}-q^l+1} + (-1)^J \sum_{a \in \mathbb{F}_q^\times} \left((T^{q^d} + a)^{J-1} \frac{T^{q^d} + a}{T+a} \right)^{q^l} (T+a). \end{aligned} \tag{30}$$

Now, similarly to the proof of Theorem 4.2, we get the coefficient $c_{(q-1)Jq^{d+l}+q^l-1}$. \square

Remark 4.2. Corollary 7 of [2, p. 115] gives the expression of the coefficient $c_{(q-1)q^{d+l}+q^l-1}$ up to a factor $P_d(T)$ of degree $q^l - q$. In Theorem 4.3 this polynomial $P_d(T)$ is determined to be $\frac{T^{q^l} - T}{T^q - T}$.

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