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On the vertices belonging to all edge metric bases

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ABSTRACT

An edge metric basis of a connected graph G is a smallest possible set of vertices S of G satisfying the following: for any two edges e, f of G there is a vertex $s \in S$ such that the distances from s to e and f differ. The cardinality of an edge metric basis is the edge metric dimension of G. In this article we consider the existence of vertices in a graph G such that they must belong to each edge metric basis of G, and we call them *edge basis forced vertices*. On the other hand, we name *edge void vertices* those vertices which do not belong to any edge metric basis. Among other results, we first deal with the computational complexity of deciding whether a given vertex is an edge basis forced vertex or an edge void vertex. We also establish some tight bounds on the number of edge basis forced vertices of a graph, as well as, on the number of edges in a graph having at least one edge basis forced vertex. Moreover, we show some realization results concerning which values for the integers n, k and f allow to confirm the existence of a graph G with n vertices, f edge basis forced vertices and edge metric dimension k.

1. Introduction

Along our exposition, all the graphs $G = (V(G), E(G))$ considered are simple, undirected and connected unless otherwise specifically stated. For two given vertices $x, y \in V(G)$, we shall denote the length of a shortest path between x and y, that is, the *distance* between x and y, by $d_G(x, y)$ (or simply $d(x, y)$). Let $R \subseteq V(G)$. If for all distinct vertices $x, y \in V(G)$ there exists an $r \in R$ such that $d(r, x) \neq d(r, y)$, then R is a *resolving set* of G. The smallest cardinality among all resolving sets of G is the *metric dimension* of G, and it is denoted by $\dim(G)$. A resolving set of cardinality $\dim(G)$ is called a *metric basis* of G. In addition to these concepts, it is said that a vertex r *resolves* the vertices x, y (or x, y are *resolved* by r) if $d(r, x) \neq d(r, y)$. Also, a set R resolves a set S if each pair of vertices of S is resolved by a vertex of R. In this sense, a resolving set of a graph G is a set that resolves the whole set $V(G)$.

The notion of metric dimension in graphs is a classical one in the graph theory introduced in the 1970's. The first steps in this direction appeared separately and independently in the articles [11,19]. However, the topic remained almost unstudied until the article [5] appeared. This work practically rediscovered the concept and brought it again to the research scene; and after this, it turned out to be of a high interest. Consequently, there are lots of investigations on the metric dimension of graphs and related topics so far. For more information we suggest the reader to check the two surveys [15,20]. Some examples of significant works that have recently appeared are [3,6,12,17].

The resolving sets (and related concepts) of a graph have the property of uniquely recognizing the vertices of a graph by using distances. Such property has proved to be very useful in several problems concerning locating and/or monitoring the

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nodes of a network; for instance, identifying failures or intruders, or recognizing data information. A couple of interesting examples are, for instance, the following ones: in [21], the authors presented a model that uses a metric dimension related concept to embed biological data sequences into a network and, in [2], a model for generating error correcting codes from a generalization of resolving sets was presented.

On the other hand, one of the variations of the classical metric dimension that has highly attracted the attention of researchers in the last years is the one called the edge metric dimension, introduced in [13]. To this end, we need the distance between a vertex $v \in V(G)$ and an edge $e = u_1u_2 \in E(G)$ to be defined as $d_G(v, e) = \min\{d(v, u_1), d(v, u_2)\}$ (or $d(v, e)$ for short). Let $R \subseteq V(G)$. If for all distinct edges $e, f \in E(G)$ there exists an $r \in R$ such that $d(r, e) \neq d(r, f)$, then R is an *edge resolving set* of G. The minimum cardinality of an edge resolving set of G is the *edge metric dimension* of G, which is denoted by $\mathrm{edim}(G)$. An edge resolving set of cardinality $\mathrm{edim}(G)$ is called an *edge metric basis* of G. Similarly to the metric dimension concepts, a pair of edges $e, f \in E(G)$ is *resolved* by a vertex $u \in V(G)$ if $d(u, e) \neq d(u, f)$.

Such variations of the metric dimension was introduced in connection with adapting the monitoring property of resolving sets to that of monitoring the connection between vertices. This idea of monitoring the edges of a graph has been also considered in other related investigations, which shows the interest into controlling the connections between vertices. For example, in [7], the authors Foucaud, Kao, (R.) Klasing, Miller and Ryan presented a graph-theoretic concept in the area of network monitoring which was focused on edges. This work has been extended in [22], co-authored again by R. Klasing, which shows a connection of our work with the investigation of Dr. Ralf Klasing, to whom this special issue is dedicated.

The spectrum of investigations on the metric dimension and related parameters in graphs is indeed very wide, including several different approaches to obtain combinatorial properties of the resolving sets or metric bases of graphs. One research direction has been concentrated into the existence of graphs with a unique metric basis (see for instance [1,16]). Closely related to this direction, a relaxed situation of the uniqueness of metric bases comes while considering the existence of metric bases with "partially unique parts". That is, the existence of vertices in a graph that must belong to every metric basis of the graph. The first contributions in this direction appeared relatively long ago in [4], where authors proved that for all integers k, r with $k \geq 2$ and $0 \leq r \leq k$, there is a graph G whose metric dimension equals k and has r vertices belonging to each metric basis of G. The contributions along these lines remained unattended for about two decades, until the article [9] was published, which brought back the attention to this topic. In this work, the vertices that must belong to every metric basis of a graph were called *basis forced vertices*. Some continuations of this work are for instance [8,10]. In particular, the recent work [8], shows a tight bound on the largest number of basis forced vertices that a given graph can have.

Having all these arguments in mind, in this investigation we focus our attention into adapting the notion of basis forced vertices while considering the variation of the metric dimension dealing with recognizing edges, that is, the edge metric dimension. We next formalize the main concept of our work.

Definition 1. A vertex $v \in V(G)$ is an *edge basis forced vertex* of G if it is included in every edge metric basis of G.

From now on, in order to avoid confusion between edge basis forced vertices and the basis forced vertices (regarding the classical metric bases), a vertex that belong to every (classical) metric basis shall be called a *v-basis forced vertex*. Moreover, note that a given vertex of a graph can be either an edge basis forced vertex and not a v-basis forced vertex or a v-basis forced vertex and not an edge basis forced vertex or both things at the same time or none of them. Examples of graphs having edge basis forced vertices, as well as, all possible combinations above are drawn in Fig. 1.

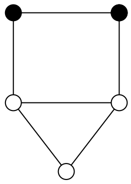
We remark that edge basis forced vertices play a crucial role in edge monitoring processes, since they are indeed mandatory (in the smallest edge resolving sets) to identify some pairs of edges in a graph. In this sense, identifying and/or counting them is worthy of considering as they can be understood as those key points in a graph that any (smallest edge) monitoring device must always include.

1.1. Other terminology and notation

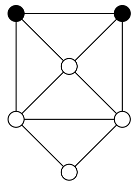
In contrast with the notion of edge basis forced vertex, we say that a vertex $v \in V(G)$ is an *edge void vertex* of G if v does not belong to any edge metric basis of G. An analogous concept (that of *void vertices* or *v-void vertices*) is already known from [9] as those vertices that do not belong to any (classical) metric basis. We may remark here that deciding whether a given vertex is a v-basis forced or a v-void vertex of a graph belongs to the class of decision problems equivalent to an NP-complete problem (as shown in [9]).

In order to simplify our arguments, we also say that an edge e is *resolved (by a set R)* if for each $f \in E(G) \setminus \{e\}$ the edges e and f are resolved by some vertex $u \in R$. Notice that, if a set R is an edge resolving set of G, then it resolves all the edges of G (as defined above).

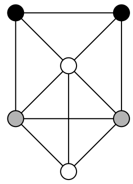
Other terminologies and notations are as follows. If R is a set of vertices, $v \in R$ and $u \notin R$, then we write $R[v \leftarrow u]$ to represent the set $(R \setminus \{v\}) \cup \{u\}$. The *join* of two graphs G and H is the graph $G \vee H$, where $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G), h \in V(H)\}$. The *open neighborhood* of a vertex $v \in V(G)$ is the set of vertices adjacent to v, and is denoted by $N_G(v)$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The subindex G could be removed in the notations above if G is clear from the context. Two vertices $u, v \in V(G)$ are called *false* or *true twins* if $N(u) = N(v)$ or $N[u] = N[v]$, respectively. Moreover, u and v are called *twins* if they are false or true twins. Notice that if u and v are twins and $w \in N(u) \cap N(v)$, then $d(x, uw) = d(x, vw)$ for any $x \in V(G) \setminus \{u, v\}$. Therefore, at least one of the twins u and v has to be chosen in any edge resolving set of G.



(a) Graph with two edge basis forced vertices (in black) and no v-basis forced vertices. The classical and edge metric dimension are both 2.



(b) Graph with two v-basis forced vertices (in black) and no edge basis forced vertices. The classical metric dimension is 2, whereas $\text{edim}(G) = 4$.



(c) Graph with two v-basis forced vertices (in black) and four edge basis forced vertices (two black plus two gray). The classical metric dimension is 2, whereas $\text{edim}(G) = 4$.

Fig. 1. Graphs having v-basis forced vertices and edge basis forced vertices.

1.2. Some preliminary results

The next known results shall be useful throughout our exposition.

Theorem A ([24]). *We have* $\text{edim}(G) = |V(G)| - 1$ *if and only if for any distinct* $v_1, v_2 \in V(G)$ *there exists* $w \in V(G)$ *such that* $v_1w, v_2w \in E(G)$ *and* w *is adjacent to all non-mutual neighbors of* v_1 *and* v_2.

Theorem B ([24]). *If for any vertex* $v \in V(G)$ *there exists another vertex* $w \in V(G)$ *such that* $V(G) \setminus N(v) \subseteq N(w)$*, then* $\text{edim}(G \vee K_1) = |V(G)|$. *Otherwise,* $\text{edim}(G \vee K_1) = |V(G)| - 1$.

Proposition C ([13]). *Let* G *be a connected graph of order* n. *If there is a vertex* $v \in V(G)$ *of degree* $n-1$*, then either* $\text{edim}(G) = n - 1$ *or* $\text{edim}(G) = n - 2$.

A set $S \subseteq V(G)$ is a *total dominating set* of G if every vertex of G has a neighbor in S. The graph class $\mathcal{G}$ contains all graphs G such that for all vertices $v \in V(G)$ there exists an edge $vw \in E(G)$ such that $\{v, w\}$ is a minimum total dominating set of G. This implies also that all $G \in \mathcal{G}$ are connected. With this in mind, we could also define $\mathcal{G}$ as follows: a graph G is in $\mathcal{G}$ if it is connected and for all vertices $v \in V(G)$ there exists an edge $vw \in E(G)$ such that $N[\{v, w\}] = V(G)$.

Theorem D ([18]). *For all nontrivial (not necessarily connected) graphs* G *and* H*, we have*

$$\text{edim}(G \vee H) = \begin{cases} |V(G)| + |V(H)| - 1, & \text{if } G \in \mathcal{G} \text{ or } H \in \mathcal{G}, \\ |V(G)| + |V(H)| - 2, & \text{if } G, H \notin \mathcal{G}. \end{cases}$$

Theorem E ([23]). *Let* G *be a connected graph of order* n. *If* $\overline{G}$ *has at least three components, then* $\text{edim}(G) = n - 1$.

1.3. Structure of the article

Since this article is indeed written in order to be part of a special issue celebrating the research achievements of Dr. Ralf Klasing, and he has an important part of his scientific career addressed to solve computational aspects of some combinatorial problems, we first deal with the computational complexity of the decision problem concerning whether a given vertex is an edge basis forced vertex or an edge void vertex. This is done in Section 2. In Section 3 we establish some tight bounds for the number of edge basis forced vertices of a graph, as well as, for the number of edges in graph having at least one edge basis forced vertex. Next, in Section 4 we present comparisons between the amount of edge basis forced and v-basis forced vertices in graphs. Further on, in Section 5 we show realization results concerning which values of the integers n, k and f allow to confirm the existence of a graph G with n vertices, f edge basis forced vertices and $\text{edim}(G) = k$. We close our exposition with open questions that can be of interest for future investigation.

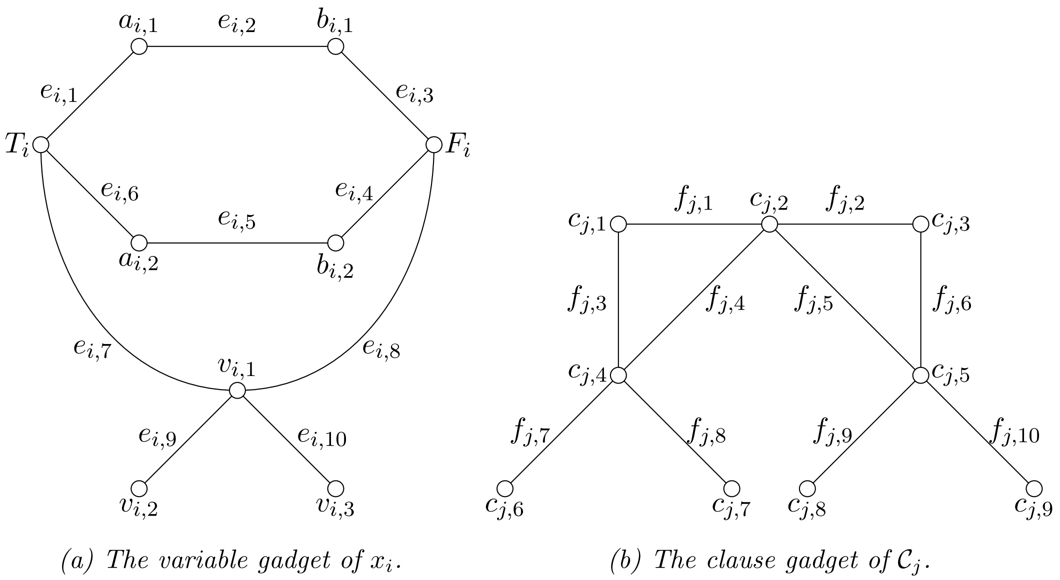


Fig. 2. The variable and clause gadgets of the reduction illustrated.

2. Complexity results

In this section, we consider the algorithmic complexity of determining whether a given vertex belongs to all or to no edge metric basis of a graph. In particular, we show that the first problem is co-NP-hard and the latter problem is NP-hard. The proofs are based on a polynomial-time reduction from the well-known 3-SAT problem. The obtained complexities are analogous to the ones given in [9] for the case of v-basis forced vertices and v-void vertices. Previously, in [13], it has been shown that given an arbitrary graph $G = (V, E)$ and an integer k, the problem of deciding whether $\mathrm{edim}(G) \le k$ is NP-complete. In what follows, we first present an alternative reduction of the 3-SAT problem to the problem of deciding whether $\mathrm{edim}(G) \le k$. Then, based on this new reduction, the above mentioned complexity results are shown by applying an approach inspired by the reductions used in [9] for the v-basis forced and v-void vertices.

For the 3-SAT problem, denote the set of variables by $X = \{x_1, x_2, \dots, x_n\}$ and the set of literals by $U = \{x_1, x_2, \dots, x_n, \overline{x_1}, \overline{x_2}, \dots, \overline{x_n}\}$, where $\overline{x_i}$ denotes the negation of the variable x_i. Let F be an instance of the 3-SAT problem; more precisely, let F be a formula $F = \mathcal{C}_1 \wedge \mathcal{C}_2 \wedge \cdots \wedge \mathcal{C}_m$, where each clause $\mathcal{C}_j$ contains exactly three literals, i.e., each clause is of the form $\mathcal{C}_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$, where $u_{j,1}, u_{j,2}, u_{j,3} \in U$, such that the variables of the literals are distinct. Based on the given formula F, we form a graph $G_F = (V_F, E_F)$ as follows:

- For each variable $x_i \in X$, we construct a variable gadget of x_i with vertices $a_{i,1}$, $a_{i,2}$, $b_{i,1}$, $b_{i,2}$, $v_{i,1}$, $v_{i,2}$, $v_{i,3}$, T_i and F_i and edges $e_{i,k}$ (with $k \in \{1, \dots, 10\}$) as given in Fig. 2(a).
- For each clause $\mathcal{C}_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$, we construct a clause gadget $\mathcal{C}_j$ with vertices $c_{j,1}, c_{j,2}, \dots, c_{j,9}$ and edges $f_{j,k}$ (with $k \in \{1, \dots, 10\}$) as given in Fig. 2(b). Moreover, if $u_{j,k} = x_i$ (where $k \in \{1, 2, 3\}$), then $c_{j,3}$ is adjacent to F_i, otherwise $u_{j,k} = \overline{x_i}$ and $c_{j,3}$ is adjacent to T_i. In addition, $c_{j,3}$ is adjacent to both T_i and F_i for all variables x_i not occurring in $\mathcal{C}_j$ and $c_{j,1}$ is adjacent to both T_i and F_i for all variables x_i (whether occurring in $\mathcal{C}_j$ or not).

For the rest of the section, we assume that the formula F is such that the number of variables is $n > 3$ and each variable occurs in at least one clause. Obviously, this assumption does not restrict the generality of our result since the 3-SAT problem remains NP-complete with this minor restriction. Indeed, the unused variables could just be omitted from the given instance of the 3-SAT problem and, for $n \le 3$ variables, the problem can be solved in constant time.

It is clear that the graph G_F can be constructed in polynomial time. In the following lemma, we first give some preliminary results concerning the reduction.

Lemma 2. *Let F be an instance of the 3-SAT problem with the number of variables $n > 3$ and where each variable occurs in at least one clause. Then the following observations hold:*

(i) *If R is an edge resolving set of G_F, then at least one of the vertices $a_{i,1}$, $a_{i,2}$, $b_{i,1}$ and $b_{i,2}$ as well as one of $v_{i,2}$ and $v_{i,3}$ of each variable gadget belongs to R.*

(ii) *If R is an edge resolving set of G_F, then at least one of the vertices $c_{j,6}$ and $c_{j,7}$ as well as one of $c_{j,8}$ and $c_{j,9}$ of each clause gadget belongs to R.*

(iii) *If R is an edge resolving set of G_F, then $|R| \geq 2n+2m$. In other words, we have* $\mathrm{edim}(G_F) \geq 2n+2m$.

Furthermore, let R be a subset of $V(G_F)$ consisting of the vertices indicated by the conditions (i) and (ii), i.e., R satisfies (i)–(ii) and $|R| = 2n+2m$. Then the following statements hold:

(iv) *For any $r \in R$ not included in a variable gadget corresponding to the literals of the clause $\mathcal{C}_j$, we have $d(f_{j,1}, r) = d(f_{j,2}, r)$.*
(v) *The set R is an edge resolving set of G_F if and only if the pairs $f_{j,1}$ and $f_{j,2}$ of edges are resolved for each clause gadget.*

Proof. For the proofs of the observations (i)–(iii), assume that R is an edge resolving set of G_F.

(i) Suppose to the contrary that $R \cap \{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}\} = \emptyset$ for some variable gadget. This implies that $d(e_{i,2}, r) = d(e_{i,5}, r)$ for all $r \in R$, since the variable gadget is connected to the rest of the graph through the vertices T_i and F_i. Hence, a contradiction follows. Furthermore, $R \cap \{v_{i,2}, v_{i,3}\} \neq \emptyset$, by Lemma 10, since $v_{i,2}$ and $v_{i,3}$ are (false) twins.
(ii) Suppose to the contrary that $R \cap \{c_{j,6}, c_{j,7}\} = \emptyset$ for some $j = 1, \ldots, m$. This immediately leads to a contradiction, since $d(f_{j,7}, u) = d(f_{j,8}, u)$ for all $u \in V(G_F) \setminus \{c_{j,6}, c_{j,7}\}$. Similarly, it can be shown that $R \cap \{c_{j,8}, c_{j,9}\} \neq \emptyset$.
(iii) By (i) and (ii), we immediately obtain $|R| \geq 2n+2m$.

Let R be a subset of $V(G_F)$ such that the conditions (i)–(ii) are satisfied and $|R| = 2n+2m$. Without loss of generality, we may assume that $v_{i,2}$ $c_{j,6}$ and $c_{j,8}$ belong to R (for all i and j). Furthermore, denote the vertex in $R \cap \{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}\}$ by z_i. For the proof of (iv), we make the following observations:

- Clearly, we have $d(f_{j,1}, c_{j,6}) = d(f_{j,1}, c_{j,8}) = 2$ and $d(f_{j,2}, c_{j,6}) = d(f_{j,2}, c_{j,8}) = 2$.
- For any other clause gadget $\mathcal{C}_{j'}$, there exists a variable x_i not appearing in $\mathcal{C}_j$, since the number of variables $n > 3$. Hence, by considering a path passing through the variable gadget of x_i, we obtain that $d(f_{j,1}, c_{j',6}) = d(f_{j,1}, c_{j',8}) = 4$ and $d(f_{j,2}, c_{j',6}) = d(f_{j,2}, c_{j',8}) = 4$.
- For any variable x_i not appearing in the clause $\mathcal{C}_j$, we have $d(f_{j,1} z_i) = d(f_{j,2}, z_i) = 2$ and $d(f_{j,1}, v_{i,2}) = d(f_{j,2}, v_{i,2}) = 3$.

Thus, the statement (iv) follows.

(v) Obviously, if R is an edge resolving set of G_F, then the pairs $f_{j,1}$ and $f_{j,2}$ of edges are resolved for each clause gadget. For the other direction, assume that the pairs $f_{j,1}$ and $f_{j,2}$ of edges are resolved for each clause gadget. In what follows, we show that all the other pairs of edges are also resolved by R:

- Consider first the edges of a clause gadget corresponding to $\mathcal{C}_j$. Observe that the only edges of G_F at distance 1 from $c_{j,6}$ are $f_{j,3}, f_{j,4}$ and $f_{j,8}$. The edge $f_{j,4}$ is resolved from $f_{j,3}$ and $f_{j,8}$ since $d(f_{j,4}, c_{j,8}) = 2$ and $d(f_{j,3}, c_{j,8}) = 3 = d(f_{j,8}, c_{j,8})$. Furthermore, $d(f_{j,8}, v_{i,2}) = 4 = d(f_{j,3}, v_{i,2}) + 1$ for any i. Similarly, it can be shown that the edges $f_{j,5}, f_{j,6}$ and $f_{j,10}$ are resolved (since $c_{j,3}$ is adjacent to T_i and F_i corresponding to each variable not appearing in $\mathcal{C}_j$). Moreover, the edges $f_{j,7}$ and $f_{j,9}$ are resolved due to the fact that $d(f_{j,7}, c_{j,6}) = 0 = d(f_{j,9}, c_{j,8})$. Finally, $f_{j,1}$ and $f_{j,2}$ are resolved from each edge e outside the clause gadget since $d(f_{j,k}, c_{j,6}) = 2 = d(f_{j,k}, c_{j,8})$ (with $k \in \{1, 2\}$) and $d(e, c_{j,6}) \geq 3$ or $d(e, c_{j,8}) \geq 3$.
- Consider then the edges between variable and clause gadgets. Observe that the edges $f_{j,1}, f_{j,2}$ and $f_{j,5}$ as well as the ones from $c_{j,1}$ to a variable gadget are the only edges of the graph at distance 2 from $c_{j,6}$. The edges $c_{j,1}T_i$ and $c_{j,1}F_i$ are resolved from the edges of the clause gadget due to the fact that the distances of $f_{j,1}, f_{j,2}$ and $f_{j,5}$ to $c_{j,8}$ are smaller than $d(c_{j,1}T_i, c_{j,8}) = d(c_{j,1}F_i, c_{j,8}) = 3$. Furthermore, the pair $c_{j,1}T_i$ and $c_{j,1}F_i$ of edges is resolved by z_i as the distances of the edges to z_i are equal to 1 and 2. Finally, $c_{j,1}T_i$ and $c_{j,1}F_i$ are resolved from $c_{j,1}T_{i'}$ and $c_{j,1}F_{i'}$ for any $i' \neq i$ since the distance of $c_{j,1}T_{i'}$ and $c_{j,1}F_{i'}$ to $v_{i,2}$ is greater than 2. Similarly, it can be shown that the edges $c_{j,3}T_i$ and $c_{j,3}F_i$ (if they exist) are resolved from any other edge of the graph and if both $c_{j,3}T_i$ and $c_{j,3}F_i$ exist, then they are mutually resolved from each other.
- By the previous observations, the edges in the variable gadgets are resolved from all the edges in the clause gadgets as well as from the ones between the clause and variable gadgets. In order to show that the edges in the variable gadget corresponding to x_i are resolved from all the edges in the variable gadgets, we first assume that $z_i = a_{i,1}$. The distances of the edges $e_{i,k}$ to $a_{i,1}$ and $v_{i,2}$ are listed in Table 1. Observe that $d(e_{i,k}, a_{i,1}) \leq 2$ for each $k \in \{1, \ldots, 10\}$. This immediately implies that the edges corresponding to the variable x_i are resolved from the ones in other variable gadgets since their distance to $a_{i,1}$ is greater than two. By Table 1, the only pairs of edges requiring further investigation with respect to them being resolved are $e_{i,3}$ and $e_{i,6}$ as well as $e_{i,8}$ and $e_{i,10}$. Obviously, the edges $e_{i,8}$ and $e_{i,10}$ are resolved since $d(e_{i,8}, c_{1,6}) = 3 \neq 4 = d(e_{i,10}, c_{1,6})$. For the pair $e_{i,3}$ and $e_{i,6}$, recall that each variable occurs in some clause, say in $\mathcal{C}_j$. Therefore, if x_i appears in the clause $\mathcal{C}_j$, then $d(e_{i,3}, c_{j,8}) = 3 \neq 4 = d(e_{i,6}, c_{j,8})$, otherwise $\overline{x_i}$ occurs in $\mathcal{C}_j$ and $d(e_{i,6}, c_{j,8}) = 3 \neq 4 = d(e_{i,3}, c_{j,8})$. The cases with z_i being equal to $a_{i,2}$, $b_{i,1}$ or $b_{i,2}$ can be handled analogously.

Thus, in conclusion, we have shown that R is an edge resolving set of G_F. □

In the following theorem, we show that the satisfiability of a formula F can be determined based on the edge metric dimension of G_F.

Theorem 3. *Let F be an instance of the 3-SAT problem with the number of variables $n > 3$ and such that each variable occurs in at least one clause. Then the formula F is satisfiable if and only if $\mathrm{edim}(G_F) = 2n+2m$, where n is the number of variables and m the number of clauses of F.*

Table 1
The distances of $e_{i,k}$ to $a_{i,1}$ and $v_{i,2}$.

u	$a_{i,1}$	$v_{i,2}$
$d(e_{i,1}, u)$	0	2
$d(e_{i,2}, u)$	0	3
$d(e_{i,3}, u)$	1	2
$d(e_{i,4}, u)$	2	2
$d(e_{i,5}, u)$	2	3
$d(e_{i,6}, u)$	1	2
$d(e_{i,7}, u)$	1	1
$d(e_{i,8}, u)$	2	1
$d(e_{i,9}, u)$	2	0
$d(e_{i,10}, u)$	2	1

Proof (⇒)**.** Assume first that F is satisfiable and that A is a satisfying truth assignment of F. Construct then a set $R \subseteq V_F$ as follows: R consists of all the vertices $v_{i,2}$ ($i \in \{1, \dots, n\}$) as well as $c_{j,6}$ and $c_{j,8}$ ($j \in \{1, \dots, m\}$), and if the assignment of x_i is *true*, then $a_{i,1}$ belongs to R, otherwise x_i = *false* and then $b_{i,1}$ is in R. Notice that the cardinality of R is equal to $2n + 2m$. Since R satisfies the conditions (i) and (ii) of Lemma 2, it is an edge resolving set of G_F if and only if the pairs $f_{j,1}$ and $f_{j,2}$ of edges are resolved for each clause gadget. Recalling A is a satisfiable truth assignment of F, the value of C_j is *true* and, hence, the value of at least one of the literals $u_{j,1}$, $u_{j,2}$ and $u_{j,3}$, say $u_{j,1}$, is *true*. Therefore, if $u_{j,1} = x_i$, then $a_{i,1} \in R$ and $d(f_{j,2}, a_{i,1}) = 3 \neq 2 = d(f_{j,1}, a_{i,1})$, otherwise $u_{j,1} = \overline{x_i}$, $b_{i,1} \in R$ and $d(f_{j,2}, b_{i,1}) = 3 \neq 2 = d(f_{j,1}, b_{i,1})$. Hence, the edges $f_{j,1}$ and $f_{j,2}$ are resolved by $a_{i,1}$ or $b_{i,1}$. Thus, in conclusion, R is an edge resolving set of G_F and $\mathrm{edim}(G_F) = 2n + 2m$ by Lemma 2(iii).

(⇐) Assume then that $\mathrm{edim}(G_F) = 2n + 2m$ and that R is an edge resolving set of G_F with $2n + 2m$ vertices. By Lemma 2(i)–(iii), the subset $\{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}\}$ contains exactly one vertex of R. Form then a truth assignment A of F as follows: if $a_{i,1}$ or $a_{i,2}$ belongs to R, then set x_i = *true*, otherwise $b_{i,1}$ or $b_{i,2}$ belongs to R and set x_i = *false*. Since R is an edge resolving set of G_F, it also resolves all the pairs $f_{j,1}$ and $f_{j,2}$ of edges. Hence, by Lemma 2(iv), there exist a variable $x_{i'}$ in the clause C_j and a vertex $r \in R \cap \{a_{i',1}, a_{i',2}, b_{i',1}, b_{i',2}\}$ such that $d(f_{j,1}, r) = 2 \neq 3 = d(f_{j,2}, r)$. This further implies that C_j has value *true* under A. Thus, A is a satisfiable truth assignment of F. □

The following result is an immediate corollary of the previous theorem.

Corollary 4. *If k is a positive integer, then deciding whether $\mathrm{edim}(G) \le k$ for a given graph G is an NP-complete problem.*

Proof. Clearly, the problem of deciding whether $\mathrm{edim}(G) \le k$ for G belongs to NP. Furthermore, it is NP-complete due to Theorem 3. □

Based on the previous reduction, we show in the following theorem that determining whether a given vertex is an edge basis forced or a void vertex is algorithmically difficult.

Theorem 5. *Let G be a graph and u be a vertex of G.*

(i) *Deciding whether u is an edge basis forced vertex of G is a co-NP-hard problem.*
(ii) *Deciding whether u is an edge void vertex of G is an NP-hard problem.*

Proof. In order to prove the first claim (i), we show that the problem of deciding whether a given 3-SAT formula is *not satisfiable* – a co-NP-complete problem – can be reduced in polynomial time to the problem of determining if a given vertex is an edge basis forced one of a graph. For the second claim (ii), we similarly prove that the problem of deciding whether a given 3-SAT formula is *satisfiable* – an NP-complete problem – can be reduced in polynomial time to the problem of determining if a given vertex is an edge basis void one of a graph.

Let F be an instance of the 3-SAT problem with the number of variables $n > 3$ and such that each variable occurs in at least one clause. Based on the given formula F, we form a graph $G' = (V', E')$ as follows:

- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint copies of the graph G_F constructed in the previous reduction. Denote the vertices of the graphs by $a^k_{i,1}$, $a^k_{i,2}$, $b^k_{i,1}$, $b^k_{i,2}$, $v^k_{i,1}$, $v^k_{i,2}$, $v^k_{i,3}$, T^k_i, F^k_i and $c^k_{j,1}, \dots, c^k_{j,10}$ as well as the edges by $e^k_{i,1}, \dots, e^k_{i,10}$ and $f^k_{j,1}, \dots, f^k_{j,10}$, where $k \in \{1, 2\}$.
- Furthermore, an edge is added from each $c^1_{j,1}$ and $c^1_{j,3}$ to T^2_i and F^2_i for all $i \in \{1, \dots, n\}$. Analogously, an edge is added from each $c^2_{j,1}$ and $c^2_{j,3}$ to T^1_i and F^1_i for all $i \in \{1, \dots, n\}$.
- Finally, add a vertex w such that it is adjacent to $c^k_{j,3}$ for all $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$.

It is immediate that the graph $G' = (V', E') = (V_1 \cup V_2 \cup \{w\}, E')$ can be constructed in polynomial time. Before diving into the proofs of (i) and (ii), we need to discuss some preliminary results. As in Lemma 2, it can be shown that if R is an edge resolving set of G', then (a) $R \cap \{a^k_{i,1}, a^k_{i,2}, b^k_{i,1}, b^k_{i,2}\} \neq \emptyset$, (b) $R \cap \{v^k_{i,2}, v^k_{i,3}\} \neq \emptyset$ (c) $R \cap \{c^k_{j,6}, c^k_{j,7}\} \neq \emptyset$ and (d)

$R \cap \{c^k_{j,8}, c^k_{j,8}\} \neq \emptyset$ for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and $k \in \{1, 2\}$. Hence, for an edge resolving set R of G', we have $|R| \geq 4n + 4m$, i.e., $\mathrm{edim}(G') \geq 4n + 4m$. In what follows, we show (based on similar ideas as in the proofs of Lemma 2 and Theorem 3) that F is a satisfiable formula if and only if $\mathrm{edim}(G') = 4n + 4m$.

($\Rightarrow$) Let us first show that if the formula F is satisfiable, then $\mathrm{edim}(G') = 4n+4m$. Let A be a satisfiable truth assignment of F. Construct a set R as follows: R consists of all the vertices $v^k_{i,2}$, $c^k_{j,6}$ and $c^k_{j,8}$, where $k \in \{1, 2\}$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, and if the assignment of x_i is *true* in A, then $a^1_{i,1}$ and $a^2_{i,1}$ belong to R, otherwise $b^1_{i,1}$ and $b^2_{i,1}$ are in R. We immediately notice that R contains exactly $4n + 4m$ vertices. In what follows, we show that R is an edge resolving set of G'. For this purpose, we first observe (based on similar reasoning as in Lemma 2) that all the pairs of distinct edges of G' except $f^k_{j,1}$ and $f^k_{j,2}$ are resolved by some element of R:

- As in the proof of Lemma 2, it can be deduced that the pairs of edges involving an edge from a clause gadget can be resolved by R unless the considered pair is $f^k_{j,1}$ and $f^k_{j,2}$. In particular, the edge $c^k_{j,3}w$ is resolved from the edges of the clause gadget by $c^k_{j,6}$ or $c^k_{j,8}$.
- Similar to the proof of Lemma 2, each pair of edges involving an edge between a clause and variable gadget can be resolved by R. In particular, the edge $c^k_{j,3}w$ is resolved from the previous edges by any $v^k_{i,2} \in R$.
- Analogous to the proof of Lemma 2, it can be shown that an edge of a variable gadget is resolved from any other edge of G'.
- Finally, considering the edges $c^k_{j,3}w$, we have $d(c^k_{j,3}w, c^k_{j,8}) = 2$ and $d(c^k_{j,3}w, c^{k'}_{j',8}) = 3$ for all $(k', j') \neq (k, j)$. Therefore, for $(k, j) \neq (k', j')$, the pairs $c^k_{j,3}w$ and $c^{k'}_{j',3}w$ are resolved by $c^k_{j,8}$ (or $c^{k'}_{j',8}$). Furthermore, by the previous cases, the edge $c^k_{j,3}w$ cannot be mixed with other edges of the graph.

Thus, it is enough to consider the pairs $f^k_{j,1}$ and $f^k_{j,2}$ of edges. It is immediate that the distance of $f^k_{j,1}$ to any vertex of $R \cap \{a^k_{i,1}, a^k_{i,2}, b^k_{i,1}, b^k_{i,2}\}$ in the variable gadgets corresponding to the literals $u_{j,1}$, $u_{j,2}$ and $u_{j,3}$ is equal to 2. However, by the fact that A is a satisfying truth assignment of F and by the construction of R, the distance of $f^k_{j,2}$ to some vertex of $R \cap \{a^k_{i,1}, a^k_{i,2}, b^k_{i,1}, b^k_{i,2}\}$ in the variable gadgets corresponding to $u_{j,1}$, $u_{j,2}$ and $u_{j,3}$ is equal to 3. Thus, in conclusion, R is an edge resolving set of G' and $\dim(G') = 4n + 4m$.

($\Leftarrow$) Let us then show that if the edge metric dimension of G' is equal to $4n + 4m$, then the formula F is satisfiable. Let R be an edge resolving set of G' with $4n + 4m$ vertices. By the observations (a)–(d), we know that for each $k \in \{1, 2\}$ and $i \in \{1, \dots, n\}$ exactly one of the vertices $a^k_{i,1}$, $a^k_{i,2}$, $b^k_{i,1}$ and $b^k_{i,2}$ as well as exactly one of $v^k_{i,2}$ and $v^k_{i,3}$ belongs to R and for each $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$ exactly one of the vertices $c^k_{j,6}$ and $c^k_{j,7}$ as well as exactly one of $c^k_{j,8}$ and $c^k_{j,9}$ belongs to R. Thus, as $|R| = 4n + 4m$, we may deduce that $w \notin R$. Form a truth assignment A of F as follows: if $a^1_{i,1}$ or $a^1_{i,2}$ belongs to R, then set the variable x_i to be *true*, otherwise $b^1_{i,1}$ or $b^1_{i,2}$ belongs to R and set x_i to be *false*. In what follows, we show that the truth assignment A satisfies the formula F. Suppose to the contrary that a clause $\mathcal{C}_j$ is not satisfied by A. This implies that the distance of $f^1_{j,1}$ and $f^1_{j,2}$ to any vertex of $R \cap \{a^k_{i,1}, a^k_{i,2}, b^k_{i,1}, b^k_{i,2}\}$ in the variable gadgets corresponding to $u_{j,1}$, $u_{j,2}$ and $u_{j,3}$ is equal to 2. Furthermore, it is straightforward to verify that the distances of $f^1_{j,1}$ and $f^1_{j,2}$ to any other vertex of R in the variable gadgets are also equal and that their distances are equal to 4 to any other vertices of R (in the clause gadgets). Thus, they are not resolved by any element of R (a contradiction). Thus, the truth assignment A satisfies the formula F.

Let us next show that $\mathrm{edim}(G') \leq 4n + 4\ m + 1$ regardless of the existence of a satisfiable truth assignment for the formula F. For this purpose, let R be a set consisting of w and all the vertices $a^k_{i,1}$, $v^k_{i,2}$, $c^k_{j,6}$ and $c^k_{j,8}$, where $k \in \{1, 2\}$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Clearly, the cardinality of R is equal to $4n + 4m + 1$. As above, it can be shown that a pair of distinct edges is resolved even without taking into account the vertex w unless the pair is $f^k_{j,1}$ and $f^k_{j,2}$. However, it is immediate that this pair is resolved since $d(f^k_{j,1}, w) = 2 \neq 1 = d(f^k_{j,2}, w)$. Therefore, the set R is an edge resolving set of G' and $\mathrm{edim}(G') \leq 4n + 4m + 1$.

(i) Now we are ready to prove that w is an edge basis forced vertex of G' if and only if the formula F is not satisfiable. Observe first that if w is an edge basis forced vertex of G', then by the observations (a)–(d), we obtain that $\mathrm{edim}(G') \geq 4n+4m+1$. Therefore, we have $\mathrm{edim}(G') = 4n+4\ m+1$ and, as shown above, F cannot be satisfiable. For the other direction, assume that F is not satisfiable. Hence, we have $\mathrm{edim}(G') \neq 4n + 4m$ implying $\mathrm{edim}(G') = 4n + 4m + 1$. Suppose to the contrary that w is not an edge basis forced vertex of G' and there exists a metric basis R of G' with $4n + 4m + 1$ vertices such that $w \notin R$. By the observations (a)–(d), we obtain that either G_1 or G_2 contains exactly $2n + 2m$ vertices of R; without loss of generality, we may assume that G_1 is does. Notice that $f^1_{j,1}$ and $f^1_{j,2}$ are resolved for all $j \in \{1, \dots, m\}$ since R is a metric basis of G'. However, for any $u \in V(G_2)$, we have $d(f_{j,1}, u) = d(f_{j,2}, u)$. Therefore, as above, it can be shown that F is satisfiable (a contradiction). Therefore, w is an edge basis forced vertex of G'. Thus, there exists a polynomial-time reduction of the complement of the 3-SAT problem to the problem of deciding whether a given vertex is an edge basis forced vertex. Hence, the studied problem is co-NP-hard.

(ii) Let us then show that w is an edge void vertex of G' if and only if the formula F is satisfiable. Observe first that if F is satisfiable, then $\mathrm{edim}(G') = 4n+4m$. Hence, if R is any metric basis of G' (with cardinality $4n+4m$), then w does not belong to R by the observations (a)–(d). Therefore, w is an edge void vertex of G'. For the other direction, assume that w is an edge void vertex of G'. Recall that F is satisfiable if and only if $\mathrm{edim}(G') = 4n + 4m$. Suppose to the contrary that F

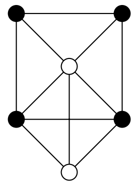


Fig. 3. The graph G_6 with 6 vertices that attains the bound in Theorem 7.

is not satisfiable, i.e., $\text{edim}(G') \neq 4n + 4m$. This implies that $\text{edim}(G') = 4n + 4m + 1$. Let R be a metric basis of G' with cardinality $4n + 4m + 1$ and $w \notin R$. As above, we obtain that G_1 or G_2 contains exactly $2n + 2m$ vertices of R; without loss of generality, we may assume that G_1 is such a graph. Analogous to the case (i), we can show that F is satisfiable based on the fact that $w \notin R$ (a contradiction). Hence, if w is an edge void vertex, then F is satisfiable. Thus, there exists a polynomial-time reduction of the 3-SAT problem to the problem of deciding whether a given vertex is an edge void vertex. Hence, the studied problem is NP-hard. □

3. Some related bounds

Based on the NP-hardness results from Section 2, one notices that bounding the number of edge basis forced vertices of a given graph clearly deserves to be considered. To this end, we first recall the following fact. If $v \in V(G)$ is any vertex, then the set $V(G) \setminus \{v\}$ is both a resolving set and an edge resolving set of a connected graph G. This observation yields us the following upper bound on the number of edge basis forced vertices of a connected graph.

Proposition 6. *If G is a connected graph of order n, then G can have at most $n - 2$ edge basis forced vertices. Moreover, if G contains an edge basis forced vertex, then* $\text{edim}(G) \leq n - 2$.

Proof. Let $v \in V(G)$ be an edge basis forced vertex of G. Suppose that $\text{edim}(G) = n - 1$. Now $V(G) \setminus \{v\}$ is an edge metric basis, a contradiction. Thus, $\text{edim}(G) \leq n - 2$ when G contains an edge basis forced vertex. Consequently, the graph G contains at most $n - 2$ edge basis forced vertices. □

We will later (in Section 5) show that the bounds of Proposition 6 are sharp for connected graphs (of sufficiently large order). In [9] (see also [1]), it was shown that a connected graph can have at most $\lfloor (n-1)/2 \rfloor$ v-basis forced vertices, and this result was further improved to $\lfloor 2(n-1)/5 \rfloor$ in [8]. Hence, we may observe that the upper bounds on the number of v-basis forced vertices *differ significantly* from upper bounds on the number of edge basis forced vertices. The bound $\lfloor 2(n-1)/5 \rfloor$ is tight as a construction attaining it has been presented in [1,9].

On the other hand, we can also bound the number of edges in a graph G when there are edge basis forced vertices in such graph. Similar contributions for the v-basis forced vertices were previously given in [9]. Namely, it was shown that if a connected graph G has f v-basis forced vertices, then $|E(G)| \leq \frac{n(n-1)}{2} - 2f$, i.e. the maximum number of edges depends on the number of v-basis forced vertices. In contrast, the analogous fact for edge basis forced vertices does not hold. Indeed, we will prove an upper bound for the number of edges when edge basis forced vertices are present that is independent of the number of edge basis forced vertices. The upper bound we establish is sharp, which we will see from a construction in Section 5.

Theorem 7. *Let G be a connected graph with $n \geq 4$ vertices and at least one edge basis forced vertex. Then*

$$|E(G)| \leq \frac{n(n-1)}{2} - n + 2.$$

Proof. Suppose to the contrary that $|E(G)| > \frac{n(n-1)}{2} - n + 2$. Hence, the complement graph $\overline{G}$ has at most $n - 3$ edges, and thus at least three connected components. Now, according to Theorem E, we have $\text{edim}(G) = n - 1$ and G has no edge basis forced vertices due to Proposition 6, a contradiction. □

One example of a graph achieving the bound above is shown in Fig. 3. Notice that the graph G_6 has order 6, maximum degree 5 and 11 edges. Thus, by Proposition C, we know that either $\text{edim}(G_6) = 5$ or $\text{edim}(G_6) = 4$. Also, it is easy to verify that the four black vertices form an edge resolving set, and so $\text{edim}(G_6) = 4$. In addition, if we search for any other edge metric basis (of cardinality 4), we conclude that the only possibility is the one of the black vertices. Thus, such an edge metric basis is unique. Notice that, on the other hand, this graph is also an example that shows the tightness of the bound from Proposition 6.

We might recall that the graph given in Fig. 3 was previously used in [9], as a graph having v-basis forced vertices, and as the *only* graph that attains a similar bound (as the one of Theorem 7) for the number of edges in a graph having v-basis forced vertices (see [9, Theorem 23]). A natural question would be then if this is also the only graph achieving the bound of Theorem 7. However, this is not the case, as we shall further prove in Section 5, which shows a contrast between v-basis forced vertices and edge basis forced vertices. The next section is focused on such contrasts.

4. Edge basis forced vertices versus v-basis forced vertices

Based on the fact that the largest possible number of edge basis forced vertices in a graph of order n is $n-2$, and the largest possible number of v-basis forced vertices is $\lfloor 2(n-1)/5 \rfloor$ (as mentioned in the previous section) one might think that the latter value is always smaller than the former one. We next show that this is not the case. Namely, we show the existence of graphs having much more edge basis forced vertices than v-basis forced vertices, as well as, the opposite situation. To this end, we need the following constructions.

If $k \geq 3$, then let G_k be a graph of order $k+2^k$ with $V(G_k) = A \cup B$, where A and B are pairwise disjoint sets with $|A| = k$ and $|B| = 2^k$. Assume $A = \{1, 2, \dots, k\}$ and that $B = \{b_1, b_2, \dots, b_{2^k}\}$ is the collection of all possible subsets of A. The edge set of G_k is as follows. The sets A and B induce cliques in G_k. Also, a vertex $i \in A$ is adjacent to $b_j \in B$ if the integer i belongs to the set b_j. We observe that the graph G_k has order $k + 2^k$ and diameter two, since there is set in B which is equal to A, and clearly, such a vertex is adjacent to every vertex of G_k. From now on, we assume that such vertex is b_1. Moreover, there is a unique vertex in B which is not adjacent to any vertex of A, that is, the set $\emptyset \subset A$. From now on we assume that such vertex is b_{2^k}. A very similar construction has been already described in [4]. Now, again for $k \geq 3$, let G_k^- obtained from G_k by removing the vertex $b_{2^k} = \emptyset \in B$, and all the edges incident with it. Notice that also such graph has diameter two and order $k + 2^k - 1$.

We next describe in Proposition 8 some properties of these two graphs G_k and G_k^-. The proofs of the properties are similar to the ones presented in [4] and therefore we omit them here.

Proposition 8. *If $k \geq 3$ is an integer, then*

(i) $\dim(G_k) = \dim(G_k^-) = k$, *and*
(ii) *the set A is the unique metric basis in both graphs G_k and G_k^-.*

We next consider the edge metric dimension of these two graphs defined above.

Proposition 9. *If $k \geq 3$ is an integer, then*

(i) $\mathrm{edim}(G_k) = |V(G_k)| - 2 = k + 2^k - 2$,
(ii) $\mathrm{edim}(G_k^-) = |V(G_k^-)| - 1 = k + 2^k - 2$,
(iii) *the set $V(G_k) \setminus \{b_1, b_{2^k}\}$ is the unique edge metric basis of G_k, and*
(iv) *there is no edge basis forced vertex in G_k^-.*

Proof. Since both G_k and G_k^- have a vertex of degree equal to the order of the graph minus one, by Proposition C, we have that $|V(G_k)|-2 \leq \mathrm{edim}(G_k) \leq |V(G_k)|-1$ and that $|V(G_k^-)|-2 \leq \mathrm{edim}(G_k^-) \leq |V(G_k^-)|-1$. We consider first the graph G_k. We claim that the set $S = V(G_k) \setminus \{b_1, b_{2^k}\}$ is an edge resolving set of G_k. Let e, f be two edges from G_k. If the end-vertices of e, f are pairwise different, then at least two vertices from them are in S, and clearly both of them resolve the edges e, f. Hence, we may assume w.l.o.g. that $e = xy$ and $f = xz$ where $y \neq z$. If y or z belongs to S, say $y \in S$, then y resolves e, f. It remains to consider the case when $y, z \notin S$, and so, this means w.l.o.g. that $y = b_1$ and $z = b_{2^k}$. Since $b_{2^k} = z$ has no neighbors in A, we must have that $x = b_j$ for some $b_j \in B$. Since $b_j \neq b_1$, there must be at least one $x' \in A$ that is not a neighbor of x. Since $d(x', e) = 1$ and $d(x', f) = 2$, it holds that x' resolves e, f. Therefore, S is an edge resolving set, and so $\mathrm{edim}(G_k) \leq |V(G_k)| - 2$, which leads to the equality in (i).

Consider now the graph G_k^-. Suppose $\mathrm{edim}(G_k^-) = |V(G_k^-)| - 2$ and let S' be an edge metric basis. Assume x, y are the two vertices outside S'. First note that if $b_1 \in S'$, then the two edges xb_1, yb_1 are not resolved by any vertex of S', which is not possible. Hence, $b_1 \notin S'$ and w.l.o.g. let $x = b_1$. We have now two situations.

- $y \in A$. Let $y' \in B$ be a neighbor of y such that $y' \neq x$. Hence, for the edges $e = y'y$ and $f = y'x$ holds that $d(y', e) = 0 = d(y', f)$ and $d(w, e) = 1 = d(w, f)$ for every $w \in S' \setminus \{y'\}$. Thus, e and f are not resolved by S', which is not possible.
- $y \in B$. Let $y'' \in A$ be a neighbor of y. Now, for the edges $e = y''y$ and $f = y''x$ holds that $d(y'', e) = 0 = d(y'', f)$ and $d(w, e) = 1 = d(w, f)$ for every $w \in S' \setminus \{y''\}$. Thus, e and f are not resolved by S', which is also not possible.

Consequently, we must have $\mathrm{edim}(G_k^-) \geq |V(G_k^-)| - 1$, and therefore the desired equality of item (ii) follows.

Similar arguments to the ones used in the proof of item (ii) show that if S is an edge metric basis (of cardinality $|V(G_k)|-2 = k+2^k-2$) of G_k, then the only possibility for the vertices not in S are b_1 and b_{2^k}. Thus, clearly $V(G_k) \setminus \{b_1, b_{2^k}\}$ is the unique metric basis of G_k. Finally, (iv) follows from item (ii) and Proposition 6. $\square$

As a consequence of the result above, we observe that the graph G_k has a larger number of edge basis forced vertices than of v-basis forced vertices, while the graph G_k^- has a larger number of v-basis forced vertices than of edge basis forced vertices, which makes that the number of such types of vertices is in general not comparable. This situation might not be exactly surprising based on the fact that indeed the metric dimension and edge metric dimension are not comparable, see for instance [14], where infinite families of graphs G satisfying either $\dim(G) > \mathrm{edim}(G)$ or $\mathrm{edim}(G) > \dim(G)$ are given.

5. Realization results for edge basis forced vertices

Let n, k and f be nonnegative integers. We are herein focused on the following question: For which values of n, k and f does there exist a graph G with n vertices, f edge basis forced vertices and $\mathrm{edim}(G) = k$? We must clearly have $f \le k \le n$ and $k \ge 1$. Since every set $V(G) \setminus \{v\}$ is an edge resolving set, we have $k \le n-1$. If $f = 0$ and n is fixed, then consider a graph $G_{n-k-1,k+1}$ (of order n) where $k+1$ leaves are added to one end of a path of order $n-k-1$. The fact that (as twins) at least k of the leaves have to be chosen in any edge resolving set rather straightforwardly leads to the observation that $\mathrm{edim}(G_{n-k-1,k+1}) = k$ and that the graph has no edge basis forced vertices (see also [13, Remark 3]). Moreover, the complete graph K_n works as an example in the case with $f = 0$ and $k = n-1$. Thus, let us focus on the case where $f \ge 1$.

Recall that according to Proposition 6, we have $1 \le f \le k \le n-2$ when the graph has at least one edge basis forced vertex. The case where $k = 1$ and $f = 1$ is not possible. Indeed, the path P_n is the only graph with $\mathrm{edim}(G) = 1$ [13], but either vertex at the end of the path always gives an edge metric basis. Hence, there exist no graphs with $k = 1$ and $f = 1$. Thus, the only possible combinations of n, f and k are such that $2 \le k \le n-2$ and $1 \le f \le k$. We will show that all such combinations are realizable (for sufficiently large n).

Recall that according to Theorem A we have $\mathrm{edim}(G) = n-1$ if and only if for any distinct $v_1, v_2 \in V(G)$ there exists $w \in V(G)$ such that $v_1w, v_2w \in E(G)$ and w is adjacent to all non-mutual neighbors of v_1 and v_2. If the graph we consider contains edge basis forced vertices, then $\mathrm{edim}(G) \le n-2$. By Theorem A, the graph G contains a pair of distinct vertices v_1 and v_2 such that no mutual neighbor of v_1 and v_2 is adjacent to all of their non-mutual neighbors. In the following lemma, we will prove that such vertex pairs give us all edge resolving sets of the form $V(G) \setminus \{v_1, v_2\}$.

Lemma 10. *Let $v_1, v_2 \in V(G)$. The set $V(G) \setminus \{v_1, v_2\}$ is an edge resolving set of G if and only if there does not exist $w \in N(v_1) \cap N(v_2)$ such that w is adjacent to all non-mutual neighbors of v_1 and v_2.*

Proof. Assume that there does not exist $w \in N(v_1) \cap N(v_2)$ such that w is adjacent to all non-mutual neighbors of v_1 and v_2. We will show that the set $R = V(G) \setminus \{v_1, v_2\}$ is an edge resolving set of G. The edges with both endpoints in R are clearly resolved. There is at most one edge with neither endpoint in R, and that edge is also clearly resolved from the others. The only pairs we need to check are of the form uv_1, uv_2 where $u \in R$. Notice that now $u \in N(v_1) \cap N(v_2)$. Since u is not adjacent to all non-mutual neighbors of v_1 and v_2, there exists a vertex x that is a neighbor of either v_1 or v_2 but not both. If $x \in N(v_1)$, then $d(x, uv_1) = 1$ and $d(x, uv_2) = 2$, and the edges uv_1 and uv_2 are resolved by x. Similarly, if $x \in N(v_2)$, the edges are resolved by x. Therefore, R is an edge resolving set of G.

Assume then that there exists a vertex $w \in N(v_1) \cap N(v_2)$ such that w is adjacent to all non-mutual neighbors of v_1 and v_2. We will show that now the set $R = V(G) \setminus \{v_1, v_2\}$ is not an edge resolving set of G. To that end, consider the edges wv_1 and wv_2. Except for w, all mutual neighbors of v_1 and v_2 are at distance 1 from both wv_1 and wv_2. Moreover, we have $d(x, wv_1) = d(x, w) = d(x, wv_2)$ for all $x \in N[w] \cap R$ since $v_1, v_2 \notin R$. Because all non-mutual neighbors of v_1 and v_2 are neighbors of w, they cannot resolve wv_1 and wv_2 either. Thus, no vertex in $N(v_1) \cup N(v_2) \cup N[w]$ resolves the pair wv_1, wv_2. Consequently, no vertex in $R \setminus (N(v_1) \cup N(v_2) \cup N[w])$ resolves that pair (as the shortest path from wv_i to such vertices passes through $N(v_1) \cup N(v_2) \cup N[w]$), and thus the set R is not an edge resolving set of G. □

Recall that at least one of the (false or true) twins v_1 and v_2 belongs to each edge resolving set of G. As immediate corollary of the previous lemma, we obtain the following extension of this idea which gives a characterization for the pairs of vertices of which at least one has to be chosen in each edge resolving set.

Corollary 11. *Let $v_1, v_2 \in V(G)$. At least one of the vertices v_1 and v_2 belongs to each edge resolving set if and only if there exists $w \in N(v_1) \cap N(v_2)$ such that w is adjacent to all non-mutual neighbors of v_1 and v_2.*

5.1. Leaf and path additions

Before delving into the constructions, we prove two useful lemmas regarding how adding a leaf to a graph affects the edge metric dimension.

Lemma 12. *Let $v \in V(G)$ be such that v is in some edge metric basis of G. Let G' be the graph we obtain by adding a leaf u to v.*

(i) *We have* $\mathrm{edim}(G') = \mathrm{edim}(G)$*.*
(ii) *Furthermore, R' is an edge metric basis of G' if and only if R' is an edge metric basis of G that resolves uv from all other edges or $R' = R[v \leftarrow u]$, where R is an edge metric basis of G such that $v \in R$.*
(iii) *In particular, if w is an edge basis forced vertex of G such that $w \neq v$, then w remains edge basis forced in G'.*

Proof. Let $R \subseteq V(G)$ be an edge metric basis of G such that $v \in R$ (there is such a set by the assumption). We will show that the set $R[v \leftarrow u]$ is an edge resolving set of G'. Notice that $d_{G'}(u, w) = d_{G'}(v, w) + 1$ for all $w \in V(G)$. Therefore, if $d_G(v, e) \neq d_G(v, f)$ for some $e, f \in E(G)$, then

$$d_{G'}(u, e) = d_G(v, e) + 1 \neq d_G(v, f) + 1 = d_{G'}(u, f).$$

Thus, all pairs of edges in $E(G') \setminus \{uv\}$ are resolved by $R[v \leftarrow u]$. Since u clearly resolves all pairs uv and e where $e \in E(G') \setminus \{uv\}$, the set $R[v \leftarrow u]$ is an edge resolving set of G'. Consequently, $\mathrm{edim}(G') \leq \mathrm{edim}(G)$.

Let $R \subseteq V(G')$ be an edge metric basis of G'. If $u \notin R$, then $R \subseteq V(G)$ and R is clearly an edge resolving set of G. Assume that $u \in R$. We will show that the set $R[u \leftarrow v]$ is an edge resolving set of G. If $d_{G'}(u, e) \neq d_{G'}(u, f)$ for some $e, f \in E(G)$, then

$$d_G(v, e) = d_{G'}(u, e) - 1 \neq d_{G'}(u, f) - 1 = d_G(v, f),$$

and the set $R[u \leftarrow v]$ is indeed an edge resolving set of G. Consequently, $\mathrm{edim}(G) \leq \mathrm{edim}(G')$, and thus $\mathrm{edim}(G') = \mathrm{edim}(G)$.

($\Rightarrow$) In order to show the claimed equivalence in (ii), we first assume that R' is an edge metric basis of G'. The proof now divides into the following cases based on whether u belongs to R' or not:

- If $u \notin R'$, then it is immediate $R' \subseteq V(G)$ is an edge resolving set of G such that uv is resolved from all other edges. Furthermore, as $\mathrm{edim}(G') = \mathrm{edim}(G)$, R' is an edge metric basis of G.
- If $u \in R'$, then $R = R'[u \leftarrow v]$ is an edge resolving set of G by the second paragraph of the proof and, consequently, R is an edge metric basis of G as $\mathrm{edim}(G') = \mathrm{edim}(G)$. Thus, $R' = R[v \leftarrow u]$, where R is an edge metric basis of G such that $v \in R$.

($\Leftarrow$) For the other direction, if R' is an edge metric basis of G that resolves uv from all other edges, then it is immediate that R' is an edge metric basis of G'. Hence, we may assume that $R' = R[v \leftarrow u]$, where R is an edge metric basis of G such that $v \in R$. By the first paragraph of the proof, R' is an edge resolving set of G' and, furthermore, an edge metric basis as $\mathrm{edim}(G') = \mathrm{edim}(G)$.

In order to prove (iii), let w be an edge basis forced vertex of G such that $w \neq v$. Suppose to the contrary that there exists an edge metric basis of G' not including w. Based on the equivalence of (ii), we divide into the following cases:

- If $R' \subseteq V(G)$, then R' is an edge metric basis of G such that $w \notin R'$ (a contradiction).
- If $R' = R[v \leftarrow u]$, where R is an edge metric basis of G such that $v \in R$, then R is an edge metric basis of G such that $w \notin R$ (a contradiction).

Thus, w is also an edge basis forced vertex of G'. □

If we add a leaf to an edge basis forced vertex, then that vertex will no longer be an edge basis forced vertex (because the set $R[v \leftarrow u]$ is an edge metric basis that does not contain v by the previous proof). The added leaf sometimes becomes an edge basis forced vertex and sometimes not. The following lemma fully characterizes these two cases.

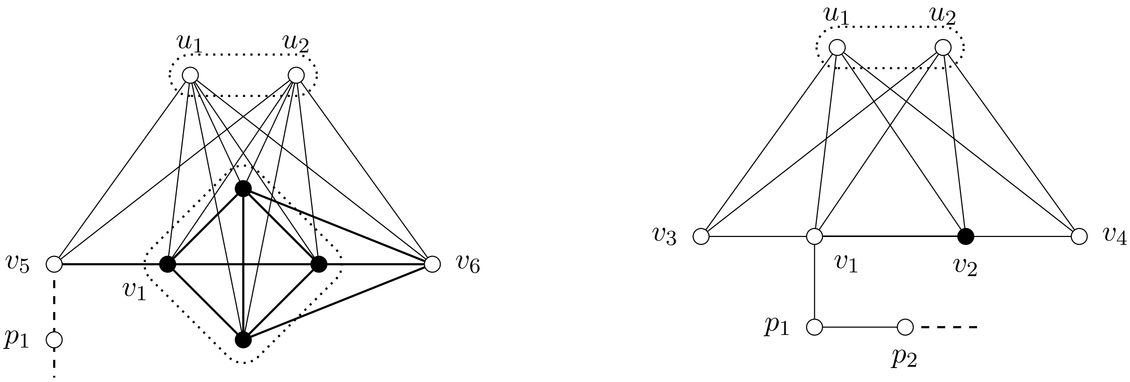
Lemma 13. *Let $v \in V(G)$ be an edge basis forced vertex of G. Let G' be the graph we obtain by adding a leaf u to v. The vertex u is an edge basis forced vertex of G' if and only if for all edge metric bases R of G there exists an $x \in N_G(v)$ such that $d(r, x) \geq d(r, v)$ for all $r \in R$.*

Proof. Assume that the added leaf u is an edge basis forced vertex of G'. All metric bases of G' are of the form $R[v \leftarrow u]$ where R is an edge metric basis of G due to Lemma 12. Since R itself is not an edge metric basis of G', there are two edges e and e' such that $d_{G'}(r, e) = d_{G'}(r, e')$ for all $r \in R$. Clearly, e or e' is uv, say, $e' = uv$. The vertex v is an edge basis forced vertex of G, and thus $v \in R$ and $d_{G'}(v, uv) = 0$. This implies that $e = vx$ for some $x \in N_{G'}(v)$, $x \neq u$. Notice that $d(r, uv) = d(r, v)$ for all $r \in R$. Now we have $d(r, vx) = d(r, v)$ and thus $d(r, x) \geq d(r, v)$ for all $r \in R$.

Assume then that u is not an edge basis forced vertex of G'. Due to Lemma 12, there exists a set $R \subseteq V(G)$ that is an edge metric basis of both G and G'. The set R resolves all pairs of edges in G', in particular, R resolves all pairs uv, vx where $x \in N_{G'}(v)$, $x \neq u$. However, we again have $d(r, uv) = d(r, v)$ for all $r \in R$. Now $d(r, uv) \neq d(r, vx)$ implies that $d(r, vx) \neq d(r, v)$, and thus $d(r, x) < d(r, v)$ for some $r \in R$, a contradiction. □

We will use Lemmas 12 and 13 to construct graphs with edge basis forced vertices from other graphs in order to realize the same combination of f and k for different n. We do this not only by adding single leaves but paths as well. Adding a path to a graph can be seen as adding leaves iteratively. Let v be the vertex we want to add the path to. In order to use Lemma 12, we need to have $v \in R$ for some edge metric basis R of the original graph. When we have added the first leaf u, the set $R[v \leftarrow u]$ is an edge metric basis of the new graph G' that contains u, which is needed in order to use Lemma 12 again to add the next leaf to u. Thus, Lemma 12 can also be used to add a path to a graph and still preserve the edge metric dimension (and edge metric bases for the most part as will be discussed in more detail for the specific constructions given in Section 5.2) of the original graph.

We will also use Lemma 13 when adding a path to a graph. When we add a path to an edge basis forced vertex, our goal is not only increase n, but also to get rid of one edge basis forced vertex. Let v be an edge basis forced vertex of G. If we add a path with at least two vertices to v, then the obtained graph will have one less edge basis forced vertices. Indeed, if we add one leaf u to v, then u may or may not be an edge basis forced vertex of the obtained graph depending on the condition presented in Lemma 13. However, notice that $N(u) = \{v\}$ and $d(w, v) < d(w, u)$ for all $w \neq u$. Thus, even if u is an edge basis forced vertex after adding one leaf, the second added leaf can never be an edge basis forced vertex as long as $\mathrm{edim}(G) \geq 2$ (which is always true for a graph with edge basis forced vertices). We will use this when we construct graphs with just one edge basis forced vertex in particular.



(a) The graph $G_{4,2,m}$, $m \geq 0$. (b) The graph $G^{\ell}_{2,2}$, $\ell \geq 2$.

Fig. 4. Examples of the graphs $G_{s,t,m}$ and $G^{\ell}_{s,t}$. The black vertices are edge basis forced vertices. The number of vertices in the dotted area varies with s and t.

5.2. Constructions

We are now ready to begin constructing graphs to prove which combinations are realizable. Our first construction takes care of the large majority of the possible combinations.

Define the graph $G_{s,t,m}$ as follows (see Fig. 4(a) for an example of the graph $G_{s,t,m}$ where $s = 4$ and $t = 2$). Let G_s, $s \geq 2$, be such that $V(G_s) = \{v_1, \ldots, v_{s+2}\}$, the vertices $v_1, \ldots, v_s$ form a clique, v_{s+1} is adjacent to v_1, and v_{s+2} is adjacent to $v_2, \ldots, v_s$ (but *not* to v_1). The graph $G_{s,t}$, $t \geq 1$, is defined as $G_{s,t} = G_s \vee \overline{K_t}$, where $V(\overline{K_t}) = \{u_1, \ldots, u_t\}$. Note that $|V(G_{s,t})| = s + t + 2$. The graph $G_{s,t,m}$, $m \geq 0$, (if $m = 0$, then $G_{s,t,m} = G_{s,t}$) is constructed from $G_{s,t}$ and P_m, $V(P_m) = \{p_1, \ldots, p_m\}$, by adding an edge between v_{s+1} and p_1.

In the following lemmas n is the order of the graph in each particular case.

Lemma 14. *Let $s \geq 2$, $t \geq 1$, $m \geq 0$. The graph $G_{s,t,m}$ has s edge basis forced vertices and* $\mathrm{edim}(G_{s,t,m}) = s + t = n - m - 2$.

Proof. Let us first consider the case where $m = 0$. We will show that $\mathrm{edim}(G_{s,t}) = s + t$ by using Theorems B and D. Notice that for v_{s+1} there does not exist a vertex $w \in V(G_s)$ such that $V(G_s) \setminus N(v_{s+1}) \subseteq N(w)$. Indeed, in order to cover both v_{s+1} and v_{s+2}, the vertex w should be adjacent to both but no such vertex exists. Thus, in the case where $t = 1$, $\mathrm{edim}(G_{s,1}) = \mathrm{edim}(G_s \vee K_1) = |V(G_s)| - 1 = s + 1$ due to Theorem B. We can prove the claim for $t \geq 2$ by using Theorem D. Notice that the edge v_1v_{s+1} is the only edge in G_s with v_{s+1} as an endpoint. The set $\{v_1, v_{s+1}\}$ does not dominate v_{s+2}. Thus, $G_s \notin \mathcal{G}$. Since $\overline{K_t}$ is also clearly not in $\mathcal{G}$, we have $\mathrm{edim}(G_{s,t}) = \mathrm{edim}(G_s \vee \overline{K_t}) = s + 2 + t - 2 = s + t$ due to Theorem D.

We will then show that the edge basis forced vertices of $G_{s,t}$ are precisely the vertices $v_1, \ldots, v_s$.

Suppose to the contrary that there exists an edge metric basis R of $G_{s,t}$ that does not contain v_i where $i \in \{1, \ldots, s\}$. Let $v_j = v_1$ if $i \neq 1$ and $v_j = v_2$ if $i = 1$. Consider the edges v_jv_i and v_ju_k for some $u_k \in V(\overline{K_t})$. Notice that $d(w, v_jv_i) = d(w, v_ju_k)$ for all $w \in V(G_{s,t}) \setminus \{v_i, u_k\}$. Since $v_i \notin R$, we have $u_k \in R$ for all $k \in \{1, \ldots, t\}$. Now $v_l \notin R$ for some $l \neq i$, since $|R| = s + t = n - 2$. However, now $d(w, u_kv_l) = d(w, u_kv_i)$ for all $w \in V(G_{s,t}) \setminus \{v_i, v_l\} = R$, a contradiction. Therefore, the vertices $v_1, \ldots, v_s$ are edge basis forced vertices of $G_{s,t}$.

To show that no other vertices are edge basis forced vertices, consider the sets $R_1 = V(G_{s,t}) \setminus \{v_{s+1}, u_j\}$ and $R_2 = V(G_{s,t}) \setminus \{v_{s+2}, u_j\}$ where $u_j \in V(\overline{K_t})$. Both of these sets are edge metric bases of $G_{s,t}$ due to Lemma 10. Indeed, the only common neighbor of v_{s+1} and u_j is v_1, but v_1 is not adjacent to $v_{s+2} \in N(u_j)$. Similarly, there does not exist any vertex in $N(v_{s+2}) \cap N(u_j) = \{v_2, \ldots, v_s\}$ that would be adjacent to $v_{s+1} \in N(u_j)$. Therefore, the sets R_1 and R_2 are edge metric bases of $G_{s,t}$.

Let us then consider the case where $m \neq 0$. When we consider the path P_m to be added to $G_{s,t}$ one leaf at a time, we obtain $\mathrm{edim}(G_{s,t,m}) = \mathrm{edim}(G_{s,t}) = s + t = n - m - 2$ due to Lemma 12 and the edge metric basis R_2 above. Recall that v_2 is an edge basis forced vertex, so it is in all edge metric bases of $G_{s,t}$. Now $d(v_2, v_{s+1}p_1) = 2$, $d(v_2, p_ip_{i+1}) = i + 2$ and $d(v_2, e) \leq 1$ for all $e \in V(G_{s,t})$. Consequently, every edge metric basis of $G_{s,t}$ is also an edge metric basis of $G_{s,t,m}$; in particular R_1 and R_2 are edge metric bases of $G_{s,t,m}$. Therefore, by Lemma 12(iii), the edge basis forced vertices of $G_{s,t,m}$ are exactly the same as those of $G_{s,t}$. □

Lemma 14 shows that when $n \geq 5$ and $2 \leq f < k \leq n - 2$, all combinations of n, f and k are realizable. Notice that the graph $G_{s,1}$ has $\frac{s(s-1)}{2} + 2s + 2 = \frac{n(n-1)}{2} - n + 2$ edges, which shows that the upper bound of Theorem 7 is sharp for all $n \geq 5$.

We have already realized the majority of the possible combinations of n, f, k with Lemma 14. What we have left are the extremal cases where $f = 1$ or $f = k$. We will first turn our attention to the case where $f = 1$. By modifying the graph $G_{s,t}$ we can take care of most of the combinations where $f = 1$ as the following lemma shows.

Lemma 15. *Let $G^{\ell}_{2,t}$ ($t \geq 1$, $\ell \geq 2$) be the graph obtained from $G_{2,t}$ and P_ℓ by adding an edge between p_1 and v_1. The graph $G^{\ell}_{2,t}$ contains exactly one edge basis forced vertex, and* $\text{edim}(G^{\ell}_{2,t}) = t + 2$.

Proof. We consider the graph $G^{\ell}_{2,t}$ to be constructed from $G_{2,t}$ by adding the vertices of the path P_ℓ one by one as leaves starting from v_1; the graph $G^{\ell}_{2,2}$ is illustrated in Fig. 4(b). The graph $G_{2,t}$ has two edge basis forced vertices, v_1 and v_2 and $\text{edim}(G_{2,t}) = t + 2$, according to Lemma 14. By iterating Lemma 12 we obtain $\text{edim}(G^{\ell}_{2,t}) = t + 2$. The first added leaf p_1 might be (and, by Lemma 13, actually is) an edge basis forced vertex, but (recall that we have $\ell \geq 2$) the second added leaf p_2 is not according to Lemma 13 since $d(w, v_1) < d(w, p_1)$ for all $w \neq p_1$. The vertex v_2 is an edge basis forced vertex of $G^{\ell}_{2,t}$ due to Lemma 12(iii) and no other edge basis forced vertices exist due to similar arguments as in the proof of Lemma 14. □

We have $n = t + \ell + 4$ for $G^{\ell}_{2,t}$. Thus, all combinations of f, k, n where $f = 1$, $n \geq 7$ and $3 \leq k \leq n - 4$ are realizable due to Lemma 15. We will then cover the cases where $f = 1$ and $k = n - 3$ or $n - 2$ by constructing a new graph family.

Let $W_{s,t,y}$ be the graph constructed as follows (see Fig. 5(a)). Let $s, t \geq 2$ and $y \in \{0, 1\}$. Denote the vertices of the complete graph K_{s+t+1} as follows: $V(K_{s+t+1}) = \{x, u_1, \ldots, u_s, w_1, \ldots, w_t\}$. The graph $W_{s,t,0}$ is the graph obtained by adding two vertices v_1 and v_2 to K_{s+t+1} such that v_1 is adjacent to all u_i (and no others) and v_2 is adjacent to all w_i (and no others). The graph $W_{s,t,1}$ is obtained from $W_{s,t,0}$ by adding a leaf v_3 to u_1.

Lemma 16. *Let $s, t \geq 2$ and $y \in \{0, 1\}$. The vertex x is the only edge basis forced vertex of $W_{s,t,y}$ and* $\text{edim}(W_{s,t,y}) = s+t+1 = n - y - 2$.

Proof. Let us first consider the case where $y = 0$, i.e. the leaf v_3 is not present.

We will first show that $\text{edim}(W_{s,t,0}) \geq s + t + 1 = n - 2$. For this purpose, let R be an edge resolving set of $W_{s,t,0}$. The vertices u_i are twins, and thus $|R \cap \{u_1, \ldots, u_s\}| \geq s - 1$. Analogously, we have $|R \cap \{w_1, \ldots, w_t\}| \geq t - 1$. Furthermore, $R \cap \{v_1, u_j\} \neq \emptyset$ for any $j \in \{1, \ldots, s\}$ since we may choose $w = u_i$ ($i \neq j$) in Corollary 11. Similarly, $R \cap \{v_2, w_j\} \neq \emptyset$ for any $j \in \{1, \ldots, t\}$. Thus, in conclusion, we obtain that (i) $|R \cap \{v_1, u_1, \ldots, u_s\}| \geq s$ and (ii) $|R \cap \{v_2, w_1, \ldots, w_t\}| \geq t$. Consequently, $\text{edim}(W_{s,t,0}) \geq s + t = n - 3$.

Suppose that $\text{edim}(W_{s,t,0}) = s+t = n-3$, and let R be an edge metric basis of $W_{s,t,0}$ with $s+t$ vertices. Now $x \notin R$ due to (i) and (ii). This implies that $u_i \in R$ for any i since by choosing $w = u_j$ ($i \neq j$) in Corollary 11, we obtain that $R \cap \{x, u_i\} \neq \emptyset$. Similarly, it can be shown that $w_i \in R$ for any i. Thus, in conclusion, we have $R = \{u_1, \ldots, u_s\} \cup \{w_1, \ldots, w_t\}$; in particular, v_1 and v_2 do not belong to R. Hence, $R \cap \{v_1, x\} = \emptyset$ contradicting Corollary 11 (with the choice of $w = u_1$). Therefore, $\text{edim}(W_{s,t,0}) \geq s + t + 1 = n - 2$ and if there exists an edge resolving set of cardinality $s + t + 1$, then x is in all such sets, i.e., it is edge basis forced.

What remains to be shown is that $\text{edim}(W_{s,t,0}) \leq s + t + 1$ and that no vertex other than x is an edge basis forced vertex. We will prove both of these claims by finding suitable edge resolving sets with the aid of Lemma 10. Let $R_1 = V(W_{s,t,0}) \setminus \{v_1, w_i\}$ and $R_2 = V(W_{s,t,0}) \setminus \{v_2, u_j\}$ where $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$. Notice that no mutual neighbor of v_1 and w_i is adjacent to $v_2 \in N(w_i)$, and no mutual neighbor of v_2 and u_j is adjacent to v_1. Thus, both R_1 and R_2 are edge resolving sets of $W_{s,t,0}$ for any i and j. In conclusion, $\text{edim}(W_{s,t,0}) = s + t + 1 = n - 2$ and x is the only edge basis forced vertex of $W_{s,t,0}$.

Let us then consider the case where $y = 1$. Notice that the vertex u_1 is in some metric bases of $W_{s,t,0}$. Thus, we have $\text{edim}(W_{s,t,1}) = \text{edim}(W_{s,t,0}) = s + t + 1 = n - 3$ according to Lemma 12. The vertex x is also an edge basis forced vertex of $W_{s,t,1}$ by Lemma 12(iii). In what follows, we will show that there are no other edge basis forced vertices. Recall that according to Lemma 12 the set $R[u_1 \leftarrow v_3]$ is an edge metric basis of $W_{s,t,1}$ for any edge metric basis R of $W_{s,t,0}$ that contains u_1. The sets $V(W_{s,t,0}) \setminus \{v_1, w_i\}$, $i \in \{1, \ldots, t\}$, and $V(W_{s,t,0}) \setminus \{v_2, u_j\}$, $j \neq 1$, are such edge metric bases of $W_{s,t,0}$ (due to R_1 and R_2 above). Consequently, none of the vertices v_1, v_2, w_i and u_j are edge basis forced vertices for any $i \in \{1, \ldots, t\}$ or $j \in \{1, \ldots, s\}$. To show that v_3 is not an edge basis forced vertex, consider the set $R = V(W_{s,t,1}) \setminus \{u_1, v_2, v_3\} = V(W_{s,t,0}) \setminus \{u_1, v_2\}$. Since the set R is an edge metric basis of $W_{s,t,0}$, it is an edge metric basis of $W_{s,t,1}$ if the edge v_3u_1 is resolved from all other edges in $W_{s,t,1}$ (see Lemma 12). The vertices u_1 and v_2 are not adjacent, and thus the edge v_3u_1 is the only edge with neither endpoint in R. Therefore, the set R is an edge metric basis of $W_{s,t,1}$, and v_3 is not an edge basis forced vertex of $W_{s,t,1}$. □

All combinations where $f = 1$ and $3 \leq k \leq n - 2$ are thus realizable by Lemmas 15 and 16 for all $n \geq 8$. The combination where $f = 1$ and $k = 2$ will be considered after the following lemma, because we will base the construction on the same graph as the following graph which gives us the realization result to almost all cases where $f = k$.

Let H be the house graph, that is, $V(H) = \{v_1, \ldots, v_5\}$, $E(H) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_5, v_4v_5\}$. Let H_t, $t \geq 0$ (if $t = 0$, then $H_t = H$) be the graph obtained from H and $\overline{K_t}$, $V(\overline{K_t}) = \{u_1, \ldots, u_t\}$, by connecting each vertex of $\overline{K_t}$ to v_2, v_3 and v_4 with an edge. Let $H_{t,m}$ ($t, m \geq 0$) (if $m = 0$, then $H_{t,m} = H_t$) be the graph obtained from H_t and P_m, $V(P_m) = \{p_1, \ldots, p_m\}$, by adding an edge between v_1 and p_1. See Fig. 5(b) for an example of the graph $H_{t,m}$ (with $t = 2$). Notice that the case $f = k$ means that the graph has a unique edge metric basis.

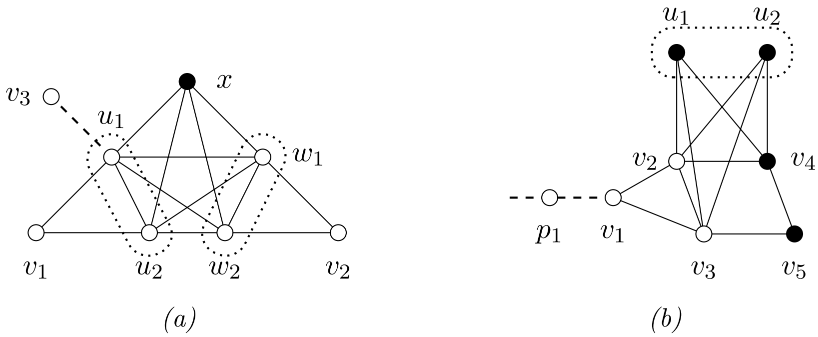


Fig. 5. Examples of the graphs $W_{s,t,y}$ and $H_{t,m}$ with $s = t = 2$. The black vertices are edge basis forced vertices. The number of vertices in the dotted area varies with s and t.

Lemma 17. *Let $t, m \geq 0$. The graph $H_{t,m}$ has a unique edge metric basis $\{v_4, v_5, u_1, \ldots, u_t\}$. In particular, $\text{edim}(H_{t,m}) = t + 2 = n - m - 3$ and $H_{t,m}$ has $t + 2$ edge basis forced vertices.*

Proof. We will first prove that the set $R = \{v_4, v_5, u_1, \ldots, u_t\}$ is an edge resolving set of $H_{t,m}$. It is enough to consider pairs of edges where both endpoints are not in R and pairs of edges with one common endpoint in R. The edges with neither endpoint in R, namely v_1v_2, v_1v_3, v_2v_3, v_1p_1 and p_ip_{i+1} where $i \in \{1, \ldots, m\}$ (if $m > 0$), are resolved from each other by v_4 and v_5. Pairs of edges with the same endpoint in R and the other endpoints not in R are also resolved by v_5. Indeed, all such pairs are of the form u_iv_2, u_iv_3 where $i \in \{1, \ldots, t\}$ (no such pairs exist if $t = 0$), and we have $d(v_5, u_iv_2) = 2 \neq 1 = d(v_5, u_iv_3)$. All other edges are resolved by their endpoints. Thus, the set R is an edge resolving set of $H_{t,m}$ and $\text{edim}(H_{t,m}) \leq t + 2$.

We will then show that $\text{edim}(H_{t,m}) \geq t+2$. Choosing $w = v_2$ in Corollary 11, we observe that every edge resolving set of $H_{t,m}$ must contain v_3 or v_4. Similarly, with the choice $w = v_3$, it follows that v_2 or v_5 belongs to each edge resolving set. Therefore, every edge resolving set of $H_{t,m}$ contains at least two vertices that are not in $V(\overline{K_t})$.

Assume that R is an edge resolving set that contains exactly two vertices not in $V(\overline{K_t})$. We claim that $|R| = t + 2$. We will first show that those two vertices must be v_4 and v_5. Suppose to the contrary that $v_4 \notin R$. By the previous paragraph, $v_3 \in R$, and either v_2 or v_5 belongs to R. If $v_2 \in R$ (and $v_5 \notin R$), then the edges v_1v_2 and v_2v_4 are not resolved, and otherwise $v_5 \in R$ (and $v_2 \notin R$) leads to a contradiction due to the edges v_1v_3 and v_2v_3. Suppose then that $v_4 \in R$ and $v_5 \notin R$ (and thus $v_2 \in R$ and $v_3 \notin R$). Now, the set R cannot resolve the pair v_1v_2, v_2v_3, a contradiction. Thus, $v_4, v_5 \in R$ and no other vertices v_i or p_i are in R. Suppose that $u_i \notin R$ for some $i \in \{1, \ldots, t\}$. Now the edges v_4v_2 and v_4u_i are not resolved; the distance from v_5 and u_j, $j \neq 1$, to both of these edges is 1. Consequently, $R = \{v_4, v_5, u_1, \ldots, u_t\}$ and $|R| = t + 2$.

Assume then that R is an edge resolving set of $H_{t,m}$ that contains at least three vertices not in $V(\overline{K_t})$. Suppose that $|R| \leq t + 2$. Now there exists some u_i, say u_1, that is not in R. The other vertices in $V(\overline{K_t})$ must be in R since they are twins. Thus, without loss of generality, the set R consists of the vertices u_i where $i \in \{2, \ldots, t\}$ (if $t \geq 2$) and exactly three vertices not in $V(\overline{K_t})$. Notice that u_1 or v_5 belongs to R by Corollary 11 (choose $w = v_3$). Since $u_1 \notin R$, we have $v_5 \in R$. Similarly, u_1 or v_2 belongs to R by choosing $w = v_3$ in Corollary 11, and thus $v_2 \in R$. Recall that any edge resolving set must also contain v_3 or v_4. However, none of the vertices $v_2, \ldots, v_5$ resolve the pair u_1v_2, v_1v_2, and neither does any u_i other than u_1. Therefore, if an edge resolving set R of $H_{t,m}$ contains at least three vertices not in $V(\overline{K_t})$, then $|R| \geq t + 3$.

In conclusion, $\text{edim}(H_{t,m}) = t + 2$. Moreover, the unique metric basis of $H_{t,m}$ is the set $\{v_4, v_5, u_1, \ldots, u_t\}$, and thus the vertices $v_4, v_5, u_1, \ldots, u_t$ are edge basis forced vertices of $H_{t,m}$. □

Combinations of n, f and k where $2 \leq f = k \leq n - 3$ are thus realizable due to Lemma 17. The only remaining case besides the one where $f = 1$ and $k = 2$ is the case where $f = k = n - 2$. The following two lemmas prove that both of these cases are realizable.

Lemma 18. *Let H^ℓ ($\ell \geq 2$) be the graph obtained from the house graph H and P_ℓ by adding an edge between p_1 and v_5. The graph H^ℓ contains exactly one edge basis forced vertex, and $\text{edim}(H^\ell) = 2$.*

Proof. According to Lemma 17 the graph H has the unique edge metric basis $\{v_4, v_5\}$. By Lemmas 12 and 13 the edge metric bases of H^ℓ are the sets $\{v_4, p_i\}$, $i \in \{1, \ldots, \ell\}$, and the only edge basis forced vertex of H^ℓ is v_4. □

Let F_s, where $s \geq 2$, be constructed as follows (see Fig. 6(b)). Denote $V(K_s) = \{u_1, \ldots, u_s\}$. Then F_s is obtained from K_s by adding four vertices v_1, v_2, v_3 and w, and the following edges: v_1u_i for all i, v_2u_i for all $i \neq 1$, v_1v_3, v_2v_3, wu_i for all i, and wv_j for all j.

Lemma 19. *Let $s \geq 2$. We have $\text{edim}(F_s) = n - 2$ and F_s has a unique edge metric basis, i.e. $n - 2$ edge basis forced vertices.*

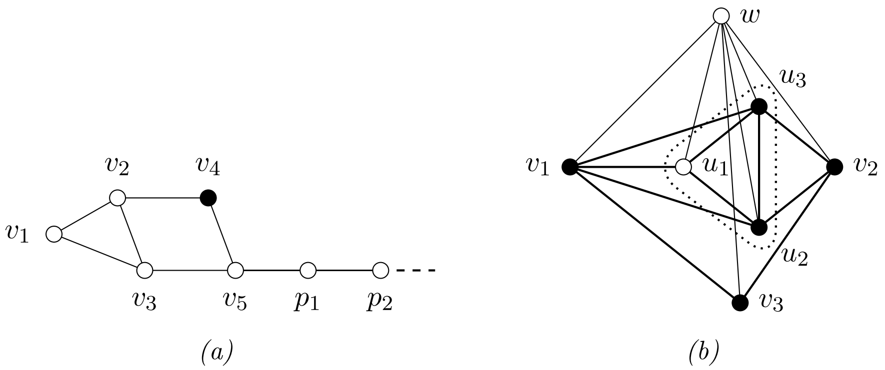


Fig. 6. Examples of the graphs H^{ℓ}, $\ell \geq 2$ and F_s where $s = 2$. The black vertices are edge basis forced vertices.

Proof. Notice that w is a universal vertex, i.e. $F_s = (F_s - w) \vee w$. According to Theorem B, we have $\text{edim}(F_s) = n - 2$, since $V(F_s - w) \setminus N(u_1) = \{u_1, v_2, v_3\}$ and no vertex of $F_s - w$ is adjacent to all three of these.

Let R be an edge metric basis of F_s. Suppose that $w \in R$. Since $|R| = n - 2$, there exist $x, y \in V(F_s) \setminus \{w\}$ such that $x, y \notin R$. However, this contradicts Lemma 10 when w is chosen as the mutual neighbor of x and y. Thus, we have $w \notin R$. By carefully choosing the mutual neighbor in Corollary 11, we obtain the following observations:

- for all $i \neq 1$, $u_i \in R$ or $w \in R$ (v_2 chosen as their mutual neighbor)
- for all $i \in \{1, 2\}$, $v_i \in R$ or $w \in R$ (u_2 chosen as their mutual neighbor)
- $v_3 \in R$ or $w \in R$ (v_1 chosen as their mutual neighbor)

Thus, as $w \notin R$ and $|R| = n - 2$, we have $R = V(F_s) \setminus \{w, u_1\}$. $\square$

Recall that if a graph G of order n contains $f \geq 1$ edge basis forced vertices, then we have $2 \leq \text{edim}(G) \leq n - 2$ and $1 \leq f \leq k$. As a summary of Lemmas 14–19 (and the discussions above), we obtain the following theorem, which states that all the previous values can be realized for sufficiently large n.

Theorem 20. *If n, k and f are integers such that $n \geq 8$, $2 \leq k \leq n - 2$ and $1 \leq f \leq k$, then there exists a graph G such that $n = |V(G)|$, $k = \text{edim}(G)$ and f is the number of edge basis forced vertices.*

6. Concluding remarks

We have presented in this work several results concerning the vertices of a graph that must always be included in any edge metric basis of such a graph. We next remark some possible research lines that can be considered as a continuation of our investigation.

- It would be of interest to characterize the classes of graphs having the smallest possible, as well as the largest possible, number of edge basis forced vertices.
- Since the number of edge basis forced vertices and v-basis forced vertices of graphs are in general not comparable, it might be interesting to classify the graphs having smaller number of edge basis forced vertices than of v-basis forced vertices, and vice versa.
- There can be vertices in a graph that are edge basis forced vertices and v-basis forced vertices at the same time. In this sense, a possible investigation concerning the existence of these kind of vertices would be worthwhile.
- Can we characterize the graphs that have neither edge basis forced vertices nor edge void vertices? A similar question can be posed for v-basis forced vertices and v-void vertices.

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Data availability

No data was used for the research described in the article.

References

[1] B. Bagheri Gh, M. Jannesari, B. Omoomi, Unique basis graphs, Ars Combin. 129 (2016) 249–259.
[2] R.F. Bailey, I.G. Yero, Error-correcting codes from k-resolving sets, Discuss. Math. Graph Theory 39 (2) (2019) 341–355.
[3] C. Biró, B. Novick, D. Olejnikova, Metric dimension of growing infinite graphs, J. Comb. 15 (2) (2024) 159–177.
[4] P.S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, Period. Math. Hungar. 46 (1) (2003) 9–15.
[5] G. Chartrand, L. Eroh, M.A. Johnson, O. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) 99–113.
[6] B. Foster-Greenwood, C. Uhl, Metric dimension of a direct product of three complete graphs, Electron. J. Combin. 31 (2) (2024) 22, Paper No. 2.13.
[7] F. Foucaud, S.-S. Kao, R. Klasing, M. Miller, J. Ryan, Monitoring the edges of a graph using distances, Discrete Appl. Math. 319 (2022) 424–438.
[8] A. Hakanen, V. Junnila, T. Laihonen, H. Miikonen, I.G. Yero, On a tight bound for the maximum number of vertices that belong to every metric basis, in: Lecture Notes on Computer Science, LNCS, vol. 15536, 2025, pp. 171–184.
[9] A. Hakanen, V. Junnila, T. Laihonen, I.G. Yero, On vertices contained in all or in no metric basis, Discrete Appl. Math. 319 (2022) 407–423.
[10] A. Hakanen, V. Junnila, T. Laihonen, I.G. Yero, On the unicyclic graphs having vertices that belong to all their (strong) metric bases, Discrete Appl. Math. 353 (2024) 191–207.
[11] F. Harary, R. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[12] M.A. Henning, S. Klavžar, I.G. Yero, Resolvability and convexity properties in the sierpiński product of graphs. Mediterr, J. Math. 21 (1) (2024) 17, Paper No. 3.
[13] A. Kelenc, N. Tratnik, I.G. Yero, Uniquely identifying the edges of a graph: The edge metric dimension, Discrete Appl. Math. 251 (2018) 204–220.
[14] M. Knor, S. Majstorović, A.T. Masa Toshi, R. Škrekovski, I.G. Yero, Graphs with the edge metric dimension smaller than the metric dimension, Appl. Math. Comput. 401 (2021) 126076.
[15] D. Kuziak, I.G. Yero, Metric dimension related parameters in graphs: A survey on combinatorial, computational and applied results, 2021, Preprint.
[16] M. Mohagheghi Nehzad, F. Rahbarnia, M. Mirzavaziri, R. Ghanbari, Solis graphs and uniquely metric basis graphs, Iran. J. Math. Sci. Inform. 17 (2) (2022) 191–212.
[17] M. Mora, M.L. Puertas, On the metric representation of the vertices of a graph, Bull. Malays. Math. Sci. Soc. 46 (6) (2023) 16, Paper No. 187.
[18] I. Peterin, I.G. Yero, Edge metric dimension of some graph operations, Bull. Malays. Math. Sci. Soc. 43 (2020) 2465–2477.
[19] P.J. Slater, Leaves of trees, in: Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ. Boca Raton, Fla. 1975), 1975, pp. 549–559, Congressus Numerantium, No. XIV, Winnipeg, Man. Utilitas Math.
[20] R.C. Tillquist, R.M. Frongillo, M.E. Lladser, Getting the lay of the land in discrete space: A survey of metric dimension and its applications, SIAM Rev. 65 (4) (2023) 919–962.
[21] R.C. Tillquist, M.E. Lladser, Low-dimensional representation of genomic sequences, J. Math. Biol. 79 (1) (2019) 1–29.
[22] C. Yang, G. Yang, S.-Y. Hsieh, Y. Mao, R. Klasing, Monitoring the edges of a graph using distances with given girth, J. Comput. System Sci. 143 (2024) 103528.
[23] E. Zhu, A. Taranenko, Z. Shao, J. Xu, On graphs with the maximum edge metric dimension, Discrete Appl. Math. 257 (2019) 317–324.
[24] N. Zubrilina, On the edge dimension of a graph, Discrete Math. 341 (7) (2018) 2083–2088.