Fuzzy Sets and Systems 528 (2026) 109706

Contents lists available at ScienceDirect

Fuzzy Sets and Systems

journal homepage: www.elsevier.com/locate/fss

General and left-continuous operators on lattice-based sums[★]

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ARTICLE INFO

2000 MSC:
03C80
03E72
06A15
06B23

Keywords:
T-norm
Ordinal sum
Horizontal sum
Lattice sum

ABSTRACT

Lattice-based sum provides a procedure to obtain posets and lattices from families of posets and lattices, respectively. Establishing sufficient conditions to ensure the lattice structure was the most significant challenge achieved in previous works. Next steps are to consider structures with general operators defined on the lattices of the family, introduce a sum of these operators on the obtained lattice-based sum and study the properties preserved by this new definition. We will prove that the natural definition preserve, in general, the monotonicity, associativity, commutativity, etc. This paper also introduces a new mechanism focused on preserving the left-continuity property of the operators defined on the lattices. This new approach also preserves the associativity and the infimum of non-empty subsets, and takes into account (infinite) complete lattices, unlike the previous works.

1. Introduction

Edge computing and distributive systems are examples of the nowadays necessity of working in local for obtaining information of the global system. A fundamental part of the different formal mechanisms used in these approaches is to establish the algebraic structure to be dealt with (in the diverse local perspectives), as well as how they can be combined (in the global system). These possible structures need to contemplate a range of values sufficient to take into account all the casuistry of the problem to be examined, many of which require incomparable values, as when considering four-valued logic or intervals of truth values [1,6,7]. Furthermore, it is becoming increasingly clear that the operators defined in these structures need to be flexible, without the restriction required, for example, by the use of t-norms.

The approach given in [12] introduced a mechanism, adapting the well-known ordinal sum of t-norms to sum of posets and lattices, focusing on preserving their properties, which was called lattice-based sum. The next step was to consider t-norms on the family of lattices, which was studied in [11]. However, as we commented above, more flexible operators are required in real-use cases avoiding, for example, the associativity. Hence, it is also interesting to introduce the definition of sum in general and study what specific independent properties are preserved. This has been the first goal of this paper, discovering interesting properties. For example, proving that the monotonicity is only preserved if the operator is less than the infimum operator; showing that the

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[★] Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR, project FEDER-UCA-2024-A2-04 funded by Programa Operativo FEDER Andalucía 2021-2027 and by Consejería de Universidad, Investigación e Innovación, Junta de Andalucía and by the project PR2023-009 funded by University of Cádiz.

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https://doi.org/10.1016/j.fss.2025.109706
Received 23 April 2025; Received in revised form 31 October 2025; Accepted 26 November 2025

associativity of the underlying operators is only required to reach the associativity of the obtained sum. As a consequence of these results we obtain the sufficient conditions to obtain t-norms.

We will also show that the left-continuity property, which is fundamental to obtain residuated implications [9,16], is not preserved with the lattice-based sum definition given in [11]. However, this property is fundamental to model knowledge systems with imperfect information. Recall that, for example, the most used fuzzy extension of the well-known Modus Ponens of classical logic is given by the residuated implications [14]. Therefore, this paper is also focused on introducing a new mechanism to preserve distributive lattices and left-continuous operators. The associativity property and the infimum-preserving property of non-empty subsets will also be studied. The consideration of complete lattices is also an added value to previous works, which only takes into account (finite) lattices. This is a first approach in which the indices lattice cannot have an arbitrary number of incomparable elements. This general case will be studied in future extensions of this work.

The structure of the paper is the following: Section 2 includes the basic definitions considered in this paper. The notions necessary to define the lattice-based sum are given in Section 3. In Section 4, some approaches of ordinal sums of t-norms are extended together with several properties associated with the operators. A new lattice-based sum preserving the distributivity of the given lattice is defined in Section 5 together with the study of its algebraic structure. Moreover, different properties based on the operators defined on each lattice of the considered family are analyzed in Section 6. Finally, Section 7 presents a series of conclusions and research perspectives for future work.

2. Preliminaries

In order to make this paper self-contained, we provide in this section the classical definitions of ordinal sums of partially ordered sets (posets) [2], of semigroups [4], of t-norms [16,17,25], and of horizontal sums of posets and the relation with the ordinal sum of t-norms [25].

The ordinal sum of two disjoint posets was presented by Birkhoff in [2]. Next, we present an immediate generalization introduced in [25].

Definition 1. Consider a non-empty linearly ordered index set $(I, \leq_I)$, and a family of pairwise disjoint posets $(X_i, \preceq_i)_{i\in I}$. The ordinal sum $(X, \preceq) = \oplus_{i\in I}(X_i, \preceq_i)$ is defined as the set $X = \cup_{i\in I} X_i$, equipped with the following relation $\preceq$:

$$x \preceq y \quad \text{if and only if} \quad \begin{cases} x, y \in X_i \text{ and } x \preceq_i y \\ \text{or} \\ x \in X_i, y \in X_j \text{ and } i <_I j \end{cases} \tag{1}$$

This relation is indeed an ordering relation.

Proposition 1 ([25]). *With all the assumptions of the previous definition, the ordinal sum $(X, \preceq)$ is a partially ordered set.*

The definition above can be generalized in order to relax the condition of disjointness.

Proposition 2 ([25]). *Given a non-empty linearly ordered index set $(I, \leq_I)$, and a family of posets $(X_i, \preceq_i)_{i\in I}$, such that for all $i, j \in I$ with $i < j$ either X_i and X_j are disjoint or $X_i \cap X_j = \{x_{ij}\}$ where x_{ij} is the maximum on X_i and the minimum on X_j and, for each $k \in I$, with $i < k < j$, we have $X_k = \{x_{ij}\}$. The ordinal sum $(X, \preceq) = \oplus_{i\in I}(X_i, \preceq_i)$, where $X = \cup_{i\in I} X_i$ and $\preceq$ is defined as in Eq. (1), is a partially ordered set.*

The concept of ordinal sum is also applied in the context of abstract semigroups. Now, we recall a definition from [4], which generalize another one of [5] concerning the ordinal sum of two disjoint semigroups.

Definition 2 ([4]). Let $(I, \leq)$ be a non-empty linearly ordered index set, $(X_i)_{i\in I}$ a family of pairwise disjoint sets, and $((X_i, \&_i))_{i\in I}$ a family of semigroups. Put $X = \cup_{i\in I} X_i$ and define the binary operation $\&$ on X by:

$$x \,\&\, y = \begin{cases} x \,\&_i\, y & \text{if } (x, y) \in X_i \times X_i \\ x & \text{if } (x, y) \in X_i \times X_j \text{ and } i < j \\ y & \text{if } (x, y) \in X_i \times X_j \text{ and } j < i \end{cases} \tag{2}$$

Then we say that $(X, \&) = \oplus_{i\in I}(X_i, \&_i)$ is the *ordinal sum of the family of semigroups.*

Proposition 3 ([4]). *With all the assumptions of the previous definition, the ordinal sum $(X, \&)$ is also a semigroup, i.e., $\&$ is an associative operation on X.*

As a consequence, the ordinal sum of a family of pairwise disjoint semigroups is a certain way of constructing a new semigroup from a family of semigroups. The following result provides a generalization of the definition above, relaxing the condition of disjointness.

Theorem 1 ([4]). *Let $(I, \leq)$ be a non-empty linearly ordered index set and $(G_i)_{i\in I}$, with $G_i = (X_i, \&_i)$, a family of semigroups. Assume that, for all $i, j \in I$, with $i < j$, the sets X_i and X_j are either disjoint or that $X_i \cap X_j = \{x_{ij}\}$, where x_{ij} is both the unit element of G_i and the annihilator of G_j, and where, for each $k \in I$, with $i < k < j$, we have $X_k = \{x_{ij}\}$. Put $X = \cup_{i\in I} X_i$ and define the binary operation $\&$ on X as in Eq. (2). Then $(X, \&)$ is a semigroup. The semigroup $(X, \&)$ is commutative if and only if the semigroup $(X_i, \&_i)$ is commutative, for all $i \in I$.*

Both types of ordinal sums can be unified for ordered semigroups (means that semigroup operation is nondecreasing in both coordinates).

Corollary 1. *With all the assumptions of the previous theorem considering a family of ordered semigroups* $((X_i, \preceq_i, \&_i))_{i\in I}$ *and assuming that if, for* $i < j$, $X_i \cap X_j = \{x_{ij}\}$, *then* x_{ij} *is the maximum on* X_i *and the minimum on* X_j. *The ordinal sum* $(X, \preceq, \&) = \oplus_{i\in I}(X_i, \preceq_i, \&_i)$ *is a ordered semigroup, where* $X = \cup_{i\in I} X_i$ *and* $\preceq$, $\&$ *are defined as in Eqs. (1) and (2), respectively.*

There are also other possible relaxations of the disjointness of carriers in Proposition 3, see e.g. [15], but then it is not guaranteed that the resulting ordinal sum will be an extension (there is no embedding of the original semigroup into the resulting ordinal sum).

The theorem above is later applied in order to generalize ordinal sum for t-norms and t-subnorms [15,16,26]. Here, we recall the classical definition of t-norm and of ordinal sum of t-norms defined on the unit interval.

Definition 3. A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a *triangular norm on* $[0,1]$ if the following conditions are fulfilled, for all $x, y, z \in [0,1]$:

1. $T(x,y) = T(y,x)$,
2. If $x \leq y$ then $T(x,z) \leq T(y,z)$,
3. $T(x,1) = x$,
4. $T(x, T(y,z)) = T(T(x,y), z)$.

Definition 4. Let $(I, \preceq)$ be a linearly ordered index set, $\{]a_i, b_i[\}_{i\in I}$ a family of pairwise disjointed subintervals of $[0,1]$, and $(T_i)_{i\in I}$ a family of t-norms on $[0,1]$; the operator $T = \{\langle a_i, b_i, T_i\rangle\}_{i\in I} : [0,1]\times[0,1] \to [0,1]$ defined, for each $x, y \in [0,1]$, as:

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) & \text{if } x, y \in [a_i, b_i],\ \text{with } i \in I \\ \min\{x,y\} & \text{otherwise} \end{cases} \tag{3}$$

is called *ordinal sum* of the family $(T_i)_{i\in I}$ on $[0,1]$.

Note that ordinal sums of t-norms on $[0,1]$ can be seen as a special case of Corollary 1 (for deeper discussion see [16]).

These definitions have been generalized in order to consider a general bounded lattice instead of the unit interval. There exist two equivalent definitions of the notion of lattice, as an ordered set and as an algebraic structure (see [3,10]). Now, we will recall both definitions and the equivalence.

Definition 5. Let $(L, \preceq)$ be a non-empty ordered set. If there exist the supremum and the infimum of each pair of elements $x, y \in P$, then $(L, \preceq)$ is called a *lattice*.

We may adopt an alternative viewpoint, we can present a lattice as an algebraic structure $(L, \wedge, \vee)$.

Definition 6. A nonempty set L together with two binary operations $\wedge$ and $\vee$ on L is called a *lattice* if the operation satisfy the following identities, for all $x, y, z \in L$:

L1:	(a)	$x \wedge y = y \wedge x$	
	(b)	$x \vee y = y \vee x$	(commutative laws)
L2:	(a)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	
	(b)	$x \vee (y \vee z) = (x \vee y) \vee z$	(associative laws)
L3:	(a)	$x \wedge x = x$	
	(b)	$x \vee x = x$	(idempotent laws)
L4:	(a)	$x \wedge (x \vee y) = x$	
	(b)	$x \vee (x \wedge y) = x$	(absorption laws)

The operations $\vee$ and $\wedge$ are read as *join* and *meet*, respectively.

Both definition are equivalent because if we have a lattice $(L, \preceq)$ as ordered set, we define the mappings $\wedge$ and $\vee$ as $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$, which satisfy the properties in Definition 6. On the other hand, if we have the lattice $(L, \wedge, \vee)$, then we define the order $\preceq$ as: $x \preceq y$ if and only if $x \wedge y = x$ (which is equivalent to $x \vee y = y$), and this order satisfies that there exist the supremum and the infimum of each pair of elements on L.

Recall also that if there is an element $\top$ ($\bot$) in L such that for each $x \in L$ it holds $x \preceq \top$ ($\bot \preceq x$), then $\top$ ($\bot$) is called top (bottom) element of the lattice $(L, \preceq)$. Lattices possessing both the top and the bottom elements are then called *bounded lattices*. Similarly, the *bounded posets* are defined.

From [25] we extract the definitions of t-norm on bounded lattices and of ordinal sum of t-norms defined on subintervals of a bounded lattice.

Definition 7. Let $(L, \preceq, \bot, \top)$ be a bounded lattice. A binary operation $T : L \times L \to L$ is a *triangular norm on* L if the following conditions are fulfilled, for all $x, y, z \in L$:

1. $T(x,y) = T(y,x)$,
2. If $x \preceq y$ then $T(x,z) \preceq T(y,z)$,
3. $T(x,\top) = x$,

4. $T(x, T(y, z)) = T(T(x, y), z)$.

Definition 8. Given a bounded lattice $(L, \preceq, \bot, \top)$, and $a, b \in L$, a *subinterval* $[a, b]$ *of* L is a sublattice of L defined as:

$$[a, b] = \{x \in L \mid a \preceq x \preceq b\}$$

Definition 9. Given a bounded lattice $(L, \preceq, \bot, \top)$, a linearly ordered index set $(I, \leq)$, a family of pairwise disjointed subintervals of L, $\{]a_i, b_i[\}_{i\in I}$, and a family of t-norms $(T_i)_{i\in I}$ on the corresponding intervals $\{[a_i, b_i]\}_{i\in I}$; the operator $T = \{\langle a_i, b_i, T_i\rangle\}_{i\in I} : L \times L \to L$ defined, for each $x, y \in L$, as:

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } x, y \in [a_i, b_i], \text{ with } i \in I \\ \inf\{x, y\} & \text{otherwise} \end{cases} \tag{4}$$

is called *ordinal sum of the family* $(T_i)_{i\in I}$ *on the bounded lattice* L.

The problem is that, in general, the ordinal sum T defined by Eq. (4) is not a t-norm. In [19,25] sufficient and necessary conditions were studied in order to prove Eq. (4) provides t-norms. Next, one of these results will be presented. Before that, we recall an alternative of the ordinal sum of posets in order to construct new posets, which is the horizontal sum of posets [10,25].

Definition 10. Given a non-empty index set I and $(G_i)_{i\in I}$, with $G_i = (X_i, \preceq_i, \bot, \top)$, a family of bounded posets, where $X_i \cap X_j = \{\bot, \top\}$ whenever $i \neq j$, the *horizontal sum of the family* $(G_i)_{i\in I}$ is the tuple $(X, \preceq, \bot, \top)$ where $X = \cup_{i\in I} X_i$ and $x \preceq y$ if and only if there is $i \in I$ such that $x, y \in X_i$ and $x \preceq_i y$.

From this kind of structures we obtain a sufficient and necessary condition to ensure that the ordinal sum of t-norms (Eq. (4)) is a t-norm.

Proposition 4 ([25]). *Consider some bounded lattice* $(L, \preceq, \bot, \top)$. *Then the following are equivalent:*

1. *For any linearly ordered index set* $(I, \leq)$, *the ordinal sum* T *as defined by* Eq. (4) *is a t-norm for arbitrary families of pairwise disjoint subintervals* $(]a_i, b_i[)_{i\in I}$ *and for arbitrary t-norms* T_i *on the corresponding* $[a_i, b_i]$.
2. L *is a horizontal sum of chains.*

In this paper, we will also extend some results in [12] to complete lattices. Hence, we can consider infinite subsets to ensure the existence of least upper bounds and greatest lower bounds. In the following example, the supremum of a particular infinite set is computed.

Example 1. Given a complete lattice $(\Lambda, \sqsubseteq)$, every chain $\{\alpha_i \in \Lambda \mid i \in I\}$, with I being an infinite index set, has a supremum α_s. Moreover, we will consider lattices $(L_{\alpha_i}, \preceq_{\alpha_i})$ with at least two incomparable elements denoted as $x_i, y_i \in L_{\alpha_i}$, for all $i \in I$.

Now, we take into account the set $X = \{x_i \mid i \in I\} \cup \{y_i \mid i \in I\}$. Therefore, an upper bound of X cannot be an element of L_{α_i}, with $i \in I$, because another element in X exists in $L_{\alpha_{i+1}}$ greater than it. Hence, the upper bounds of X are the elements in L_β, with $\beta \in \Lambda$ and $\alpha_s \sqsubseteq \beta$, whose least element is $\bot_{\alpha_s}$. Thus, $\sup X = \bot_{\alpha_s}$.

3. Lattice-based sums

This section recalls the main definitions and results given in [12]. Based on an index set Λ equipped with a lattice structure, the authors in [12] generalized the idea of Birkhoff's ordinal sums of posets. From now on, given a poset $(P_\alpha, \preceq_\alpha)$, if its maximum (minimum) element exists, it will be denoted as $\top_\alpha$ $(\bot_\alpha)$. Furthermore, we will write $x \parallel_\alpha y$ when two elements x and y of P_α are incomparable, of simply $x \parallel y$ to elements of arbitrary posets [10].

Definition 11. Given a non-empty lattice $(\Lambda, \sqsubseteq)$, a Λ-*based poset family* is a family of posets $\{(P_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ that satisfies, for each $\alpha, \beta \in \Lambda$, with $\alpha \neq \beta$, that $P_\alpha \cap P_\beta = \varnothing$, or one of the following two conditions:

1. $P_\alpha \cap P_\beta = \{x_{\alpha\beta}\}$, with $\alpha \sqsubseteq \beta$, where $x_{\alpha\beta} = \top_\alpha = \bot_\beta$ and, for each $\varepsilon \in \Lambda$, with $\alpha \sqsubset \varepsilon \sqsubset \beta$, we have $P_\varepsilon = \{x_{\alpha\beta}\}$ and verifies, given $\delta, \gamma \in \Lambda$, with $\delta \parallel \gamma$, $\delta \sqsubset \beta$ and $\alpha \sqsubset \gamma$, that: $P_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ or $P_\gamma = \{\bot_{\sup\{\delta,\gamma\}}\}$.
2. $1 \leq \left|P_\alpha \cap P_\beta\right| \leq 2$ with $\alpha \parallel \beta$, and for all $x_{\alpha\beta} \in P_\alpha \cap P_\beta$ we have $x_{\alpha\beta} = \top_\alpha = \top_\beta = \bot_{\sup\{\alpha,\beta\}}$ or $x_{\alpha\beta} = \bot_\alpha = \bot_\beta = \top_{\inf\{\alpha,\beta\}}$.

Notice that when the maximum or minimum elements of a particular poset have been considered in the previous definition, it is assumed that this element exists in the poset. Moreover, we have updated the name used in [12] in order to emphasize the structure of the pairs considered in the family. From now on, we will consider a family $\left\{\left(L_\alpha, \preceq_\alpha\right)\right\}_{\alpha\in\Lambda}$ of lattices, that is, a Λ-based lattice family. Next, we present a technical definition.

Definition 12. Given a Λ-based lattice family $\left\{\left(L_\alpha, \preceq_\alpha\right)\right\}_{\alpha\in\Lambda}$ and $x \in \bigcup_{\alpha\in\Lambda} L_\alpha$, we say that an element $\alpha^* \in \Lambda$ is *a maximal index of* x (respectively, $\alpha_* \in \Lambda$ is a *minimal index of* x) if α^* is a maximal (respectively, a minimal) element of the set $I_x = \left\{\alpha \in \Lambda \mid x \in L_\alpha\right\}$. Denote by $I_x^{\max}$ and $I_x^{\min}$ the set of all maximal and minimal indices of x, respectively.

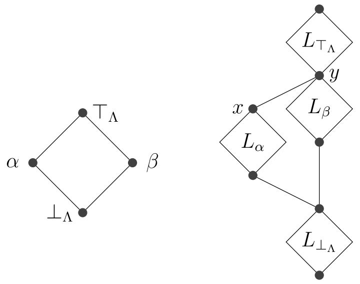

Fig. 1. Lattice $(\Lambda, \sqsubseteq)$ and Λ-based lattice sum of Example 2.

Given a Λ-based lattice family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$, an ordering $\preceq$ was defined on $\bigcup_{\alpha\in\Lambda} L_\alpha$ in [12]. Furthermore, given $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$, with $x \neq y$, we have that x *and* y *are incomparable* (simply denoted as[1] $x \parallel y$) if for all $\alpha, \beta \in \Lambda$ such that $x \in L_\alpha$ and $y \in L_\beta$ we have $\alpha \parallel \beta$, or $\alpha = \beta$ and $x \parallel_\alpha y$, that is, x and y are incomparable in L_α. A particular condition on the sets I_x will be required, which is detailed next.

Definition 13. Given a Λ-based lattice family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ we say that it satisfies the *(M) condition* if:
(M): for all $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, the sets $I_x = \{\delta \in \Lambda \mid x \in L_\delta\}$ and $I_y = \{\beta \in \Lambda \mid y \in L_\beta\}$ have both maximal and minimal elements.

Definition 14. A semibounded Λ-based lattice family is a Λ-based lattice family that satisfies, for all $\alpha, \beta \in \Lambda$ with $\alpha \parallel \beta$, the set $L_{\inf\{\alpha,\beta\}}$ has a top element and the set $L_{\sup\{\alpha,\beta\}}$ has a bottom element.

Notice that this definition is needed because the lattices do not need to be complete or bounded from the beginning. Clearly, families of complete, or only bounded, lattices implies the semibounded property. Now, it is possible to define the supremum and infimum of two elements on a Λ-based lattice family.

Definition 15.
Let $(\Lambda, \sqsubseteq)$ be a non-empty lattice. A Λ*-based lattice sum* of a semibounded Λ-based lattice family with (M) condition $\{(L_\alpha, \wedge_\alpha, \vee_\alpha)\}_{\alpha\in\Lambda}$ is a triple $(L_\mathrm{s}, \wedge_\mathrm{s}, \vee_\mathrm{s})$ where $L_\mathrm{s} = \cup_{\alpha\in\Lambda} L_\alpha$ and the mappings $\wedge_\mathrm{s}, \vee_\mathrm{s}$ on L_s are defined, for each $x, y \in L_\mathrm{s}$, as follows:

$$x \wedge_\mathrm{s} y = \begin{cases} x \wedge_\alpha y & \text{if } x, y \in L_\alpha, \text{ with } \alpha \in \Lambda \\ x & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha, \beta \in \Lambda \text{ and } \alpha \sqsubset \beta \\ y & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha, \beta \in \Lambda \text{ and } \beta \sqsubset \alpha \\ \top_{\inf\{\alpha^*,\beta^*\}} & \text{otherwise} \end{cases}$$

where $\alpha^* \in I_x^{\max}$ and $\beta^* \in I_y^{\max}$.

$$x \vee_\mathrm{s} y = \begin{cases} x \vee_\alpha y & \text{if } x, y \in L_\alpha, \text{ with } \alpha \in \Lambda \\ y & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha, \beta \in \Lambda \text{ and } \alpha \sqsubset \beta \\ x & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha, \beta \in \Lambda \text{ and } \beta \sqsubset \alpha \\ \bot_{\sup\{\alpha_*,\beta_*\}} & \text{otherwise} \end{cases}$$

where $\alpha_* \in I_x^{\min}$ and $\beta_* \in I_y^{\min}$.

Notice that the fourth case in both definitions is lightly different from the definition given in [12]. This case is more accurate than the original one because it really captures the real definition, since this four case is only considered when the indices α and β are incomparable. However, the notation $x \parallel y$ also captures the case when $\alpha = \beta$ and $x \parallel_\alpha y$.

Remark that all possibilities in definitions above are considered. For example, if $x \in L_\alpha$, $y \in L_\beta$, with $\alpha \parallel \beta$, but x and y are not incomparable, that is, there exist $\alpha', \beta' \in \Lambda$ such that $x \in L_{\alpha'}, y \in L_{\beta'}$ and α', β' are comparable, then we cannot apply the fourth case, but we have one of the three first cases, since α' and β' are comparable.

Moreover, note that this definition includes the ordinal sum and the horizontal sum as special cases.

Example 2. We will show an example about the comment above. Let $(\Lambda, \sqsubseteq)$ be the lattice on the left side of Fig. 1 and the Λ-based lattice sum on the right side Fig. 1. We have that $x \in L_\alpha$, $y \in L_\beta$, with $\alpha \parallel \beta$, but x and y are not incomparable, because $x \in L_\alpha, y \in L_{\top_\Lambda}$ and $\alpha, \top_\Lambda$ are comparable ($\alpha \sqsubseteq \top_\Lambda$), then we cannot apply the fourth case, but the second case, and we obtain that $x \wedge_\mathrm{s} y = x$.

Theorem 2 ([12]). *A* Λ*-based lattice sum* $(L_s, \wedge_s, \vee_s)$ *is a lattice.*

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[1] In order to simplify the notation we have use the same symbols to incomparable elements in $(\bigcup_{\alpha\in\Lambda} L_\alpha, \preceq)$ and in $(\Lambda, \sqsubseteq)$, since no confusion exists due to, for instance, different symbols to the elements are considered.

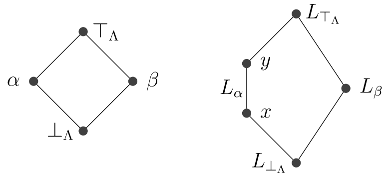


Fig. 2. Lattice $(\Lambda, \sqsubseteq)$ and Λ-based lattice sum of Example 3.

Although Λ and the lattices in the family be distributive, the lattice-based sum obtained might not be distributive, as the following example shows.

Example 3. Given the lattice $(\Lambda, \sqsubseteq)$ depicted on the left side of Fig. 2, if we consider singletons as lattices for each element except to α, in which we take the linear set of two elements $L_\alpha = \{x, y\}$, with $x \preceq_\alpha y$, we clearly obtain that the lattice-based sum obtained (Fig. 2) is not distributive, since it is the well-known N_5 lattice [10]. The proof is reproduced below[2]

$$(x \vee_s \beta) \wedge_s y = \top_\Lambda \wedge_s y = y \neq x = x \vee_s \bot_\Lambda = (x \wedge_\alpha y) \vee_s \bot_\Lambda = (x \wedge_s y) \vee_s (\beta \wedge_s y)$$

4. Lattice-based sums of generalized t-norms

Based on Clifford's ordinal sums of semigroups [4], ordinal sums of t-norms and related types of operations were introduced and discussed in many papers and monographs, for example in [16,18,23,24,27–29]. In this section, we will increase the flexibility of these approaches, extending them in two directions: the non-empty index set will be the career Λ of a lattice $(\Lambda, \sqsubseteq)$, and discussed operations can be more general than the classical t-norms, also improving the approach presented in [11]. Moreover, we will prove that, if these operators are associative, then the operator of the Λ-based lattice sum is associative, and we obtain a similar result if these operators are t-norms on lattices.

Definition 16. Given a Λ-based lattice sum $(L_s, \preceq_s)$ of a semibounded Λ-based lattice family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ satisfying the (M) condition, and an operator $\&_\alpha : L_\alpha \times L_\alpha \to L_\alpha$, for each $\alpha \in \Lambda$, satisfying that, if $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$, with $x_{\alpha\beta} = \top_\alpha = \bot_\beta$, then $\bot_\beta$ is the annihilator of $(L_\beta, \&_\beta)$, for all $\alpha, \beta \in \Lambda$. The family $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$ is called *$\&_\Lambda$-lattice family*. Furthermore, its *$\&_\Lambda$-lattice sum* is the triple $(L_s, \preceq_s, \&_s)$, where $\&_s : L_s \times L_s \to L_s$ is a binary operator defined, for all $x, y \in L_s$, with $x \in L_\alpha$, $y \in L_\beta$ and $\alpha, \beta \in \Lambda$, as:

$$x \,\&_s\, y = \begin{cases} x \,\&_\alpha\, y & \text{if } \alpha = \beta \\ x \wedge_s y & \text{otherwise} \end{cases}$$

where $\wedge_s$ is the meet operator introduced in Definition 15.

It is easy to check that a $\&_\Lambda$-lattice sum of general operators defined on bounded lattices is a generalization of an ordinal sum of t-norms on $[0, 1]$, because, given a linearly ordered index set $(I, \leq)$, a family of pairwise disjointed subintervals $\{]a_i, b_i[\}_{i\in I}$ of $[0, 1]$ and a family of t-norms $(T_i)_{i\in I}$ on [0,1], if we consider $\Lambda = I$ and $L_i = [a_i, b_i]$, the $\&_\Lambda$-lattice sum of $(T_i)_{i\in I}$ is the ordinal sum of the t-norms $(T_i)_{i\in I}$ defined in Eq. (3).

Moreover, we have that the operator $\&_s$ is equal to:

$$x \,\&_s\, y = \begin{cases} x \,\&_\alpha\, y & \text{if } \alpha = \beta \\ x & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha \sqsubset \beta \\ y & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \beta \sqsubset \alpha \\ \top_{\inf\{\alpha^*, \beta^*\}} & \text{otherwise} \end{cases}$$

for all $x \in L_\alpha$, $y \in L_\beta$, with $\alpha, \beta \in \Lambda$, and $\alpha^* \in I_x^{\max}$ and $\beta^* \in I_y^{\max}$. Remark that in the last case we could write $\top_{\alpha\otimes\beta}$ instead of $\top_{\inf\{\alpha,\beta\}}$, where $\otimes$ is a binary operator defined on Λ. However, the associativity of $\otimes$ does not imply the associativity of $\&_s$, as the following example shows.

Example 4. In the lattice that appears in Fig. 3 we can define the commutative and associative operation $\otimes$ given in the left table of the lattice.

$\otimes$	$\bot_\Lambda$	γ	α	β	δ	$\top_\Lambda$
$\bot_\Lambda$	$\bot_\Lambda$	$\bot_\Lambda$	$\bot_\Lambda$	$\bot_\Lambda$	$\bot_\Lambda$	$\bot_\Lambda$
γ	$\bot_\Lambda$	$\bot_\Lambda$	γ	$\bot_\Lambda$	$\bot_\Lambda$	γ
α	$\bot_\Lambda$	γ	α	γ	γ	α
β	$\bot_\Lambda$	$\bot_\Lambda$	γ	γ	$\bot_\Lambda$	β
δ	$\bot_\Lambda$	$\bot_\Lambda$	γ	$\bot_\Lambda$	δ	δ
$\top_\Lambda$	$\bot_\Lambda$	γ	α	β	δ	$\top_\Lambda$

---

[2] It should be noted that the lattices corresponding to an index that differs from α are singletons. To simplify the notation, we will use the same symbol to denote either the index or the element of the associated lattice.

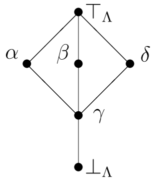


Fig. 3. Lattice $(\Lambda, \sqsubseteq)$.

From this lattice we can construct a family $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$ satisfying that $L_i \cap L_j = \emptyset$, for all $i, j \in \Lambda$. If we assume that $\top_{\alpha\otimes\beta}$ instead of $\top_{\inf\{\alpha,\beta\}}$, in the definition of $\&_s$, then we obtain, given $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\delta$, that:

$$\begin{aligned}(x \&_s y) \&_s z &= \top_{\alpha\otimes\beta} \&_s z = \top_\gamma \&_s z = \top_\gamma\\ x \&_s (y \&_s z) &= x \&_s \top_{\beta\otimes\delta} = x \&_s \top_{\bot_\Lambda} = \top_{\bot_\Lambda}\end{aligned}$$

which are not equals. Therefore, $\&_s$ will not be associative.

Therefore, although some other associative operators could provide the associativity of $\&_s$, we will consider the infimum on $(\Lambda, \sqsubseteq)$ hereon. Before to introduce some general properties of operator $\&_s$, we must ensure that it is well defined.

Proposition 5. *Given a $\&_\Lambda$-lattice sum $(L_s, \preceq_s, \&_s)$ associated with the $\&_\Lambda$-lattice family $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$, we have that the operator $\&_s$ is well defined.*

Proof. The proof is similar to well-defined proof for the meet operator $\wedge_s$ given to Theorem 2 in [12]. ☐

Now, we will present some properties about $\&_s$, in order to prove that this is a generalization of the ordinal sum on the sense of Clifford for t-norms.

Proposition 6. *Let $(L_s, \preceq_s, \&_s)$ be a $\&_\Lambda$-lattice sum. The operators $\&_\alpha$ are associative (commutative), for all $\alpha \in \Lambda$, if and only if the operator $\&_s$ is associative (commutative).*

Proof. It is similar to the proof of Theorem 2 given in [12], to prove that the meet operator is associative (commutative). ☐

From Proposition 6, we obtain as a particular case Theorem 1 (recalled in Section 2, from [4]). As a consequence, in the linear index set case, we have that, if $(\Lambda, \sqsubseteq)$ is linear, then the family of semigroups in Theorem 1 is a $\&_\Lambda$-lattice family, and, therefore, Proposition 6 proves Theorem 1.

Some other interesting properties are given in the following results.

Proposition 7. *Let $(L_s, \preceq_s, \&_s)$ be a $\&_\Lambda$-lattice sum. For all $\alpha \in \Lambda$, $x \&_\alpha y \preceq_\alpha x \wedge_\alpha y$, if and only if $\&_s$ satisfies $x \&_s y \preceq_s x \wedge_s y$, for all $x, y \in L_s$.*

Proof. Firstly, we consider that $x \&_\alpha y \preceq_\alpha x \wedge_\alpha y$, for all $\alpha \in \Lambda$ and $x, y \in L_s$. Let $x \in L_\alpha$, $y \in L_\beta$ be, with $\alpha, \beta \in \Lambda$, and we will consider the two possible cases to $x \&_s y$:

- If $\alpha = \beta$, then $x \&_s y = x \&_\alpha y \preceq_\alpha \inf\{x, y\}$, by hypothesis.
- If $\alpha \neq \beta$, then $x \&_s y = x \wedge_s y$, by definition.

The other implication is given considering x, y in the same lattice L_α, since, in this case, $x \&_\alpha y = x \&_s y \preceq_s x \wedge_s y = x \wedge_\alpha y$. ☐

Corollary 2. *Given a $\&_\Lambda$-lattice sum $(L_s, \preceq_s, \&_s)$, we have that $\bot_{\bot_\Lambda}$ is the annihilator of $\&_{\bot_\Lambda}$ if and only if $\bot_{\bot_\Lambda}$ is the annihilator of $\&_s$.*

Now, we will present a result about the increasing property of the operator $\&_s$del, in order to introduce sufficient conditions to obtain that $\&_s$ is a t-norm.

Proposition 8. *Given a $\&_\Lambda$-lattice sum $(L_s, \preceq_s, \&_s)$, where $\&_\alpha$ is increasing on the left side and $x \&_\alpha y \preceq_\alpha y$, for all $x, y \in L_\alpha$ and $\alpha \in \Lambda$, the operator $\&_s$ is increasing on the left side.*

Proof. Given $x, y, z \in L_s$, with $x \preceq_s y$, we will show that $x \&_s z \preceq_s y \&_s z$. We know that $\alpha, \beta, \delta \in \Lambda$ exist such that $\alpha \sqsubseteq \beta$, $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\delta$, and we will consider the following possibilities:

If $\alpha = \delta$, then we take into account the two cases

- If $\beta = \delta$, then we consider the following cases $x \&_s z = x \&_\alpha z \preceq_s y \&_\alpha z = y \&_s z$.
- If $\beta \neq \delta$, then $\delta \sqsubset \beta$ and, by hypothesis, we obtain that $x \&_s z = x \&_\alpha z \preceq_\alpha z = y \&_s z$.

If $\alpha \neq \delta$, then we have

- If $\beta = \delta$, then $\alpha \sqsubset \beta$ and $x \preceq_s y \&_\beta z$, and consequently $x \&_s z = x \preceq_s y \&_\beta z = y \&_s z$.
- If $\beta \neq \delta$, then we obtain that $x \&_s z = x \wedge_s z \preceq_s y \wedge_s z = y \&_s z$, by the monotonicity of $\wedge_s$.

which prove the monotonicity of $\&_s$. ☐

An analogous result holds for the right side. Hence, we obtain the following corollary.

Corollary 3. *Given a $\&_\Lambda$-lattice sum $(L_s, \preceq_s, \&_s)$, where $\&_\alpha$ is increasing on both arguments and $x \&_\alpha y \preceq_\alpha x \wedge_\alpha y$ for all $x, y \in L_\alpha$ and $\alpha \in \Lambda$, the operator $\&_s$ is increasing on both arguments.*

The following theorem introduces a sufficient condition to assure when the operator of a $\&_\Lambda$-lattice sum is a t-norm.

Theorem 3. *Let $(L_s, \preceq_s, \&_s)$ be a $\&_\Lambda$-lattice sum, where $\&_\alpha$ is associative, commutative, increasing on both arguments, $x \&_\alpha y \preceq_\alpha x \wedge_\alpha y$ for all $x, y \in L_\alpha$ and $\alpha \in \Lambda$, and $\top_{\top_\Lambda}$ is the neutral element of $\&_{\top_\Lambda}$. Then, the operator $\&_s$ is a t-norm.*

Proof. The operator $\&_s$ is commutative, associative and increasing in both arguments by Propositions 6 and 8, respectively.

To prove the boundary condition with the top of the lattice, we need to prove that $x \&_s \top_{\top_\Lambda} = x$.

If $x \in L_\alpha$, with $\alpha \in \Lambda$ and $\alpha \sqsubset \top_\Lambda$, then we straightforwardly obtain $x \&_s \top_{\top_\Lambda} = x$.

Otherwise, $x \&_s \top_{\top_\Lambda} = x \&_{\top_\Lambda} \top_{\top_\Lambda}$, which is equal to x because $\top_{\top_\Lambda}$ is the neutral element of $(L_{\top_\Lambda}, \&_{\top_\Lambda})$. ☐

Notice that the top element in each lattice L_α, with $\alpha \in \Lambda$ does not need to be the neutral element of $\&_\alpha$, for all $\alpha \in \Lambda \setminus \{\top_\Lambda\}$. A first consequence of the previous results is that a $\&_\Lambda$-lattice sum is a generalization, in two directions, of the classical ordinal sum of t-norms. On the one hand, we consider a bounded lattice instead of a bounded linear set of indexes and, on the other hand, we have a $\&_\Lambda$-lattice sum of general operators defined on bounded lattices instead of an ordinal sum of t-norms on $[0, 1]$.

As we previously commented, the contribution of this section extends the results given in other approaches. For example, the definitions of sums given in [11,19,25] are particular cases of the operator $\&_s$ of the $\&_\Lambda$-lattice sum, as it can be seen from the definitions and results recalled in the preliminary section.

5. Distributive-preserving lattice-based sum

One of the most significant properties is the left-continuous property, since it implies the existence of a residuated implication, which is associated with the fuzzy extension of the classical modus ponens [8,14], and so it is a fundamental property in rule-based systems. However, this property is not easy to obtain. For example, despite that every operator $\&_\alpha$ is left-continuous, the lattice-based sum of the family $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$ does not need to be left-continuous, in general, as the following example shows.

Example 5. Given the lattices in Example 3 and the infimum operator in each sublattice (all singleton except to L_α, which has two elements), we have that the operator $\&_s$ of the obtained $\&_\Lambda$-lattice sum is not left-continuous. The proof is similar to the one given in Example 3 to obtain the non-distributivity of the obtained lattice-based sum, which is rewritten next.

$$(x \vee_s \beta) \&_s y = \top_\Lambda \&_s y = y \neq x = x \vee_s \bot_\Lambda = (x \&_\alpha y) \vee_s \bot_\Lambda = (x \&_s y) \vee_s (\beta \&_s y)$$

In order to avoid the aforementioned problem, the Cartesian product of incomparable indices will be considered in the lattice-based sum, obtaining a new mechanism to aggregate lattices.

First of all, we will study the case when a non-empty lattice $(\Lambda, \sqsubseteq)$, with only two incomparable elements $\delta, \gamma \in \Lambda$, is considered. Notice that, in this case every $\alpha \in \Lambda \setminus \{\delta, \gamma\}$ satisfies that

$$\alpha \sqsubseteq \inf\{\delta, \gamma\} \quad \text{or} \quad \sup\{\delta, \gamma\} \sqsubseteq \alpha$$

Otherwise, we will obtain more than two incomparable elements. Hence, the down-set of $\inf\{\delta, \gamma\}$ and the up-set of $\sup\{\delta, \gamma\}$ are chains. The following definition adapts the notion of Λ-based poset family (Definition 11) to the new framework. Since we are focused on sums preserving the distributivity property, we will use the acronym "dp" in the name of the adapted notion.

Definition 17. Given a non-empty lattice $(\Lambda, \sqsubseteq)$, with only two incomparable elements $\delta, \gamma \in \Lambda$, a Λ-based dp-poset family is a family of posets $\{(P_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$, that satisfies, for each $\alpha, \beta \in \Lambda \setminus \{\delta, \gamma\}$, with $\alpha \neq \beta$, that

- $P_\alpha \cap P_\beta = \varnothing$, or
- $P_\alpha \cap P_\beta = \{x_{\alpha\beta}\}$, with $\sup\{\delta, \gamma\} \sqsubseteq \alpha \sqsubseteq \beta$ or $\alpha \sqsubseteq \beta \sqsubseteq \inf\{\delta, \gamma\}$, where $\top_\alpha$, $\bot_\beta$ exist, $x_{\alpha\beta} = \top_\alpha = \bot_\beta$ and, for each $\varepsilon \in \Lambda$, with $\alpha \sqsubset \varepsilon \sqsubset \beta$, we have that $P_\varepsilon = \{x_{\alpha\beta}\}$.

In this new family, an ordering relation can be defined.

Definition 18.

Let $(\Lambda, \sqsubseteq)$ be a non-empty lattice with only two incomparable elements $\delta, \gamma \in \Lambda$, and $\{(P_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ a Λ-based dp-poset family. The Λ-based dp-poset sum of this family is the pair $(P_d, \preceq_d)$, where $P_d = \bigcup\{P_\alpha \mid \alpha \in \Lambda \setminus \{\delta, \gamma\}\} \cup P_\delta \times P_\gamma$ and the relation $\preceq_d$ is defined as: $x \preceq_d y$ if one of the following cases holds, where $\alpha, \beta \in \Lambda \setminus \{\delta, \gamma\}$.

$$\begin{cases} x \preceq_\alpha y, \text{ with } x, y \in P_\alpha \\ x \in P_\alpha, y \in P_\beta, \text{ with } \alpha \sqsubset \beta \\ x \in P_\delta \times P_\gamma, y \in P_\beta, \text{ with } \sup\{\delta, \gamma\} \sqsubseteq \beta \\ x \in P_\alpha, y \in P_\delta \times P_\gamma, \text{ with } \alpha \sqsubseteq \inf\{\delta, \gamma\} \\ x \preceq_{\delta,\gamma} y, \text{ with } x, y \in P_\delta \times P_\gamma \end{cases}$$

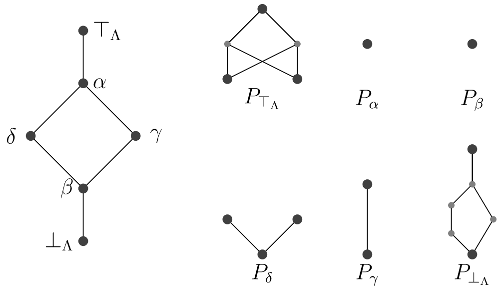


Fig. 4. Lattice $(\Lambda, \sqsubseteq)$ and the posets associated with the indices of Λ in Example 6.

The following result proves that this relation is an ordering relation.

Proposition 9. *With all the assumptions of Definition 18, the relation $\preceq_{\mathrm{d}}$ is an ordering relation, that is, $(P_{\mathrm{d}}, \preceq_{\mathrm{d}})$ is a partial ordered set.*

Proof. Clearly, $\preceq_{\mathrm{d}}$ is reflexive. Moreover, for the antisymmetry we assume that $x \preceq_{\mathrm{d}} y$ and $y \preceq_{\mathrm{d}} x$. We will start analyzing the assumption $x \preceq_{\mathrm{d}} y$ for each case in the definition of the relation $\preceq_{\mathrm{d}}$.

- If $x \preceq_\alpha y$, with $x, y \in P_\alpha, \alpha \notin \{\delta, \gamma\}$, then the expression $y \preceq_{\mathrm{d}} x$ implies $y \preceq_\alpha x$, and so $x = y$ from the asymmetry property of $\preceq_\alpha$.
- If $x \in P_\alpha, y \in P_\beta$, with $\alpha \sqsubset \beta$, then $y \preceq_{\mathrm{d}} x$ implies there exist $\alpha', \beta' \in \Lambda$ with $\beta' \sqsubseteq \alpha'$ such that $y \in P_{\beta'}$ and $x \in P_{\alpha'}$. Thus, $x \in P_\alpha \cap P_{\alpha'}$ and $y \in P_\beta \cap P_{\beta'}$. Now, we can distinguish different possibilities between the indexes. If $\beta' \sqsubseteq \alpha' \sqsubseteq \alpha \sqsubset \beta$, by the definition of Λ-based dp-poset sum, we have $P_\beta \cap P_{\beta'} = \{y\}$, where $\top_{\beta'} = \bot_\beta = y$ and also $P_\alpha = \{y\}$ since $\beta' \sqsubseteq \alpha \sqsubset \beta$. Therefore, we can conclude that $P_\alpha \cap P_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta} = x = y$. We obtain the same result following a similar reasoning for the other cases: $\alpha \sqsubset \beta \sqsubseteq \beta' \sqsubseteq \alpha'$ and $\alpha \sqsubseteq \beta' \sqsubseteq \alpha' \sqsubseteq \beta$.
- If $x \in P_\delta \times P_\gamma, y \in P_\beta$, with $\sup\{\delta, \gamma\} \sqsubseteq \beta$, then the expression $y \preceq_{\mathrm{d}} x$ implies that $\beta \sqsubseteq \inf\{\delta, \gamma\}$, which is a contradiction. Hence, this case is not possible if both inequalities hold.
- If $x \in P_\alpha, y \in P_\delta \times P_\gamma$, with $\alpha \sqsubseteq \inf\{\delta, \gamma\}$, we obtain a contradiction as above.
- Finally, if $x \preceq_{\delta,\gamma} y$, with $x, y \in P_\delta \times P_\gamma$, then the expression $y \preceq_{\mathrm{d}} x$ implies $y \preceq_{\delta,\gamma} x$ and by the considered pointwise ordering of two ordering relations we obtain $x = y$.

Now, we will prove the transitivity. Given $x, y, z \in P_{\mathrm{d}}$, such that $x \preceq_{\mathrm{d}} y$ and $y \preceq_{\mathrm{d}} z$, we need to prove $x \preceq_{\mathrm{d}} z$. Clearly, if there exists $\alpha \in \Lambda \setminus \{\delta, \gamma\}$ such as $x, y, z \in P_\alpha$ or $x, y, z \in P_\delta \times P_\gamma$, then we obtain the result by the transitivity of the involved orderings. Moreover, if x, y, z belong to three different elements in $\Lambda \setminus \{\delta, \gamma\}$, we obtain the result by the ordering in Λ. Otherwise, at least two of the elements are associated with two different indices, and so we have the following cases, where $\alpha, \beta \in \Lambda \setminus \{\delta, \gamma\}$ and $\alpha \sqsubset \beta$.

- If $x \in P_\alpha$ and $z \in P_\beta$, then $x \preceq_{\mathrm{d}} z$, both if $y \in P_\alpha$ or $y \in P_\beta$.
- If $x \in P_\alpha$, $y \in P_\beta$, and $z \in P_\alpha$, then the given inequalities imply $P_\alpha \cap P_\beta = \{x_{\alpha\beta}\}$, where $x \preceq_\alpha x_{\alpha\beta} = \top_\alpha = \bot_\beta = y = z$, that is, $x \preceq_{\mathrm{d}} z$.
- If $\alpha \sqsubseteq \inf\{\delta, \gamma\}$, the only remaining possibility is $x \in P_\alpha$ and $z \in P_\delta \times P_\gamma$, which implies $x \preceq_{\mathrm{d}} z$, in both cases of y, that is, $y \in P_\alpha$ or $y \in P_\delta \times P_\gamma$.
- If $\sup\{\delta, \gamma\} \sqsubseteq \alpha$, then $x \preceq_{\mathrm{d}} z$ holds similarly to the last previous case.

□

Remark 1. Notice that, if $x \in P_{\mathrm{d}}$ and an infinite chain I exists in Λ, such that $x \in P_{\alpha_i}$ for all $\alpha_i \in I$, then by Definition 17 we have that $x = x_{\alpha_i\alpha_{i+1}}$, with $P_{\alpha_{i+1}} = \{x_{\alpha_i\alpha_{i+1}}\}$, for all $\alpha_i \in I$. Therefore, the ordinal limit ω is an element in Λ, which must satisfy $P_\alpha \cap P_\omega = \emptyset$ or $P_\alpha \cap P_\omega = \{x_{\alpha_i\alpha_{i+1}}\}$.

The following examples help to illustrate that a Λ-based dp-poset sum together with the partial order $\preceq_{\mathrm{d}}$ forms a poset.

Example 6. Consider the non-empty lattice $(\Lambda, \sqsubseteq)$ and the family of posets $\{(P_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ given in Fig. 4. The family of posets associated with the structure in Fig. 5 is a Λ-based dp-poset family, P_{d}. Moreover, considering the partial order defined in Definition 18, we have $(P_{\mathrm{d}}, \preceq_{\mathrm{d}})$ is a Λ-based dp-poset sum and a poset as Proposition 9 states.

As we can observe, the two incomparable index δ and γ have the Cartesian product of their respective posets associated with them, preserving the consistency of the order $\sqsubseteq$ in the lattice Λ. Furthermore, it is easy to check that the partial order relation $\preceq_{\mathrm{d}}$ is well defined since the consistency and the order of the indices are preserved through both all the elements of each poset and two elements from different posets. For example, if we consider $x \in P_\delta \times P_\gamma$ and $y \in P_\alpha$, we have that $\sup\{\delta, \gamma\} = \alpha$, and therefore, we obtain $x \preceq_{\mathrm{d}} y$.

The new construction of posets provides the comparability of every two elements belonging to different posets.

Proposition 10. *With all the assumptions of Definition 18, given $x, y \in P_{\mathrm{d}}$, if neither $x, y \in P_\delta \times P_\gamma$ nor $\alpha \in \Lambda \setminus \{\delta, \gamma\}$ exists, such as $x, y \in P_\alpha$, then $x \preceq_{\mathrm{d}} y$ or $y \preceq_{\mathrm{d}} x$.*

Proof. The result arises from Definition 18 and the fact that only two incomparable elements exist in Λ. □

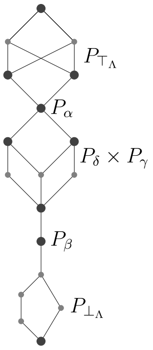


Fig. 5. The Λ-based dp-poset sum $(P_{\mathrm{d}}, \preceq_{\mathrm{d}})$ in Example 6.

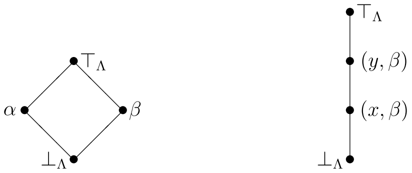


Fig. 6. Lattice $(\Lambda, \sqsubseteq)$ and Λ-based dp-lattice sum of Example 7.

The following definition introduces the meet and join operators on the sum P_{d}, when the posets are lattices. Note that we will consider a family of lattices $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ instead of posets in Definition 17 and will be called Λ-based dp-lattice family. From now on, we will denote such a family as $\mathcal{L}_\Lambda$.

Definition 19.

Let $(\Lambda, \sqsubseteq)$ be a non-empty lattice with only two incomparable elements $\delta, \gamma \in \Lambda$. A *distributive-preserving Λ-based lattice sum of a Λ-based dp-lattice family* $\mathcal{L}_\Lambda$(Λ-based dp-lattice sum, in short) is a triple $(L_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$ where $L_{\mathrm{d}} = \bigcup\{L_\alpha \mid \alpha \in \Lambda \setminus \{\delta,\gamma\}\} \cup L_\delta \times L_\gamma$ and the mappings $\wedge_{\mathrm{d}}, \vee_{\mathrm{d}}$ on L_{d} are defined, for each $x, y \in L_{\mathrm{d}}$, as follows:

$$x \wedge_{\mathrm{d}} y = \begin{cases} x \wedge_\alpha y & \text{if } x, y \in L_\alpha, \text{ with } \alpha \notin \{\delta,\gamma\} \\ x & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha \sqsubset \beta \\ y & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \beta \sqsubset \alpha \\ x & \text{if } x \in L_\delta \times L_\gamma, y \in L_\beta, \text{ with } \sup\{\delta,\gamma\} \sqsubseteq \beta \\ y & \text{if } x \in L_\alpha, y \in L_\delta \times L_\gamma, \text{ with } \sup\{\delta,\gamma\} \sqsubseteq \alpha \\ x & \text{if } x \in L_\alpha, y \in L_\delta \times L_\gamma, \text{ with } \alpha \sqsubseteq \inf\{\delta,\gamma\} \\ y & \text{if } x \in L_\delta \times L_\gamma, y \in L_\beta, \text{ with } \beta \sqsubseteq \inf\{\delta,\gamma\} \\ x \wedge_{\delta,\gamma} y & \text{otherwise, that is, } x, y \in L_\delta \times L_\gamma \end{cases}$$

where, given $(x_1, x_2), (y_1, y_2) \in L_\delta \times L_\gamma$, we have that $(x_1, x_2) \wedge_{\delta,\gamma} (y_1, y_2) = (x_1 \wedge_\delta y_1, x_2 \wedge_\gamma y_2)$.

$$x \vee_{\mathrm{d}} y = \begin{cases} x \vee_\alpha y & \text{if } x, y \in L_\alpha \\ y & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \alpha \sqsubset \beta \\ x & \text{if } x \in L_\alpha, y \in L_\beta, \text{ with } \beta \sqsubset \alpha \\ y & \text{if } x \in L_\delta \times L_\gamma, y \in L_\beta, \text{ with } \sup\{\delta,\gamma\} \sqsubseteq \beta \\ x & \text{if } x \in L_\alpha, y \in L_\delta \times L_\gamma, \text{ with } \sup\{\delta,\gamma\} \sqsubseteq \alpha \\ y & \text{if } x \in L_\alpha, y \in L_\delta \times L_\gamma, \text{ with } \alpha \sqsubseteq \inf\{\delta,\gamma\} \\ x & \text{if } x \in L_\delta \times L_\gamma, y \in L_\beta, \text{ with } \beta \sqsubseteq \inf\{\delta,\gamma\} \\ x \vee_{\delta,\gamma} y & \text{otherwise, that is, } x, y \in L_\delta \times L_\gamma \end{cases}$$

where, given $(x_1, x_2), (y_1, y_2) \in L_\delta \times L_\gamma$, we have that $(x_1, x_2) \vee_{\delta,\gamma} (y_1, y_2) = (x_1 \vee_\delta y_1, x_2 \vee_\gamma y_2)$.

Next, an illustrative example is presented.

Example 7. Following with the distributive lattice $(\Lambda, \sqsubseteq)$ in Example 3 (left side of Fig. 6) and the lattices associated with the elements (singleton except to the lattice L_α, which is the linear lattice of two elements), we have that the Λ-based dp-lattice sum is the one depicted in Fig. 6, which clearly is distributive.

The following result proves that the construction in Definition 19 is just a lattice.

Theorem 4. *With all the assumptions of Definition 19, the Λ-based dp-lattice sum $(L_d, \wedge_d, \vee_d)$ is a lattice.*

Proof. We will analyze the operator $\wedge_{\mathrm{d}}$, the proof of the other one follows analogously. First of all, we need to check whether $\wedge_{\mathrm{d}}$ is well-defined. The only cases that need to be analyzed are when two elements $\alpha, \beta \in \Lambda \setminus \{\delta, \gamma\}$ exist such that $x \in L_\alpha$ and $y \in L_\alpha \cap L_\beta$.

We will assume that $\alpha \sqsubset \beta$, and the other case can be proved similarly. Therefore, by Definition 17, $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$, with $\sup\{\delta,\gamma\} \sqsubseteq \alpha \sqsubseteq \beta$ or $\alpha \sqsubseteq \beta \sqsubseteq \inf\{\delta,\gamma\}$, where $x_{\alpha\beta} = \top_\alpha = \bot_\beta$. Thus, $y = \top_\alpha$ and consequently we obtain the same result in any case, that is, $x \wedge_{\mathrm{d}} y = x \wedge_\alpha y = x$.

Now, from the definition of $\wedge_{\mathrm{d}}$ and $\preceq_{\mathrm{d}}$, we trivially obtain that $x \wedge_{\mathrm{d}} y \preceq_{\mathrm{d}} x$ and $x \wedge_{\mathrm{d}} y \preceq_{\mathrm{d}} y$, that is, $x \wedge_{\mathrm{d}} y$ is a lower bound of $\{x, y\}$.

Moreover, we will prove that $x \wedge_{\mathrm{d}} y$ is the greatest lower bound of $\{x, y\}$. Hence, given $z \in L_{\mathrm{d}}$, such as $z \preceq_{\mathrm{d}} x$ and $z \preceq_{\mathrm{d}} y$, we will prove that $z \preceq_{\mathrm{d}} x \wedge_{\mathrm{d}} y$.

If there exists $\alpha \in \Lambda \setminus \{\delta,\gamma\}$, such as $x, y \in L_\alpha$, or $x, y \in L_\delta \times L_\gamma$, then the result is clearly obtained.

In the remaining six cases, we obtain from $x \wedge_{\mathrm{d}} y$ one of the two values, that is, either x or y. Therefore, we clearly obtain that $z \preceq_{\mathrm{d}} x \wedge_{\mathrm{d}} y$. ☐

Notice that, by Remark 1, it has not been necessary to require the (M) condition in Definition 19. If the lattice of indices is complete, then the Λ-based dp-lattice sum is complete. This result also extends the ones given in [12], in which complete lattices were not considered.

Theorem 5. *With all the assumptions of Definition 19, and assuming that* $(\Lambda, \sqsubseteq, \bot_\Lambda, \top_\Lambda)$ *and* $(L_\alpha, \wedge_\alpha, \vee_\alpha)$ *are also complete, for all* $\alpha \in \Lambda$, *then the* Λ*-based dp-lattice sum* $(L_d, \wedge_d, \vee_d)$ *is a complete lattice.*

Proof. Given $X \subseteq L_d$, if $X = \varnothing$, we clearly obtain that $\inf X = \top_{\top_\Lambda}$ and $\sup X = \bot_{\bot_\Lambda}$, because $(\Lambda, \sqsubseteq)$, $(L_{\bot_\Lambda}, \preceq_{\bot_\Lambda})$ and $(L_{\top_\Lambda}, \preceq_{\top_\Lambda})$ are bounded. Otherwise, if $\Lambda_X = \{\alpha \in \Lambda \mid X \cap L_\alpha \neq \varnothing\}$ is finite, then we apply Theorem 4 to ensure that it has supremum.

If Λ_X is infinite, we consider the sets

$$\begin{aligned} \Lambda^u_\delta &= \{\alpha \in \Lambda \mid \delta \sqsubset \alpha\} \\ \Lambda^l_\delta &= \{\alpha \in \Lambda \mid \alpha \sqsubset \delta\} \end{aligned}$$

Analogously to Λ_X, we define the sets $\Lambda^u_{\delta,X}$, $\Lambda^l_{\delta,X}$. Now, we consider two cases:

1. If $\Lambda^u_{\delta,X}$ is infinite, then if there exists α_0 such that $X \cap L_{\alpha_0} \neq \varnothing$ and $X \cap L_\alpha = \varnothing$, for all $\alpha \in \Lambda$, with $\alpha_0 \sqsubset \alpha$, then we obtain that

$$\bigvee\nolimits_d X = \bigvee\nolimits_{\alpha_0} (X \cap L_{\alpha_0})$$

   Otherwise, an infinite index I exists such that $\{\alpha_i \in \Lambda \mid i \in I\} \subset \Lambda$, $\alpha_i \sqsubset \alpha_{i+1}$, $L_{\alpha_i} \cap X \neq \varnothing$, for all $i \in I$, and $\bigvee_d X = \bigvee_d \{L_{\alpha_i} \cap X \mid i \in I\}$. Since $(\Lambda, \sqsubseteq)$ is a complete lattice, we consider, in particular, $\alpha_s = \sup\{\alpha_i \mid i \in I\}$ exists, and so

$$\bigvee\nolimits_d X = \bigvee\nolimits_{\alpha_s} (X \cap L_{\alpha_s})$$

2. if $\Lambda^u_{\delta,X}$ is empty, then
   (a) if $X \cap L_\delta \times L_\gamma \neq \varnothing$, then

$$\bigvee\nolimits_d X = \bigvee\nolimits_{\delta,\gamma} (X \cap L_\delta \times L_\gamma)$$

   (b) Otherwise, a similar procedure as in step 1, now with respect to $\Lambda^l_{\delta,X}$, can be applied to reach the result.

Therefore, since the lattices L_α, for all $\alpha \in \Lambda$, are complete, we have that all the previously mentioned suprema exist. Consequently, L_d is a complete lattice. ☐

The following theorem ensures that the distributivity is also preserved.

Theorem 6. *With all the assumptions of Definition 19, if the lattices of* $\mathcal{L}_\Lambda$ *are distributive, then the* Λ*-based dp-lattice sum* $(L_d, \wedge_d, \vee_d)$ *is also distributive.*

Proof. Notice that, since $(\Lambda, \sqsubseteq)$ only has two incomparable elements, then it is distributive. Moreover, $L_\delta \times L_\gamma$ is distributive because of L_δ and L_γ are distributive. As a consequence, L_{d} is a chain of distributive lattices. Therefore, L_{d} is distributive. ☐

Hence, we have analyzed the main properties of a Λ-based dp-lattice sum. In the following section we will enrich this structure with a general operator focused on preserving the left-continuous property.

6. Lattice-based sum of operators towards the left-continuous property

In this section, we will consider an extra operator defined on every lattice of the given family, from which a new general operator will be defined. Different properties of this new operator will be analyzed, paying special attention to the left-continuity property. From here on, we will denote a family of algebras $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$ associated with a Λ-based dp-lattice family and $\&_\alpha : L_\alpha \times L_\alpha \to L_\alpha$ being a binary operator, for every $\alpha \in \Lambda$, as $\mathcal{L}_{\&_\Lambda}$ and we will call it $\&_\Lambda$*-lattice algebra*.

Definition 20. Given a non-empty lattice $(\Lambda, \sqsubseteq)$ with only two incomparable elements $\delta, \gamma \in \Lambda$ and a $\&_\Lambda$-lattice family of algebras $\mathcal{L}_{\&_\Lambda}$, we say that the *$\&_\Lambda$-dp sum* of the family $\mathcal{L}_{\&_\Lambda}$ with respect to $(L_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$ is the triple $(L_{\mathrm{d}}, \preceq_{\mathrm{d}}, \&_{\mathrm{d}})$ where $(L_{\mathrm{d}}, \preceq_{\mathrm{d}})$ is a Λ-based dp-lattice sum of the family $\mathcal{L}_\Lambda$ and the operator $\&_{\mathrm{d}} : L_{\mathrm{d}} \times L_{\mathrm{d}} \to L_{\mathrm{d}}$ is defined, for each $x, y \in L_{\mathrm{d}}$, as follows:

$$x \,\&_{\mathrm{d}}\, y = \begin{cases} \&_\alpha(x, y) & \text{if } x, y \in L_\alpha, \text{ with } \alpha \notin \{\delta, \gamma\} \\ \&_{\delta,\gamma}(x, y) & \text{if } x, y \in L_\delta \times L_\gamma \\ x \wedge_{\mathrm{d}} y & \text{otherwise} \end{cases}$$

where, given $(x_1, x_2), (y_1, y_2) \in L_\delta \times L_\gamma$, we have that $\&_{\delta,\gamma}((x_1, x_2), (y_1, y_2)) = (\&_\delta(x_1, y_1), \&_\gamma(x_2, y_2))$.

This operator clearly adapts the usual definition of ordinal sum to the new construction of lattice-based sum focused on preserving the distributivity of lattices. In this case, the problem is to ensure the sup-preserving property of the operator.

Due to only two elements are incomparable in Λ, then the original lattice must be distributive. This fact will be fundamental to solve the aforementioned problem. First of all, we will prove that the monotonicity of the operators in the family is preserved by its lattice-based sum.

Proposition 11. *With all the assumptions of Definition 20, the operator $\&_d$ of the $\&_\Lambda$-dp-lattice sum of a family $\mathcal{L}_{\&_\Lambda}$, where $\&_\alpha$ is monotonic in the left (right) argument and $x \,\&_\alpha\, y \preceq_\alpha y$ ($x \,\&_\alpha\, y \preceq_\alpha x$), for all $x, y \in L_\alpha$, $\alpha \in \Lambda$, is also monotonic in the left (right) argument, and $x \,\&_d\, y \preceq_d y$ ($x \,\&_d\, y \preceq_d x$), for all $x, y \in L_d$.*

Proof. Given $x, y, z \in L_{\mathrm{d}}$, such that $x \preceq_{\mathrm{d}} y$, we need to prove that $x \,\&_{\mathrm{d}}\, z \preceq_{\mathrm{d}} y \,\&_{\mathrm{d}}\, z$.

- If $x, z \in L_\alpha$, with $\alpha \notin \{\delta, \gamma\}$, then, due to $x \preceq_{\mathrm{d}} y$,
  - $y \in L_\alpha$, from which we trivially obtain the result; or
  - $y \in L_\beta$, with $\beta \notin \{\delta, \gamma\}$ and $\alpha \sqsubset \beta$, we obtain that

    $$x \,\&_{\mathrm{d}}\, z = x \,\&_\alpha\, z \overset{(*)}{\preceq_\alpha} z = y \wedge_{\mathrm{d}} z = y \,\&_{\mathrm{d}}\, z$$

    where $(*)$ holds by hypothesis; or
  - $y \in L_\delta \times L_\gamma$, with $\alpha \sqsubseteq \inf\{\delta, \gamma\}$, then we also obtain the result following a similar chain as in the previous case.
- If $x, z \in L_\delta \times L_\gamma$, then we only have one of the following cases, if $x \preceq_{\mathrm{d}} y$.
  - $y \in L_\delta \times L_\gamma$, from which we trivially obtain the result; or
  - $y \in L_\alpha$, with $\alpha \notin \{\delta, \gamma\}$ and $\sup\{\delta, \gamma\} \sqsubseteq \alpha$, then $x \,\&_{\mathrm{d}}\, z = x \,\&_{\delta,\gamma}\, z \overset{(*)}{\preceq_{\delta,\gamma}} z = y \wedge_{\mathrm{d}} z = y \,\&_{\mathrm{d}}\, z$ where $(*)$ holds because $\&_{\delta,\gamma}$ is defined by component and by hypothesis.
- If $x \in L_\delta \times L_\gamma$ and $z \in L_\alpha$, with $\alpha \notin \{\delta, \gamma\}$, then
  - if $y \in L_\delta \times L_\gamma$, we clearly obtain the result in both cases: $\sup\{\delta, \gamma\} \sqsubseteq \alpha$ or $\alpha \sqsubseteq \inf\{\delta, \gamma\}$.
  - if $y \in L_\beta$, with $\beta \notin \{\delta, \gamma\}$ and $\sup\{\delta, \gamma\} \sqsubseteq \beta$, then $x \,\&_{\mathrm{d}}\, z = x \wedge_{\mathrm{d}} z$, which clearly is less or equal to y (by $x \preceq_{\mathrm{d}} y$), z and $y \,\&_\beta\, z$, that are the only possibilities to $y \,\&_{\mathrm{d}}\, z$.
- If $x \in L_\alpha$, with $\alpha \notin \{\delta, \gamma\}$, and $z \in L_\delta \times L_\gamma$, then
  - if $y \in L_\delta \times L_\gamma$, then $\alpha \sqsubseteq \inf\{\delta, \gamma\}$ and we obtain the result because: $x \,\&_{\mathrm{d}}\, z = x \wedge_{\mathrm{d}} z = x \in L_\alpha$, and $y \,\&_{\mathrm{d}}\, z = y \,\&_{\delta,\gamma}\, z \in L_\delta \times L_\gamma$.
  - if $y \in L_\beta$, with $\beta \notin \{\delta, \gamma\}$, then $\alpha \sqsubseteq \beta$, and so we obtain the result independently of the relation to $\sup\{\delta, \gamma\}$ and $\inf\{\delta, \gamma\}$.
    - ∗ if $\alpha \sqsubseteq \beta \sqsubseteq \inf\{\delta, \gamma\}$, then

      $$x \,\&_{\mathrm{d}}\, z = x \wedge_{\mathrm{d}} z = x \preceq_{\mathrm{d}} y = y \wedge_{\mathrm{d}} z = y \,\&_{\mathrm{d}}\, z$$

    - ∗ if $\sup\{\delta, \gamma\} \sqsubseteq \alpha \sqsubseteq \beta$, then

      $$x \,\&_{\mathrm{d}}\, z = x \wedge_{\mathrm{d}} z = z = y \wedge_{\mathrm{d}} z = y \,\&_{\mathrm{d}}\, z$$

    - ∗ if $\alpha \sqsubseteq \inf\{\delta, \gamma\} \sqsubseteq \beta$, then

      $$x \,\&_{\mathrm{d}}\, z = x \wedge_{\mathrm{d}} z = x \preceq_{\mathrm{d}} z = y \,\&_{\mathrm{d}}\, z$$

Thus, $\&_{\mathrm{d}}$ is monotonic. Furthermore, $x \,\&_{\mathrm{d}}\, y \preceq_{\mathrm{d}} \min\{x, y\}$, for all $x, y \in L_{\mathrm{d}}$, because every part of the definition of $\&_{\mathrm{d}}$ satisfies this property. □

The following result shows that the new construction of lattice-based sum preserves the supremum of finite sets.

Theorem 7. *With all the assumptions of Definition 20, where $(\Lambda, \sqsubseteq)$ is finite, the operator $\&_{\mathrm{d}}$ of the $\&_\Lambda$-dp-lattice sum of a family $\mathcal{L}_{\&_\Lambda}$, where $\&_\alpha$ preserves the supremum of finite sets on the left (right) side and $x \,\&_\alpha\, y \preceq_{\mathrm{d}} y$ ($x \,\&_\alpha\, y \preceq_{\mathrm{d}} x$), for all $x, y \in L_\alpha$, $\alpha \in \Lambda$, also preserves the supremum of finite sets on the left (right) side.*

Proof. Given $x, y, z \in L_{\mathrm{d}}$, we will prove that:

$$(x \vee_{\mathrm{d}} y) \,\&_{\mathrm{d}}\, z = (x \,\&_{\mathrm{d}}\, z) \vee_{\mathrm{d}} (y \,\&_{\mathrm{d}}\, z) \tag{5}$$

following the next cases:

- If $x, y \in L_\alpha$, with $\alpha \in \Lambda \setminus \{\delta, \gamma\}$, and

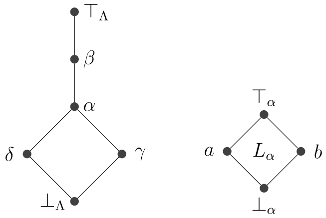


Fig. 7. Lattice $(\Lambda, \sqsubseteq)$ and lattice $(L_\alpha, \preceq_\alpha)$ of Example 8.

  - $z \in L_\alpha$, then Equality (5) holds by the hypothesis concerning $\&_\alpha$.
  - $z \in L_\beta$, with $\alpha \sqsubset \beta$ or $\beta \sqsubset \alpha$, we obtain Equality (5) clearly.
  - $z \in L_\delta \times L_\gamma$ and, if $\alpha \sqsubseteq \inf\{\delta, \gamma\}$, then

$$(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} z = x \vee_\alpha y = (x \wedge_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \wedge_{\mathrm{d}} z) = (x \&_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} z)$$

    Otherwise, if $\sup\{\delta, \gamma\} \sqsubseteq \alpha$, then

$$(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} z = z = z \vee_{\mathrm{d}} z = (x \&_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} z)$$

- If $x, y \in L_\delta \times L_\gamma$ and
  - $z \in L_\delta \times L_\gamma$, then Equality (5) holds by hypothesis concerning $\&_\delta$ and $\&_\gamma$.
  - $z \in L_\alpha$, and, if $\alpha \sqsubseteq \inf\{\delta, \gamma\}$, then

$$(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} z = z = z \vee_{\mathrm{d}} z = (x \&_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} z)$$

    Otherwise, if $\sup\{\delta, \gamma\} \sqsubseteq \alpha$, then

$$(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} z = x \vee_\alpha y = (x \wedge_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \wedge_{\mathrm{d}} z) = (x \&_{\mathrm{d}} z) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} z)$$

- If neither $x, y \in L_\delta \times L_\gamma$ nor $\alpha \in \Lambda \setminus \{\delta, \gamma\}$ exists, such as $x, y \in L_\alpha$, then by Proposition 10 we obtain that $x \preceq_{\mathrm{d}} y$ or $y \preceq_{\mathrm{d}} x$. We will assume without loss of generality, that $x \preceq_{\mathrm{d}} y$. In this case, we have that $x \vee_{\mathrm{d}} y = y$. Moreover, by the monotonicity on the left side of the operators in the family (they preserve the supremum of finite sets), hypothesis and Proposition 11, we also have that $\&_{\mathrm{d}}$ is also monotonic on the left side. Therefore, $(x \&_{\mathrm{d}} z) \preceq_{\mathrm{d}} (y \&_{\mathrm{d}} z)$, and we obtain Equality (5).

  The property concerning the right side of the operators follows similarly.

□

Notice that the assumption that $x \&_\alpha y \preceq_{\mathrm{d}} y$ $(x \&_\alpha y \preceq_{\mathrm{d}} x)$, for all $x, y \in L_\alpha$, $\alpha \in \Lambda$, is necessary, as the following example shows.

Example 8. Let us consider the lattice $(\Lambda, \sqsubseteq)$ together with the lattice $(L_\alpha, \preceq_\alpha)$ depicted in Fig. 7. Given an operator $\&_\alpha$ defined on L_α, we will show that the condition $x \&_\alpha z \preceq_{\mathrm{d}} z$, for all $z \in L_\alpha$, must hold, if we include the structure $(L_\alpha, \preceq_\alpha, \&_\alpha)$ in the family $\mathcal{L}_{\&_\Lambda}$ and we require that $\&_d$ be left-continuous.

Specifically, if the following equalities holds

$$\begin{aligned}
(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} b &= (x \&_{\mathrm{d}} b) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} b) \\
(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} a &= (x \&_{\mathrm{d}} a) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} a) \\
(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} \perp_\alpha &= (x \&_{\mathrm{d}} \perp_\alpha) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} \perp_\alpha)
\end{aligned}$$

for $x \in L_\alpha$ and $y \in L_\beta$, then we obtain from the first one: $(x \vee_{\mathrm{d}} y) \&_{\mathrm{d}} b = y \&_{\mathrm{d}} b = b$; and $(x \&_{\mathrm{d}} b) \vee_{\mathrm{d}} (y \&_{\mathrm{d}} b) = (x \&_\alpha b) \vee_\alpha b$. Consequently, in order to obtain the equality, we have: $(x \&_\alpha b) \preceq_\alpha b$. From the second and third equalities, we deduce analogously that $(x \&_\alpha a) \preceq_\alpha a$, and $(x \&_\alpha \perp_\alpha) \preceq_\alpha \perp_\alpha$. The inequality with respect to $\top_\alpha$ holds trivially. Thus, $x \&_\alpha z \preceq_\alpha z$ for all $x, z \in L_\alpha$.

In order to ensure that the new construction of lattice-based sum preserves the left-continuous property we need that the lattice $(\Lambda, \sqsubseteq)$ be complete.

Theorem 8. *With all the assumptions of Definition 20, where $(\Lambda, \sqsubseteq)$ is a complete lattice, $\mathcal{L}_{\&_\Lambda}$ is an $\&_\Lambda$-lattice family of algebras, satisfying the lattices of $\mathcal{L}_\Lambda$ are complete lattices and the operators $\&_\alpha$ are left-continuous on the left (right) side and satisfy $x \&_\alpha y \preceq_d y$ $(x \&_\alpha y \preceq_d x)$, for all $x, y \in L_\alpha$, $\alpha \in \Lambda$. Then, the $\&_\Lambda$-dp-lattice sum $(L_{\mathrm{d}}, \preceq_{\mathrm{d}}, \&_{\mathrm{d}})$ of $\mathcal{L}_{\&_\Lambda}$ verifies that $(L_{\mathrm{d}}, \preceq_{\mathrm{d}})$ is a complete lattice and $\&_{\mathrm{d}}$ is also left-continuous on the left (right) side.*

Proof. We need to proof that the operator $\&_{\mathrm{d}} : L_{\mathrm{d}} \times L_{\mathrm{d}} \to L_{\mathrm{d}}$, given in Definition 20 is left-continuous on the left side, that is, it preserves supremums of arbitrary sets on the left argument.

Clearly, $\perp_{\perp_\Lambda} \&_{\mathrm{d}} z = \perp_{\perp_\Lambda}$, for all $z \in L_{\mathrm{d}}$. Now, given $X \subseteq L_{\mathrm{d}}$, with $X \neq \varnothing$, and $z \in L_{\mathrm{d}}$, we will prove that:

$$\left(\bigvee\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}} z = \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X\} \tag{6}$$

following the next cases, where

$$\begin{aligned}\Lambda^u_\delta &= \{\alpha \in \Lambda \mid \delta \sqsubset \alpha\}\\ \Lambda^l_\delta &= \{\alpha \in \Lambda \mid \alpha \sqsubset \delta\}\\ \Lambda_X &= \{\alpha \in \Lambda \mid X \cap L_\alpha \neq \varnothing\}\end{aligned}$$

and $\Lambda^u_{\delta,X}$, $\Lambda^l_{\delta,X}$ are defined similarly.

1. If $\Lambda^u_{\delta,X}$ is non-empty, then
   (a) If there exists α_0 such that $X \cap L_{\alpha_0} \neq \varnothing$ and $X \cap L_\beta = \varnothing$, for all $\beta \in \Lambda$, with $\alpha_0 \sqsubset \beta$, then
   - If $z \in L_{\alpha_0}$, then Eq. (6) holds because $\&_{\alpha_0}$ is left-continuous.
   - If $z \in L_\alpha$, with $\alpha \in \Lambda \setminus \{\delta, \gamma\}$ and $\alpha \sqsubset \alpha_0$, or $z \in L_\delta \times L_\gamma$, then

     $$\Big(\bigvee\nolimits_{\mathrm{d}} X\Big) \&_{\mathrm{d}} z = \Big(\bigvee\nolimits_{\alpha_0} (X \cap L_{\alpha_0})\Big) \&_{\mathrm{d}} z = z$$

     Since $x \&_{\mathrm{d}} z = z$, for all $x \in X \cap L_{\alpha_0}$, by the monotonicity of $\&_{\mathrm{d}}$ (Proposition 11), we obtain that

     $$\bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X\} = \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X \cap L_{\alpha_0}\} = z$$

     which proves Eq. (6).
   - If $z \in L_\beta$, with $\alpha_0 \sqsubset \beta$, then by the monotonicity of $\&_{\mathrm{d}}$, we obtain that

     $$\begin{aligned}\Big(\bigvee\nolimits_{\mathrm{d}} X\Big) \&_{\mathrm{d}} z &= \Big(\bigvee\nolimits_{\alpha_0} (X \cap L_{\alpha_0})\Big) \&_{\mathrm{d}} z\\ &= \bigvee\nolimits_{\alpha_0} (X \cap L_{\alpha_0})\\ &= \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X \cap L_{\alpha_0}\}\\ &= \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X\}\end{aligned}$$

   (b) Otherwise, we consider $\alpha_s = \bigvee \Lambda_X$, which exists, because $(\Lambda, \sqsubseteq)$ is complete, and satisfies $L_{\alpha_s} \cap X = \varnothing$. In this case, non element x in L_{α_i}, with $\alpha_i \in \Lambda_X$, can be an upper bound of X because we reach a contradiction with the maximality of α_s. Therefore, $\bigvee_{\mathrm{d}} X = \bot_{\alpha_s}$ and we obtain
   - If $z \in L_\beta$, with $\alpha_s \sqsubseteq \beta$, then

     $$\Big(\bigvee\nolimits_{\mathrm{d}} X\Big) \&_{\mathrm{d}} z = \bot_{\alpha_s} \&_{\mathrm{d}} z = \bot_{\alpha_s} = \bigvee\nolimits_{\mathrm{d}} X = \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X\}$$

   - Otherwise, $z \in L_\alpha$, with $\alpha \sqsubset \alpha_s$. Hence, there exists $\alpha_i \in \Lambda_X$ such that $\alpha \sqsubseteq \alpha_i$, and so $x \&_{\mathrm{d}} z = z$ for all $x \in L_{\alpha_i} \cap X$ and, by the monotonicity of $\&_{\mathrm{d}}$, we obtain that

     $$\Big(\bigvee\nolimits_{\mathrm{d}} X\Big) \&_{\mathrm{d}} z = \bot_{\alpha_s} \&_{\mathrm{d}} z = z = \bigvee\nolimits_{\mathrm{d}} \{x \&_{\mathrm{d}} z \mid x \in X\}$$

2. If $\Lambda^u_{\delta,X}$ is empty, then
   (a) If $X \cap L_\delta \times L_\gamma \neq \varnothing$, then Eq. (6) is satisfied following an analogous development to above due to $\&_\delta$ and $\&_\gamma$ are left-continuous, and the definition by components of $\&_{\delta,\gamma}$.
   (b) Otherwise, a similar reasoning to Step 1 can be developed concerning $\Lambda^l_{\delta,X}$.

☐

Notice that, the lattices of the family does not need to be distributive in order to obtain the left-continuity of the operator of the $\&_\Lambda$-dp-lattice sum. We illustrate this fact in the following example.

Example 9. We consider the complete lattice $(\Lambda, \sqsubseteq)$ given in Fig. 4 of Example 6. In order to obtain a family $\mathcal{L}_\Lambda$ of complete lattices from the family of posets given in Example 6, we can consider the Dedekind-McNeille completion of the poset $P_{\top_\Lambda}$ and P_δ with a top element. Now, we define the operators $\&_\alpha : L_\alpha \times L_\alpha \to L_\alpha$, for every $\alpha \in \Lambda$, as follows:

$$x \&_\alpha y = \begin{cases} x \wedge_\alpha y & \text{if } \alpha \neq \bot_\Lambda \\ x \&^* y & \text{otherwise} \end{cases}$$

where the operator $\&^*$ is given in Fig. 8 along with the lattice $L_{\bot_\Lambda}$ which is not distributive.

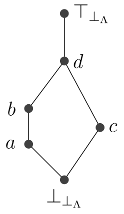
Thus, we can consider the $\&_\Lambda$-lattice family of algebras $\mathcal{L}_{\&_\Lambda} = \{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha \in \Lambda}$ where for every $\alpha \in \Lambda$, the pair $(L_\alpha, \preceq_\alpha)$ is a complete lattice and the operator $\&_\alpha$ is left-continuous and satisfies the condition $x \&_\alpha y \preceq x \wedge_\alpha y$, for all $x, y \in L_\alpha$. Therefore, by Theorem 8, we obtain that the $\&_\Lambda$-dp-lattice sum of $\mathcal{L}_{\&_\Lambda}$ is a complete lattice and the operator $\&_{\mathrm{d}}$ is left-continuous. The $\&_\Lambda$-dp-lattice sum $(L_{\mathrm{d}}, \preceq_{\mathrm{d}})$ is depicted in Fig. 9 and we can check that the left-continuity of $\&_{\mathrm{d}}$.

Other properties could be studied on this family, such as the associativity.

Proposition 12. *Let $(L_{\mathrm{d}}, \preceq_{\mathrm{d}}, \&_{\mathrm{d}})$ be a $\&_\Lambda$-dp-lattice sum of a family $\mathcal{L}_{\&_\Lambda}$. The operator $\&_{\mathrm{d}}$ is associative if and only if the operators $\&_\alpha$ in the family $\mathcal{L}_{\&_\Lambda}$ are associative.*

Proof. If $\&_{\mathrm{d}}$ is associative, then it is clear that every operator $\&_\alpha$ is associative.

Now, we consider any three elements $x, y, z \in L_{\mathrm{d}}$ where we can distinguish the following cases:



$\&^*$	$\bot_{\bot_\Lambda}$	a	b	c	d	$\top_{\bot_\Lambda}$
$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$
a	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	a
b	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	b
c	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	c	c	c
d	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	c	c	d
$\top_{\bot_\Lambda}$	$\bot_{\bot_\Lambda}$	a	b	c	d	$\top_{\bot_\Lambda}$

Fig. 8. Non-distributive lattice $L_{\bot_\Lambda}$ and the operator $\&^*$ of Example 9.

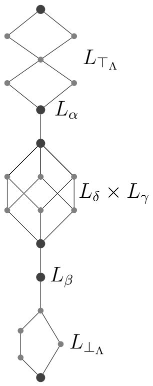


Fig. 9. The complete lattice $(L_{\mathrm{d}}, \preceq_{\mathrm{d}})$ in Example 9.

1. If $x, y, z \in L_\alpha$ for some $\alpha \in \Lambda$, then clearly the associativity holds because every operator $\&_\alpha$ is associative.
2. If $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\mu$ with $\alpha \sqsubset \beta \sqsubset \mu$, then the associative property holds since we obtain the following chain of equalities is satisfied by Theorem 4
$$(x \,\&_{\mathrm{d}}\, y) \,\&_{\mathrm{d}}\, z = (x \wedge_{\mathrm{d}} y) \wedge_{\mathrm{d}} z = x \wedge_{\mathrm{d}} (y \wedge_{\mathrm{d}} z) = x \,\&_{\mathrm{d}} (y \,\&_{\mathrm{d}}\, z)$$
3. If $x, y \in L_\alpha$ and $z \in L_\beta$ with $\alpha \sqsubset \beta$, then we have
$$(x \,\&_{\mathrm{d}}\, y) \,\&_{\mathrm{d}}\, z = (x \,\&_\alpha\, y) \wedge_{\mathrm{d}} z = x \,\&_\alpha\, y = x \,\&_{\mathrm{d}} (y \wedge_{\mathrm{d}} z) = x \,\&_{\mathrm{d}} (y \,\&_{\mathrm{d}}\, z)$$
4. If $x, z \in L_\alpha$ and $y \in L_\beta$ with $\beta \sqsubset \alpha$, then we have
$$(x \,\&_{\mathrm{d}}\, y) \,\&_{\mathrm{d}}\, z = (x \wedge_{\mathrm{d}} y) \,\&_{\mathrm{d}}\, z = y \,\&_{\mathrm{d}}\, z = y = x \wedge_{\mathrm{d}} y = x \,\&_{\mathrm{d}} (y \,\&_{\mathrm{d}}\, z)$$

The remaining cases are similar to the one above. ☐

Finally, we will show that the new lattice-based sum of family of algebras also preserves the infimum of arbitrary non-empty subsets of L_{d}. Notice that, in non-trivial lattices, it is not possible to preserve the supremum and infimum of arbitrary subsets of the given lattice [20,21].

Theorem 9. *Let $(L_d, \preceq_d, \&_d)$ be a $\&_\Lambda$-dp-lattice sum of a family $\mathcal{L}_{\&_\Lambda}$ where $(\Lambda, \sqsubseteq)$ is a complete lattice as well as every lattice of the family $\mathcal{L}_\Lambda$. If the operators $\&_\alpha$ in the family $\mathcal{L}_{\&_\Lambda}$ preserve the infimum of non-empty subsets on the left (right) side, then $\&_d$ also preserves the infimum of non-empty subsets of L_d on the left (right) side.*

Proof. Given $X \subseteq L_{\mathrm{d}}$, with $X \neq \varnothing$, and $z \in L_{\mathrm{d}}$, we will prove that:

$$\left(\bigwedge\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}}\, z = \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X\} \tag{7}$$

following the next cases, where

$$\begin{aligned}
\Lambda_\delta^u &= \{\alpha \in \Lambda \mid \delta \sqsubset \alpha\} \\
\Lambda_\delta^l &= \{\alpha \in \Lambda \mid \alpha \sqsubset \delta\} \\
\Lambda_X &= \{\alpha \in \Lambda \mid X \cap L_\alpha \neq \varnothing\}
\end{aligned}$$

and $\Lambda_{\delta,X}^u$, $\Lambda_{\delta,X}^l$ are defined similarly.

1. If $\Lambda_{\delta,X}^l$ is non-empty, then if there exists α_0 such that $X \cap L_{\alpha_0} \neq \varnothing$ and $X \cap L_\beta = \varnothing$, for all $\beta \in \Lambda$, with $\beta \sqsubset \alpha_0$, then
    - If $z \in L_{\alpha_0}$, then Eq. (7) holds because $\&_{\alpha_0}$ preserves the infimum on the left side.

- If $z \in L_\beta$, with $\beta \sqsubset \alpha_0$, then
$$\left(\bigwedge\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}}\, z = \left(\bigwedge\nolimits_{\alpha_0} (X \cap L_{\alpha_0})\right) \&_{\mathrm{d}}\, z = z$$
Since $x \,\&_{\mathrm{d}}\, z = z$, for all $x \in X \cap L_{\alpha_s}$, by the monotonicity of $\&_{\mathrm{d}}$ (Proposition 11), we obtain that
$$\bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X\} = \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X \cap L_{\alpha_0}\} = z$$
which proves Eq. (7).
- If $z \in L_\alpha$, with $\alpha_0 \sqsubset \alpha$, or $z \in L_\delta \times L_\gamma$, then by the monotonicity of $\&_{\mathrm{d}}$, we obtain that
$$\begin{aligned}\left(\bigwedge\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}}\, z &= \left(\bigwedge\nolimits_{\alpha_0} (X \cap L_{\alpha_0})\right) \&_{\mathrm{d}}\, z \\ &= \bigwedge\nolimits_{\alpha_0} (X \cap L_{\alpha_0}) \\ &= \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X \cap L_{\alpha_0}\} \\ &= \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X\}\end{aligned}$$

Otherwise, we consider $\alpha_m = \bigwedge \Lambda_X$, which exists, because $(\Lambda, \sqsubseteq)$ is complete, and satisfies $L_{\alpha_m} \cap X = \varnothing$. In this case, non element x in L_{α_i}, with $\alpha_i \in \Lambda_X$, can be a lower bound of X because we reach a contradiction with the minimality of α_m. Therefore, $\bigwedge_{\mathrm{d}} X = \top_{\alpha_m}$ and we obtain

- If $z \in L_\alpha$, with $\alpha_m \sqsubset \alpha$, then
$$\left(\bigwedge\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}}\, z = \top_{\alpha_m} \&_{\mathrm{d}}\, z = \top_{\alpha_m} = \bigwedge\nolimits_{\mathrm{d}} X = \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X\}$$
- Otherwise, $z \in L_\beta$, with $\beta \sqsubseteq \alpha_m$. Hence, there exists $\alpha_i \in \Lambda_X$ such that $\beta \sqsubseteq \alpha_i$, and so $x \,\&_{\mathrm{d}}\, z = z$ for all $x \in L_{\alpha_i} \cap X$ and, by the monotonicity of $\&_{\mathrm{d}}$, we obtain that
$$\left(\bigwedge\nolimits_{\mathrm{d}} X\right) \&_{\mathrm{d}}\, z = \top_{\alpha_m} \&_{\mathrm{d}}\, z = z = \bigwedge\nolimits_{\mathrm{d}} \{x \,\&_{\mathrm{d}}\, z \mid x \in X\}$$

2. If $\Lambda^l_{\delta,X}$ is empty, then
   (a) if $X \cap L_\delta \times L_\gamma \neq \varnothing$, then Eq. (7) is satisfied following an analogous development to above due to $\&_\delta$ and $\&_\gamma$ preserve the infimum on the left side, and the definition by components of $\&_{\delta,\gamma}$.
   (b) Otherwise, a similar reasoning to Step 1 can be developed concerning $\Lambda^u_{\delta,X}$.

□

Hence, different interesting properties have been studied obtaining, for example, that the new sum procedure preserves the left-continuous property.

7. Conclusions and future work

This paper has extended and analyzed the natural definition of the ordinal sum of t-norms on bounded lattices in order to consider general operators. Moreover, we have remarked that this extension does not preserve the left-continuous property of the underlying operators. Due to the significance of this property to the applications, which is very related to the fuzzy extension of Modus Ponens, we have also introduced a new strategy to define sums of general operators with the main goal of preserving their properties, focusing mainly on the left-continuous operator. This study is very complex when an arbitrary number of incomparable elements is considered. Hence, we have started this study when only two incomparable elements exist in the lattice, such as the well-known and useful rhombus lattice. Moreover, it is possible to extend this study to lattices with more than two incomparable elements, by couples, as the ones given in Fig. 10, or linear combination of them. However, the study is more challenging when we consider other more complex distributive lattices, such as the one showed in Fig. 11, in which there exist elements with more than one incomparable element.

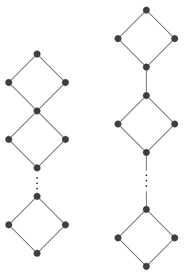


Fig. 10. Kinds of lattices with infinite incomparable elements.

In the future, we will analyze this case and try to extend the results given in this paper to any distributive lattice. Furthermore, other properties and approaches will be analyzed, such as the sum of implications [13,22]. We will also study the application of the obtained result to real-world use cases.

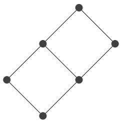


Fig. 11. Lattice with antichains with more than two elements.

Data availability

No data was used for the research described in the article.

CRediT authorship contribution statement

Roberto G. Aragón: Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Pascual Jara:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Jesús Medina:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Glossary of terms

Λ-*based poset family*: A family of posets based on the lattice Λ that satisfies the conditions of Definition 11.
Λ-*based lattice family*: A family of lattices based on the lattice Λ that satisfies the conditions of Definition 11.
Λ-*based lattice sum* $(L_s, \wedge_s, \vee_s)$, $(L_s, \preceq_s)$: Sum of a semibounded Λ-based lattice family together with the mappings $\wedge_s, \vee_s$ defined in Definition 15.
$\&_\Lambda$-*lattice family* $\{(L_\alpha, \preceq_\alpha, \&_\alpha)\}_{\alpha\in\Lambda}$: Family of algebras where $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ is a semibounded Λ-based lattice family and the operator satisfies the condition in Definition 16.
$\&_\Lambda$-*lattice sum* $(L_s, \preceq_s, \&_s)$: Sum of a $\&_\Lambda$-lattice family where the operator $\&_s$ is defined in Definition 16.

Λ-*based dp-poset family*: A family of posets based on the lattice Λ that satisfies the conditions of Definition 17.

Λ-*based dp-poset sum* $(P_d, \preceq_d)$: Sum of a Λ-based dp-poset family with the partial ordering relation defined in Definition 18.

Λ-*based dp-lattice family* $\mathcal{L}_\Lambda$: A family of lattices based on the lattice Λ that satisfies the conditions of Definition 17.

Λ-*based dp-lattice sum* $(L_d, \wedge_d, \vee_d), (L_d, \preceq_d)$: Sum of a Λ-based dp-lattice family with the mappings $\wedge_\mathrm{d}, \vee_\mathrm{d}$ defined in Definition 19.

$\&_\Lambda$-*lattice family of algebras* $\mathcal{L}_{\&_\Lambda}$: Family of algebras where $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ is a Λ-based dp-lattice family.

$\&_\Lambda$-*dp-lattice sum* $(L_d, \preceq_d, \&_d)$: Sum of a $\&_\Lambda$-lattice family of algebras with the partial ordering relation defined in Definition 18 and the operator $\&_\mathrm{d}$ defined in Definition 20.