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Attribute Implication Bases From Galois Connection Structures

M. Eugenia Cornejo | Jesús Medina | Francisco José Ocaña

Department of Mathematics, University of Cádiz, Cádiz, Spain

Correspondence: M. Eugenia Cornejo (mariaeugenia.cornejo@uca.es)

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ABSTRACT

Modeling knowledge systems by determining relationships among key variables have been and currently is a fundamental and nontrivial challenge in real-world scenarios. Many approaches have been developed to reach this goal, but many of them are heuristic and require of alternative procedures to provide robust and tractable rules. With this significant aim, attribute implications were introduced in the mathematical framework of Formal Concept Analysis. In this paper, we will introduce a novel procedure to obtain relationships among variables from any dataset in which a Galois connection has been defined. In particular, we will be focused on the general multiadjoint framework, which is one of the most general approached in which a Galois connection have been considered, although the obtained results and methodology can also be used in other well-known approaches, such as in the residuated or heterogeneous frameworks.

1 | Introduction

Formal concept analysis (FCA) has become a powerful methodology for managing and extracting knowledge from relational databases, with important theoretical foundations, such as considering generalized quantifiers in the definition of the concept-forming operators [1], relating it to other tools, such as local congruencies [2], fuzzy relation equations [3], rough set theory [4, 5], fuzzy linguistic approach [6], fuzzy closure structures [7], and practical applications in medicine [8], digital forensic [9–11], and renewable energy [12, 13]. FCA facilitates the representation of relationships between attributes and objects contained in datasets, by using specific units of information called concepts. One of the most important properties of the most significant FCA frameworks is that the concept-forming operators form a Galois connection, this fact happens from the classical case, with the derivation operators, to the most general fuzzy ones such as the multiadjoint [14], residuated [15] or heterogeneous [16] approaches.

Different fuzzy extensions of FCA have been proposed [14–22] to better accommodate the reasoning under vagueness and uncertainty. Specifically, multiadjoint concept lattices [14] have emerged as a unifying and flexible framework that integrates several fuzzy FCA approaches [17, 19, 20, 22]. This paradigm has been particularly useful in tasks such as attribute classification [23–25], reduct computation [26], lattice size reduction [27] and the computation of attribute implications [24, 28, 29].

The number of valid attribute implications obtained from contexts is often substantial, hence the need to obtain effective methodologies for their management and reduction [24, 30–36]. A remarkable challenge within this framework lies in identifying a nonredundant subset of valid attribute implications from which all others can be inferred, in other words, a basis of attribute implications. Several approaches have been proposed to address this challenge both in the classical case [37–41] and in the fuzzy residuated case [15]. The particular mechanisms, which have been considered in these approaches, are not equivalent in general.

In this paper, we introduce a new procedure inspired by Guigues and Duquenne's methodology [40], which was clarified and compared with other classical procedures in [42]. This procedure is based on the Galois connection to define the relevant notions associated with the attribute implication theory. We will start introducing the notion of validity of an attribute implication as the degree of inclusion of the consequent set of the implication in the closure of the antecedent set. Then, we will see that it is necessary to consider only informative implication, and so a mechanism to determine the most informative implications is provided. A sound and complete operational semantics is given based on the Armstrong's axioms. In addition, the notions of nonredundant attribute implication and base are presented in general. The notion of minimal nonredundant node given in [40] is extended to the general framework, which is fundamental to compute bases of attribute implications associated with a context. Finally, three possible alternatives of bases from three different point of views are determined.

The structure of the paper is as follows. The preliminary notions are included in Section 2, which mainly recalls the multiadjoint concept lattice framework and the key notion of antitone Galois connection. Section 3 introduces the notion of validity given by Guigues and Duquenne in the multiadjoint framework, which can be straightforwardly given from any approach based on a Galois connection as concept-forming operators. Moreover, the informative attribute implications are studied in this section and paying more attention to the maximal informative attribute implications in Section 4. The definition and computation of bases in this general framework is considered in Section 5. The paper finishes with the conclusions and prospects for future work.

2 | Preliminaries

The multiadjoint concept lattice fuzzy framework [14, 43] arise as a general and flexible approach capable of conveniently embed different fuzzy extensions of concept lattices present in the literature such as [17, 19, 20, 22], among others. The basic operators used in this framework are the adjoint triples. Adjoint triples [44–47] represent a formal extension of t-norms and their residuated implications [48], providing greater flexibility while preserving their fundamental properties. Characterized by minimal mathematical conditions for their operability, these operators increase their applicability in a wide range of fields. For instance, they are suitable in noncommutative and nonassociative frameworks. The formal definition of adjoint triples is presented below.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& : P_1 \times P_2 \to P_3$, $\swarrow : P_3 \times P_2 \to P_1$, $\nwarrow : P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \& y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called adjoint property.

The following proposition shows some direct properties of the residuated implications of an adjoint triple derived from the adjoint property.

Proposition 2 ([46]). *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, we have that*

1. *$\swarrow, \nwarrow$ are order-reversing in the second argument, that is, if $x_1, x_2 \in P_1$, $y_1, y_2 \in P_2$, $z \in P_3$ and $x_1 \leq_1 x_2$, $y_1 \leq_2 y_2$, then $(z \swarrow y_2) \leq_1 (z \swarrow y_1)$ and $(z \nwarrow x_2) \leq_1 (z \nwarrow x_1)$.*

2. $(\bigwedge_{z_i \in Z} z_i) \swarrow y = \bigwedge_{z_i \in Z}(z_i \swarrow y)$, *for each* $Z \subseteq P_3$ *and* $y \in P_2$, *when infimum there exists.*
3. $(\bigwedge_{z_i \in Z} z_i) \nwarrow x = \bigwedge_{z_i \in Z}(z_i \nwarrow x)$, *for each* $Z \subseteq P_3$ *and* $x \in P_1$, *when infimum there exists.*

A well-known equivalence of the residuated implications of an adjoint triple can be obtained by assuming the associativity of conjunctor, as shown in the following proposition.

Proposition 3 ([49]). *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to the poset* $(P, \leq)$*. Then, for all* $x, y, z \in P$*, the following equivalence holds:*

$$(z \swarrow y) \nwarrow x = (z \nwarrow x) \swarrow y \quad \text{if and only if} \quad \& \text{ is associative.}$$

Finally, we present the notion of forcing-implication, which play an important role in different results obtained in this paper.

Definition 4. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets such that $\top_2$ is the top element in $(P_2, \leq_2)$. We say that the mapping $\nwarrow : P_1 \times P_1 \to P_2$ is a forcing-implication if it is order-preserving in the left argument, order-reversing in the right argument and the following equivalence holds, for all $y, z \in P_1$:

$$z \nwarrow y = \top_2 \quad \text{if and only if} \quad y \leq_1 z$$

After presenting the notion of adjoint triples and their corresponding properties, we will formally introduce the multiadjoint concept lattice framework. For this purpose, the fundamental notions related to multiadjoint concept lattices are presented below [14, 43].

Definition 5.

- The tuple $(L_1, L_2, P, \leq_1, \leq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ is a multiadjoint frame where $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(P, \leq)$, for all $i \in \{1, \dots, n\}$.
- The tuple (A, B, R, σ) is a *context* where A and B are nonempty sets, R is a P-fuzzy relation $R : A \times B \to P$ and $\sigma : A \times B \to \{1, \dots, n\}$ is a mapping which associates any element in $A \times B$ with a particular adjoint triple in the multiadjoint frame.
- The concept-forming operators ${}^{\uparrow_\sigma} : L_2^B \to L_1^A$ and ${}^{\downarrow^\sigma} : L_1^A \to L_2^B$ are defined as

$$g^{\uparrow_\sigma}(a) = \inf\{R(a, b') \swarrow^{\sigma(a,b')} (g(b')) \mid b' \in B\} \tag{1}$$

$$f^{\downarrow^\sigma}(b) = \inf\{R(a', b) \nwarrow_{\sigma(a',b)} (f(a')) \mid a' \in A\} \tag{2}$$

  for all $g \in L_2^B$, $f \in L_1^A$, $a \in A$, $b \in B$, where L_2^B and L_1^A denote the set of mappings $g : B \to L_2$ and $f : A \to L_1$, respectively.
- A *concept* is a pair $\langle g, f \rangle$ satisfying that $g^{\uparrow_\sigma} = f$, $f^{\downarrow^\sigma} = g$, for all $g \in L_2^B$, $f \in L_1^A$. The fuzzy subsets g and f are usually known as the extent and intent of the concept, respectively.
- The multiadjoint concept lattice, denoted as $(\mathcal{M}, \preceq)$, associated with the multiadjoint frame and the context (A, B, R, σ) is the set:

$$\mathcal{M} = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A, g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

  where the ordering is defined by $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ if and only if $g_1 \leq_2 g_2$ (or equivalently, $f_2 \leq_1 f_1$). The sets of extents and intents of the concepts of $(\mathcal{M}, \preceq)$ will be denoted as $\mathrm{Ext}(\mathcal{M})$ and $\mathrm{Int}(\mathcal{M})$, respectively.

From now on, in order to simplify the notation, we will write $g^\uparrow$ and $f^\downarrow$ instead of $g^{\uparrow_\sigma}$ and $f^{\downarrow^\sigma}$, respectively.

It is important to mention that the pair of concepts-forming operators $({}^\uparrow, {}^\downarrow)$ forms an antitone Galois connection, which provides $(\mathcal{M}, \preceq)$ with the algebraic structure of a complete lattice [14]. Next, in order to make the paper self-contained, we recall the notion of an antitone Galois connection.

Definition 6. Let $(P_1, \preceq_1)$, $(P_2, \preceq_2)$ be two posets. We say that the pair $(^{\uparrow}, ^{\downarrow})$ of mappings $^{\downarrow} : P_1 \to P_2, ^{\uparrow} : P_2 \to P_1$ is an antitone Galois connection between P_1 and P_2 if the following properties are satisfied:

1. $^{\uparrow}$ and $^{\downarrow}$ are order-reversing.
2. $x \preceq_1 x^{\downarrow\uparrow}$ and $y \preceq_2 y^{\uparrow\downarrow}$
3. $x^{\downarrow} = x^{\downarrow\uparrow\downarrow}$ and $y^{\uparrow} = y^{\uparrow\downarrow\uparrow}$

for all $x \in P_1$ and $y \in P_2$.

This paper is focused on the study of attribute implications within the multiadjoin concept lattice framework previously introduced. Consequently, a multiadjoint frame $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$, the context (A, B, R, σ) and the multiadjoint concept lattice $(\mathcal{M}, \preceq)$ will be fixed from now on.

3 | Attribute Implications in Multiadjoint Concept Lattices

This section introduces properties and formal mechanisms which allows us to understand and analyze attribute implications within the multiadjoint framework, following the Guigues and Duquenne approach presented in [40]. First of all, it is convenient to recall the underlying philosophy behind the notion of attribute implication.

Definition 7. An *attribute implication* over A is an expression $f_2 \Leftarrow f_1$, where $f_1, f_2 \in L_1^A$ are two fuzzy subsets of attributes. The subsets f_1 and f_2 are called *antecedent* and *consequent*, respectively.

Namely, an attribute implication indicates that if an object satisfies all the attributes from the antecedent of the rule, then it will also satisfy the attributes from the consequent. Since the above notion is a syntactic definition, a degree of validity for an attribute implication is required in the given context, in order to reinforce its meaning and interpretation. Therefore, we need to establish a definition of validity for attribute implications in the multiadjoint framework. For this purpose, we will initially introduce two generalizations of the usual notion of fuzzy inclusion considering adjoint triples.

Definition 8. Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ and two fuzzy subsets of attributes $f_1, f_2 \in L_1^A$, *the degree in which f_1 is included in f_2* is defined as

$$S^1(f_1, f_2) = \inf\{f_2(a) \nwarrow f_1(a) | a \in A\}$$

Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_2, \preceq_2)$ and two fuzzy subsets of objects $g_1, g_2 \in L_2^B$, *the degree in which g_1 is included in g_2 is defined as*

$$S^2(g_1, g_2) = \inf\{g_2(b) \swarrow g_1(b) | b \in B\}$$

We can now establish a formal definition of the degree of validity of an attribute implication for a given fuzzy subset of attributes.

Definition 9. Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ and three fuzzy subsets of attributes $f_1, f_2, f_3 \in L_1^A$, *the degree in which an attribute implication $f_2 \Leftarrow f_1$ is valid in f_3* is

$$\|f_2 \Leftarrow f_1\|_{f_3} = S^1(f_2, f_3) \swarrow S^1(f_1, f_3)$$

This notion is extrapolated to a set of fuzzy subsets of attributes $F \subseteq L_1^A$, defining *the degree in which the attribute implication $f_2 \Leftarrow f_1$ is valid in F* as

$$\|f_2 \Leftarrow f_1\|_F = \bigwedge_{f \in F} \|f_2 \Leftarrow f_1\|_f$$

The notion of validity of an attribute implication was extended in [24] to the multiadjoint concept lattice framework by means of the degree of validity in the whole set of intents of the concept lattice. In this paper, we propose an alternative definition for the degree of validity, inspired by the crisp characterization of validity of an attribute implication [39, 40]. This new definition provides a straightforward extension of the usual equivalence of validity of the Boolean case. In the multiadjoint framework, the corresponding condition of validity can be formalized through the degree of inclusion of the consequent set of the rule in the closure of the antecedent set.

Definition 10. Given two subsets of attributes $f_1, f_2 \in L_1^A$, *the degree in which the attribute implication* $f_2 \Leftarrow f_1$ *is valid in the multiadjoint concept lattice* $(\mathcal{M}, \preceq)$ is defined as

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = S^1(f_2, f_1^{\downarrow\uparrow})$$

We say that $f_2 \Leftarrow f_1$ is a fully true attribute implication if $\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1$. The whole set of fully true attribute implications will be denoted as $\mathcal{F}_t$.

The following result provides a equivalence which allow us to characterize the fully true attributes implications in the multiadjoint framework, and extent it to any approach based on a Galois connection as concept-forming operators.

Proposition 11. *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$ *such that* $\nwarrow$ *is a forcing implication. Given two subsets of attributes* $f_1, f_2 \in L_1^A$, *then we have*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1 \quad \text{if and only if} \quad f_2 \preceq_1 f_1^{\downarrow\uparrow}$$

Proof. Given $f_1, f_2 \in L_1^A$, the next chain of equivalences are obtained, for all $a \in A$:

$$\begin{aligned} \|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1 &\text{if and only if } S^1(f_2, f_1^{\downarrow\uparrow}) = \top_1 \\ &\text{if and only if } f_1^{\downarrow\uparrow}(a) \nwarrow f_2(a) = \top_1 \\ &\text{if and only if } f_2(a) \preceq_1 f_1^{\downarrow\uparrow}(a) \end{aligned}$$

where the first equivalence holds by Definition 10, the second equivalence is deduced from Definition 8, and the third one holds because of $\nwarrow$ is a forcing implication. □

From a practical standpoint, this equivalence plays a fundamental role in the computation of fully true attribute implications in a formal context, as the following example shows.

Example 12. Consider the multiadjoint frame

$$([0,1]_2, \leq, (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{DL}, \nwarrow_{DL}))$$

where $[0,1]_2$ is a regular partition of the unit interval in 2 pieces, that is, $[0,1]_2 = \{0, 0.5, 1\}$ and $(\&_{DG}, \swarrow^{DG}, \nwarrow_{DG})$, $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$, $(\&_{DL}, \swarrow^{DL}, \nwarrow_{DL})$ are adjoint triples obtained from the discretization of the Gödel, product and Łukasiewicz t-norms, respectively, considering the following mappings $\&_{DT}, \swarrow^{DT}, \nwarrow_{DT} \colon [0,1]_2 \times [0,1]_2 \to [0,1]_2$ defined as

$$x \&_{DT} y = \frac{\lceil k \cdot T(x,y) \rceil}{k} \quad z \swarrow^{DT} y = \frac{\lfloor m \cdot (z \to_T y) \rfloor}{m} \quad z \nwarrow_{DT} x = \frac{\lfloor n \cdot (z \to_T x) \rfloor}{n}$$

for each $x, y, z \in [0,1]_2$, where $\lceil \_ \rceil$ is the ceiling function, $\lfloor \_ \rfloor$ is the floor function, T is the Gödel, product or Łukasiewicz t-norm and $\to$ is the corresponding residuated implication of the t-norm T.

$$T_G(x,y) = \min\{x, y\} \qquad z \to_G y = \begin{cases} 1 & if\, y \leq z \\ z & otherwise \end{cases}$$

$$T_P(x,y) = x \cdot y \qquad z \to_P y = \begin{cases} 1 & if\, y \leq z \\ z/y & otherwise \end{cases}$$

$$T_Ł(x,y) = \max\{0, x + y - 1\} \qquad z \to_Ł y = \min\{1, 1 - y + z\}$$

TABLE 1 | Relation R and mapping σ of Example 12.

R	b_1	b_2	b_3
a_1	0.5	0	0
a_2	0.5	0	0.5
a_3	1	1	0.5
a_4	1	0.5	0
a_5	1	0.5	1
σ	b_1	b_2	b_3
a_1	$\&_{DL}$	$\&_{DP}$	$\&_{DG}$
a_2	$\&_{DL}$	$\&_{DP}$	$\&_{DG}$
a_3	$\&_{DL}$	$\&_{DP}$	$\&_{DG}$
a_4	$\&_{DL}$	$\&_{DP}$	$\&_{DG}$
a_5	$\&_{DL}$	$\&_{DP}$	$\&_{DG}$

The context (A, B, R, σ) is composed of the set of attributes $A = \{a_1, a_2, a_3, a_4, a_5\}$, the set of objects $B = \{b_1, b_2, b_3\}$, the relation $R : A \times B \to [0, 1]_2$ and the mapping σ given in Table 1. The concept lattice related to this framework and context has 11 concepts whose intents are given in Figure 1.

Now, we will consider the fuzzy subset of attributes $f_1 = \{0.5/a_5\}$, and we will compute its closure, $f_1^{\downarrow\uparrow}$, in order to identify fully true attribute implications. Applying the definition of the operator $^{\downarrow}$, we have that

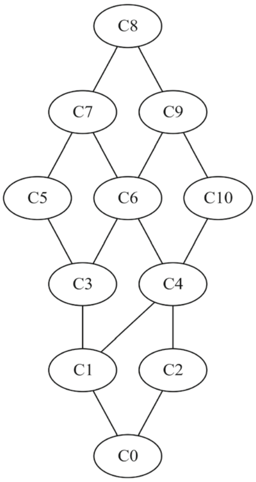
$$f_1^{\downarrow}(b_1) = \inf\{R(a, b_1) \nwarrow_{DL} f_1(a) | a \in A\} = 1$$
$$f_1^{\downarrow}(b_2) = \inf\{R(a, b_2) \nwarrow_{DP} f_1(a) | a \in A\} = 1$$
$$f_1^{\downarrow}(b_3) = \inf\{R(a, b_3) \nwarrow_{DG} f_1(a) | a \in A\} = 1$$

Then, we obtain that $f_1^{\downarrow} = \{1/b_1, 1/b_2, 1/b_3\}$. Now, applying the definition of the operator $^{\uparrow}$ to the set $f_1^{\downarrow}$, we obtain that

$$f_1^{\downarrow\uparrow}(a_1) = \inf\{R(a_1, b_1) \swarrow^{DL} f_1^{\downarrow}(b_1), R(a_1, b_2) \swarrow^{DP} f_1^{\downarrow}(b_2), R(a_1, b_3) \swarrow^{DG} f_1^{\downarrow}(b_3)\} = 0$$
$$f_1^{\downarrow\uparrow}(a_2) = \inf\{R(a_2, b_1) \swarrow^{DL} f_1^{\downarrow}(b_1), R(a_2, b_2) \swarrow^{DP} f_1^{\downarrow}(b_2), R(a_2, b_3) \swarrow^{DG} f_1^{\downarrow}(b_3)\} = 0$$
$$f_1^{\downarrow\uparrow}(a_3) = \inf\{R(a_3, b_1) \swarrow^{DL} f_1^{\downarrow}(b_1), R(a_3, b_2) \swarrow^{DP} f_1^{\downarrow}(b_2), R(a_3, b_3) \swarrow^{DG} f_1^{\downarrow}(b_3)\} = 0.5$$
$$f_1^{\downarrow\uparrow}(a_4) = \inf\{R(a_4, b_1) \swarrow^{DL} f_1^{\downarrow}(b_1), R(a_4, b_2) \swarrow^{DP} f_1^{\downarrow}(b_2), R(a_4, b_3) \swarrow^{DG} f_1^{\downarrow}(b_3)\} = 0$$
$$f_1^{\downarrow\uparrow}(a_5) = \inf\{R(a_5, b_1) \swarrow^{DL} f_1^{\downarrow}(b_1), R(a_5, b_2) \swarrow^{DP} f_1^{\downarrow}(b_2), R(a_5, b_3) \swarrow^{DG} f_1^{\downarrow}(b_3)\} = 0.5$$

Hence, we have that $f_1^{\downarrow\uparrow} = \{0.5/a_3, 0.5/a_5\}$. As a consequence, applying Proposition 11, we can conclude that $\{0.5/a_3\} \Leftarrow \{0.5/a_5\}$ is the only nontrivial fully true attribute implication that we can obtain considering $f_1 = \{0.5/a_5\}$ as antecedent.

Notice that, in the above example, the attribute implication $f_1 \Leftarrow f_1$ is fully true for all $f_1 \in [0, 1]_2^A$, since the inequality $f_1 \leq f_1^{\downarrow\uparrow}$ always holds. This type of attribute implication is considered as trivial in practice. Namely, we will use the notion of trivial attribute implication to refer to logical relationships between attributes that are considered to be self-evident or not informative, that is, attribute implications that are usually straightforward and do not provide significant or valuable information. Another kind of trivial attribute implications arises, if we have that $f_2 \preceq_1 f_1$. Trivial attribute implications are generally not of interest in data analysis. They are usually ignored or excluded from further analysis since they do not contribute to uncovering meaningful patterns or dependencies within the data. The focus in data analysis is usually on discovering nontrivial and informative attribute implications that provide valuable insights and allow us to obtain significant information from the given context. The following definition presents the concept of informative attribute implication [40].



$$
\begin{aligned}
Int(C_0) &= \langle\{1/a_1, 1/a_2, 1/a_3, 1/a_4, 1/a_5\}\rangle \\
Int(C_1) &= \langle\{0.5/a_1, 0.5/a_2, 1/a_3, 1/a_4, 1/a_5\}\rangle \\
Int(C_2) &= \langle\{1/a_2, 1/a_3, 1/a_5\}\rangle \\
Int(C_3) &= \langle\{1/a_3, 1/a_4, 1/a_5\}\rangle \\
Int(C_4) &= \langle\{0.5/a_2, 1/a_3, 1/a_5\}\rangle \\
Int(C_5) &= \langle\{1/a_3, 0.5/a_4, 0.5/a_5\}\rangle \\
Int(C_6) &= \langle\{1/a_3, 1/a_5\}\rangle \\
Int(C_7) &= \langle\{1/a_3, 0.5/a_5\}\rangle \\
Int(C_8) &= \langle\{0.5/a_3, 0.5/a_5\}\rangle \\
Int(C_9) &= \langle\{0.5/a_3, 1/a_5\}\rangle \\
Int(C_{10}) &= \langle\{0.5/a_2, 0.5/a_3, 1/a_5\}\rangle
\end{aligned}
$$

FIGURE 1 | Hasse diagram and intents of $(\mathcal{M}, \preceq)$ in Example 12.

Definition 13. Given two subsets of attributes $f_1, f_2 \in L_1^A$, we say that the attribute implication $f_2 \Leftarrow f_1$ is *informative* if $f_2 \preceq_1 f_1^{\downarrow\uparrow}$ and $f_2 \npreceq_1 f_1$.

In addition, we can define an ordering relation, in order to compare the attribute implications of the context and analyze which one provides more information.

Definition 14. Given $f_1, f_2, f_3, f_4 \in L_1^A$, we say that the attribute implication $f_2 \Leftarrow f_1$ is at least as informative as $f_4 \Leftarrow f_3$, which is denoted as $f_4 \Leftarrow f_3 \sqsubseteq f_2 \Leftarrow f_1$, if $f_1 \preceq_1 f_3$ and $f_4 \preceq_1 f_2$.

In other words, one implication is considered more informative than another when it requires less conditions (corresponding to the antecedent set) to generate a greater number of valid conclusions (corresponding to the consequent set). Notice that, $(\mathcal{F}_t, \sqsubseteq)$ is a poset, and hence, if we assume that L_1 is finite, then it has maximal elements, which will be called maximal informative attribute implications. From now on, we will consider L_1 as a finite set.

Definition 15. Given two subsets of attributes $f_1, f_2 \in L_1^A$, we say that the attribute implication $f_2 \Leftarrow f_1 \in \mathcal{F}_t$ is a maximal informative implication, if it is a maximal element of the poset $(\mathcal{F}_t, \sqsubseteq)$, that is, $f_2 \Leftarrow f_1 \not\sqsubseteq f_4 \Leftarrow f_3$ for all $f_4 \Leftarrow f_3 \in \mathcal{F}_t$. The set of all maximal informative implications of $(\mathcal{M}, \preceq)$ is denoted as $\mathbf{M}\mathcal{F}_t$.

The following result shows that the closure of the antecedent set coincides with the consequent set in every maximal informative implication.

Proposition 16. *Given two subsets of attributes $f_1, f_2 \in L_1^A$, if $f_2 \Leftarrow f_1$ is a maximal informative implication, then $f_2 = f_1^{\downarrow\uparrow}$.*

Proof. Let $f_2 \Leftarrow f_1$ be a maximal informative implication. By Definition 13, the inequality $f_2 \preceq_1 f_1^{\downarrow\uparrow}$ holds. Consequently, considering Definition 14, we deduce that the attribute implication $f_2 \Leftarrow f_1$ is less informative than $f_1^{\downarrow\uparrow} \Leftarrow f_1$, that is, $f_2 \Leftarrow f_1 \sqsubseteq f_1^{\downarrow\uparrow} \Leftarrow f_1$. Moreover, since $f_2 \Leftarrow f_1$ is a maximal informative implication, by Definition 15, we have that is a maximal element of $(\mathcal{F}_t, \sqsubseteq)$. Thus, from the last inequality, we can deduce that $f_2 = f_1^{\downarrow\uparrow}$. □

However the counterpart is not true, that is, if the equality $f_2 = f_1^{\downarrow\uparrow}$ holds for $f_1, f_2 \in L_1^A$ the implication $f_2 \Leftarrow f_1$ is not necessarily maximal informative. This fact is illustrated by the following example.

Example 17. We will consider again the multiadjoint framework and context given in Example 12, and the fuzzy subset of attributes $f_1 = \{1/a_3, 1/a_4, 1/a_5\}$. Following the same procedure as in Example 12, we obtain that $f_1^{\downarrow\uparrow} = \{0.5/a_1, 0.5/a_2, 1/a_3, 1/a_4, 1/a_5\}$. Taking into account Proposition 11, if we consider $f_2 \in L_1^A$ such that $f_2 = f_1^{\downarrow\uparrow}$, we have that $\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1$, and hence, $f_2 \Leftarrow f_1 \in \mathcal{F}_t$. Now, we will see that this attribute implication is not a maximal element of $(\mathcal{F}_t, \sqsubseteq)$, that is, there exists $f_4 \Leftarrow f_3 \in \mathcal{F}_t$ such that $f_3 \preceq_1 f_1$ and $f_2 \preceq_1 f_4$. For this purpose, we will consider the fuzzy subsets of attributes $f_3 = \{0.5/a_4\}$, which satisfies $f_3^{\downarrow\uparrow} = \{0.5/a_1, 0.5/a_2, 1/a_3, 1/a_4, 1/a_5\} = f_2$. It is easy to check that the attribute implication $f_3^{\downarrow\uparrow} \Leftarrow f_3 \in \mathcal{F}_t$, as a consequence of Proposition 11. Moreover, it is clear that $f_3 \prec_1 f_1$ and $f_2 = f_3^{\downarrow\uparrow}$. Hence, we can conclude that $f_2 \Leftarrow f_1$ is not a maximal informative implication.

The notion of maximal informative implication can be characterized as the following theorem shows.

Proposition 18. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. The attribute implication $f_1^{\downarrow\uparrow} \Leftarrow f_1$ is maximal informative if and only if for each $f_2 \in L_1^A$ such that $f_2 \prec_1 f_1$, we have that $f_1 \npreceq_1 f_2^{\downarrow\uparrow}$.*

Proof. First of all, given $f_1 \in L_1^A$, we will prove that if $f_1^{\downarrow\uparrow} \Leftarrow f_1$ is a maximal informative implication, then for every $f_2 \in L_1^A$ with $f_2 \prec_1 f_1$, it is satisfied that $f_1 \npreceq_1 f_2^{\downarrow\uparrow}$. By reductio ad absurdum, if $f_1 \preceq_1 f_2^{\downarrow\uparrow}$, then $f_2 \prec_1 f_1 \preceq_1 f_2^{\downarrow\uparrow}$. Since the pair $({}^\uparrow, {}^\downarrow)$ forms an antitone Galois connection, from Definition 6(1), we obtain that $f_2^{\downarrow\uparrow} \prec_1 f_1^{\downarrow\uparrow} \preceq_1 f_2^{\downarrow\uparrow}$, that is, $f_2^{\downarrow\uparrow} = f_1^{\downarrow\uparrow}$ and so, $f_1^{\downarrow\uparrow} \Leftarrow f_2$ contradicts the maximality of $f_1^{\downarrow\uparrow} \Leftarrow f_1$.

Now, in order to prove the opposite, we will prove the contraposition. Hence, we assume that the attribute implication $f_1^{\downarrow\uparrow} \Leftarrow f_1$ is not maximal informative. Considering Definition 15, there must be an attribute implication $f_3 \Leftarrow f_2 \in \mathcal{F}_t$ such that $f_1^{\downarrow\uparrow} \Leftarrow f_1 \sqsubseteq f_3 \Leftarrow f_2$. By Definition 14, this is equivalent to $f_2 \preceq_1 f_1$ and $f_1^{\downarrow\uparrow} \preceq_1 f_3$, which implies the following chain of inequalities

$$f_1 \overset{(i)}{\preceq_1} f_1^{\downarrow\uparrow} \preceq_1 f_3 \overset{(ii)}{\preceq_1} f_2^{\downarrow\uparrow},$$

where (i) holds by Definition 6(2), and (ii) is obtained because $f_3 \Leftarrow f_2 \in \mathcal{F}_t$ and by Proposition 11. Thus, we have shown that there exists $f_2 \in L_1^A$ such that $f_2 \preceq_1 f_1$ and $f_1 \preceq_1 f_2^{\downarrow\uparrow}$, which is the opposite of the hypothesis. □

As a consequence of the two previous results, to maximize the informativeness of this implication, it would be necessary to identify the minimal subsets $f_1 \prec f_2$ such that $f_2 = f_1^{\downarrow\uparrow}$. Formally prove this claim will be the next goal. For this purpose, we need to define the notion of subintent.

Definition 19. Let $i \in Int(\mathcal{M})$ be an intent of a concept of $(\mathcal{M}, \preceq)$. A *subintent* of i is any subset f of i such that $f^{\downarrow\uparrow} = i$, that is, whose closure is h. The set of all subintents of i will be denoted as $\mathcal{S}(i)$. We say that a subintent is *irreducible* if it is a minimal set in $\mathcal{S}(i)$. The set of all irreducible subintents of an intent i is denoted as $\mathcal{IS}(i)$, and the union of these sets, that is, $\bigcup\{\mathcal{IS}(i) | i \in Int(\mathcal{M})\}$, is denoted as $\mathcal{IS}_{\mathcal{M}}$.

From the above definition, we can establish an equivalence relation between fuzzy subsets of attributes, providing a partition of L_1^A.

Definition 20. The relation $\equiv$ on $L_1^A \times L_1^A$ is defined as $f_1 \equiv f_2$, if $f_1^{\downarrow\uparrow} = f_2^{\downarrow\uparrow}$, for all $f_1, f_2 \in L_1^A$. The equivalence classes will be denoted as $[f]_{\mathcal{M}}$, and are given by the families $[f]_{\mathcal{M}} = \mathcal{S}(f^{\downarrow\uparrow})$, for all $f \in L_1^A$.

Moreover, given an intent i, if it has more than one subintent, that is, $\mathcal{S}(i) \neq \{i\}$, then the minimal elements of the set of subintents $\mathcal{S}(i)$ are irreducible elements. As we previously advanced, these elements play an important role to detect and obtain maximal informative implications, since they are directly related to the antecedent part of these implications, as the following proposition asserts.

Proposition 21. *Given $i \in Int(\mathcal{M})$, if $\mathcal{S}(i) \neq \{i\}$, then the nonclosed sets $f \in L_1^A$ serving as antecedent parts for maximal informative implications $i \Leftarrow f$ are the irreducible subintents of i.*

Proof. If we assume that $\mathcal{S}(i) \neq \{i\}$, then there exist maximal informative implications such that their consequent is i, that is, attribute implications of the form $i \Leftarrow f$, with $f \in L_1^A$ and $f \neq i$. From Proposition 16, we have that $i = f^{\downarrow\uparrow}$, which implies that f is a subintent of i, that is, $f \in \mathcal{S}(i)$. In addition, by Proposition 18, we can deduce the antecedent f is also a minimal subset in $\mathcal{S}(i)$, and hence, by Definition 19, an irreducible subintent of i. □

As a consequence of the previous proposition, we can establish an isomorphism between the sets of irreducible subintents and the set of maximal informative implications, and hence, the last mentioned set can be defined as $\mathbf{M}\mathcal{F}_t = \{f^{\downarrow\uparrow} \Leftarrow f \mid f \in \mathcal{IS}_{\mathcal{M}}\}$. We will denote $\mathbf{M}\mathcal{F}_t(i)$ the set of maximal informative attribute implications which generates the intent i, that is, $\mathbf{M}\mathcal{F}_t(i) = \{i \Leftarrow f \mid f \in \mathcal{IS}(i)\}$.

This section finishes with an example in order to illustrate the previous notions and results associated with the attribute implications in the multiadjoint concept lattice framework.

Example 22. Consider the multiadjoint frame

$$([0,1]_2, \preceq, (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{DL}, \nwarrow_{DL}))$$

and the context (A, B, R, σ), where the set of attributes is $A = \{a_1, a_2, a_3\}$, the set of objects is $B = \{b_1, b_2, b_3\}$, the relation $R : A \times B \to [0,1]_2$ and the mapping σ are given in Table 2.

The multiadjoint concept lattice $(\mathcal{M}, \preceq)$ associated with the context (A, B, R, σ) is composed of 10 concepts whose intents are given in Figure 2.

The equivalence classes of subintents corresponding to each of intents are presented below:

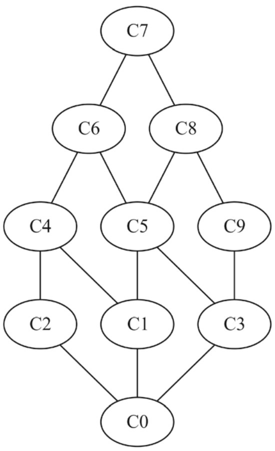
$$
\begin{aligned}
[i_0]_{\mathcal{M}} = \{ & \{1/a_2, 1/a_3\}, \{1/a_1, 1/a_2\}, \{0.5/a_1, 1/a_2, 1/a_3\}, \{1/a_1, 0.5/a_2, 1/a_3\}, \\
& \{1/a_1, 1/a_2, 0.5/a_3\}, \{0.5/a_1, 0.5/a_2, 1/a_3\}, \{0.5/a_2, 1/a_3\}, \{1/a_1, 1/a_2, 1/a_3\}\} \\
[i_1]_{\mathcal{M}} = \{ & \{1/a_1, 0.5/a_2\}, \{1/a_1, 0.5/a_2, 0.5/a_3\}\} \\
[i_2]_{\mathcal{M}} = \{ & \{1/a_3\}, \{0.5/a_1, 1/a_3\}, \{1/a_1, 1/a_3\}\} \\
[i_3]_{\mathcal{M}} = \{ & \{1/a_2, 0.5/a_3\}, \{0.5/a_1, 1/a_2\}, \{0.5/a_1, 1/a_2, 0.5/a_3\}\} \\
[i_4]_{\mathcal{M}} = \{ & \{1/a_1\}, \{1/a_1, 0.5/a_3\}\} \\
[i_5]_{\mathcal{M}} = \{ & \{0.5/a_2, 0.5/a_3\}, \{0.5/a_1, 0.5/a_2\}, \{0.5/a_1, 0.5/a_2, 0.5/a_3\}\} \\
[i_6]_{\mathcal{M}} = \{ & \{0.5/a_3\}, \{0.5/a_1\}, \{0.5/a_1, 0.5/a_3\}\} \\
[i_7]_{\mathcal{M}} = \{ & \{\}\} \\
[i_8]_{\mathcal{M}} = \{ & \{0.5/a_2\}\} \\
[i_9]_{\mathcal{M}} = \{ & \{1/a_2\}\}
\end{aligned}
$$

Notice that, the sets of subintents associated with the intents of the concepts C_7, C_8, and C_9 are formed only by their own closures, and hence, these sets do not produce any informative implication. Therefore, the irreducible subintents in the above nonsingleton equivalence classes are the following:

$$
\begin{aligned}
\mathcal{IS}(i_0) &= \{\{1/a_1, 1/a_2\}, \{0.5/a_2, 1/a_3\}\} \\
\mathcal{IS}(i_1) &= \{\{1/a_1, 0.5/a_2\}\} \\
\mathcal{IS}(i_2) &= \{\{1/a_3\}\} \\
\mathcal{IS}(i_3) &= \{\{1/a_2, 0.5/a_3\}, \{0.5/a_1, 1/a_2\}\} \\
\mathcal{IS}(i_4) &= \{\{1/a_1\}\} \\
\mathcal{IS}(i_5) &= \{\{0.5/a_2, 0.5/a_3\}, \{0.5/a_1, 0.5/a_2\}\} \\
\mathcal{IS}(i_6) &= \{\{0.5/a_3\}, \{0.5/a_1\}\}
\end{aligned}
$$

TABLE 2 | Relation R and mapping σ of Example 22.

R	b_1	b_2	b_3
a_1	1	1	0
a_2	0.5	0	1
a_3	0.5	0.5	0
σ	b_1	b_2	b_3
a_1	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_2	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_3	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$



$$\begin{aligned}
i_0 = Int(C_0) &= \langle\{1/a_1, 1/a_2, 1/a_3\}\rangle \\
i_1 = Int(C_1) &= \langle\{1/a_1, 0.5/a_2, 0.5/a_3\}\rangle \\
i_2 = Int(C_2) &= \langle\{1/a_1, 1/a_3\}\rangle \\
i_3 = Int(C_3) &= \langle\{0.5/a_1, 1/a_2, 0.5/a_3\}\rangle \\
i_4 = Int(C_4) &= \langle\{1/a_1, 0.5/a_3\}\rangle \\
i_5 = Int(C_5) &= \langle\{0.5/a_1, 0.5/a_2, 0.5/a_3\}\rangle \\
i_6 = Int(C_6) &= \langle\{0.5/a_1, 0.5/a_3\}\rangle \\
i_7 = Int(C_7) &= \langle\{\}\rangle \\
i_8 = Int(C_8) &= \langle\{0.5/a_2\}\rangle \\
i_9 = Int(C_9) &= \langle\{1/a_2\}\rangle
\end{aligned}$$

FIGURE 2 | Hasse diagram and intents of $(\mathcal{M}, \preceq)$ in Example 22.

Finally, taking into account Proposition 21, the following maximal informative attribute implications result from the above sets of irreducible subintents:

$$\begin{aligned}
\mathbf{M}\mathcal{F}_\iota(i_0) = \{&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{1/a_1, 1/a_2\}, \\
&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{0.5/a_2, 1/a_3\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_1) = \{&\{1/a_1, 0.5/a_2, 0.5/a_3\} \Leftarrow \{1/a_1, 0.5/a_2\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_2) = \{&\{1/a_1, 1/a_3\} \Leftarrow \{1/a_3\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_3) = \{&\{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}, \\
&\{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{0.5/a_1, 1/a_2\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_4) = \{&\{1/a_1, 0.5/a_3\} \Leftarrow \{1/a_1\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_5) = \{&\{0.5/a_1, 0.5/a_2, 0.5/a_3\} \Leftarrow \{0.5/a_2, 0.5/a_3\}, \\
&\{0.5/a_1, 0.5/a_2, 0.5/a_3\} \Leftarrow \{0.5/a_1, 0.5/a_2\}\} \\
\mathbf{M}\mathcal{F}_\iota(i_6) = \{&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_3\}, \\
&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_1\}\}
\end{aligned}$$

4 | Maximal Informative Attribute Implications

In formal contexts, the number of attribute implications often constitute a considerably large set, and many of these implications can indeed be redundant or trivial, making it difficult to work efficiently with the whole set of attribute implications. For this reason, we focus on exploring smaller sets of attribute implications that can provide us with the desired information without the need to compute the whole set of valid attribute implications associated with the given context. In this section, we will show that the set of maximal informative implications is indeed sufficient to generate all the fully true attribute implications that can be logically deduced from a given formal context.

The notion of informative implicational system is presented below.

Definition 23. An implicational system $\mathcal{T}$ is a set of fuzzy attribute implications over A. If an implicational system $\mathcal{T}$ composed only by informative implications will be called informative implicational system.

We now focus on two fundamental properties of an implicational system, soundness and completeness, which are essential to ensure that all attribute implications derived from the system are fully true and that every fully true attribute implication in the concept lattice can be derived from it. First, we will introduce the notion of soundness.

Definition 24. An implicational system $\mathcal{T}$ is called $\mathcal{M}$-sound if every attribute implication of $\mathcal{T}$ is fully true in $(\mathcal{M}, \preceq)$.

The soundness of an implicational system provides a consistent and logically valid representation of the dataset. The following proposition shows that the set of all maximal informative implications is sound.

Proposition 25. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Then, the set $\mathbf{M}\mathcal{F}_t = \{f^{\downarrow\uparrow} \Leftarrow f \mid f \in \mathcal{IS}_{\mathcal{M}}\}$ is $\mathcal{M}$ -sound.*

Proof. The proof straightforwardly follows from Proposition 11. □

Next, we will present the notion of completeness of an implicational system, which guarantees that all valid attribute implications in a multiadjoint concept lattice can be deduced from it. For this purpose, we need to consider a proof system with certain inference rules for attribute implications. Namely, we will use the usual syntactic inference system based on Armstrong's axioms.

Definition 26. Given three subsets of attributes $f_1, f_2, f_3 \in L_1^A$, the adaptation of the Armstrong's axioms is as follows:

- If $f_2 \prec_1 f_1$, then $\vdash f_2 \Leftarrow f_1$ (Reflexivity)
- $f_2 \Leftarrow f_1 \vdash f_2 \vee f_3 \Leftarrow f_1 \vee f_3$ (Augmentation)
- $f_3 \Leftarrow f_2$ and $f_2 \Leftarrow f_1 \vdash f_3 \Leftarrow f_1$ (Transitivity)

Different inference rules can be obtained directly from the above inference system, which will be used later on.

Proposition 27. *Given four subsets of attributes $f_1, f_2, f_3, f_4 \in L_1^A$, the following inference rules are derived from the adaptation of the Armstrong's axioms:*

1. $f_2 \Leftarrow f_1 \vdash f_2 \wedge f_3 \Leftarrow f_1$ (*Left weakening*)
2. $f_2 \Leftarrow f_1 \vdash f_2 \wedge f_4 \Leftarrow f_1 \vee f_3$ (*Information loss*)

Proof. We will prove that the above inference rules can be deduced from the adaptation of the Armstrong's axioms.

1. Let $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1$. By the reflexivity rule, we have that $f_2 \wedge f_3 \Leftarrow f_2$, for all $f_3 \in L_1^A$. Now, applying transitivity to $f_2 \wedge f_3 \Leftarrow f_2$ and $f_2 \Leftarrow f_1$, we deduce that $f_2 \wedge f_3 \Leftarrow f_1$.
2. Let $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1$. By the reflexivity rule, we obtain that $f_2 \wedge f_4 \Leftarrow f_2$ and $f_1 \Leftarrow f_1 \vee f_3$, for all $f_3, f_4 \in L_1^A$. Applying transitivity to $f_2 \wedge f_4 \Leftarrow f_2$, $f_2 \Leftarrow f_1$, and $f_1 \Leftarrow f_1 \vee f_3$, we can conclude that $f_2 \wedge f_4 \Leftarrow f_1 \vee f_3$ □

Notice that, Rule 1 is called Right weakening when the consequent is located in the right side of the implication symbol. Since the notation in this paper considers the consequent in the left side of the implication symbol, then we have called it Left weakening.

The next result shows that the inference system presented in Definition 26 is sound.

Proposition 28. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Then, Armstrong inference system is sound for any formal context.*

Proof. In order to prove the soundness of the Armstrong inference system, we will show that any rule deduced from this system is a fully true attribute implication.

1. Since the operators $^\uparrow$ and $^\downarrow$ form an antitone Galois connection, by Definition 6(2), we have that $f_1 \preceq_1 f_1^{\downarrow\uparrow}$ holds, for all $f_1 \in L_1^A$. In addition, by hypothesis, we have that $f_2 \prec_1 f_1$. From the above two inequalities, we deduce that $f_2 \prec_1 f_1^{\downarrow\uparrow}$, which, by Proposition 11, implies that $\|f_2 \Leftarrow f_1\| = \top_1$. Consequently, we conclude that $f_2 \Leftarrow f_1$ is a fully true attribute implication.
2. Let $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1 \in \mathcal{F}_t$, that is, $\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1$. Given $f_3 \in L_1^A$, we will see that the attribute implication $f_2 \vee f_3 \Leftarrow f_1 \vee f_3$ is also fully true, that is, the equality $\|f_2 \vee f_3 \Leftarrow f_1 \vee f_3\| = \top_1$ holds. As $\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1$, by Proposition 11, we have that the inequality $f_2 \preceq_1 f_1^{\downarrow\uparrow}$ holds. In addition, since $^\uparrow$ and $^\downarrow$ form an antitone Galois connection, by Definition 6(2), we have that $f_3 \preceq_1 f_3^{\downarrow\uparrow}$. Applying the supremum property, we then obtain that $f_2 \vee f_3 \preceq_1 f_1^{\downarrow\uparrow} \vee f_3^{\downarrow\uparrow}$. On the other hand, the following inequalities $f_1 \preceq_1 f_1 \vee f_3$ and $f_3 \preceq_1 f_1 \vee f_3$ also hold due to the supremum property. Then, as consequence of the monotonicity of the concept-forming operators, we deduce that $f_1^{\downarrow\uparrow} \preceq_1 (f_1 \vee f_3)^{\downarrow\uparrow}$ and $f_3^{\downarrow\uparrow} \preceq_1 (f_1 \vee f_3)^{\downarrow\uparrow}$. Once again, applying the supremum property, we have that the inequality $f_1^{\downarrow\uparrow} \vee f_3^{\downarrow\uparrow} \preceq_1 (f_1 \vee f_3)^{\downarrow\uparrow}$ holds. Therefore, we have obtained that the inequalities $f_2 \vee f_3 \preceq_1 f_1^{\downarrow\uparrow} \vee f_3^{\downarrow\uparrow}$ and $f_1^{\downarrow\uparrow} \vee f_3^{\downarrow\uparrow} \preceq_1 (f_1 \vee f_3)^{\downarrow\uparrow}$ are satisfied, from which we deduce that $f_2 \vee f_3 \preceq_1 (f_1 \vee f_3)^{\downarrow\uparrow}$. Consequently, by Proposition 11, we can conclude that the equality $\|f_2 \vee f_3 \Leftarrow f_1 \vee f_3\| = \top_1$ holds.
3. Let $f_1, f_2, f_3 \in L_1^A$ such that $f_3 \Leftarrow f_2, f_2 \Leftarrow f_1 \in \mathcal{F}_t$. We will prove that the attribute implication $f_3 \Leftarrow f_1$ is also fully true. For this purpose, we only have to check that the inequality $f_3 \preceq_1 f_1^{\downarrow\uparrow}$ is satisfied. By hypothesis, we have that $\|f_3 \Leftarrow f_2\|_{\mathcal{M}} = \top_1$ and $\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1$. By Proposition 11, this fact implies that $f_3 \preceq_1 f_2^{\downarrow\uparrow}$ and $f_2 \preceq_1 f_1^{\downarrow\uparrow}$. From the second inequality, due to the properties of the antitone Galois connections, we obtain that $f_2^{\downarrow\uparrow} \preceq_1 f_1^{\downarrow\uparrow}$. Then, from the last inequality and the first previous one obtained, we deduce that the inequality $f_3 \preceq_1 f_1^{\downarrow\uparrow}$ holds.
4. Let $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1 \in \mathcal{F}_t$. Given $f_3 \in L_1^A$, we will see that the attribute implication $f_2 \wedge f_3 \Leftarrow f_1$ is also fully true, that is, the equality $\|f_2 \wedge f_3 \Leftarrow f_1\| = \top_1$ holds. To prove this equality, as a consequence of Proposition 11, it is enough to show that the inequality $(f_2 \wedge f_3) \preceq_1 f_1^{\downarrow\uparrow}$ is satisfied. This fact follows from the following chain of inequalities:

$$f_2 \wedge f_3 \overset{(i)}{\preceq_1} f_2 \overset{(ii)}{\preceq_1} f_1^{\downarrow\uparrow}$$

where (i) holds due to the infimum property, and (ii) is obtained because $f_2 \Leftarrow f_1 \in \mathcal{F}_t$ and by Proposition 11. □

Therefore, we can ensure the correctness of the inference process when working with relational databases. In other words, these rules ensure that if we consider a set of attribute implications that are fully true in a given multiadjoint concept lattice, then any other attribute implication inferred/deduced from that set, by applying the adapted Armstrong's axioms, will also be fully true. Next definition introduces the formal notion of inference from an implicational system considering the Armstrong's axioms.

Definition 29. Given an implicational system $\mathcal{T}$, we say that an attribute implication $f_2 \Leftarrow f_1$ is A-inferred from $\mathcal{T}$, denoted as $\mathcal{T} \vdash_A f_2 \Leftarrow f_1$, if $f_2 \Leftarrow f_1$ can syntactically be obtained from $\mathcal{T}$, by using the Armstrong. We will simply write *inferred* if there is no confusion about the inference system.

Once presented the inference system, we can provide the notion of completeness of an implicational system.

Definition 30. An implicational system $\mathcal{T}$ is called $\mathcal{M}_A$-complete if every fully true attribute implication in $(\mathcal{M}, \preceq)$ is A-inferred from $\mathcal{T}$.

The following result shows that the whole set of maximal informative implications is complete.

Proposition 31. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Then, the set $\mathbf{M}\mathcal{F}_t = \{f^{\downarrow\uparrow} \Leftarrow f \mid f \in \mathcal{IS}_{\mathcal{M}}\}$ is $\mathcal{M}_A$-complete.*

Proof. To prove that the set $\mathbf{M}\mathcal{F}_t$ is $\mathcal{M}_A$-complete, we proceed as follows. We will show that, for any fully true attribute implication, there exists a maximal informative implication from which it can be deduced. Let $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1 \in \mathcal{F}_t$. By Proposition 11, this is equivalent to $f_2 \preceq_1 f_1^{\downarrow\uparrow}$. Now, two possibilities regarding the antecedent of the rule can be considered:

- f_1 is an irreducible subintent of $f_1^{\downarrow\uparrow}$, that is, $f_1 \in \mathcal{S}(f_1^{\downarrow\uparrow})$. Then, clearly $f_1^{\downarrow\uparrow} \Leftarrow f_1 \in \mathbf{M}\mathcal{F}_t$. In addition, the attribute implication $f_2 \Leftarrow f_1$ can be deduced from $f_1^{\downarrow\uparrow} \Leftarrow f_1$ considering Armstrong's axioms. From the inequality $f_2 \preceq_1 f_1^{\downarrow\uparrow}$ and the left weakening rule, we deduce that $f_2 \Leftarrow f_1$.
- f_1 is not an irreducible subintent of $f_1^{\downarrow\uparrow}$, that is, $f_1 \notin \mathcal{S}(f_1^{\downarrow\uparrow})$. In this case, an irreducible subintent f of $f_1^{\downarrow\uparrow}$ exists such that $f \prec_1 f_1$. By construction, it is clear that $f_1^{\downarrow\uparrow} \Leftarrow f \in \mathbf{M}\mathcal{F}_t$, and by augmentation with f_1, we obtain $f_1^{\downarrow\uparrow} \Leftarrow f_1$.

Therefore, we can conclude that $\mathbf{M}\mathcal{F}_t$ is $\mathcal{M}_A$-complete. □

As a consequence of the above result, any fully true attribute implication that are not maximally informative can be inferred from maximal informative implications as a logical consequence of using the adapted Armstrong inference system. Even some maximal attribute implications can also be deduced from other maximally informative implications, as the following example shows.

Example 32. Consider again the multiadjoint frame and context presented in Example 22. From the attribute implications $\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_3\}, \{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_1\} \in \mathbf{M}\mathcal{F}_t(i_6)$, by the augmentation rule, we deduce the following attribute implications $\{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}, \{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{0.5/a_1, 1/a_2\} \in \mathbf{M}\mathcal{F}_t(i_3)$, respectively. Then, we have obtained that

$$\mathbf{M}\mathcal{F}_t \vdash_A \{\{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}, \{0.5/a_1, 1/a_2, 0.5/a_3\} \Leftarrow \{0.5/a_1, 1/a_2\}\}$$

In other words, the maximal informative attribute implications obtained from the irreducible subintents of $\mathcal{IS}(i_3)$, can be deduces using the Armstrong inference system from others maximally informative implications.

Hence, the maximal attribute implications considered in the above example are redundant in practice, since they can be obtained from other maximal attribute implications in $\mathbf{M}\mathcal{F}_t$. The main goal of the following section is to introduce a mechanism to detect and remove from the set $\mathbf{M}\mathcal{F}_t$ the attribute implications that can be deduced from others.

5 | Bases of Attribute Implications in Multiadjoint Concept Lattices

As we mentioned in the previous section, we are interested in studying smaller sets of attribute implications that can provide us with the underlying information from a given context without the need to compute the whole set of attribute implications. A reasonable approach would be to require, on the one hand, that this set provides information regarding the validity of all attribute implications and, on the other hand, that it is not redundant, that is, attribute implications belonging to the set cannot be deduced from each other. In this section, we present an extension of the notion of a basis of attribute implications to the multiadoint framework and we establish the corresponding results for this purpose, following the Guigues and Duquenne approach [40].

First of all, we will introduce the notion of nonredundant implicational system.

Definition 33. Given an implicational system $\mathcal{T}$, an attribute implication $f_2 \Leftarrow f_1 \in \mathcal{T}$ is called $\mathcal{T}$-redundant if $f_2 \Leftarrow f_1$ is inferred from $\mathcal{T} \setminus \{f_2 \Leftarrow f_1\}$. An implication system $\mathcal{T}$ is *nonredundant* if none of the attribute implications $f_2 \Leftarrow f_1 \in \mathcal{T}$ is $\mathcal{T}$-redundant.

An example of a redundant attribute implication from a particular implicational system and context is presented below.

Example 34. In the multiadjoint concept lattice and context of Example 22, we will consider a implicational system $\mathcal{T}$ composed of the three attribute implications displayed in Table 3.

Now, we will see that the attribute implication I_3 can be deduced from I_1 and I_2, using the Armstrong inference system.

- By the augmentation rule applied to I_1 and I_2, we obtain the following attribute implications $\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{1/a_1, 0.5/a_2, 1/a_3\}$ and $\{1/a_1, 0.5/a_2, 1/a_3\} \Leftarrow \{0.5/a_1, 0.5/a_2, 1/a_3\}$, respectively.
- Applying transitivity to the two implications obtained above, we deduce I_3.

Then, we have proved that the implication I_3 is inferred from $\mathcal{T} \setminus \{I_3\}$, and hence, considering Definition 33, we conclude that the attribute implication I_3 is $\mathcal{T}$-redundant.

From a complete and nonredundant implicational system, we can introduce the notion of a basis of fuzzy attribute implications according to the requirements mentioned at the beginning of the section.

Definition 35. Given an implicational system $\mathcal{T}$ is a basis of attribute implications, which will be denoted as $\mathfrak{B}_{\mathcal{M}}$, if it is $\mathcal{M}$-sound, $\mathcal{M}_A$-complete and nonredundant system for $(\mathcal{M}, \preceq)$.

The above definition implies that we can obtain different bases of attribute implications for a given context. In this paper, we focus on a kind of basis introduced for a context with a finite set of attributes by Guigues and Duquenne [40] and clarified in [42], which are based on the concept of *minimal nonredundant nodes*. The idea is to systematically expand sets of attributes while ensuring that they remain minimal and nonredundant. This process, known as saturation, is a fundamental step for constructing the bases proposed by Guigues and Duquenne. This notion will be studied from now on in this section.

First of all, we will present the formal definition of pre-saturation. Before that, we need to introduce the following notation for a given informative implicational system $\mathcal{T}$. We will denote as $J_{\mathcal{T}}$ the set of antecedents of rules in $\mathcal{T}$, that is, $J_{\mathcal{T}} = \{h \in L_1^A | l \Leftarrow h \in \mathcal{T}\}$.

Definition 36. Given an informative implicational system $\mathcal{T}$ and a fuzzy subset of attributes $f \in L_1^A$, the set:

$$\overline{f}^{\mathcal{T}} = \bigvee \left(\{f\} \cup \left\{h^{\downarrow\uparrow} | h \prec_1 f, h \in J_{\mathcal{T}},\ and\ h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\right\}\right)$$

is called $\mathcal{T}$-pre-satured.

Basic properties of the pre-saturation operator are presented below. It is important to recall that we are considering that L_1 is finite.

Proposition 37. *Given an informative implicational system $\mathcal{T}$ and three fuzzy subsets of attributes $f, f_1, f_2 \in L_1^A$, we have that*

1. $f \preceq_1 \overline{f}^{\mathcal{T}} \preceq_1 f^{\downarrow\uparrow}$
2. *The pre-saturation operator is increasing, that is, if $f_1 \preceq_1 f_2$, then $\overline{f_1}^{\mathcal{T}} \preceq_1 \overline{f_2}^{\mathcal{T}}$.*

Proof.

1. Since we assume that L_1 is finite, we have that the set $\{h \in L_1^A | h \prec_1 f, h \in J_{\mathcal{T}},\ and\ h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\}$ is also finite, that is, equal to $\{h_1, \ldots, h_n\}$, with $n \in \mathbb{N}$. The first inequality $f \preceq_1 \overline{f}^{\mathcal{T}}$, follows directly from the definition of the pre-saturation operator. Now, we will prove the inequality $\overline{f}^{\mathcal{T}} \preceq_1 f^{\downarrow\uparrow}$ holds. Due to the pair $(^{\uparrow}, ^{\downarrow})$ forms an antitone

TABLE 3 | Attribute implications of the theory $\mathcal{T}$ given in Example 34.

Name	$f_2 \Leftarrow f_1$
I_1	$\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{0.5/a_2, 1/a_3\}$
I_2	$\{1/a_1, 1/a_3\} \Leftarrow \{1/a_3\}$
I_3	$\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{0.5/a_1, 0.5/a_2, 1/a_3\}$

Galois connection, from $h_i \prec_1 f$, we obtain that $h_i^{\downarrow\uparrow} \preceq_1 f^{\downarrow\uparrow}$, for all $i \in \{1, \dots, n\}$. Therefore, we have that

$$\overline{f}^{\mathcal{T}} = \bigvee(\{f\} \cup \{h_1^{\downarrow\uparrow}, \dots, h_n^{\downarrow\uparrow}\}) \preceq_1 \bigvee(\{f\} \cup \{f^{\downarrow\uparrow}\}) = \bigvee\{f, f^{\downarrow\uparrow}\} \preceq_1 f^{\downarrow\uparrow}$$

2. The proof straightforwardly follows from the definition of the pre-satured operator. □

The following proposition shows that definition of the pre-saturation operator can be characterized by terms of $\mathcal{IS}_{\mathcal{M}}$, that is, replacing $J_{\mathcal{T}}$ by $\mathcal{IS}_{\mathcal{M}}$ in Definition 36.

Proposition 38. *Given an informative implicational system $\mathcal{T}$ and $f \in L_1^A$, if $\mathcal{IS}_{\mathcal{M}} \subseteq J_{\mathcal{T}}$, then we have*

$$\overline{f}^{\mathcal{T}} = \bigvee\left(\{f\} \cup \{h^{\downarrow\uparrow} | h \prec_1 f, h \in \mathcal{IS}_{\mathcal{M}}, \text{ and } h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\}\right)$$

Proof. Let $f \Leftarrow h$ be an attribute implication in $\mathcal{T}$, such that $h \prec_1 f$ and $h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}$. If $h \in \mathcal{IS}_{\mathcal{M}}$, then, as $\mathcal{IS}_{\mathcal{M}} \subseteq J_{\mathcal{T}}$, and taking into account Definition 36, it is clear that the closure of the antecedent h must be considered in the computation of $\overline{f}^{\mathcal{T}}$. Now, we suppose that $h \notin \mathcal{IS}_{\mathcal{M}}$, that is, $h \in J_{\mathcal{T}} \setminus \mathcal{IS}_{\mathcal{M}}$. In this case, there exists an irreducible subintent $f_h \in \mathcal{IS}_{\mathcal{M}}$ such that $f_h \prec_1 h$ and $f_h^{\downarrow\uparrow} = h^{\downarrow\uparrow}$. Specifically, we have an irreducible subintent f_h satisfying $f_h \prec_1 h \prec_1 f$ and with the same closure of h. Therefore, the same contribution provide both f_h and h to the computation of $\overline{f}^{\mathcal{T}}$ ($f_h^{\uparrow\downarrow} = h^{\uparrow\downarrow}$). As a result, it is possible to avoid the consideration of the mappings in $J_{\mathcal{T}} \setminus \mathcal{IS}_{\mathcal{M}}$ in the computation of $\overline{f}^{\mathcal{T}}$. □

Moreover, we can define a pre-saturation for a given fuzzy subset of attributes with respect to a multiadjoint concept lattice, as follows.

Definition 39. Given a fuzzy subset of attributes $f \in L_1^A$, the set:

$$\overline{f} = \bigvee\left(\{f\} \cup \{h^{\downarrow\uparrow} | h \prec_1 f, h \in L_1^A, \text{ and } h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\}\right)$$

is called *pre-satured with respect to* $(\mathcal{M}, \preceq)$.

The computation of the above pre-saturation operator respect a multiadjoint concept lattice $(\mathcal{M}, \preceq)$ can be obtained directly from the set of the irreducible subintents of $(\mathcal{M}, \preceq)$, that is, considering $\mathcal{T} = \mathbf{M}\mathcal{F}_t$ $(J_{\mathcal{T}} = \mathcal{IS}_{\mathcal{M}})$ in Definition 36. This fact is shown in the following proposition.

Proposition 40. *Given a fuzzy subset of attributes $f \in L_1^A$, then we have that $\overline{f} = \overline{f}^{\mathbf{M}\mathcal{F}_t}$, that is,*

$$\overline{f} = \bigvee\left(\{f\} \cup \{h^{\downarrow\uparrow} | h \prec_1 f, h \in \mathcal{IS}_{\mathcal{M}}, \text{ and } h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\}\right)$$

Proof. It is straightforwardly follows similarly to the proof of Proposition 38. □

The following proposition presents more properties which allow us to relate the pre-saturation operator to the redundancy of implicational systems. The philosophy behind these properties is to eliminate redundant attribute implications from a given implicational system using the augmentation axiom.

Proposition 41. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$ -complete informative implicational system $\mathcal{T}$ and a fuzzy subset of attributes $f \in L_1^A$, we have that $\overline{f}^{\mathcal{T}} \Leftarrow f$ is $\mathcal{T}$ -redundant.*

Proof. As we assume that L_1 is finite, we can consider that $\{h \in L_1^A | h \prec_1 f, h \in J_{\mathcal{T}}, \text{ and } h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\} = \{h_1, \dots, h_n\}$, with $n \in \mathbb{N}$. Due to $h \preceq_1 h^{\downarrow\uparrow}$ is satisfied, for all $h \in L_1^A$, taking into account Proposition 11, it is clear that $h_i^{\downarrow\uparrow} \Leftarrow h_i \in \mathcal{F}_t$, that is, $\|h_i^{\downarrow\uparrow} \Leftarrow h_i\| = \top_1$, for all $i \in \{1, \dots, n\}$. Then, as $\mathcal{T}$ is $\mathcal{M}_A$-complete, we have that, the attribute implication $h_i^{\downarrow\uparrow} \Leftarrow h_i$ is inferred from $\mathcal{T}$, that is, $\mathcal{T} \vdash h_i^{\downarrow\uparrow} \Leftarrow h_i$, for all $i \in \{1, \dots, n\}$. Namely, it is satisfied that $\mathcal{T} \vdash h_1^{\downarrow\uparrow} \Leftarrow h_1$. Now, applying the augmentation axiom, we obtain that $\mathcal{T} \vdash f \vee h_1^{\downarrow\uparrow} \Leftarrow f \vee h_1$, that is, $\mathcal{T} \vdash \bigvee\{f, h_1^{\downarrow\uparrow}\} \Leftarrow f$. Therefore, if we sequentially apply this axiom to $h_2, \ldots, h_n$, we obtain that

$$\mathcal{T} \vdash \bigvee\{f, h_1^{\downarrow\uparrow}, \ldots, h_n^{\downarrow\uparrow}\} \Leftarrow f$$

Then, taking into account that $\bigvee\{f, h_1^{\downarrow\uparrow}, \ldots, h_n^{\downarrow\uparrow}\} = \bigvee(\{f\} \cup \{h_1^{\downarrow\uparrow}, \ldots, h_n^{\downarrow\uparrow}\}) = \overline{f}^{\mathcal{T}}$, we have that $\mathcal{T} \vdash \overline{f}^{\mathcal{T}} \Leftarrow f$, which implies that $\overline{f}^{\mathcal{T}} \Leftarrow f$ is $\mathcal{T}$-redundant. □

It is important to mention that, the pre-saturation is not a closure operator, as the following example shows.

Example 42. Consider the multiadjoint frame

$$([0,1]_2, \leq, (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{DL}, \nwarrow_{DL}))$$

and the context (A, B, R, σ), where the set of attributes is $A = \{a_1, a_2, a_3, a_4, a_5\}$, the set of objects is $B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$, the relation $R : A \times B \to [0,1]_2$ and the mapping σ are given in Table 4.

Given $f = \{0.5/a_3, 0.5/a_4\}$, we will see that the pre-saturation of this set is not a closed process, that is, $\overline{\overline{f}} \neq \overline{f}$. According to Proposition 40, we only need to consider the whole set of irreducible subintents of $(\mathcal{M}, \preceq)$ for the computation of the above pre-saturation. Following a similar process to the one presented in Example 22, we obtain that the irreducible subintents in the nonsingleton equivalence classes given by the relation $\equiv$ are the following:

$$\begin{aligned}
\mathcal{IS}(i_0) &= \{\{0.5/a_4\}\}\\
\mathcal{IS}(i_1) &= \{\{1/a_4\}\}\\
\mathcal{IS}(i_2) &= \{\{1/a_5\}, \{0.5/a_3, 0.5/a_5\}, \{0.5/a_1, 0.5/a_5\}\}\\
\mathcal{IS}(i_3) &= \{\{0.5/a_4, 1/a_5\}, \{0.5/a_3, 0.5/a_4\}, \{0.5/a_2, 0.5/a_5\},\\
&\qquad \{0.5/a_2, 0.5/a_4\}, \{0.5/a_2, 0.5/a_3\}, \{0.5/a_1, 0.5/a_4\}\}\\
\mathcal{IS}(i_4) &= \{\{1/a_4, 1/a_5\}, \{0.5/a_3, 1/a_4\}, \{0.5/a_2, 1/a_4\}, \{0.5/a_1, 1/a_4\}\}\\
\mathcal{IS}(i_5) &= \{\{1/a_1, 0.5/a_5\}\}\\
\mathcal{IS}(i_6) &= \{\{0.5/a_1, 1/a_3\}\}\\
\mathcal{IS}(i_7) &= \{\{1/a_3, 0.5/a_5\}, \{1/a_3, 0.5/a_5\}, \{0.5/a_2, 1/a_3\}, \{1/a_1, 0.5/a_4\},\\
&\qquad \{1/a_1, 0.5/a_2, 0.5/a_5\}, \{1/a_1, 0.5/a_2, 0.5/a_3\}\}\\
\mathcal{IS}(i_8) &= \{\{0.5/a_1, 1/a_2\}\}\\
\mathcal{IS}(i_9) &= \{\{1/a_3, 1/a_4\}, \{1/a_2, 0.5/a_5\}, \{1/a_2, 0.5/a_4\}, \{1/a_2, 0.5/a_3\},\\
&\qquad \{1/a_1, 1/a_4\}\}
\end{aligned}$$

where i_k, with $k \in \{0, 1, \ldots, 9\}$, are the intents given by the following fuzzy subsets of attributes:

$$\begin{aligned}
i_0 &= \{0.5/a_4, 0.5/a_5\}\\
i_1 &= \{1/a_4, 0.5/a_5\}\\
i_2 &= \{0.5/a_1, 0.5_a3, 1/a_5\}\\
i_3 &= \{0.5/a_1, 0.5/a_2, 0.5/a_3, 0.5/a_4, 1/a_5\}\\
i_4 &= \{0.5/a_1, 0.5/a_2, 0.5/a_3, 1/a_4, 1/a_5\}\\
i_5 &= \{1/a_1, 0.5/a_3, 1/a_5\}\\
i_6 &= \{1/a_1, 1/a_3\}\\
i_7 &= \{1/a_1, 0.5/a_2, 1/a_3, 0.5/a_4, 1/a_5\}\\
i_8 &= \{1/a_1, 1/a_2\}\\
i_9 &= \{1/a_1, 1/a_2, 1/a_3, 1/a_4, 1/a_5\}
\end{aligned}$$

TABLE 4 | Relation R and mapping σ of Example 42.

R	b_1	b_2	b_3	b_4	b_5	b_6
a_1	1	0	0	0	1	1
a_2	0.5	0	0	0	0	0
a_3	0.5	0.5	0	1	0.5	0.5
a_4	0	0	1	0	0	0
a_5	0	0	0.5	0	0	1
σ	b_1	b_2	b_3	b_4	b_5	b_6
a_1	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_2	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_3	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_4	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$
a_5	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$	$\&_{DG}$	$\&_{DP}$	$\&_{DL}$

Now, applying Proposition 40, we have that

$$\begin{aligned}\overline{f} &= \bigvee\left(\{f\} \cup \{h^{\downarrow\uparrow} | h \prec_1 f, h \in \mathcal{IS}_{\mathcal{M}},\ and\ h^{\downarrow\uparrow} \neq f^{\downarrow\uparrow}\}\right)\\ &= \bigvee\left(\{\{0.5/a_3, 0.5/a_4\}\} \cup \{\{0.5/a_4, 0.5/a_5\}\}\right)\\ &= \bigvee\left(\{\{0.5/a_3, 0.5/a_4\}, \{0.5/a_4, 0.5/a_5\}\}\right)\\ &= \{0.5/a_3, 0.5/a_4, 0.5/a_5\}\end{aligned}$$

Then, we obtain that $\overline{f} = \{0.5/a_3, 0.5/a_4, 0.5/a_5\}$. Now applying again the pre-saturation operator to the set $\overline{f}$, we have that

$$\begin{aligned}\overline{\overline{f}} &= \bigvee\left(\{\overline{f}\} \cup \{h^{\downarrow\uparrow} | h \prec_1 \overline{f}, h \in \mathcal{IS}_{\mathcal{M}},\ and\ h^{\downarrow\uparrow} \neq \overline{f}^{\downarrow\uparrow}\}\right)\\ &= \bigvee\left(\{\{0.5/a_3, 0.5/a_4, 0.5/a_5\}\} \cup \{\{0.5/a_4, 0.5/a_5\}, \{0.5/a_1, 0.5/a_3, 1/a_5\}\}\right)\\ &= \bigvee\left(\{\{0.5/a_3, 0.5/a_4, 0.5/a_5\}, \{0.5/a_4, 0.5/a_5\}, \{0.5/a_1, 0.5/a_3, 1/a_5\}\}\right)\\ &= \{0.5/a_1, 0.5/a_3, 0.5/a_4, 1/a_5\}\end{aligned}$$

Thus, $\overline{\overline{f}} = \{0.5/a_1, 0.5/a_3, 0.5/a_4, 1/a_5\} \neq \overline{f}$. As a consequence, we can conclude that the pre-saturation operator is not closed in this case.

Therefore, given a fuzzy subset of attributes $f \in L_1^A$, we have that its pre-saturation regarding an informative implication system $\mathcal{T}$ is an increasing and not closed process, such that is upper bounded by its own closure $f^{\downarrow\uparrow}$. Indeed, the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined as $f_0 = f$ and $f_{n+1} = \overline{f_n}^{\mathcal{T}}$, for all $n \in \mathbb{N}$, is also increasing and upper bounded by $f^{\downarrow\uparrow}$. This repeated pre-saturation application process gives as a result the saturation of the given fuzzy subset f. The formal definition of saturation is presented below.

Definition 43. Given an informative implicational system $\mathcal{T}$ and a fuzzy subset of attributes $f \in L_1^A$, the $\mathcal{T}$-saturation operator is defined as

$$\tilde{f}^{\mathcal{T}} = \bigvee\{f_n | n \in \mathbb{N}\}$$

where $f_0 = f$ and $f_{n+1} = \overline{f_n}^{\mathcal{T}}$, for all $n \in \mathbb{N}$.

In order to simply the notation, if no confusion exists with the implicational system, we will simply write saturation in the above definition. We will write $\tilde{f}$, when $\mathcal{T} = \mathbf{M}\mathcal{F}_t$.

Notice that, as we assume that L_1 is finite, the saturation of a fuzzy subset of attributes is attained in a finite number of steps, that is, given $f \in L_1^A$, and the sequence $f_0 = f$ and $f_{n+1} = \overline{f_n}^{\mathcal{T}}$, for all $n \in \mathbb{N}$, there exits $n_0 \in \mathbb{N}$ such that $f_{n+1} = f_n$, with $n_0 \le n$, where n is bounded by the distance between f and $f^{\downarrow\uparrow}$. This distance between two fuzzy subsets of attributes can be a generalization from any distance between pairs of elements of L_1. From this fact, we also deduce that the saturation is a closure operator. Similar properties to presented in Proposition 37 can be obtained for the saturation operator.

Lemma 44. *Given an informative implicational system $\mathcal{T}$ and three fuzzy subsets of attributes $f, f_1, f_2 \in L_1^A$, we have that*

1. $f \preceq_1 \tilde{f}^{\mathcal{T}} \preceq_1 f^{\downarrow\uparrow}$
2. *The saturation operator is increasing, that is, if $f_1 \preceq_1 f_2$, then the inequality $\tilde{f_1}^{\mathcal{T}} \preceq_1 \tilde{f_2}^{\mathcal{T}}$ holds.*

Proof. Follows from Definition 43 and Proposition 37. □

Moreover, we can define an equivalence relation from the definition of saturation, which provide a partition of L_1^A, as follows.

Definition 45. The relation $\cong^{\mathcal{T}}$ on $L_1^A \times L_1^A$ is defined as $f_1 \cong^{\mathcal{T}} f_2$, if $\tilde{f_1}^{\mathcal{T}} = \tilde{f_2}^{\mathcal{T}}$, for all $f_1, f_2 \in L_1^A$. The equivalence classes will be denoted as $[f]_{\cong^{\mathcal{T}}}$, for all $f \in L_1^A$.

It is convenient to mention that the above partition induced by the saturation operator is finer than the induced by intents (see Definition 20), as the following proposition proofs.

Proposition 46. *Given an informative implicational system $\mathcal{T}$ and a fuzzy subset of attributes $f \in L_1^A$, we have that $[f]_{\cong^{\mathcal{T}}} \subseteq [f]_{\mathcal{M}}$.*

Proof. Given $f_1 \in L_1^A$, we will consider $f_2 \in L_1^A$ such that $f_2 \in [f_1]_{\cong^{\mathcal{T}}}$, which is equivalent to $\tilde{f_2}^{\mathcal{T}} = \tilde{f}^{\mathcal{T}}$. We aim to prove that $f_2 \in [f_1]_{\mathcal{M}}$, i.e, $f_2^{\downarrow\uparrow} = f_1^{\downarrow\uparrow}$. By Lemma 44(1), we have that $f \preceq_1 \tilde{f}^{\mathcal{T}} \preceq_1 f^{\downarrow\uparrow}$, for all $f \in L_1^A$. In addition, due to the pair $(^{\uparrow}, ^{\downarrow})$ forms an antinone Galois connection, from the above chain of inequalities we obtain that $f^{\downarrow\uparrow} \preceq_1 (\tilde{f}^{\mathcal{T}})^{\downarrow\uparrow} \preceq_1 f^{\downarrow\uparrow}$, which implies that $(\tilde{f}^{\mathcal{T}})^{\downarrow\uparrow} = f^{\downarrow\uparrow}$, for all $f \in L_1^A$. Consequently, we have that $f_2^{\downarrow\uparrow} = (\tilde{f_2}^{\mathcal{T}})^{\downarrow\uparrow} = (\tilde{f_1}^{\mathcal{T}})^{\downarrow\uparrow} = f_1^{\downarrow\uparrow}$, that is $f_2^{\downarrow\uparrow} = f_1^{\downarrow\uparrow}$. Therefore, we can conclude $[f_1]_{\cong^{\mathcal{T}}} \subseteq [f_1]_{\mathcal{M}}$, for all $f_1 \in L_1^A$. □

The following lemma shows that the redundancy of attribute implications can also be related to the saturation process.

Lemma 47. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$-complete informative implicational system $\mathcal{T}$ and $f \in L_1^A$, we have that $\tilde{f}^{\mathcal{T}} \Leftarrow f$ is $\mathcal{T}$-redundant.*

Proof. Given $f \in L_1^A$, we will consider the sequence defined as $f_0 = f$ and $f_{n+1} = \overline{f_n}^{\mathcal{T}}$, for all $n \in \mathbb{N}$. As we assume that L_1 is finite, then the saturation $\tilde{f}^{\mathcal{T}}$ is attained in a finite number of steps, and hence, there exits $n_0 \in \mathbb{N}$ such that $f_{n+1} = f_n$, for all $n \in \mathbb{N}$, with $n_0 \le n$. Moreover, by the definition of the considered sequence and Proposition 41(1), it follows that the attribute implication $f_{n+1} \Leftarrow f_n$ is $\mathcal{T}$-redundant, for all $n \in \mathbb{N}$. Then, if we apply the transitivity axiom from the implication $f_1 \Leftarrow f_0$ to the implication $f_{n_0} \Leftarrow f_{n_0-1}$, we have that $\bigvee\{f_n | n \le n_0\} \Leftarrow f_0$ is also $\mathcal{T}$-redundant, that is, $\tilde{f}^{\mathcal{T}} \Leftarrow f$ is $\mathcal{T}$-redundant. □

As a consequence of the above result, we can ensure that attribute implications of the form $f^{\downarrow\uparrow} \Leftarrow f$ will be $\mathcal{T}$-redundant whenever $f^{\downarrow\uparrow} = \tilde{f}^{\mathcal{T}}$. Moreover, it is convenient to mention that if f is an intent of a concept, then it is verified that $f = \tilde{f}^{\mathcal{T}} = f^{\downarrow\uparrow}$. From now on, we will focus in the fixed points provided by the saturation operator that are not intents, which yield the concept of nonredundant node.

Definition 48. A fuzzy subset of attributes $f \in L_1^A$ is called nonredundant node, denoted as NRN for short, if $f = \tilde{f}^{\mathcal{T}}$ and $f \neq f^{\downarrow\uparrow}$. The set of nonredundant nodes will be denoted as $\mathcal{N}$.

The above definition implies that the saturation of a fuzzy set of attributes is either an intent, or is an NRN. This definition can be characterized in terms of the pre-saturation operator, as the following lemma shows.

Lemma 49. *Given an informative implicational system $\mathcal{T}$, a fuzzy subset of attributes $f \in L_1^A$ is an NRN if and only if $f = \overline{f}^{\mathcal{T}}$ and $f \neq f^{\downarrow\uparrow}$.*

Proof. First of all, if f is an NRN then $f = \tilde{f}^{\mathcal{T}}$ and $f \neq f^{\downarrow\uparrow}$. From the first equality and the definition of pre-saturation, we obtain that

$$f = \tilde{f}^{\mathcal{T}} = \overline{\tilde{f}^{\mathcal{T}}}^{\mathcal{T}} = \overline{f}^{\mathcal{T}}$$

Conversely, if $f = \overline{f}^{\mathcal{T}}$ and $f \neq f^{\downarrow\uparrow}$, then by the first equality, we obtain

$$f = \overline{f}^{\mathcal{T}} = \overline{\overline{f}^{\mathcal{T}}}^{\mathcal{T}}$$

that is, the iteration of the pre-saturation does not modify the mapping, and so f is a NRN. □

Notice that we can consider two types of NRN depending on whether they are irreducible or not. Therefore, we need to introduce the notion of minimal NRN.

Definition 50. Given an informative implicational system $\mathcal{T}$, a minimal nonredundant node is an NRN $f \in L_1^A$ such that for every subset $h \preceq_1 f$, if $h^{\downarrow\uparrow} = f^{\downarrow\uparrow}$ and $h = \tilde{h}^{\mathcal{T}}$, then $h = f$. The set of minimal nonredundant nodes will be denoted as $\mathcal{N}_0$.

The following results show the importance of the NRNs in the study of redundancy of implicational systems.

Proposition 51. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$-complete informative implicational system $\mathcal{T}$, a NRN $f \in \mathcal{N}$ and $h \in [f]_{\cong^{\mathcal{T}}}$, we have that $f^{\downarrow\uparrow} \Leftarrow f$ is inferred from $\mathcal{T}$ if and only if $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$, that is,*

$$\mathcal{T} \vdash f^{\downarrow\uparrow} \Leftarrow f \qquad \text{if and only if} \qquad \mathcal{T} \vdash h^{\downarrow\uparrow} \Leftarrow h$$

Proof. Consider $f \in \mathcal{N}$ and $h \in [f]_{\cong^{\mathcal{T}}}$. First of all, we will prove that if $f^{\downarrow\uparrow} \Leftarrow f$ is inferred from $\mathcal{T}$, then $h^{\downarrow\uparrow} \Leftarrow h$ is also inferred from $\mathcal{T}$. Since $h \in [f]_{\cong^{\mathcal{T}}}$, the following items holds:

- By definition of $[f]_{\cong^{\mathcal{T}}}$, we have that $\tilde{f}^{\mathcal{T}} = \tilde{h}^{\mathcal{T}}$. In addition, since $f \in \mathcal{N}$, we have that $f = \tilde{f}^{\mathcal{T}} = \tilde{h}^{\mathcal{T}}$, that is $f = \tilde{h}^{\mathcal{T}}$.
- From Proposition 46, we deduce that $h \in [f]_{\mathcal{M}}$, which implies that $f^{\downarrow\uparrow} = h^{\downarrow\uparrow}$.

Now, due to $f^{\downarrow\uparrow} \Leftarrow f$ is inferred from $\mathcal{T}$, taking into account that $f = \tilde{h}^{\mathcal{T}}$ and $f^{\downarrow\uparrow} = h^{\downarrow\uparrow}$, we obtain that $h^{\downarrow\uparrow} \Leftarrow \tilde{h}^{\mathcal{T}}$ is also inferred from $\mathcal{T}$. Moreover, by Lemma 47, we have that $\tilde{h}^{\mathcal{T}} \Leftarrow h$ is inferred from $\mathcal{T}$. Applying transitivity to $h^{\downarrow\uparrow} \Leftarrow \tilde{h}^{\mathcal{T}}$ and $\tilde{h}^{\mathcal{T}} \Leftarrow h$, we can conclude that $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$.

Conversely, we will prove that if $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$, then $f^{\downarrow\uparrow} \Leftarrow f$ is also inferred from $\mathcal{T}$. Since $h \in [f]_{\cong^{\mathcal{T}}}$, it is satisfied that

- From the definition of $[f]_{\cong^{\mathcal{T}}}$, and due to $f \in \mathcal{N}$, we obtain that $\tilde{h}^{\mathcal{T}} = \tilde{f}^{\mathcal{T}} = f$. By Lemma 44(1), we have that $h \preceq_1 \tilde{h}^{\mathcal{T}}$. Thus, we deduce that $h \preceq_1 f$.
- By Proposition 46, it follows that $f^{\downarrow\uparrow} = h^{\downarrow\uparrow}$. By Definition 6(2), the inequality $f \preceq_1 f^{\downarrow\uparrow}$ holds. Then, we obtain that $f \preceq_1 h^{\downarrow\uparrow}$.

Now, due to $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$, then, by the augmentation axiom, we deduce that $h^{\downarrow\uparrow} \vee f \Leftarrow h \vee f$ is also inferred from $\mathcal{T}$. Finally, taking into account that $h \preceq_1 f$, $f \preceq_1 h^{\downarrow\uparrow}$ and $f^{\downarrow\uparrow} = h^{\downarrow\uparrow}$, we conclude that $f^{\downarrow\uparrow} \Leftarrow f$ is inferred from $\mathcal{T}$. □

From the previous result, we obtain the following corollary.

Corollary 52. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$ -complete informative implicational system $\mathcal{T}$, a NRN $f \in \mathcal{N}$ and $h, l \in [f]_{\cong^{\mathcal{T}}}$, we have that $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$ if and only if $l^{\downarrow\uparrow} \Leftarrow l$ is inferred from $\mathcal{T}$.*

As a consequence of the above results, to construct a nonredundant implicational system, we can include at most one implication $h^{\downarrow\uparrow} \Leftarrow h$, with $h \in [f]_{\cong^{\mathcal{T}}}$, for all $f \in \mathcal{N}$. Specifically, we can only consider minimal NRNs $f \in \mathcal{N}_0$ to obtain nonredundant attribute implications of the form $h^{\downarrow\uparrow} \Leftarrow h$, with $h \in [f]_{\cong^{\mathcal{T}}}$, since nonminimal NRNs result in redundant attribute implications, as the following proposition shows.

Proposition 53. *Given a $\mathcal{M}_A$ -complete informative implicational system $\mathcal{T}$, a NRN $l \in \mathcal{N} \setminus \mathcal{N}_0$, and $f \in [l]_{\mathcal{M}}$, satisfying $f \prec_1 l$ and $f \in \mathcal{N}_0$. Let $h \in [f]_{\cong^{\mathcal{T}}}$, we have that, if $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$, then $l^{\downarrow\uparrow} \Leftarrow l$ is inferred from $\mathcal{T}$.*

Proof. Let $l \in \mathcal{N} \setminus \mathcal{N}_0$, and $f \in [l]_{\mathcal{M}}$ such that $f \prec_1 l$ and $f \in \mathcal{N}_0$. Consider $h \in [f]_{\cong^{\mathcal{T}}}$ such that $h^{\downarrow\uparrow} \Leftarrow h$ is inferred from $\mathcal{T}$, we will prove that $l^{\downarrow\uparrow} \Leftarrow l$ is also inferred from $\mathcal{T}$. Since $h \in [f]_{\cong^{\mathcal{T}}}$ and $f \in \mathcal{N}_0$, we have that $h \preceq_1 \tilde{h}^{\mathcal{T}} = \tilde{f}^{\mathcal{T}} = f$, that is, $h \preceq_1 f$, which implies that $h \preceq_1 l$, due to $f \prec_1 l$, by hypothesis. Moreover, we have that the equality $h^{\downarrow\uparrow} = l^{\downarrow\uparrow}$ holds, which follows from the following chain of equalities $h^{\downarrow\uparrow} = f^{\downarrow\uparrow} = l^{\downarrow\uparrow}$, where the first equality is a consequence of Proposition 46, and the second one is obtained from the fact that $f \in [l]_{\mathcal{M}}$. Therefore, applying the augmentation axiom we deduce that $h^{\downarrow\uparrow} \vee l \Leftarrow h \vee l$ is inferred from $\mathcal{T}$, which, taking into account the previous results obtained, implies that $l^{\downarrow\uparrow} \Leftarrow l$ is also inferred from $\mathcal{T}$. □

Therefore, nonredundant and complete implicational systems, that is, bases, are defined by considering only the classes provided by each minimal nonredundant node, that is, $[f]_{\cong^{\mathcal{T}}}$, with $f \in \mathcal{N}_0$.

Theorem 54. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$ -complete informative implicational system $\mathcal{T}$, and a family of fuzzy subsets of attributes $\mathcal{F}$ composed of only one representative class $[f]_{\cong^{\mathcal{T}}}$, such that $f \in \mathcal{N}_0$. Any implicational systems defined as*

$$\mathfrak{B} = \{h^{\downarrow\uparrow} \Leftarrow h \mid h \in \mathcal{F}\}$$

is a base, that is, $\mathcal{M}_A$ -complete and nonredundant.

Proof. First of all, we will prove that the set $\mathfrak{B}$ is $\mathcal{M}_A$-complete. For this purpose, we will show that any fully true attribute implication of $(\mathcal{M}, \preceq)$ (excluding those belonging to the base $\mathfrak{B}$) can be inferred from $\mathfrak{B}$, that is, they are $\mathfrak{B}$-redundant. Taking into account the left weakening rule, it is enough to prove this fact for fully true attribute implications of the forms $l^{\downarrow\uparrow} \Leftarrow l$, for all $l \in L_1^A$. Then, given $l \in L_1^A$, since the relation $\cong^{\mathcal{T}}$ forms a partition of L_1^A, we can ensure that there exists a satured set $f \in L_1^A$ such that $l \in [f]_{\cong^{\mathcal{T}}}$. Hence, we need to consider the following possibilities regarding the satured set f:

- If $f \in \mathcal{N} \setminus \mathcal{N}_0$, then there exists $m \prec_1 f$ such that $m \in \mathcal{N}_0$ and $m^{\downarrow\uparrow} = f^{\downarrow\uparrow}$. Therefore, by definition of $\mathfrak{B}$ there exists $h \in [m]_{\cong^{\mathcal{T}}} \cap \mathcal{F}$, such that $m^{\downarrow\uparrow} \Leftarrow m \in \mathfrak{B}$. Hence, by Proposition 53, the attribute implication $l^{\downarrow\uparrow} \Leftarrow l$ is $\mathfrak{B}$-redundant.
- If $f \in \mathcal{N}_0$, by the definition of the set $\mathcal{F}$, there exists $h \in \mathcal{F}$, such that $h \in [f]_{\cong^{\mathcal{T}}}$ and $h^{\downarrow\uparrow} \Leftarrow h \in \mathfrak{B}$. Therefore, by Corollary 52, we deduce that the attribute implication $l^{\downarrow\uparrow} \Leftarrow l$ is $\mathfrak{B}$-redundant.
- If $f \notin \mathcal{N}$ is because of $\tilde{f}^{\mathcal{T}} = f^{\downarrow\uparrow}$, which implies that

$$\tilde{l}^{\mathcal{T}} \overset{(i)}{=} \tilde{f}^{\mathcal{T}} \overset{(ii)}{=} f^{\downarrow\uparrow} \overset{(iii)}{=} l^{\downarrow\uparrow}$$

  where (i) holds due to $l \in [f]_{\cong^{\mathcal{T}}}$, (ii) follows from the fact that $f \notin \mathcal{N}$ (see Definition 48), and (iii) is satisfied because $[f]_{\cong^{\mathcal{T}}} \subseteq [f]_{\mathcal{M}}$ (see Proposition 46) and $l \in [f]_{\cong^{\mathcal{T}}}$.

Now, we will show that the set $\mathfrak{B}$ is nonredundant. For this purpose, we will prove that given $h^{\downarrow\uparrow} \Leftarrow h \in \mathfrak{B}$, this attribute implication cannot be inferred from $\mathfrak{B} \setminus \{h^{\downarrow\uparrow} \Leftarrow h\}$. Consider $h^{\downarrow\uparrow} \Leftarrow h \in \mathfrak{B}$, since $h \in \mathcal{F}$, then there exists $f \in \mathcal{N}_0$, such that $h \in [f]_{\cong^{\mathcal{T}}}$. Specifically, h is the only antecedent in $\mathfrak{B}$ representing the class $[f]_{\cong^{\mathcal{T}}}$. Then, for every antecedent $l \in J_{\mathfrak{B} \setminus \{h^{\downarrow\uparrow} \Leftarrow h\}}$, it is satisfied that $l \notin [f]_{\cong^{\mathcal{T}}}$, which implies that $\tilde{l}^{\mathcal{T}} \neq \tilde{f}^{\mathcal{T}} = \tilde{h}^{\mathcal{T}}$. As a consequence, we obtain that the attribute implication $h^{\downarrow\uparrow} \Leftarrow h$ cannot be deduced from $\mathfrak{B} \setminus \{h^{\downarrow\uparrow} \Leftarrow h\}$. Therefore, we conclude that the set $\mathfrak{B}$ is nonredundant.

□

In other words, only one representative of each class must be chosen for each minimal node to obtain a complete and minimal base. As a consequence, different bases can be obtained for a given context. For example, we differentiate between bases where the antecedents of the implications are exclusively minimal nodes and the ones that where the antecedents of the implications are only irreducible subintents.

Proposition 55. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\nwarrow$ is a forcing implication. Given a $\mathcal{M}_A$ -complete informative implicational system $\mathcal{T}$, and a family of irreducible subintents $\mathcal{F}_I$ that are selected of every class $[f]_{\cong}$, where $f \in \mathcal{N}_0$. The sets of attributes implications defined as*

$$\mathfrak{B}_{\mathcal{N}_0} = \{f^{\downarrow\uparrow} \Leftarrow f \mid f \in \mathcal{N}_0\}$$
$$\mathfrak{B}_I = \{l^{\downarrow\uparrow} \Leftarrow l \mid l \in \mathcal{F}_I\}$$

are $\mathcal{M}_A$ -complete and nonredundant.

Proof. Follows directly from Theorem 54. □

This section finishes with a particular example which illustrates the above results.

Example 56. Consider the multiadjoint frame and context proposed in Example 22. The following equivalence classes $[f]_{\cong}$ are obtained, where f corresponds with the maximal set in its equivalence class:

$$[\{1/a_1, 1/a_2, 1/a_3\}]_{\cong} = \{\{1/a_2, 1/a_3\}, \{0.5/a_1, 1/a_2, 1/a_3\}, \{1/a_1, 1/a_2, 1/a_3\}\}$$
$$[\{1/a_1, 1/a_2, 0.5/a_3\}]_{\cong} = \{\{1/a_1, 1/a_2\}, \{1/a_1, 1/a_2, 0.5/a_3\}\}$$
$$[\{1/a_1, 0.5/a_2, 1/a_3\}]_{\cong} = \{\{0.5/a_2, 1/a_3\}, \{0.5/a_1, 0.5/a_2, 1/a_3\}, \{1/a_1, 0.5/a_2, 1/a_3\}\}$$
$$[\{1/a_1, 0.5/a_2, 0.5/a_3\}]_{\cong} = \{\{1/a_1, 0.5/a_2\}, \{1/a_1, 0.5/a_2, 0.5/a_3\}\}$$
$$[\{0.5/a_1, 1/a_2, 0.5/a_3\}]_{\cong} = \{\{0.5/a_1, 1/a_2\}, \{1/a_2, 0.5/a_3\}, \{0.5/a_1, 1/a_2, 0.5/a_3\}\}$$
$$[\{0.5/a_1, 0.5/a_2, 0.5/a_3\}]_{\cong} = \{\{0.5/a_1, 0.5/a_2\}, \{0.5/a_2, 0.5/a_3\}, \{0.5/a_1, 0.5/a_2, 0.5/a_3\}\}$$
$$[\{1/a_1, 1/a_3\}]_{\cong} = \{\{1/a_1, 1/a_3\}\}$$
$$[\{1/a_1, 0.5/a_3\}]_{\cong} = \{\{1/a_1\}, \{1/a_1, 0.5/a_3\}\}$$
$$[\{0.5/a_1, 1/a_3\}]_{\cong} = \{\{1/a_3\}, \{0.5/a_1, 1/a_3\}\}$$
$$[\{0.5/a_1, 0.5/a_3\}]_{\cong} = \{\{0.5/a_1, 0.5/a_3\}\}$$
$$[\{0.5/a_1\}]_{\cong} = \{\{0.5/a_1\}\}$$
$$[\{0.5/a_2\}]_{\cong} = \{\{0.5/a_2\}\}$$
$$[\{0.5/a_3\}]_{\cong} = \{\{0.5/a_3\}\}$$
$$[\{1/a_2\}]_{\cong} = \{\{1/a_2\}\}$$

According to the Proposition 46, we obtain a partition of the equivalence classes of subintents presented in Example 22 from the above equivalence classes of satured sets as follows:

$$[i_0]_{\mathcal{M}} = [\{1/a_1, 1/a_2, 1/a_3\}]_{\cong} \cup [\{1/a_1, 1/a_2, 0.5/a_3\}]_{\cong} \cup [\{1/a_1, 0.5/a_2, 1/a_3\}]_{\cong}$$
$$[i_1]_{\mathcal{M}} = [\{1/a_1, 0.5/a_2, 0.5/a_3\}]_{\cong}$$
$$[i_2]_{\mathcal{M}} = [\{1/a_1, 1/a_3\}]_{\cong} \cup [\{0.5/a_1, 1/a_3\}]_{\cong}$$
$$[i_3]_{\mathcal{M}} = [\{0.5/a_1, 1/a_2, 0.5/a_3\}]_{\cong}$$
$$[i_4]_{\mathcal{M}} = [\{1/a_1, 0.5/a_3\}]_{\cong}$$
$$[i_5]_{\mathcal{M}} = [\{0.5/a_1, 0.5/a_2, 0.5/a_3\}]_{\cong}$$
$$[i_6]_{\mathcal{M}} = [\{0.5/a_1, 0.5/a_3\}]_{\cong} \cup [\{0.5/a_1\}]_{\cong} \cup [\{0.5/a_3\}]_{\cong}$$
$$[i_8]_{\mathcal{M}} = [\{0.5/a_2\}]_{\cong}$$
$$[i_9]_{\mathcal{M}} = [\{1/a_2\}]_{\cong}$$

Therefore, we have that the set of NRNs nodes is given by:

$$\mathcal{N} = \{\{1/a_1, 1/a_2, 0.5/a_3\}, \{1/a_1, 0.5/a_2, 1/a_3\}, \{0.5/a_1, 1/a_3\}, \{0.5/a_1\}, \{0.5/a_3\}\}$$

It is easy to see that every fuzzy subset in $\mathcal{N}$ is minimal, and hence, we get that $\mathcal{N}_0 = \mathcal{N}$. Moreover, the subsets $\{1/a_1, 1/a_2, 0.5/a_3\}, \{1/a_1, 0.5/a_2, 1/a_3\}$ and $\{0.5/a_1, 1/a_3\}$ are not irreducible subintents, which, by Proposition 55, implies that we can consider two different kind of minimal bases. Specifically, every base is composed by the following implications:

$$\begin{aligned}
\mathfrak{B}_{\mathcal{N}_0} = \{&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{1/a_1, 1/a_2, 0.5/a_3\},\\
&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{1/a_1, 0.5/a_2, 1/a_3\},\\
&\{1/a_1, 1/a_3\} \Leftarrow \{0.5/a_1, 1/a_3\},\\
&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_1\},\\
&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_3\}\}\\
\mathfrak{B}_I = \{&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{1/a_1, 1/a_2\},\\
&\{1/a_1, 1/a_2, 1/a_3\} \Leftarrow \{0.5/a_2, 1/a_3\},\\
&\{1/a_1, 1/a_3\} \Leftarrow \{1/a_3\},\\
&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_1\},\\
&\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_3\}\}
\end{aligned}$$

From the above sets, we can obtain any fully true attribute implications of $(\mathcal{M}, \preceq)$, using the inference rules presented in Definition 26 and Proposition 27. For example, we consider the attribute implication $\{0.5/a_1\} \Leftarrow \{1/a_2, 0.5/a_3\} \in \mathcal{F}_t$ (see Example 22). By the reflexivity rule, we have that $\{0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}$. Applying transitivity to $\{0.5/a_1, 0.5/a_3\} \Leftarrow \{0.5/a_3\}$ (belonging to $\mathfrak{B}_{\mathcal{N}_0}$ and $\mathfrak{B}_I$) and $\{0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}$, we obtain that $\{0.5/a_1, 0.5/a_3\} \Leftarrow \{1/a_2, 0.5/a_3\}$. Then, by the weakening rule, we can deduce that $\{0.5/a_1\} \Leftarrow \{1/a_2, 0.5/a_3\}$.

This paper introduces a mechanism for the computation of bases of attribute implications, based on the minimal nonredundant nodes (NRNs), following the Guigues and Duquenne approach. As previously was commented, alternative approaches to extracting bases in the fuzzy setting have also been proposed in the literature, such as in the residuated [15] or heterogeneous formal context [50] frameworks, which clearly provide different perspectives. An in-depth comparison will be given in the future.

6 | Conclusions and Future Work

This paper has introduced a general procedure to compute bases of attribute implication from datasets, which can be used by different approaches based on Galois connections. It has been instantiated on the multiadjoint concept lattice framework, introducing the sufficient condition to prove that a fully true attribute implication is equivalent to its consequent is included in the closure of the antecedent of the attribute implication, which is the well-known characterization of validity in the classical case given by Guigues and Duquenne in [40] and Ganter and Wille in [39].

The different notions and results in [40] have been extended to the general framework determining the most significant subsets of attributes to compute informative implications and obtaining mechanisms to obtain bases of attribute implications from three different perspectives.

In the future, a detailed relationship with other particular attribute implication approaches will be studied, such as the use of pseudo-intents and quasi-closed sets in fuzzy settings [7, 15] or heterogeneous attribute implication [50]. Moreover, we will design the algorithms based on the procedures introduced in this paper, study their complexity, and apply them to real-world scenarios, such as those presented for digital forensic [51–53] and renewable energies in [54–56].

Author Contributions

M. Eugenia Cornejo: conceptualization, investigation, writing – original draft, methodology, validation, writing – review and editing, formal analysis, project administration, supervision, funding acquisition. **Jesús Medina:** conceptualization, investigation, writing – original draft, methodology, validation, writing – review and editing, funding acquisition, formal analysis, project administration, supervision. **Francisco José Ocaña:** investigation, writing – original draft, methodology, validation, writing – review and editing, formal analysis, project administration, conceptualization.

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Conflicts of Interest

The authors declare no conflicts of interest.

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