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# Analysis of the Generalized Ostrovsky Equation in the Propagation of Surface and Internal Waves in Rotating Fluids

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Received: 17 June 2024 | Revised: 4 April 2025 | Accepted: 20 April 2025

Funding: The support of University of Cádiz is gratefully acknowledged by the author.

Keywords: conservation laws | Ostrovsky equation | reductions | symmetries

## ABSTRACT

The Ostrovsky equation models long, weakly nonlinear waves, explaining the propagation of surface and internal waves in a rotating fluid. The study focuses on the generalized Ostrovsky equation. Introduced by Levandosky and Liu, this equation demonstrates the existence of solitary waves through variational methods. This paper investigates the generalized Ostrovsky equation using Lie symmetry group method and low local conservation laws, essential for analyzing differential equations and describing conserved physical and chemical processes. Specific cases reduce it to the Ostrovsky or generalized Korteweg–de Vries (KdV) equations. Detailed calculations of local conservation laws, classical point symmetries, and symmetry reductions are provided, offering invariant solutions and Lie symmetry groups. This research advances the understanding of differential equations and their applications in modeling scientific phenomena.

## 1 | Introduction

The study of nonlinear internal waves in a rotating ocean has been an important area of research in oceanography and fluid dynamics. Ostrovsky equation was introduced by Ostrovsky in [1] as a model for long waves which are weakly nonlinear, to explain the propagation of the surface and internal waves in a fluid of reference which describe a rotating movement. This paper aims to discuss the generalized Ostrovsky equation by using Lie symmetry group method and low local conservation laws for this equation. Conservation laws are highly significant in the analysis of differential equations because they describe chemical and physical processes with conserved quantities. We study conservation laws for these equations and we resort to the invariance and multiplier perspective by using the Euler–Lagrange

operator. In this paper, we consider the generalized Ostrovsky equation.

$$u_{tx} - \beta u_{xxxx} + (g(u)_x)_x = \alpha f(u) \quad (1)$$

where  $\alpha$  and  $\beta$  are the real dispersion coefficients and the functions  $f$  and  $g$  are  $C^2$  functions. This equation was introduced by Levandosky and Liu in [2] when  $f$  is the identity function and they proved the existence of solitary waves, called ground states, by employing variational methods. Specifically, in the case  $g(u) = u^2$  and  $f(u) = u$ , we obtain the Ostrovsky equation

$$u_{tx} - \beta u_{xxxx} + ((u^2)_x)_x = \alpha u \quad (2)$$

The function  $u(t, x)$  denotes the free surface of a liquid,  $\alpha \in \mathbb{R}$  is a measure of rotational effects due to Coriolis force, and  $\beta \in \mathbb{R}$

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determines the dispersion. Some special cases of the Ostrovsky equation has been investigated by several authors [3–7]. In [8], the authors prove that solutions of this equation converge to solutions of the Korteweg–de Vries (KdV) equation. In [9], Varlamov and Liu proved that (1) has special characteristics in the space  $X_s$  for  $s > 3/2$ . Tsugawa [10] extended the results of Varlamov and Liu for  $s > -3/4$ . In the absence of rotation, that is, when  $\alpha = 0$ , we integrate and we obtain the generalized KdV equation:

$$u_t - \beta u_{xxx} + (g(u))_x = 0 \tag{3}$$

The original Ostrovsky equation models the propagation of nonlinear waves in rotating media, such as oceans and the atmosphere. However, this formulation may be insufficient to describe scenarios where dispersion and nonlinearity vary more complexly. The generalized version allows for additional terms that can represent more realistic physical effects, such as spatial and temporal variations in dispersion or more sophisticated wave interactions. Moreover, studying this version expands knowledge of nonlinear differential equations and their exact solutions, which can be applied in both numerical and theoretical fluid dynamics models.

In the current paper, we have studied (1) using Lie symmetry reductions, symmetry group, and a lot of conservative laws for them [11–13]. Partial differential equations describe several scientific processes, such as the heat equation [14] and other chemical or physical processes [15, 16]. Numerous solutions of differential equations have been obtained by several methods [17], such as the direct method [18], simplest equation method [19], the tanh method [20], or the homogeneous balanced method [21]. An important and efficient method to study differential equations is the Lie group method [22, 23]. In Olver [24] and Bluman and Kumei [25], we find a precise description about symmetry groups and Lie’s theory. Symmetries help us to obtain invariable solutions of differential equations which have been reduced to other equations with fewer independent variables [26, 27]. By means of symmetry groups, we can obtain solutions of partial differential equations through other solutions.

This paper is ordered as follows: Firstly, we calculate local conservation laws of low order admitted by (1) by employing the multipliers theory, and we obtain a wide variety of low-order conservation laws for Equation (1). After that, we obtain the classical point symmetries admitted by Equation (1) in Section 3. By means of the point symmetries, we calculate symmetry reductions of the equation in the following section. In addition, we obtain other invariant solutions of the generalized equation, and we obtain the Lie symmetry groups. Finally, we present some concluding comments.

## 2 | Conservation Laws

Conservations laws have been utilized to obtain solutions of partial differential equations and in the expansion of different numerical methods [28–30]. They are used in proving the existence and uniqueness of solutions. To evaluate conserved fluxes and densities, we study conservation laws for Equation (1), and we have used the invariance and multiplier perspective with the

use of the Euler–Lagrange operator. Then, we have obtained conservation laws for Equation (1), namely, the following continuity equation:

$$(D_t T + D_x X)|_\epsilon = 0, \tag{4}$$

in terms of the total derivative operators holding for the solutions of (1). The spatial flux is noted by  $X$ , and the conserved density is noted by  $T$ , depending of  $u, x, t$  and some derivatives of the function  $u$ . Also,  $D_x$  and  $D_t$  represent the total derivative functions with respect  $x$  and  $t$ , respectively.

A multiplier  $Q$  is a function  $Q(t, x, u, u_t, u_x, \dots)$  that satisfies that, for solutions of Equation (1),

$$(u_{tx} - \beta u_{xxxx} + (g(u))_x - \alpha f(u))Q$$

is a divergence expression. Every significant conservation law emerges from multipliers. If we move away of the group of solutions of Equation (1), all significant conservation laws (4) are equivalent to conservation laws with the following characteristic form:

$$(D_t \hat{T} + D_x \hat{X}) = (u_{tx} - \beta u_{xxxx} + (g(u))_x - \alpha f(u))Q, \tag{5}$$

where  $(\hat{T}, \hat{X})$  varies from  $(T, X)$  by a banal conserved current.

We solve the following determining equation to find the set of multipliers:

$$\frac{\delta}{\delta u}(u_{tx} - \beta u_{xxxx} + (g(u))_x - \alpha f(u))Q = 0, \tag{6}$$

where  $\frac{\delta}{\delta u}$  represents the Euler–Lagrange factor  $\hat{E}[u]$  defined by

$$\hat{E}[u] := \frac{\partial}{\partial u} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}}. \tag{7}$$

By considering (5–7), we derive the following local conservation laws for Equation (1):

**Theorem 1.** *Under some arbitrary constants, the generalized Ostrovsky equation (1) admits the following local conservation laws of low order.*

*Case 1. In the case  $f(u) = c_1 u + c_2$ , with  $f \neq 0$ , we have the conservation law multiplier  $Q = g'(u)u_x - \beta u_{xxx} + u_t$ , and we get the following local conservation law:*

$$\begin{aligned} & D_t \left( \frac{c_1 \alpha}{2} u^2 + \alpha c_2 u \right) \\ & + D_x \left( \frac{1}{2} u_t^2 + \beta \left( -\frac{1}{2} c_1 u_x^2 \alpha + (c_1 u + c_2) \alpha u_{xx} - u_t u_{xxx} \right) \right. \\ & + \frac{\beta}{2} u_{xxx}^2 - \alpha \int (c_1 u + c_2) g'(u) du \\ & \left. + \frac{1}{2} g'(u)^2 u_x^2 - u_x g'(u) (\beta u_{xxx} - u_t) \right) = 0. \end{aligned} \tag{8}$$

*Case 2. In the case  $f \equiv 0$ , we obtain local conservation laws under the following conservation law multipliers:*

a.  $Q_1 = x$ , give us the following local conservation law:

$$D_t (x u_x) + D_x (x g'(u) u_x - g(u) + \beta (-x u_{xxx} + u_{xx})) = 0. \tag{9}$$

b.  $Q_2 = F(t)$ , with the conservation law:

$$D_t(F(t)u_x) + D_x(-F(t)\beta u_{xxx} + F(t)g'(u)u_x - uF'(t)) = 0. \quad (10)$$

c.  $Q_3 = u_{xxx} - \frac{1}{\beta}u_t - \frac{1}{\beta}u_x g'(u)$ ,  $\beta \neq 0$  with the conservation law:

$$D_t(0) + D_x\left(-\frac{1}{2\beta}(\beta u_{xxx} - g'(u)u_x - u_t)^2\right) = 0. \quad (11)$$

Case 3. In the case  $g(u) = c_1u + c_2$ , with  $c_1, c_2 \in \mathbb{R}$ , we obtain local conservation laws under the following conservation law multipliers:

a.  $Q_1 = u_x$  and the local conservation law:

$$D_t\left(\frac{1}{2}u_x^2\right) + D_x\left(-u_x\beta u_{xxx} - \alpha \int f(u)du + \frac{1}{2}(\beta u_{xx}^2 + c_1u_x^2)\right) = 0. \quad (12)$$

b.  $Q_2 = u_t$ , with the conservation law:

$$D_t\left(-\frac{1}{2}(\beta u_{xxx} + u_x^2c_1) - \alpha \int f(u)du\right) + D_x\left(-u_t\beta u_{xxx} + \beta u_{tx}u_{xx} + \frac{1}{2}u_t^2 + c_1u_tu_x\right) = 0. \quad (13)$$

c.  $Q_3 = u_{xxx}$ , in the case  $f \equiv c_3 \in \mathbb{R}$ , with the conservation law:

$$D_t\left(-\frac{1}{2}u_{xx}^2\right) + D_x\left(-\frac{1}{2}\beta u_{xxx}^2 - c_3\alpha u_{xx}\frac{1}{2}c_1u_{xx}^2 + u_{tx}u_{xx}\right) = 0. \quad (14)$$

d.  $Q_4 = \beta u_{xxx} + u_t$ , in the case  $f \equiv c_3 \in \mathbb{R}$ , which give us the following local conservation law:

$$D_t\left(-\beta u_{xx}^2 - \frac{1}{2}u_x^2 - c_3\alpha u\right) + D_x\left(-\frac{\beta^2}{2}u_{xxx}^2 + \frac{\beta}{2}(u_{xx}^2 + 4u_{tx}u_{xx} - 2u_tu_{xxx}) + \frac{1}{2}u_t^2 + u_tu_x - c_3\alpha\beta u_{xx}\right) = 0. \quad (15)$$

Case 4. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$ ,  $n \neq 0$ ,  $n \neq -1$ ,  $\alpha \neq 0$ ,  $f(u) = c_3$ , with  $c_3 \neq 0$ , under the conservation law multiplier  $Q = F_2(t, \gamma) + F_1(-\gamma)$ , with  $\gamma = u^n u_x - \beta u_{xxx} + u_t$ , we obtain the following local conservations law:

$$D_t(0) + D_x\left(\int -\beta(F_2(t, \gamma) + F_1(-\gamma))du_{xxx}\right) = 0. \quad (16)$$

Case 5. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$ ,  $n \neq 0$ ,  $n \neq -1$ ,  $\alpha \neq 0$ , and  $f(u) = c_1u^{n+1} + c_2u + c_3$  with  $c_1 \neq 0$ , we obtain local conservation laws under the following conservation law multipliers:

a.  $Q_1 = (n+1)\alpha c_1u_x + u_{xxx} - \frac{1}{\beta}u^n u_x - \frac{1}{\beta}u_t$ ,  $\beta \neq 0$  give us the following local conservation law:

$$D_t\left(\frac{\alpha}{\beta}\left(\frac{n+1}{2}c_1u_x^2\beta + \frac{1}{n}c_1u^n + \frac{1}{2}c_2u^2 + c_3u\right)\right) + D_x\left(-\frac{1}{2}(n+1)\alpha\beta c_1u_xu_{xxx} - \frac{1}{2}\beta u_{xxx}^2 + u^n u_xu_{xxx} + u_tu_{xxx} + \frac{n+1}{2}\alpha\beta c_1u_{xx}^2 - \alpha c_1u^{n+1}u_{xx} - \alpha c_2uu_{xx} - \alpha c_3u_{xx} + (n+1)\alpha c_1u^n u_x^2 + \frac{1}{2}\alpha c_2u_x^2 - \frac{1}{2\beta}u^{2n}u_x^2 - \frac{1}{\beta}u^n u_tu_x - \frac{1}{2\beta}u_t^2 - \frac{n+1}{n+2}\alpha^2 c_1^2 u^{n+2} - \left(\frac{n+1}{2}\right)\alpha^2 c_1 c_2 u^2 - (n+1)\alpha^2 c_1 c_3 u + \frac{\alpha c_1}{2\beta(n+1)}u^{2(n+1)} + \frac{\alpha c_2}{\beta(n+2)}u^{n+2} + \frac{\alpha c_3}{\beta(n+1)}u^{n+1}\right) = 0. \quad (17)$$

b.  $Q_2 = \left(e^{\sqrt{n+1}\alpha^{3/2}c_1^{3/2}\beta t}\right)^{n+1} e^{x\sqrt{(n+1)\alpha c_1}e^{\frac{\sqrt{n+1}\sqrt{\alpha c_1}t}{(n+1)\sqrt{c_1}}}}$ , with the conservation law:

$$D_t\left(u_x e^{\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2 t + (n+1)c_1 x + c_2 t)}\right) + D_x\left(-\frac{n+1}{\sqrt{c_1}}e^{\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2 t + (n+1)c_1 x + c_2 t)} \cdot \left(c_1^{3/2}\alpha\beta u_x + \frac{\sqrt{c_1}}{n+1}(\beta u_{xxx} - u_x u^n) - \frac{\sqrt{\alpha(n+1)}}{n+1}\left(c_1\beta u_{xx} - \frac{c_1 u^{n+1}}{n+1} - \frac{c_2 u}{n+1}\right)\right)\right) = 0. \quad (18)$$

c.  $Q_3 = \left(e^{-\sqrt{n+1}\alpha^{3/2}c_1^{3/2}\beta t}\right)^{n+1} e^{-x\sqrt{(n+1)\alpha c_1}e^{\frac{\sqrt{n+1}\sqrt{\alpha c_1}t}{(n+1)\sqrt{c_1}}}}$ , with the conservation law:

$$D_t\left(u_x e^{-\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2 t + (n+1)c_1 x + c_2 t)}\right) + D_x\left(-\frac{n+1}{\sqrt{c_1}}e^{-\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2 t + (n+1)c_1 x + c_2 t)} \cdot \left(c_1^{3/2}\alpha\beta u_x + \frac{\sqrt{c_1}}{n+1}(\beta u_{xxx} - u_x u^n) - \frac{\sqrt{\alpha(n+1)}}{n+1}\left(c_1\beta u_{xx} - \frac{c_1 u^{n+1}}{n+1} - \frac{c_2 u}{n+1}\right)\right)\right) = 0. \quad (19)$$

Case 6. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$ ,  $n \neq 0$ ,  $n \neq -1$ ,  $\alpha \neq 0$ , and  $f(u) = c_2u + c_3$ , with  $c_2 \neq 0$ , under the conservation law multiplier  $Q = -u^n u_x + \beta u_{xxx} - u_t$ , we obtain one local conservation law:

$$\begin{aligned}
 & D_t \left( \frac{c_2 \alpha}{2} u^2 + \alpha c_3 u \right) \\
 & + D_x \left( -\frac{1}{2} u_t^2 + \beta \left( \frac{1}{2} c_2 u_x^2 \alpha - c_3 \alpha u_{xx} + u_t u_{xxx} \right) - \frac{\beta}{2} u_{xxx}^2 \right. \\
 & - \alpha \beta c_2 u u_{xx} + u_x (\beta u_{xxx} - u_t) u^n + \frac{\alpha c_3}{n+1} u^{n+1} + \frac{\alpha c_2}{n+1} u^{n+2} \\
 & \left. - \frac{1}{2} u_x^2 u^{2n} \right) = 0. \tag{20}
 \end{aligned}$$

Case 7. In the case  $g(u) = \frac{1}{n+1} u^{n+1}$ ,  $n \neq 0$ ,  $n \neq -1$ ,  $\alpha \neq 0$ ,  $f(u) = c_3$ , with  $c_3 \neq 0$ , under the conservation law multiplier  $Q = F_2(t, \gamma) + F_1(-\gamma)$ , with  $\gamma = u^n u_x - \beta u_{xxx} + u_t$ , we obtain the following local conservations law:

$$D_t(0) + D_x \left( \int -\beta(F_2(t, \gamma) + F_1(-\gamma)) du_{xxx} \right) = 0. \tag{21}$$

Case 8. In the case  $g(u) = \frac{1}{n+1} u^{n+1}$ ,  $n \neq 0$ ,  $n \neq -1$ , and  $\alpha = 0$ , under the conservation law multipliers  $Q_1 = x$ ,  $Q_2 = F(t)$ , and  $Q_3 = u_{xxx} - \frac{1}{b} u_t - \frac{1}{b} u^n u_x$ , we obtain the following local conservations laws, respectively:

$$\begin{aligned}
 & D_t(-u) + D_x(xu^n u_x - \beta x u_{xxx} + \beta u_{xx} \\
 & + xu_t - \frac{1}{n+1} u^{n+1}) = 0, \\
 & D_t(F(t)u_x) + D_x(-F(t)\beta u_{xxx} + F(t)u^n u_x - F'(t)u) = 0, \\
 & D_t(0) + D_x \left( -\frac{1}{2\beta} (-u^n u_x + \beta u_{xxx} - u_t)^2 \right) = 0, \tag{22}
 \end{aligned}$$

and the multiplier  $Q_4 = 1$  with the local conservation laws:

$$\begin{aligned}
 & D_t(u_x) + D_x(-\beta u_{xxx} + u^n u_x) = 0, \\
 & D_t(0) + D_x(-\beta u_{xxx} + u^n u_x + u_t) = 0. \tag{23}
 \end{aligned}$$

Case 9. In the case  $g(u) = u$ ,  $\alpha = 0$ , we obtain extra local conservation laws under the following conservation law multipliers:

a.  $Q_1 = x$ , give us the following local conservation law:

$$D_t(xu_x) + D_x(-\beta x u_{xxx} + \beta u_{xx} + xu_x - u) = 0. \tag{24}$$

b.  $Q_2 = \frac{1}{2} x^2 - tx$ , with the conservation law:

$$\begin{aligned}
 & D_t \left( -\frac{1}{2} (x(2t-x)u_x) \right) \\
 & + D_x \left( \frac{\beta}{2} (-u_{xxx} x^2 + 2x(tu_{xxx} + u_{xx}) \right. \\
 & \left. - 2tu_{xx} - 2u_x) - txu_x + tu \right) = 0. \tag{25}
 \end{aligned}$$

c.  $Q_3 = -\frac{x}{6}(3t^2 - 3tx + x^2)$ , with the conservation law:

$$\begin{aligned}
 & D_t \left( -\frac{x}{6} (3t^2 - 3tx + x^2) u_x \right) \\
 & + D_x \left( \frac{\beta}{6} (u_{xxx} x^3 + x^2 (-3tu_{xxx} - 3u_{xx}) \right. \\
 & + x(3t^2 u_{xxx} + 6tu_{xx} + 6u_x)) \\
 & + \beta \left( -\frac{1}{2} u_{xx} t^2 - tu_x - u \right) - \frac{1}{2} u_x t^2 x \\
 & \left. + \frac{1}{2} u_x t x^2 - \frac{1}{6} u_x x^3 + \frac{1}{2} u t^2 \right) = 0. \tag{26}
 \end{aligned}$$

d.  $Q_4 = \frac{2}{3} x u_x + t \beta u_{xxx} + u_t + \frac{1}{3} u_x$ , with the conservation law:

$$\begin{aligned}
 & D_t \left( \frac{1}{3} x u_x^2 + \frac{1}{3} t (-3\beta u_{xx}^2 - u_x^2) \right) \\
 & + D_x \left( \frac{t}{6} (-3\beta^2 u_{xxx}^2 + \beta (-2u_x u_{xxx} - 6u_t u_{xxx} \right. \\
 & \left. - 16 \left( -\frac{u_{xx}}{4} - \frac{3}{4} u_{tx} \right) u_{xx} \right) + u_x^2 + 6u_t u_x + 3u_t^2) = 0. \tag{27}
 \end{aligned}$$

e.  $Q_5 = F_2(t, \delta) + F_1(t)$ , with  $\delta = -\beta u_{xxx} + u_t + u_x$ , give us the conservation law:

$$\begin{aligned}
 & D_t(F_1(t)u_x) + D_x \\
 & \left( \int -\beta(F_2(t, \delta) + F_1(t)) du_{xxx} + F_1(t)u_x - F_1'(t)u \right) = 0. \tag{28}
 \end{aligned}$$

f.  $Q_6 = \frac{1}{3}(2t+x)u_x + tu_t$ , with the conservation law:

$$\begin{aligned}
 & D_t \left( \frac{1}{6} (x-t)u_x^2 - \frac{1}{2} \beta t u_{xx}^2 \right) \\
 & + D_x \left( \frac{1}{6} \beta t (-6u_t u_{xxx}) - 4u_x u_{xxx} + 6u_{tx} u_{xx} + 2u_{xx}^2 \right) \\
 & + \frac{1}{6} \beta ((-2xu_{xxx} + 2u_{xx})u_x + xu_{xx}^2) \\
 & + t \left( \frac{u_t^2}{2} + u_t u_x + \frac{u_x^2}{3} \right) + \frac{u_x^2 x}{6} = 0. \tag{29}
 \end{aligned}$$

They are the absolute collection of local conservation laws accepted by Equation (1) under the constants  $\alpha$  and  $\beta$ .

*Proof.* Equation (4) is satisfied when the generalized Ostrovsky equation (1) holds. The usual form of low-order multipliers of Equation (1) is represented by

$$Q(t, x, u, u_t, u_x, u_{xx}, u_{xxx}).$$

The determining equation (6) gives us determining systems. We solve them and we obtain the previous solution multipliers given in each case, which provide us conserved fluxes and densities of Equation (1).  $\square$

### 3 | Point Symmetries of the Generalized Ostrovsky Equation

A Lie symmetry of determined partial differential equations is an operator that transforms solutions into other solutions. The mathematician, Sophus Lie, developed a theory in the 1870s to study the symmetries of partial differential equations, using what are now known as Lie groups. Now, we apply the classical Lie method to calculate symmetry reductions of the generalized Ostrovsky equation (1), and we consider the following one-parameter group:

$$\begin{aligned}
 \hat{t} &= t + \varepsilon \tau(t, x, u) + \Theta(\varepsilon^2), \\
 \hat{x} &= x + \varepsilon \xi(t, x, u) + \Theta(\varepsilon^2), \\
 \hat{u} &= u + \varepsilon \eta(t, x, u) + \Theta(\varepsilon^2),
 \end{aligned}$$

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &= \frac{\partial u}{\partial t} + \varepsilon \eta^t(t, x, u) + \Theta(\varepsilon^2), \\ \frac{\partial \hat{u}}{\partial x} &= \frac{\partial u}{\partial x} + \varepsilon \eta^x(t, x, u) + \Theta(\varepsilon^2), \\ \frac{\partial^2 \hat{u}}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon \eta^{xx}(t, x, u) + \Theta(\varepsilon^2), \\ &\vdots \end{aligned} \tag{30}$$

where  $\varepsilon$  is a small group parameter and  $\eta$ ,  $\tau$ , and  $\xi$  represent the infinitesimals of symmetry transformations, corresponding to the dependent and independent variables, respectively, where

$$\begin{aligned} \eta^t &= D_t(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta^x &= D_x(\eta) - u_{xx} D_x(\xi) - u_t D_x(\tau), \\ \eta^{xx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{tx} D_x(\tau), \\ \eta^{xxx} &= D_x(\eta^{xxx}) - u_{xxxx} D_x(\xi) - u_{txx} D_x(\tau), \\ \eta^{xxxx} &= D_x(\eta^{xxxx}) - u_{xxxxx} D_x(\xi) - u_{txxx} D_x(\tau), \\ &\vdots \end{aligned}$$

The operators  $D_t$  and  $D_x$  represent the total derivative functions with respect to  $t$  and  $x$ , respectively, which are defined as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \dots \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \dots \end{aligned} \tag{31}$$

The generators associated to the Lie algebra are given by the generator  $X$ , represented by

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \tag{32}$$

where

$$\tau = \left. \frac{d\hat{t}}{d\varepsilon} \right|_{\varepsilon=0}, \quad \xi = \left. \frac{d\hat{x}}{d\varepsilon} \right|_{\varepsilon=0}, \quad \eta = \left. \frac{d\hat{u}}{d\varepsilon} \right|_{\varepsilon=0}.$$

If the generator (32) gives us a symmetry of (1), then  $X$  satisfies the symmetry condition

$$pr^{(4)} X(\Delta)|_{\varepsilon=0} = 0 \tag{33}$$

where  $\Delta = u_{tx} - \beta u_{xxxx} + ((g(u))_x)_x - \alpha f(u)$  and  $pr^{(4)} X$  represents the fourth prolongation of (32)

$$pr^{(4)} X = X + \sum_J \phi^J(t, x, u^{(4)}) \frac{\partial}{\partial u_J} \tag{34}$$

where  $\phi^J(t, x, u^{(4)}) = D_J(\phi - \tau u_t - \xi x) + \xi u_{Jx} + \tau u_{Jt}$ , and  $J = (j_1, \dots, j_k)$ , for  $1 \leq j_k \leq 2$  and  $1 \leq k \leq 4$ .

Precisely, the fourth prolongation of the vector field  $X$  is given by

$$pr^{(4)} X = X + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{tx} \frac{\partial}{\partial u_{tx}} + \phi^{xxxx} \frac{\partial}{\partial u_{xxxx}} \tag{35}$$

where  $\phi^x$ ,  $\phi^{xx}$ ,  $\phi^{tx}$ ,  $\phi^{xxxx}$  are expressed as a function of  $\tau$ ,  $\xi$ ,  $\eta$  and the derivatives of  $\eta$ . By using (33), we obtain the infinitesimals  $\tau(t, x, u)$ ,  $\xi(t, x, u)$ , and  $\eta(t, x, u)$ . From (33) and (35), the invariance condition reads as

$$\begin{aligned} (g''(u)u_{xx} + g'''(u)u_x^2 - \alpha f'(u))\eta + 2g''(u)u_x \phi^x \\ + g'(u)\phi^{xx} + \phi^{tx} - \beta \phi^{xxxx} = 0, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \phi^x &= D_x \eta - u_t D_x \tau - u_x D_x \xi, \\ \phi^{tx} &= D_x \phi^t - u_{tt} D_x \tau - u_{tx} D_x \xi, \\ \phi^{xx} &= D_x \phi^x - u_{tx} D_x \tau - u_{xx} D_x \xi, \\ \phi^{xxxx} &= D_x \phi^{xxx} - u_{txxx} D_x \tau - u_{xxxx} D_x \xi, \end{aligned} \tag{37}$$

and  $D_x$  and  $D_t$  are the total differential operators with respect to  $x$  and  $t$ , respectively, given by (31).

By using (33), we calculate the point symmetries of Equation (1). A point symmetry of Equation (1) is a one-parameter group of transformations depending on  $(t, x, u)$  with generator (32). The prolongation of the Lie group leaves invariant Equation (1). The vector field (32) yields a point symmetry of Equation (1) when it satisfies the Lie symmetry condition (33), which gives rise to the following linear system of determining differential equations for the infinitesimals  $\xi(t, x, u)$ ,  $\tau(t, x, u)$ ,  $\eta(t, x, u)$ , the functions  $g(u)$ ,  $f(u)$ , and the real parameters  $\alpha$  and  $\beta$ :

$$\begin{aligned} \tau_x = 0, \quad \tau_u = 0, \quad \eta_{xu} = 0, \quad \eta_{uu} = 0, \quad \xi_u = 0, \\ \xi_{xx} = 0, \quad 3\xi_x - \tau_t = 0, \quad 2\xi_x g'(u) + \eta g'(u) - \xi_t = 0, \\ \eta_{uu} - \xi_{tx} + 2\eta_x g''(u) = 0, \quad \eta_u g''(u) + 2\xi_x g''(u) + \eta g'''(u) = 0, \\ -\eta_{tx} + \alpha \eta f'(u) - \eta_{xx} g'(u) - \alpha \eta_u f(u) + 4\alpha \xi_x f(u) + \beta \eta_{xxxx} = 0, \end{aligned} \tag{38}$$

Solving (38), we obtain some conditions for the functions and parameters, and we have the following theorem with the infinitesimal generators of the generalized Ostrovsky equation (1):

**Theorem 2.** *The point symmetries of Equation (1) are defined by the following independent operators:*

Case 1. *In the case  $g(u) = \frac{1}{n+1}u^{n+1}$  and  $f(u) = cu^{2n+1}$  with  $n \neq 0$ ,  $n \neq -1$ , the point symmetries of Equation (1) are defined by the generators*

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = 3t\partial_t + x\partial_x - \frac{2}{n}u\partial_u \tag{39}$$

Case 2. *In the case  $g(u) = \frac{1}{n+1}u^{n+1}$  and  $f(u) = cu$ , with  $n \neq 0$ ,  $n \neq -1$ ,  $\alpha \neq 0$ , the point symmetries of Equation (1) are defined by the generators  $V_1 = \partial_t$  and  $V_2 = \partial_x$ .*

Case 3. *In the case  $g(u) = u$ , there are point symmetries of Equation (1) in the case  $\alpha \neq 0$  and  $f(u) = cu$ , or in the case  $\alpha = 0$ , and the generators are given by*

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_4 = u\partial_u \tag{40}$$

Case 4. In the case  $g(u) = c_1 e^u$  and  $f(u) = c_2 e^{2u}$ , the point symmetries are given by

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_5 = 3t\partial_t + x\partial_x - 2\partial_u \quad (41)$$

Case 5. In the general case for  $g(u)$  and  $f(u)$ , the point symmetries are given by  $V_1$  and  $V_2$ .

We can easily check that the generators (39–41) are closed under the Lie bracket.

Now, we calculate the adjoint table for  $V_i$  and  $V_j$ , for  $i, j = 1, \dots, 5$ , given by the following expression:

$$\begin{aligned} \text{Ad}(\exp(\varepsilon V))W_0 &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{com}V)^n(W_0) \\ &= W_0 - \varepsilon[V, W_0] + \frac{\varepsilon^2}{2}[V, [V, W_0]] - \dots \end{aligned}$$

By using the adjoint representation, we obtain conjugated subalgebras through equivalence classes, which represent every subalgebra of the Lie algebra. The optimal systems in each case are given, respectively, by

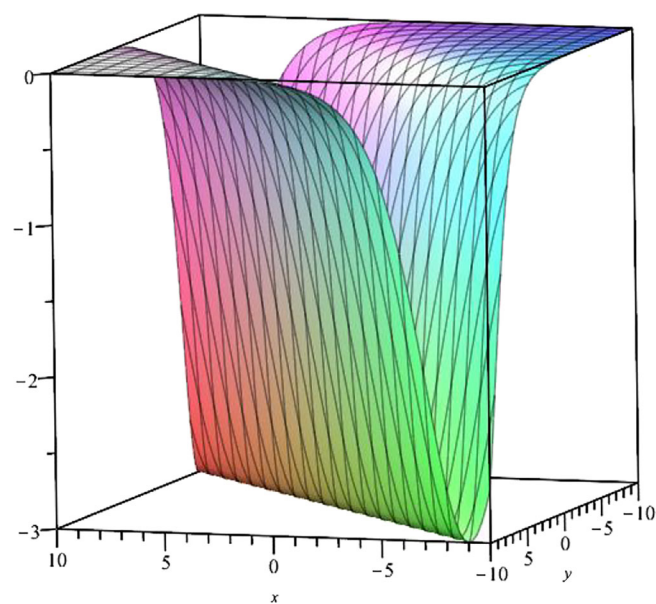
$$\begin{aligned} &\{\mu V_1 + \lambda V_2, V_3\}, \{\mu V_1 + \lambda V_2\}, \{\mu V_1 + \lambda V_2 + \gamma V_4\}, \\ &\{\mu V_1 + \lambda V_2, V_5\} \text{ and } \{\mu V_1 + \lambda V_2\}. \end{aligned}$$

Now, we use (39–41) to determinate the symmetry groups  $g(t, x, u)$  associated to the generator (32) and calculate new solutions for Equation (1) associated with them. To obtain the symmetry group, we use the following system of initial problems:

$$\frac{\partial \hat{t}}{\partial \varepsilon} = \tau(t, x, u), \quad \frac{\partial \hat{x}}{\partial \varepsilon} = \xi(t, x, u), \quad \frac{\partial \hat{u}}{\partial \varepsilon} = \eta(t, x, u) \quad (42)$$

and

$$(\hat{t}, \hat{x}, \hat{u})|_{\varepsilon=0} = (t, x, u) \quad (43)$$



**FIGURE 1** | Exact solution of (60) for  $\lambda = -1, k = 1$ , and  $b = 1$ . [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

where  $g(t, x, u) = (\hat{t}, \hat{x}, \hat{u})$ . From (42) and (43), we get the following theorem with the corresponding Lie symmetry group.

**Theorem 3.** The Lie symmetry groups  $G_i, i = 1, \dots, 5$ , generated by  $V_i, i = 1, \dots, 5$ , are specified by

$$G_1(t, x, u) = (t + \varepsilon, x, u) \quad \text{time translation} \quad (44)$$

$$G_2(t, x, u) = (t, x + \varepsilon, u) \quad \text{space translation across the } x \text{ axis} \quad (45)$$

$$G_3(t, x, u) = (e^{3\varepsilon}t, e^\varepsilon x, e^{-\frac{2\varepsilon}{n}}u) \quad \text{scaling group} \quad (46)$$

$$G_4(t, x, u) = (t, x, e^\varepsilon u) \quad \text{exponential dilation} \quad (47)$$

$$G_5(t, x, u) = (e^{3\varepsilon}t, e^\varepsilon x, u - 2\varepsilon) \quad \text{shift}, \quad (48)$$

where  $\varepsilon$  is the parameter of the group.

Solutions of Equation (1) are transformed into solution through the use of symmetry groups. So, on the assumption that  $u = f(t, x)$  is a solution of the generalized equation (1), we obtain new solutions for Equation (1) by means of diverse symmetry groups. Then, by using the previous groups  $G_i, i = 1, \dots, 5$ , we calculate the appropriate new solutions:

$$\hat{u}_1 = f(t - \varepsilon, x) \quad (49)$$

$$\hat{u}_2 = f(t, x - \varepsilon) \quad (50)$$

$$\hat{u}_3 = f(te^{-3\varepsilon}, xe^{-\varepsilon})e^{-\frac{2\varepsilon}{n}} \quad (51)$$

$$\hat{u}_4 = f(te^{-\varepsilon}, x)e^\varepsilon \quad (52)$$

$$\hat{u}_5 = f(te^{-3\varepsilon}, xe^{-\varepsilon}) - 2\varepsilon \quad (53)$$

## 4 | Symmetry Reductions and Exact Solutions

In this part, we use the optimal system computed in the anterior subsection, and we get the following symmetry reductions of Equation (1).

Case 1a. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$  and  $f(u) = cu^{2n+1}$  with  $n \neq 0, n \neq -1$ , we use the generator  $V_3$ , and we get

$$u = h(w)t^{-\frac{2}{3n}} \quad (54)$$

where  $w = xt^{-\frac{1}{3}}$ . By considering (54) and (1), we have a reduced ordinary reduced equation given by

$$-\frac{1}{3}wh'' + nh^{n-1}(h')^2 - \beta h^{(iv)} + h^n h'' - \frac{2+n}{3n}h' = 0 \quad (55)$$

Case 1b. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$  and  $f(u) = cu^{2n+1}$  with  $n \neq 0$ ,  $n \neq -1$ , from the generator  $\lambda V_1 + V_2$ , we have

$$u = h(w) \tag{56}$$

where  $w = x - \lambda t$ . By the substitution of (56) into (1), we have an ordinary differential equation given by

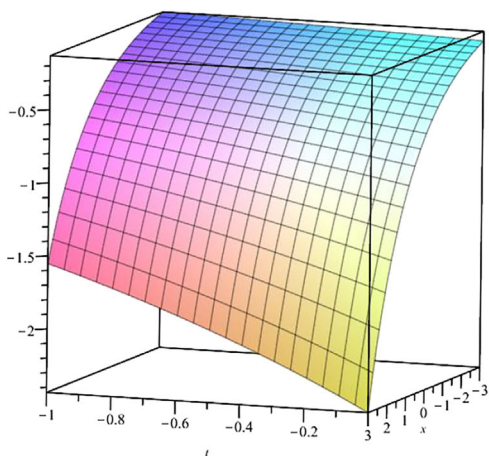
$$-\alpha h^{2n+1} - \lambda h'' + nh^{n-1}(h')^2 - \beta h^{iv} + h^n h'' = 0 \tag{57}$$

We integrate with respect to  $w$  twice, multiply by  $h'$ , and integrate again with respect to  $w$  for  $\alpha = 0$ . We consider zero the constants of integration, and we have the following ordinary differential equation:

$$\frac{\lambda}{2}(n+1)(n+2)h^2 - h^{n+2} + \beta(n+1)(n+2)(h')^2 = 0 \tag{58}$$

with solution for  $n = 1$

$$h(w) = 3\lambda \left( 1 + \tan \left( (k-w) \frac{\sqrt{18b\lambda}}{12b} \right)^2 \right) \tag{59}$$



where  $k \in \mathbb{R}$ . This gives us the following solution for Equation (1):

$$u(t, x) = 3\lambda \left( 1 + \tan \left( (k-x+\lambda t) \frac{\sqrt{18b\lambda}}{12b} \right)^2 \right) \tag{60}$$

In Figure 1, we consider (60) with  $\lambda = -1, k = 1$ , and  $b = 1$ .

The solution for  $n = 2$  is given by

$$h(w) = \frac{-24b\lambda e^{-\frac{2b\lambda}{2b}(k-w)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(k-w)}} \tag{61}$$

and

$$h(w) = \frac{-24b\lambda e^{-\frac{2b\lambda}{2b}(k-w)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(w-k)}} \tag{62}$$

where  $k \in \mathbb{R}$ . This gives us the following solution for Equation (1):

$$u(t, x) = \frac{-24b\lambda e^{-\frac{2b\lambda}{2b}(k-x+\lambda t)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(k-x+\lambda t)}} \tag{63}$$

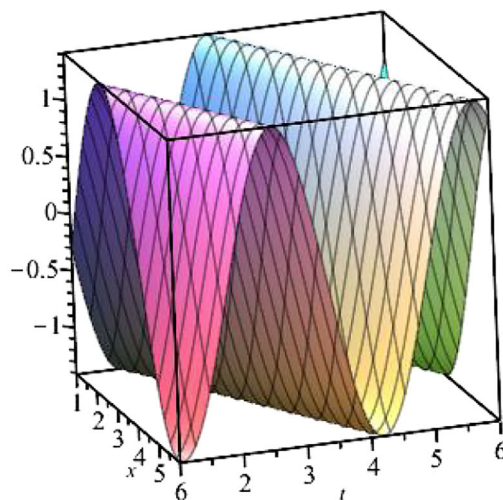


FIGURE 2 | Exact solutions of (63) and (64). [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

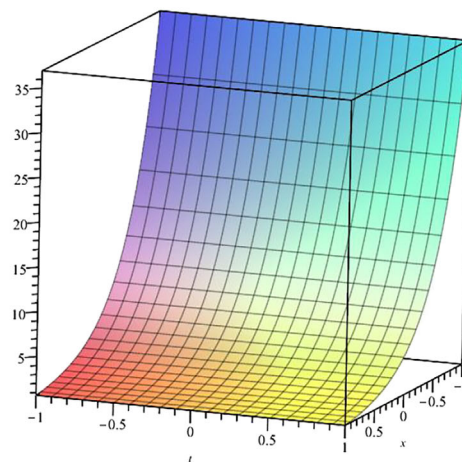
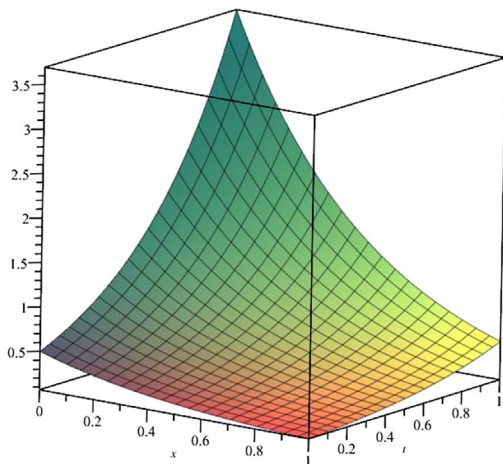


FIGURE 3 | Exact solution (85) and (86). [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

and

$$u(t, x) = \frac{-24b\lambda e^{-\frac{-2b\lambda}{2b}(k-x+\lambda t)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(-k+x-\lambda t)}} \quad (64)$$

In Figure 2, we consider solution (63) with  $\lambda = -1, k = 1$ , and  $b = 3$  and solution (64) with  $\lambda = -2, k = 1$ , and  $b = 1$ , respectively.

Case 2. In the case  $g(u) = \frac{1}{n+1}u^{n+1}$  and  $f(u) = cu$ , with  $n \neq 0, n \neq -1, \alpha \neq 0$ , by using the generator  $\lambda V_1 + V_2$ , we get the similarity variables

$$w = x - \lambda t, \quad u = h(w), \quad (65)$$

and the ODE

$$-\lambda h'' + nh^{n-1}(h')^2 - \beta h^{iv} + h^n h'' - \alpha h = 0 \quad (66)$$

Case 3. In the case  $g(u) = u$  and  $f(u) = cu$ , by using the generator  $V_1 + \lambda_2 V_2 + \lambda_4 V_4$ , we obtain

$$w = x - \lambda_2 t, \quad u = h(w)e^{\lambda_4 t}, \quad (67)$$

and the ordinary differential equation

$$\lambda_4 h' - \lambda_2 h'' - \beta h^{iv} + h'' - \alpha h = 0 \quad (68)$$

• We have the following solution of (68) in the general case of  $\alpha, \beta, \lambda_2$ , and  $\lambda_4$

$$h(w) = \sum_{k=1}^4 C_k e^{\text{RootOf}(\beta Z^4 + (\lambda_2 - 1)Z^2 - \lambda_4 Z + \alpha, \text{index}=k)w} \quad (69)$$

where  $C_k \in \mathbb{R}, k = 1 \dots 4$ , and the corresponding solution of (1)

$$u(t, x) = \sum_{k=1}^4 C_k e^{\text{RootOf}(\beta Z^4 + (\lambda_2 - 1)Z^2 - \lambda_4 Z + \alpha, \text{index}=k)(x - \lambda_2 t)} \quad (70)$$

where  $C_k \in \mathbb{R}, k = 1 \dots 4$ .

• In the case  $\alpha + 16\beta + 4\lambda_2 - 4 + 2\lambda_4 = 0$ , we obtain the following solutions of (68):

$$h(w) = \frac{c_1 \text{sech}(w)^2}{(1 + \tanh(w))^2} \quad (71)$$

$$h(w) = \frac{c_2 \text{csch}(w)^2}{(1 + \coth(w))^2} \quad (72)$$

and these give us the following solutions for Equation (1), respectively,

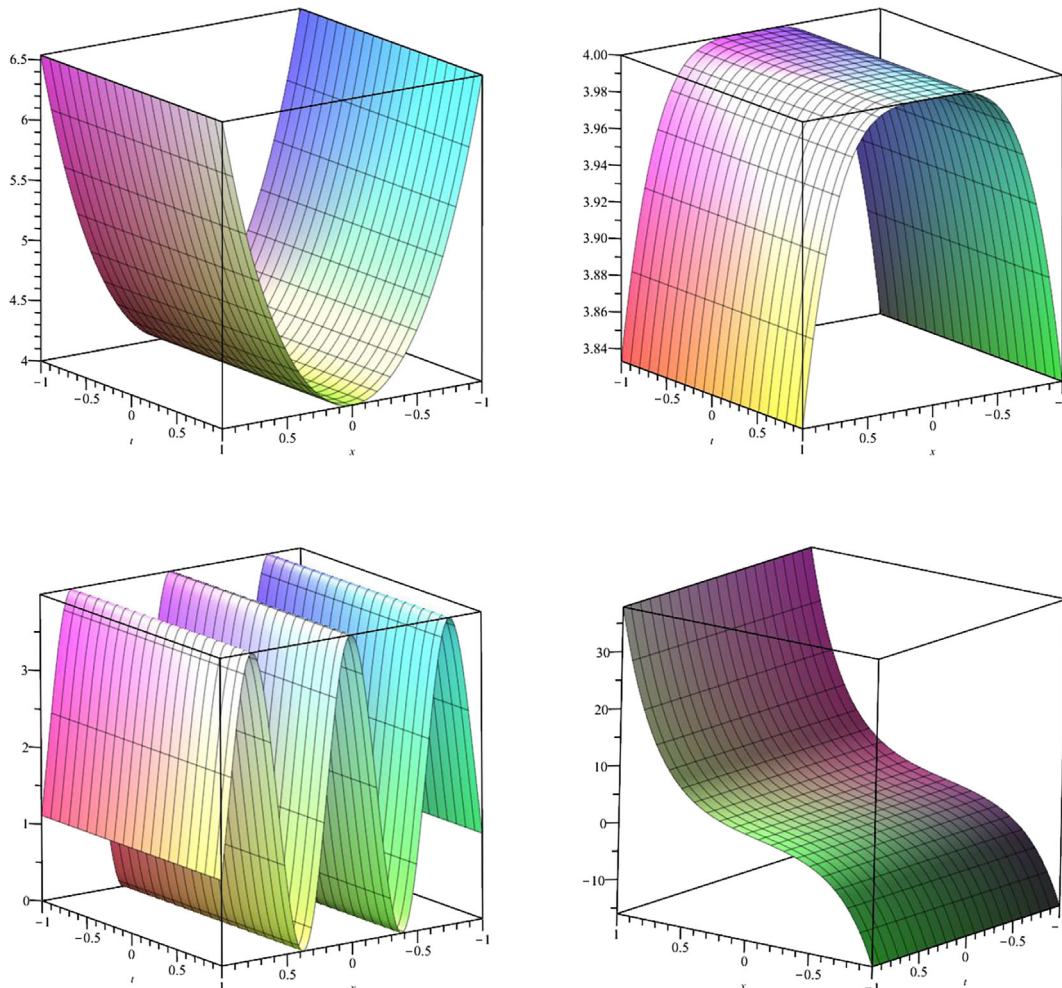


FIGURE 4 | Exact solution of Equation (1) given by Equation (76). [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

$$u(t, x) = \frac{c_1 \operatorname{sech}(x - \lambda_2 t)^2}{(1 + \tanh(x - \lambda_2 t))^2} \quad (73)$$

$$u(t, x) = \frac{c_2 \operatorname{csch}(x - \lambda_2 t)^2}{(1 + \operatorname{coth}(x - \lambda_2 t))^2} \quad (74)$$

where  $c_1, c_2 \in \mathbb{R}$ . In Figure 3, we consider solution (73) for  $c_1 = 0.5, \lambda_2 = 1$  and solution (74) for  $c_2 = 1, \lambda_2 = 0.001$ , respectively.

- In the case  $\lambda_4 = 0$ , we obtain the following general solution of (68):

$$h(w) = C_1 e^{-\frac{\sqrt{-2\beta(\lambda_2 - 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} w}{2b}} + C_2 e^{\frac{\sqrt{-2\beta(\lambda_2 - 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} w}{2b}} + C_3 e^{-\frac{\sqrt{-2\beta(-\lambda_2 + 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} w}{2b}} + C_4 e^{\frac{\sqrt{-2\beta(-\lambda_2 + 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} w}{2b}}, \quad (75)$$

where  $c_i \in \mathbb{R}, i = 1 \dots 3$ .

This give us the following solution for Equation (1):

$$u(t, x) = C_1 e^{-\frac{\sqrt{-2\beta(\lambda_2 - 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} (x - \lambda_2 t)}{2b}} + C_2 e^{\frac{\sqrt{-2\beta(\lambda_2 - 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} (x - \lambda_2 t)}{2b}} + C_3 e^{-\frac{\sqrt{-2\beta(-\lambda_2 + 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} (x - \lambda_2 t)}{2b}} + C_4 e^{\frac{\sqrt{-2\beta(-\lambda_2 + 1 + \sqrt{-4\alpha\beta + \lambda_2^2 - 2\lambda_2 + 1})} (x - \lambda_2 t)}{2b}}. \quad (76)$$

In Figure 4, we consider solution (76) in the cases

$$C_i = 1, i = 1 \dots 4, \lambda_2 = -1, \beta = 1, \alpha = -1,$$

$$C_i = 1, i = 1 \dots 4, \lambda_2 = 1, \beta = 2, \alpha = 3,$$

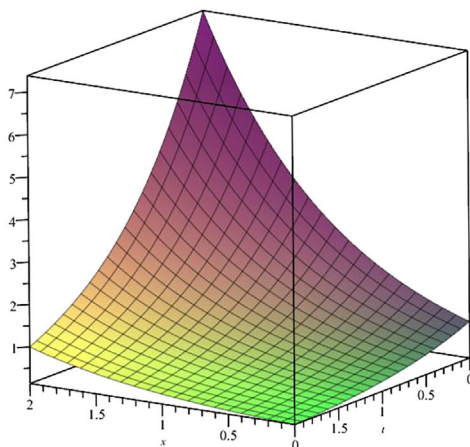
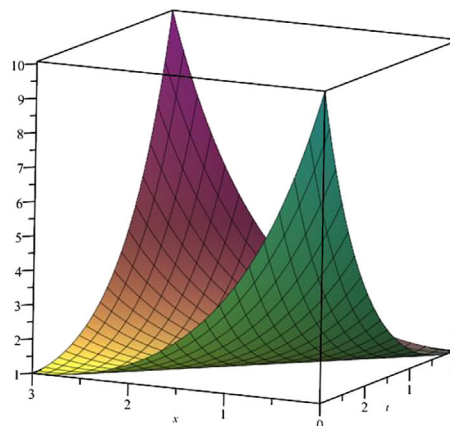
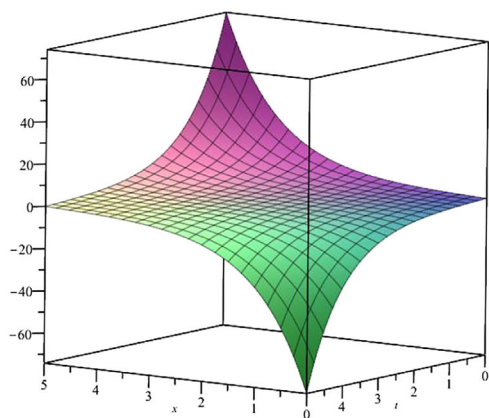
$$C_i = 1, i = 1 \dots 4, \lambda_2 = 70, \beta = 1, \alpha = 1,$$

$$C_1 = C_2 = 1, C_3 = -0.5, C_4 = 1, \lambda_2 = -50, \beta = 4, \alpha = -4,$$

respectively.

- In the case  $\lambda_4 = 0$ , we obtain the following solution of (68):

$$h(w) = c_1 \sinh(c_2 w) + c_3 \cosh(c_2 w) \quad (77)$$



**FIGURE 5** | Exact solution of Equation (1) given by Equation (78). [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

where  $c_2^2\lambda_2 + \beta c_2^4 - c_2^2 + \alpha = 0$ , with the following solution for Equation (1):

$$u(t, x) = c_1 \sinh(c_2(x - \lambda_2 t)) + c_3 \cosh(c_2)(x - \lambda_2 t) \quad (78)$$

In Figure 5, we consider solution (78) in the case  $c_1 = c_2 = 1, c_3 = 0, \lambda_2 = 1, \beta = 1$ ; the case  $c_1 = 0, c_2 = 1, c_3 = 1, \lambda_2 = 1, \beta = 1$ ; and the case  $c_i = 1, i = 1 \dots 3, \lambda_2 = 1, \beta = 1$ , respectively.

- In the case  $\lambda_4 = 0$ , we obtain the following solution of (68)

$$h(w) = c_1 \sin(c_2 w) + c_3 \cos(c_2 w) \quad (79)$$

where  $c_2^2\lambda_2 + \beta c_2^4 - c_2^2 - \alpha = 0$ , with the following solution for Equation (1):

$$u(t, x) = c_1 \sin(c_2(x - \lambda_2 t)) + c_3 \cos(c_2(x - \lambda_2 t)) \quad (80)$$

In Figure 6, we consider solution (80) in the case  $c_1 = c_2 = 1, c_3 = 1, \lambda_2 = 10, \beta = 1$ ; the case  $c_1 = 0, c_2 = c_3 = 1, \lambda_2 = 0.5, \beta = 2$ ; and the case  $c_1 = 100, c_2 = 1, c_3 = 0, \lambda_2 = 0, \beta = 1$ , respectively.

- In the case  $\alpha = \lambda_4 = 0$ , we obtain the following solutions if  $b(\lambda_2 - 1)$  is a positive, negative, or zero real

number, respectively,

$$h(w) = k_1 \sin\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right) + k_2 \cos\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right) \quad (81)$$

$$h(w) = k_1 e^{\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right)} + k_2 e^{\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right)} \quad (82)$$

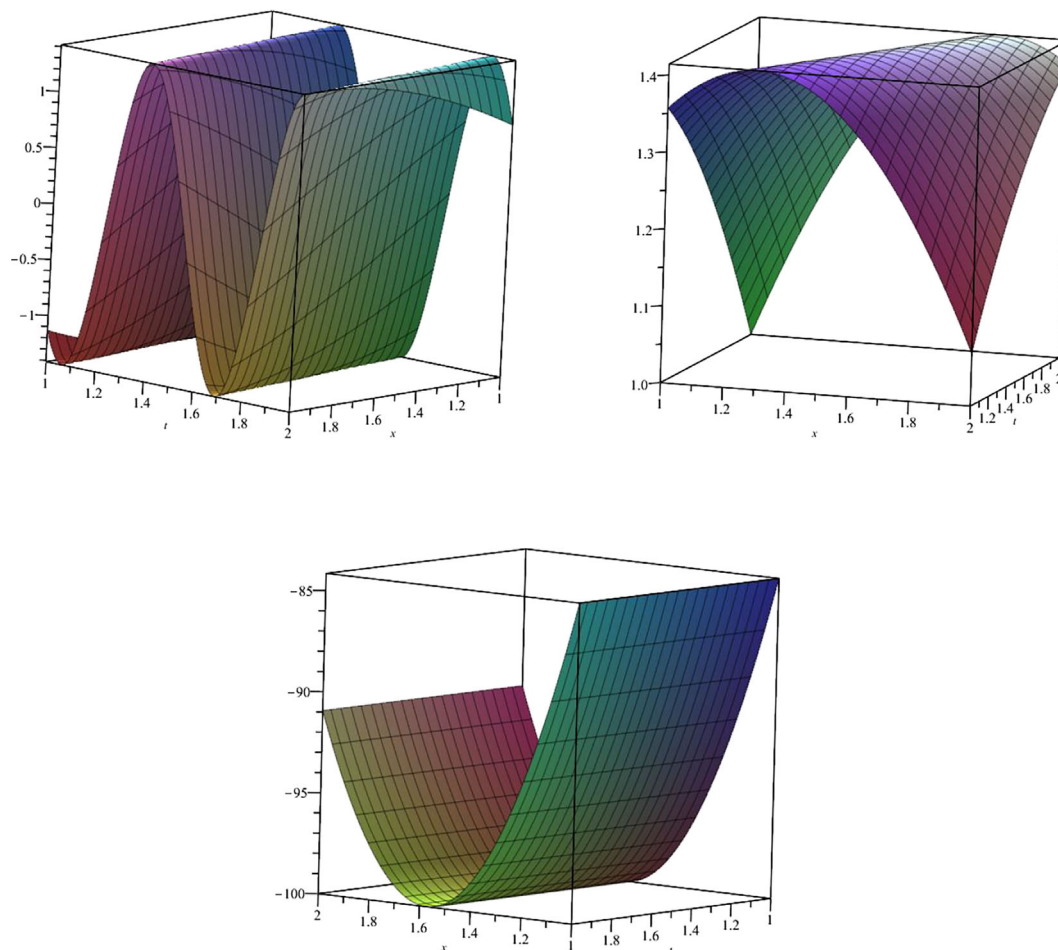
$$h(w) = k_1 w + k_2 \quad (83)$$

and these give us the following solutions for Equation (1), respectively,

$$u(t, x) = k_1 \sin\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right) + k_2 \cos\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right), \quad (84)$$

$$u(t, x) = k_1 e^{\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right)} + k_2 e^{\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right)} \quad (85)$$

$$u(t, x) = k_1(x - t) + k_2 \quad (86)$$



**FIGURE 6** | Exact solution of Equation (1) given by Equation (80). [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

In Figure 7, we consider solution (84) with  $k_1 = k_2 = 1, \lambda_2 = 2, b = 1$ ; solution (85) with  $k_1 = 1, k_2 = 1, \lambda_2 = -2, b = 1$ ; and solution (86) with  $k_1 = 1, k_2 = 1, \lambda_2 = 1$ , respectively.

Case 4a. In the case  $g(u) = c_1 e^u$  and  $f(u) = c_2 e^{2u}$ , by using the generator  $V_3$ , we have

$$u = h(w) - \frac{2}{3} \ln(t) \tag{87}$$

where  $w = xt^{-\frac{1}{3}}$ , and we have the ordinary reduced equation given by

$$\begin{aligned} &-\frac{1}{3}wh'' + c_1h''e^h + c_1(h')^2e^h \\ &- \alpha c_2e^{2h} - \beta h^{iv} - \frac{1}{3}h' = 0 \end{aligned} \tag{88}$$

Case 4b. In the case  $g(u) = c_1 e^u$  and  $f(u) = c_2 e^{2u}$  from the generator  $\lambda V_1 + V_2$ , one has

$$u = h(w) \tag{89}$$

where  $w = x - \lambda t$ . By substituting (89) into (1), we have the ordinary differential equation given by

$$-\lambda h'' - \beta h^{iv} + c_1 h'' e^h + c_1 (h')^2 e^h - \alpha c_2 e^{2h} = 0 \tag{90}$$

For  $c_2 = 0$ , we have the following solutions of (90) where  $\lambda$  is positive, negative, or  $\lambda = 0$ , respectively,

$$h(w) = k_1 \sin\left(w\sqrt{\frac{\lambda}{\beta}}\right) + k_2 \cos\left(w\sqrt{\frac{\lambda}{\beta}}\right) \tag{91}$$

$$h(w) = k_1 e^{iw\sqrt{\frac{\lambda}{\beta}}} + k_2 e^{-iw\sqrt{\frac{\lambda}{\beta}}} \tag{92}$$

$$h(w) = k_1 + k_2 w \tag{93}$$

and we get the following solutions of Equation (1):

$$u(t, x) = k_1 \sin\left((x - \lambda t)\sqrt{\frac{\lambda}{\beta}}\right) + k_2 \cos\left((x - \lambda t)\sqrt{\frac{\lambda}{\beta}}\right) \tag{94}$$

$$u(t, x) = k_1 e^{i(x-\lambda t)\sqrt{\frac{\lambda}{\beta}}} + k_2 e^{-i(x-\lambda t)\sqrt{\frac{\lambda}{\beta}}} \tag{95}$$

$$u(t, x) = k_1 + k_2 x \tag{96}$$

Case 5. In the general case for  $g(u)$  and  $f(u)$ , from the generator  $\lambda V_1 + V_2$ , we have  $u = h(w)$ ,  $w = x - \lambda t$ , and we have the reduced differential equation given by

$$-\lambda h'' - \beta h^{iv} + g''(h')^2 + g'h'' - \alpha f = 0 \tag{97}$$

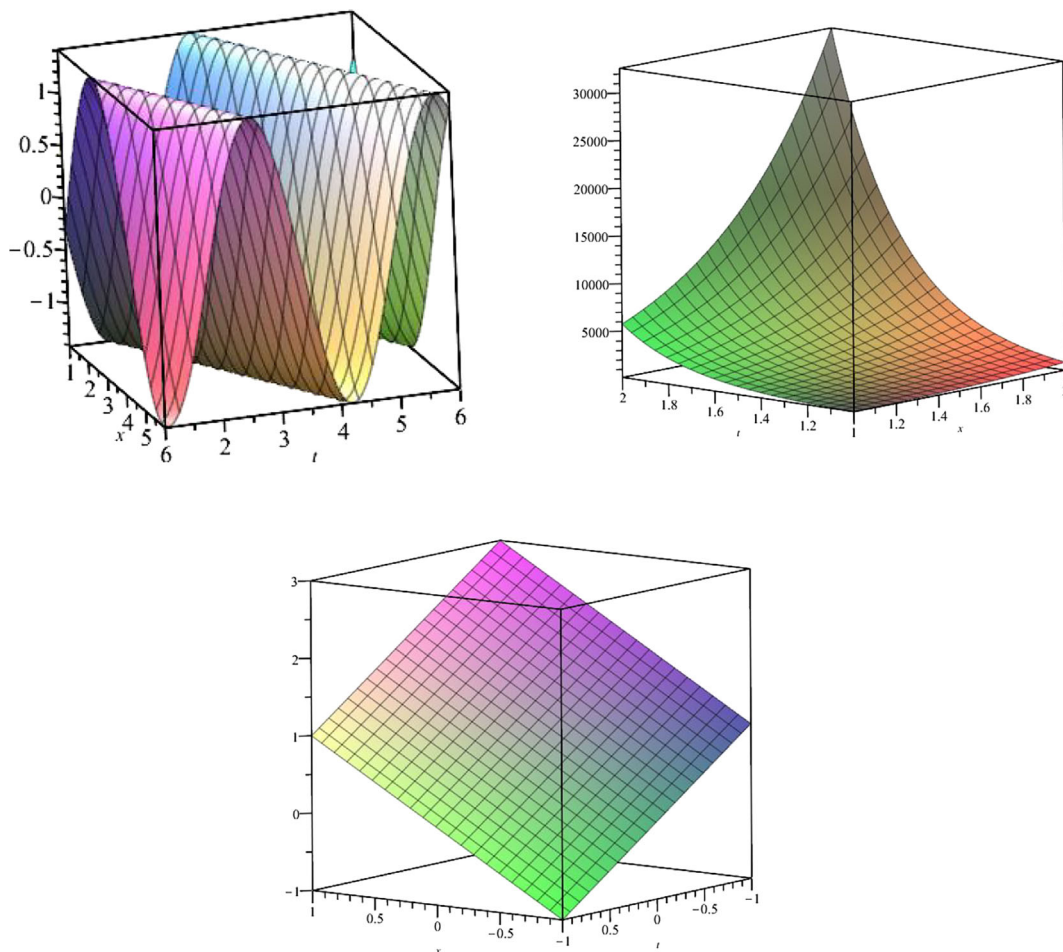


FIGURE 7 | Exact solution (84–86). [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

and integrating twice respect  $w$ , we obtain

$$-\lambda h - \beta h'' + g - \alpha \int \left( \int f dw \right) dw = 0 \quad (98)$$

For  $f, g$  identity functions or constant functions, we have similar solutions to (94–96).

## 5 | Concluding Remarks

We have studied the generalized Ostrovsky equation (1) with real dispersion coefficients, from the point of view of symmetry analysis. First of all, we have derived conservation laws for the subjacent equation through multiplier approach conservation theorem, and we have recourse to the invariance and multiplier perspective by using the Euler–Lagrange operator. We have obtained several local conservation laws for Equation (1). Secondly, we have calculated Lie point symmetries of the equation, and subsequently, we have performed symmetry reductions. Also, we have obtained traveling wave solutions of significance importance by using the vector fields. Finally, we have obtained Lie symmetry groups generated by means of the vector field and new solutions for Equation (1) associated to them. The results obtained in this study contribute to the understanding of wave propagation in rotating fluids. The derivation of exact solutions and conservation laws allows for a better characterization of wave regimes, facilitating the development of more efficient and accurate numerical models. These solutions can serve as benchmark tests for computational simulations, helping to assess the accuracy of numerical schemes used in oceanic or atmospheric models. Furthermore, the identification of symmetries and conservation laws provides useful tools for simplifying complex equations and exploring new problem formulations in fluid dynamics.

### Author Contributions

**Sol Sáez:** investigation; writing – original draft.

### Acknowledgments

The support of University of Cádiz is gratefully acknowledged by the author.

### Conflicts of Interest

The author declares no conflicts of interest.

### Data Availability Statement

Data sharing is not applicable—no new data generated, or the article describes entirely theoretical research.

### References

1. L. A. Ostrovskii, “Nonlinear Internal Waves in a Rotating Ocean,” *Okeanologiya* 18 (1978): 181–191.
2. S. Levandosky and Y. Liu, “Stability of Solitary Waves of a Generalized Ostrovsky Equation,” *SIAM Journal on Mathematical Analysis* 38, no. 3 (2006): 985–1011.
3. S. C. Anco and M. L. Gandarias, “Symmetries, Conservation Laws, and Generalized Travelling Waves for a Forced Ostrovsky Equation,” preprint, arXiv, December 23, 2021, <https://doi.org/10.48550/arXiv.2105.06415>.

4. M. L. Gandarias and M. S. Bruzon, “Traveling Wave Solutions for a Generalized Ostrovsky Equation,” *Mathematical Methods in the Applied Sciences* 41 (2018): 5840–5850.

5. M. L. Gandarias and M. S. Bruzon, “Symmetry Analysis and Exact Solutions of Some Ostrovsky Equations,” *Theoretical and Mathematical Physics* 168, no. 1 (2011): 898–911.

6. S. Levandosky, “On the Stability of Solitary Waves of a Generalized Ostrovsky Equation,” *Analysis and Mathematical Physics* 2 (2012): 407–437.

7. X. Yan and W. Yan, “The Cauchy Problem for the Generalized Ostrovsky Equation With Negative Dispersion,” preprint, arXiv, March 31, 2021, <https://doi.org/10.48550/arXiv.2104.00549>.

8. Y. Liu and V. Varlamov, “Stability of Solitary Waves and Weak Rotation Limit for the Ostrovsky Equation,” *Journal of Differential Equations* 203 (2004): 159–183.

9. V. Varlamov and Y. Liu, “Solitary Waves and Fundamental Solution for Ostrovsky Equation,” *Journal of Differential Equations* 203 (2004): 159–183.

10. K. Tsugawa, “Well-Posedness and Weak Rotation Limit for the Ostrovsky Equation,” *Journal of Differential Equations* 247 (2009): 3163–3180.

11. S. Srivastava and M. Kumar, “Nonclassical Symmetries, Optimal Classification, and Dynamical Behavior of Similarity Solutions of (3+1)-Dimensional Burgers Equation,” *Chinese Journal of Physics* 89 (2024): 404–416.

12. M. Kumar and R. Kumar, “On New Similarity Solutions of the Boiti–Leon–Pempinelli System,” *Communications in Theoretical Physics* 61, no. 1 (2014): 121.

13. M. Kumar and S. Srivastava, “Dynamics of the Breather and Solitary Waves in Plasma Using the Generalized Variable Coefficient Gardner Equation,” *International Journal of Computer Mathematics* 101, no. 4 (2024): 465–482.

14. G. W. Bluman and J. D. Cole, “The General Similarity Solutions of the Heat Equation,” *Journal of Mathematics and Mechanics* 18 (1969): 1025–1042.

15. J. W. Wilder, B. F. Edwards, D. A. Vasquez, and G. I. Sivashinsky, “Derivation of a Nonlinear Front Evolution Equation for Chemical Waves Involving Convection,” *Physica D* 73 (1994): 217–226.

16. X. Yang and E. Weinan, “Nonlinear Evolution Equation for the Stress-Driven Morphological Instability,” *Journal of Applied Physics* 91 (2002): 9414–9422.

17. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables* (Dover Publications, 1972).

18. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, 2004).

19. N. A. Kudryashov, “Simplest Equation Method to Look for Exact Solutions of Nonlinear Differential Equation,” *Chaos, Solitons and Fractals* 24 (2005): 1217–1231.

20. W. Malfliet, “The Tanh Method: A Tool for Solving Certain Classes of Nonlinear Evolution and Wave Equations,” *Journal of Computational and Applied Mathematics* 164–165 (2004): 529–541.

21. X. Zhao, L. Wang, and W. Sun, “The Repeated Homogeneous Balance Method and Its Applications to Nonlinear Partial Differential Equations,” *Chaos, Solitons and Fractals* 28 (2006): 448–453.

22. A. Ouhadan and E.H. El Kinani, “Lie Symmetries and Preliminary Classification of Group-Invariant Solutions of Thomas equation,” preprint, arXiv, April 1, 2005, <https://doi.org/10.48550/arXiv.math-ph/0412043>.

23. S. Sáez, "Symmetry Analysis of a Generalized 2D Zakharov-Kuznetsov-Burgers Equation With a Variable Dissipation," *Journal of Mathematical Chemistry* 58, no. 10 (2020): 2415–2424.
24. P. J. Olver, *Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics*, vol. 107, 2nd ed. (Springer-Verlag, 1993).
25. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer-Verlag, 1989).
26. M. Kumar and A. K. Tiwari, "On Group-Invariant Solutions of Konopelchenko–Dubrovsky Equation by Using Lie Symmetry Approach," *Nonlinear Dynamics* 94 (2018): 475–487.
27. M. Kumar, S. Srivastava, and D. V. Tanwar, "Lie Symmetries, Soliton Dynamics, Conservation Laws and Stability Analysis of Bogoyavlensky–Konoplechenko System," *Optical and Quantum Electronics* 56 (2024): 1087.
28. S. C. Anco and G. Bluman, "Direct Construction of Conservation Laws From Field Equations," *Physical Review Letters* 78 (1997): 2869–2873.
29. S. C. Anco and G. Bluman, "Direct Construction Method for Conservation Laws of Partial Differential Equations. I. Examples of Conservation Law Classifications," *European Journal of Applied Mathematics* 13 (2002): 545–566.
30. S. C. Anco and G. Bluman, "Direct Construction Method for Conservation Laws of Partial Differential Equations. II. General Treatment," *European Journal of Applied Mathematics* 13 (2002): 567–585.