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Theories, models and bases of attribute implications in multi-adjoint concept lattices with hedges

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Abstract
Modelling knowledge systems is a fundamental challenge nowadays. Logic rules are essential for this goal in order to, for instance, design decision-support systems in artificial intelligence. However, the determination of these logic rules is a complex problem. In this line, Formal Concept Analysis provides a mechanism to automatically compute rules from a relational data set, which are called attribute implications. The difficulty arises from the potentially large number of valid implications derivable from a given data set, as well as the presence of redundant implications. This paper will focus on the theoretical development of the notions of theory, model and the entailment relations between them, which will allow us to elaborate the notion of basis of attribute implications in the multi-adjoint concept lattice framework enriched with truth-stressing hedges. We will explore one of the most common types of bases, which is constructed by means of the so-called pseudo-intents.

Keywords Attribute implications · Theories · Models · Bases · Multi-adjoint concept lattices

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1 Introduction

Formal Concept Analysis (FCA) affords both a theoretical and practical tools for knowledge extraction and representation of relational databases, by constructing specific units of

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information which are known as concepts. For instance, FCA provides a theoretical foundation for analyzing the impact of local congruences on concept lattices (Aragón et al. 2022), defining quantified concept-forming operators (Cornejo et al. 2022), detecting redundant fuzzy relation equations (Lobo et al. 2023), studying fuzzy closure structures as formal concepts of certain Galois connections (Ojeda-Hernández et al. 2023), and handling fuzzy linguistic information from the hierarchical and structural perspectives of attribute partial ordered structure diagrams (Pang et al. 2023). With regard to FCA applications, differential diagnosis of similar diseases (Alcalde and Burusco 2019), weather data analysis (Cornejo et al. 2022), digital forensics (Aragón et al. 2022; Ojeda-Hernández et al. 2024; Sokol et al. 2023) and concept cognitive learning (Ren et al. 2024) are of particular relevance. Fuzzy sets and fuzzy logic were introduced by Lotfi A. Zadeh with the main focus of handling uncertainty in data sets (Madrid and Ojeda-Aciego 2023; Medina and Antonio Torné-Zambrano yyy; Nithya et al. 2023; Wang et al. 2024), which is a usual feature in real data. Many fuzzy extensions of FCA have been proposed in the literature, for instance, **L**-concept lattices (Bělohlávek 2004), L-fuzzy concept lattices (Burusco and Fuentes-González 1994, 1998), concept lattices in non-commutative frameworks (Georgescu and Popescu 2003, 2002), generalized concept lattices (Krajči 2005) and multi-adjoint concept lattices (Medina et al. 2009), in order to model the logical reasoning with vague or imprecise information. Specifically, multi-adjoint concept lattices (Medina et al. 2009) arises as a new, general and more flexible approach capable of accommodating the different fuzzy extensions presented in Bělohlávek (2004), Burusco and Fuentes-González (1998), Georgescu and Popescu (2003), Krajči (2005). Attribute classification (Antoni et al. 2021), reduct computation (Cornejo et al. 2018), lattice size reduction (Cornejo et al. 2017) and the computation of attribute implications (Cornejo et al. 2024) are important tasks which have been effectively addressed within the multi-adjoint paradigm. In what regards to attribute implications, a theoretical development about their main notions and results has been investigated in Cornejo et al. (2024), Cornejo et al. (2016), Liñeiro-Barea et al. (2015), where the advantages provided by the multi-adjoint concept lattice with truth-stressing hedges (Bělohlávek and Vychodil 2012; Hájek 1998, 2001; Lakoff 1973) framework have been incorporated for the treatment of these tools.

On the other hand, model theory is a fundamental area of mathematical logic concerned with the interplay between formal languages and mathematical structures. Model theory enables us to examine the connection between a set of formulas (theory), which express certain propositions in a formal language, and the sets of systems (models) in which these formulas hold. By studying how models interact with formal theories, model theory helps us to discover and understand different properties of formal systems (Marker 2002).

Formally defining the notions of model and theory in FCA is fundamental to model data sets through attribute implications, which is one of the major challenges in this framework. This challenge has been addressed from various perspectives. Several works have proposed axiomatic systems and reasoning procedures for handling graded dependencies and entailment (Bělohlávek et al. 2016; Kuhr and Vychodil 2015; Vychodil 2016), while others have focused on the construction and optimization of bases of attribute implications, including minimal or non-redundant representations (Cordero et al. 2013; Rodríguez-Jiménez et al. 2014; Vychodil 2016). The integration of linguistic hedges has allowed for refined interpretations of fuzzy attribute implications (Cornejo et al. 2024; Vychodil 2016). Complementary to these developments, interactive and user-guided methods have also been explored for knowledge discovery from graded contexts (Glodeanu 2016).

Contexts usually contain a massive number of valid attribute implications, and there may also be attribute implications that are trivial or redundant. Then, the efficient finding of an optimal set of valid attribute implications from a given context represents a significant challenge. In other words, instead of considering the whole set of valid attribute implications associated with the context, we are interested in a non-redundant subset of them from which all the others valid attribute implications can be deduced, that is, a basis of attribute implications. Several approaches to the extraction of bases of attribute implications have been proposed in the classical case. The method presented by Guigues and Duquenne (1986), which relies on the notion of non-redundant nodes, and the approach introduced by Ganter (2010), Ganter and Wille (1999), based on the concept of pseudo-intents, stand out for their theoretical importance. Further refinements and structural insights have been provided in Bertet et al. (2018), Rodríguez-Lorenzo et al. (2017), where new strategies to reduce redundancy and improve computational performance are explored. On the other hand, the problem of extracting bases of attribute implications has also been addressed in the fuzzy setting. A significant contribution in this field is attributed to Belohlávek and Vychodil (2016), who extended the notion of attribute implication bases to the fuzzy framework of residuated concept lattices.

In this paper, we will present a generalization of the basic notions and properties regarding to theories of attribute implications, models and the entailment between these structures presented in Belohlávek and Vychodil (2016), to the multi-adjoint concept lattice with truth-stressing hedges framework, complementing the results presented in Cornejo et al. (2024), Cornejo et al. (2016), Liñeiro-Barea et al. (2015). This paper also focuses on extending the notion of basis of attribute implications in the multi-adjoint framework with truth-stressing hedges, exploring the particular kind of basis established from pseudo-intents, usually called Duquenne-Guigues-basis. These advances will allow the use of the notions of theory and models, together with the computation of bases, in the multi-adjoint concept lattice framework, and the use of this flexible framework to design intelligence systems that model (real) data sets with imperfect information.

The paper is organized as follows. Section 2 recalls the basic notions and properties associated with the considered multi-adjoint formal concept analysis framework, and the operators involved in it, that is, adjoint triples and truth-stressing hedges. Section 3 includes the theoretical notions and properties which are essential for understanding and analyzing attribute implications in the multi-adjoint concept lattice framework with hedges. Section 4 is devoted to the study of theories of attribute implications, models and the entailment relation between these structures. Section 5 investigates the notion of a basis in the multi-adjoint framework with hedges, together with the construction of a particular basis by means of the so-called pseudo-intents. Finally, the paper finishes with some conclusions and prospects for future work.

2 Preliminaries

In this section, we will present the necessary preliminary notions and results about the multi-adjoint formal concept analysis framework, as well as the operators involved in it, in order to make the paper self-contained.

2.1 Adjoint triples and truth-stressing hedges

T-norms and their residuated implications (Hájek 2001) were generalized by adjoint triples in Cornejo et al. (2013), Cornejo et al. (2015), Cornejo et al. (2021), which are general operators preserving their main properties and requiring only the minimal mathematical conditions for ensuring the operability. Consequently, adjoint triples are more flexible and so they can be applied in a wider range of frameworks, including non-commutative and/or non-associative settings. The formal definition of adjoint triple is presented below.

Definition 1 Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence, called *adjoint property*, is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Several properties of the residuated implications of an adjoint triple can be obtained from the adjoint property, as the following proposition shows.

Proposition 2 *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, we have that:*

(1) $\swarrow, \nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.
(2) $(\bigwedge_{z_i \in Z} z_i) \swarrow y = \bigwedge_{z_i \in Z}(z_i \swarrow y)$, for each $Z \subseteq P_3$ and $y \in P_2$, when infimum there exists.
(3) $(\bigwedge_{z_i \in Z} z_i) \nwarrow x = \bigwedge_{z_i \in Z}(z_i \nwarrow x)$, for each $Z \subseteq P_3$ and $x \in P_1$, when infimum there exists.

The following proposition present the conditions that the posets and the adjoint conjunctor must satisfy in order to ensure boundary conditions of the residuated implications of an adjoint triple.

Proposition 3 *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$.*

(1) If $P_2 \subseteq P_3$ and P_1 has a maximum $\top_1$ as a left identity element for $\&$, that is, the equality $\top_1 \& y = y$ holds, for all $y \in P_2$, then we obtain:

$$\top_1 = z \swarrow y \quad \textit{if and only if} \quad y \leq_3 z$$

for all $y \in P_2$ and $z \in P_3$.
(2) If $P_1 \subseteq P_3$ and P_2 has a maximum $\top_2$ as a right identity element for $\&$, that is, the equality $x \& \top_2 = x$ is satisfied, for all $x \in P_1$, then we have:

$$\top_2 = z \nwarrow x \quad \textit{if and only if} \quad x \leq_3 z$$

for all $x \in P_1$ and $z \in P_3$.
(3) If $P_1 = P_3$ and P_2 has a maximum $\top_2$, then we obtain:

$$z \swarrow \top_2 = z \quad \textit{if and only if} \quad x \,\&\, \top_2 = x, \ \textit{for all } x \in P_1 \ \textit{and} \ z \in P_3.$$

The following equivalences are satisfied when we assume that the conjunctor of the adjoint triple is associative (Cornejo et al. 2020).

Proposition 4 *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the poset $(P, \leq)$ such that the operator $\&$ is associative. The following equalities hold, for all $a, b, x, y, z \in P$:*

(1) $(z \swarrow y) \nwarrow x = (z \nwarrow x) \swarrow y$

(2) $(x \nwarrow b) \nwarrow y = x \nwarrow (b \,\&\, y)$

An overview about adjoint triples can be found in Cornejo and Medina (2021), Cornejo et al. (2013), Cornejo et al. (2015), Cornejo et al. (2021), where different properties and examples are presented. Next example will be helpful to illustrate the developed theory in the paper.

Example 5 Let $n, m, k \in \mathbb{N}^+$ such that $[0,1]_n, [0,1]_m, [0,1]_k$ are regular partitions of $[0,1]$ in n, m and k pieces, respectively. For instance, $[0,1]_4 = \{0, 0.25, 0.5, 0.75, 1\}$ splits the unit interval into four pieces. Given a left-continuous t-norm $T\colon [0,1] \times [0,1] \to [0,1]$ and its residuated implication $\to\colon [0,1] \times [0,1] \to [0,1]$, we can define the following three operators $\&_{DT}\colon [0,1]_n \times [0,1]_m \to [0,1]_k$, $\swarrow^{DT}\colon [0,1]_k \times [0,1]_m \to [0,1]_n$ and $\nwarrow_{DT}\colon [0,1]_k \times [0,1]_n \to [0,1]_m$, for all $x \in [0,1]_n$, $y \in [0,1]_m$ and $z \in [0,1]_k$, as follows:

$$x \,\&_{DT}\, y = \frac{\lceil k \cdot T(x,y) \rceil}{k} \tag{1}$$

$$z \swarrow^{DT} y = \frac{\lfloor m \cdot (z \to y) \rfloor}{m} \tag{2}$$

$$z \nwarrow_{DT} x = \frac{\lfloor n \cdot (z \to x) \rfloor}{n} \tag{3}$$

where $\lfloor \_ \rfloor$ is the floor function and $\lceil \_ \rceil$ is the ceiling function. Specifically, $(\&_{DT}, \swarrow^{DT}, \nwarrow_{DT})$ is an adjoint triple whose conjunctor could be neither commutative nor associative. The discretizations of Gödel, product and Łukasiewicz adjoint triples are denoted as $(\&_{DG}, \swarrow^{DG}, \nwarrow_{DG})$, $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$ and $(\&_{L}, \swarrow^{DL}, \nwarrow_{DL})$, respectively, and they computed by replacing

$$T_G(x,y) = \min\{x,y\} \qquad z \to_G y = \begin{cases} 1 & \text{if } y \le z \\ z & \text{otherwise} \end{cases}$$

$$T_P(x,y) = x \cdot y \qquad z \to_P y = \begin{cases} 1 & \text{if } y \le z \\ z/y & \text{otherwise} \end{cases}$$

$$T_Ł(x,y) = \max\{0, x+y-1\} \quad z \to_Ł y = \min\{1, 1-y+z\}$$

in Eqs. (1),(2) and (3). □

The notion of forcing-implication will also play an important role in different results obtained in this paper.

Definition 6 Let $(P_1, \le_1), (P_2, \le_2)$ be two posets such that $\top_2$ is the top element in $(P_2, \le_2)$. We say that the mapping $\nwarrow\colon P_1 \times P_1 \to P_2$ is a *forcing-implication* if it is order-preserving in the left argument, order-reversing in the right argument and the following equivalence holds, for all $y, z \in P_1$:

$$z \nwarrow y = \top_2 \quad \text{if and only if} \quad y \le_1 z$$

Linguistic hedges provide enhanced flexibility in the modelling of desired information. They have been studied in different frameworks due to their relevance, as in approximate reasoning (Klir and Yuan 1995), fuzzy logic (Cornejo et al. 2022; Esteva et al. 2013; Hájek 1998, 2001), fuzzy relation equations (Bartl et al. 2012) and formal concept analysis (Bělohlávek and Vychodil 2012). For our purpose, we will consider adjoint triples in order to extend the notion of truth-stressing hedge presented in Bělohlávek and Vychodil (2012).

Definition 7 Let $(P, \leq, \top)$ be an upper bounded poset and $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(P, \leq)$. A *truth-stressing* $\swarrow$*-hedge* is a unary function $*\colon P \to P$ satisfying the following properties:

$$*(\top) = \top \qquad \text{(Boundary condition)}$$

$$*(x) \leq x \qquad \text{(Subdiagonal condition)}$$

$$*(z \swarrow y) \leq *(z) \swarrow *(y) \qquad \text{($\swarrow$-regularity condition)}$$

$$*(*(x)) = *(x) \qquad \text{(Idempotency condition)}$$

for each $x, y, z \in P$. Similarly, a *truth-stressing* $\nwarrow$*-hedge* is defined on $(P, \leq, \top)$.

Notice that, due to the $\swarrow$-regularity condition required in the above definition, if we consider an adjoint triple with respect to three different upper bounded posets $(P_1, \leq_1, \top_1)$, $(P_2, \leq_2, \top_2)$ and $(P_3, \leq_3, \top_3)$, these posets must be related by inclusion. Therefore, we have assumed that $P_1 = P_2 = P_3 = P$, in order to simplify the definition of truth-stressing hedge.

Two boundary cases of truth-stressing hedges are given by the identity (the greatest truth-stressing $\swarrow$-hedge) and the globalization (the least one truth-stressing $\swarrow$-hedge) (Takeuti and Titani 1987). The globalization hedge is defined on an upper bounded poset $(P, \leq, \top)$, for each $x \in P$, as:

$$*(x) = \begin{cases} \top & \text{if} \quad x = \top \\ \bot & \text{otherwise} \end{cases}$$

The following proposition gives sufficient conditions over the conjuctor of an adjoint triple which guarantee the monotonicity of truth-stressing hedges (Cornejo et al. 2024).

Proposition 8 *Given an adjoint triple* $(\&, \swarrow, \nwarrow)$ *with respect to an upper bounded poset* $(P, \leq, \top)$*, we have that:*

(1) If the equality $x \,\&\, \top = x$ *holds, for all* $x \in P$*, and* $*\colon P \to P$ *is a truth-stressing* $\nwarrow$*-hedge, then* $*$ *is order-preserving.*
(2) If the equality $\top \,\&\, y = y$ *holds, for all* $y \in P$*, and* $*\colon P \to P$ *is a truth-stressing* $\swarrow$*-hedge, then* $*$ *is order-preserving.*

Order-preserving mappings always satisfy the following inequalities (Davey and Priestley 2002).

Lemma 9 *Given a complete lattice* $(L, \preceq)$*, an order-preserving truth-stressing* $\nwarrow$*-hedge* $*\colon L \to L$ *and a subset* $\{x_i \in L \mid i \in I\}$*, then:*

- $\bigvee_{i\in I} *(x_i) \preceq *\left(\bigvee_{i\in I} x_i\right)$
- $*\left(\bigwedge_{i\in I} x_i\right) \preceq \bigwedge_{i\in I} *(x_i)$

2.2 Multi-adjoint concept lattices with truth-stressing hedges

The fuzzy framework of multi-adjoint concept lattices (Díaz-Moreno et al. 2014; Medina et al. 2009) emerges as a new, more general and flexible environment that accommodates other fuzzy approaches present in the literature such as Bělohlávek (2004), Burusco and Fuentes-González (1998), Georgescu and Popescu (2003), Krajči (2005), among others. This paper focuses on the multi-adjoint concept lattice framework enriched with truth-stressing hedges. Then, in order to introduce formally this framework, we need to define the following families of truth-stressing hedges on the sets of attributes and objects of a given context, that is:

$$*_A = \{*_a \colon L_1 \to L_1 \mid *_a \text{ is a truth-stressing } \swarrow\text{-hedge, for all } a \in A\}$$
$$*_B = \{*_b \colon L_2 \to L_2 \mid *_b \text{ is a truth-stressing } \nwarrow\text{-hedge, for all } b \in B\}$$

Next, we introduce the main notions related to the multi-adjoint concept lattices considering truth-stressing hedges.

Definition 10 • The tuple $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$ is a *multi-adjoint frame* where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(P, \leq)$, for all $i \in \{1, \ldots, n\}$.

- The tuple $(A_{*_A}, B_{*_B}, R, \sigma)$ is a *context* where A and B are non-empty sets, $*_A$ is a family of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $*_B$ is a family of arbitrary truth-stressing $\nwarrow$-hedges on L_2, R is a P-fuzzy relation $R \colon A \times B \to P$ and $\sigma \colon A \times B \to \{1, \ldots, n\}$ is a mapping which associates any element in $A \times B$ with a particular adjoint triple in the multi-adjoint frame.
- The *concept-forming operators* ${}^{\uparrow_{*\sigma}} \colon L_2^B \to L_1^A$ and ${}^{\downarrow^{*\sigma}} \colon L_1^A \to L_2^B$ are defined as:

$$g^{\uparrow_{*\sigma}}(a) = \inf\{R(a, b') \swarrow^{\sigma(a,b')} *_{b'}(g(b')) \mid b' \in B\} \tag{4}$$
$$f^{\downarrow^{*\sigma}}(b) = \inf\{R(a', b) \nwarrow_{\sigma(a',b)} *_{a'}(f(a')) \mid a' \in A\} \tag{5}$$

  for all $g \in L_2^B$, $f \in L_1^A$, $a \in A$, $b \in B$.
- A *concept* is a pair $\langle g, f\rangle$ satisfying that $g^{\uparrow_{*\sigma}} = f$, $f^{\downarrow^{*\sigma}} = g$, for all $g \in L_2^B$, $f \in L_1^A$. The fuzzy subsets g and f are usually known as *the extent and intent of the concept*, respectively.
- The *multi-adjoint concept lattice with truth-stressing hedges*, denoted as $(\mathcal{M}_*, \preceq)$, associated with the multi-adjoint frame and the context $(A_{*_A}, B_{*_B}, R, \sigma)$ is the set:

$$\mathcal{M}_* = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A, g^{\uparrow_{*\sigma}} = f, f^{\downarrow^{*\sigma}} = g\}$$

  where the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (or equivalently, $f_2 \preceq_1 f_1$).

From now on, in order to simplify the notation, we will write $g^{\uparrow_*}$ and $f^{\downarrow^*}$ instead of $g^{\uparrow_{*\sigma}}$ and $f^{\downarrow^{*\sigma}}$, respectively. The sets of extents and intents of the concepts of $(\mathcal{M}_*, \preceq)$ will be denoted as $Ext(\mathcal{M}_*)$ and $Int(\mathcal{M}_*)$, respectively. Notice that, the following equivalences are obtained:

$$g \in Ext(\mathcal{M}_*) \text{ if and only if } g = g^{\uparrow_*\downarrow^*}$$
$$f \in Int(\mathcal{M}_*) \text{ if and only if } f = f^{\downarrow^*\uparrow_*}$$

It is important to mention that unlike in the multi-adjoint concept lattice without truth-stressing hedges, in this case the concept-forming operators defined above do not form a Galois connection, as it was shown in [Cornejo et al. (2024), Example 15]. Next, in order to make the paper self-contained, we recall the notion of an antitone Galois connection as its properties.

Definition 11 Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets. We say that the pair $({}^\uparrow, {}^\downarrow)$ of mappings ${}^\downarrow \colon P_1 \to P_2$, ${}^\uparrow \colon P_2 \to P_1$ is an *antitone Galois connection* between P_1 and P_2 if the following properties are satisfied:

(1) $^{\uparrow}$ and $^{\downarrow}$ are order-reversing.
(2) $x \leq_1 x^{\downarrow\uparrow}$ and $y \leq_2 y^{\uparrow\downarrow}$
(3) $x^{\downarrow} = x^{\downarrow\uparrow\downarrow}$ and $y^{\uparrow} = y^{\uparrow\downarrow\uparrow}$

for all $x \in P_1$ and $y \in P_2$.

However, Cornejo et al. (2024) presented different properties that these operators satisfy, which are closely related to the well-known properties of antitone Galois connections. In this paper, we will only introduce the properties that are necessary for our purposes. Before introducing the aforementioned properties, it is necessary to consider the fuzzy subset of attributes $\bar{*}_A(f)\colon A \to L_1$ and the fuzzy subset of objects $\bar{*}_B(g)\colon B \to L_2$, defined as:

$$\bar{*}_A(f)(a) = *_a(f(a))$$
$$\bar{*}_B(g)(b) = *_b(g(b))$$

for all $*_a \in *_A$, $*_b \in *_B$, $a \in A$, $b \in B$, $f \in L_1^A$ and $g \in L_2^B$.

Proposition 12 *Let $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$ be a multi-adjoint frame, (A, B, R, σ) and $(A_{*_A}, B_{*_B}, R, \sigma)$ be contexts where $*_A$ is a family of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $*_B$ is a family of arbitrary truth-stressing $\nwarrow$-hedges on L_2. If $L_1 = L_2 = P$, $x \,\&_i\, \top_1 = x$ and $\top_1 \,\&_i\, y = y$, for all $x, y \in L_1$ and $i \in \{1, \ldots, n\}$, then the concept-forming operators $^{\uparrow_*}$ and $^{\downarrow^*}$ satisfy the following properties:*

(1) If $g_1 \preceq_1 g_2$ then $g_2^{\uparrow_*} \preceq_1 g_1^{\uparrow_*}$
If $f_1 \preceq_1 f_2$ then $f_2^{\downarrow^*} \preceq_1 f_1^{\downarrow^*}$
(2) $\bar{*}_B(g) \preceq_2 g^{\uparrow_*\downarrow^*}$
$\bar{*}_A(f) \preceq_1 f^{\downarrow^*\uparrow_*}$
(3) $g^{\uparrow_*\downarrow^*} = g^{\uparrow_*\downarrow^*\uparrow_*\downarrow^*}$
$f^{\downarrow^*\uparrow_*} = f^{\downarrow^*\uparrow_*\downarrow^*\uparrow_*}$

for all $g, g_1, g_2 \in L_1^B$ and $f, f_1, f_2 \in L_1^A$.

Notice that, by Proposition 12(2), we can ensure that the composition $^{\downarrow^*\uparrow_*}$ is not a closure operator in general, a fact that is fundamental to compute a basis of attribute implications. However, Proposition 12(3) is enough to facilitate the computation of the concepts in this framework and the construction of implicational bases. A sufficient condition under which these operators form a closure operator will be provided in this paper. Once again, in order to make the paper self-contained, the notion of closure operator is recalled below (Davey and Priestley 2002).

Definition 13 Let $(P, \leq)$ be a poset. A mapping $c\colon P \to P$ is called a *closure operator* if the following properties are satisfied, for each $x, y \in P$:

(1) $x \leq c(x)$
(2) If $x \leq y$ then $c(x) \leq c(y)$
(3) $c(c(x)) = c(x)$

Henceforth, a multi-adjoint frame $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$, two families $*_A$ and $*_B$ of arbitrary truth-stressing $\swarrow$-hedges on L_1 and $\nwarrow$-hedges on L_2, respectively, the context $(A_{*_A}, B_{*_B}, R, \sigma)$ and the multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ will be fixed.

3 Attribute implications. Definition and properties

This section is devoted to present theoretical notions and properties about attribute implications in multi-adjoint concept lattices with truth-stressing hedges. Attribute implications follow the philosophy of the if-then rules establishing relationships between sets of attributes. Specifically, an attribute implication indicates that if an object satisfies all the attributes from the antecedent of the rule, it also satisfies the attributes of the consequent. The syntactic definition of attribute implication is recalled below.

Definition 14 An *attribute implication* over A is an expression $f_2 \Leftarrow f_1$, where $f_1, f_2 \in L_1^A$ are two fuzzy subsets of attributes. The set of all attribute implications over A will be denoted as $\mathfrak{I}_A$.

The semantic interpretation of an attribute implication is supported by the notion of validity in the multi-adjoint frame, which is based on an extension of the usual fuzzy inclusion considering adjoint triples.

Definition 15 Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ and two fuzzy subsets of attributes $f_1, f_2 \in L_1^A$, *the degree in which* f_1 *is included in* f_2 is defined as:

$$S^1(f_1, f_2) = \bigwedge_{a \in A} (f_2(a) \nwarrow f_1(a))$$

Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_2, \preceq_2)$ and two fuzzy subsets of objects $g_1, g_2 \in L_2^B$, *the degree in which* g_1 *is included in* g_2 is defined as:

$$S^2(g_1, g_2) = \bigwedge_{b \in B} (g_2(b) \swarrow g_1(b))$$

Different technical properties associated with the degree of inclusion are shown in the following propositions.

Lemma 16 *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that the conjunctor & is associative. Given two subsets of attributes $f_1, f_2 \in L_1^A$ and two subsets of objects $g_1, g_2 \in L_1^B$, we have that:*

(1) $S^1(f_1, f_2) \preceq_1 S^2(f_2^{\downarrow}, f_1^{\downarrow})$
(2) $S^2(g_1, g_2) \preceq_1 S^1(g_2^{\uparrow}, g_1^{\uparrow})$

Proof Given $f_1, f_2 \in L_1^A$, the inequality $S^1(f_1, f_2) \preceq_1 S^2(f_2^{\downarrow}, f_1^{\downarrow})$ is deduced directly from the following chain of equalities and inequalities:

$$\begin{aligned}
S^1(f_1, f_2) &\overset{(i)}{=} \bigwedge_{a\in A} (f_2(a) \nwarrow f_1(a)) \\
&\overset{(ii)}{\preceq_1} \bigwedge_{a\in A} \left(f_2^{\downarrow\uparrow}(a) \nwarrow f_1(a)\right) \\
&\overset{(iii)}{=} \bigwedge_{a\in A} \left(\bigwedge_{b\in B} (R(a,b) \swarrow f_2^{\downarrow}(b)) \nwarrow f_1(a)\right) \\
&\overset{(iv)}{=} \bigwedge_{a\in A} \bigwedge_{b\in B} \left((R(a,b) \swarrow f_2^{\downarrow}(b)) \nwarrow f_1(a)\right) \\
&\overset{(v)}{=} \bigwedge_{a\in A} \bigwedge_{b\in B} \left((R(a,b) \nwarrow f_1(a)) \swarrow f_2^{\downarrow}(b)\right) \\
&\overset{(vi)}{=} \bigwedge_{b\in B} \left(\bigwedge_{a\in A} (R(a,b) \nwarrow f_1(a)) \swarrow f_2^{\downarrow}(b)\right) \\
&\overset{(vii)}{=} \bigwedge_{b\in B} \left(f_1^{\downarrow}(b) \swarrow f_2^{\downarrow}(b)\right) \\
&\overset{(viii)}{=} S^2(f_2^{\downarrow}, f_1^{\downarrow})
\end{aligned}$$

for all $a \in A$ and $b \in B$, where (i) and $(viii)$ are satisfied by Definition 15, (ii) is obtained from Definition 11(2) and taking into account that $\nwarrow$ is ordering-preserving on the first argument, (iii) and (vii) hold by the definition of the concept-forming operators $^{\uparrow}$ and $^{\downarrow}$, (iv) and (vi) are obtained because $\nwarrow$ and $\swarrow$ preserve the infimum on the consequent, and (v) holds by Proposition 4(1).

The proof of the inequality $S^2(g_1, g_2) \preceq_1 S^1(g_2^{\uparrow}, g_1^{\uparrow})$ is analogously obtained. □

Similar properties are proven when the concept-forming operators with hedges are considered.

Corollary 17 *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$ *such that the conjunctor & is associative. Given two subsets of attributes* $f_1, f_2 \in L_1^A$ *and two subsets of objects* $g_1, g_2 \in L_1^B$*, we have that:*

(1) $S^1(\bar{*}_A(f_1), \bar{*}_A(f_2)) \preceq_1 S^2(f_2^{\downarrow^*}, f_1^{\downarrow^*})$
(2) $S^2(\bar{*}_B(g_1), \bar{*}_B(g_2)) \preceq_1 S^1(g_2^{\uparrow *}, g_1^{\uparrow *})$

Proof Given $f_1, f_2 \in L_1^A$, the following chain of inequalities holds:

$$S^1(\bar{*}_A(f_1), \bar{*}_A(f_2)) \overset{(i)}{\preceq_1} S^2(\bar{*}_A(f_2)^{\downarrow}, \bar{*}_A(f_1)^{\downarrow}) \overset{(ii)}{=} S^2(f_2^{\downarrow^*}, f_1^{\downarrow^*})$$

for all $a \in A$, where (i) is obtained from Lemma 16 and (ii) is satisfied because $\bar{*}_A(f)^{\downarrow} = f^{\downarrow^*}$ for all $f \in L_1^A$, which follows from the definition of the operator $^{\downarrow^*}$. The proof of the inequality $S^2(\bar{*}_B(g_1), \bar{*}_B(g_2)) \preceq_1 S^1(g_2^{\uparrow *}, g_1^{\uparrow *})$ holds analogously. □

The following technical result indicates the behaviour of these degrees of inclusion when a truth-stressing hedge is applied to them.

Proposition 18 *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$*, satisfying* $x \,\&\, \top_1 = x$*, for all* $x \in L_1$*, and* $*\colon L_1 \to L_1$ *a truth-stressing* $\nwarrow$*-hedge. Given two subsets of attributes* $f_1, f_2 \in L_1^A$*, we have that:*

$$*(S^1(f_1, f_2)) \preceq_1 S^1(*(f_1), *(f_2))$$

Proof Given $f_1, f_2 \in L_1^A$, we obtain that:

$$\begin{aligned} *(S^1(f_1, f_2)) &\overset{(i)}{=} *\left(\bigwedge_{a \in A} (f_1(a) \nwarrow f_2(a))\right) \\ &\overset{(ii)}{\preceq_1} \bigwedge_{a \in A} (*(f_1(a) \nwarrow f_2(a))) \\ &\overset{(iii)}{\preceq_1} \bigwedge_{a \in A} (*(f_1(a)) \nwarrow *(f_2(a))) \\ &\overset{(iv)}{\preceq_1} S^1(*(f_1), *(f_2)) \end{aligned}$$

for all $a \in A$, where (i) and (iv) follow from Definition 15, (ii) holds by Lemma 9, and (iii) because of the regularity condition in Definition 7. □

Moreover, we include another property associated with the degree of inclusion S^1, which will be used later.

Proposition 19 *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to* $(L_1, \preceq_1)$ *such that* $\nwarrow$ *is a forcing-implication. The following equivalence holds:*

$$f_1 \preceq_1 f_2 \quad \textit{if and only if} \quad S^1(f_1, f_2) = \top_1$$

Proof Given $f_1, f_2 \in L_1^A$, we deduce the next chain of equivalences, for all $a \in A$:

$$\begin{aligned} f_1(a) \preceq_1 f_2(a) \quad &\text{if and only if} \quad f_2(a) \nwarrow f_1(a) = \top_1 \\ &\text{if and only if} \quad \bigwedge_{a' \in A} (f_2(a') \nwarrow f_1(a')) = \top_1 \\ &\text{if and only if} \quad S^1(f_1, f_2) = \top_1 \end{aligned}$$

where the first equivalence is satisfied because $\nwarrow$ is a forcing-implication, the second equivalence holds by the infimum property and the third one is obtained by Definition 15. □

Analogous results to those presented in Propositions 18 and 19 can be obtained for the degree of inclusion S^2.

After introducing the definition of the above degrees of inclusion and some technical properties which will be used later, we recall the degrees of validity of an attribute implication, given in Cornejo et al. (2024), which provide the semantic interpretation of this expression.

Definition 20 Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ and three fuzzy subsets of attributes $f_1, f_2, f_3 \in L_1^A$:

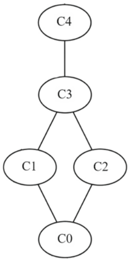
- $\| f_2 \Leftarrow f_1 \|_{f_3}$ is the degree in which $f_2 \Leftarrow f_1$ is valid in $f_3 \in L_1^A$, defined as:

$$\| f_2 \Leftarrow f_1 \|_{f_3} = S^1(f_2, f_3) \swarrow *(S^1(f_1, f_3))$$

  where $*\colon L_1 \to L_1$ is a truth-stressing $\nwarrow$-hedge.

Table 1 Relation R and mapping σ of Example 21

R	a_1	a_2	a_3
b_1	0.6	0.2	0.2
b_2	0.8	0.4	0.6
b_3	0.6	0.6	0.2
σ	a_1	a_2	a_3
b_1	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$
b_2	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$
b_3	$\&_{P}$	$\&_{DP}$	$\&_{DP}$



$$\begin{aligned}
C_0 &= \langle\{0.25/b_2\}, \{1/a_1, 1/a_2, 1/a_3\}\rangle\\
C_1 &= \langle\{1/b_2, 0.25/b_3\}, \{0.8/a_1, 0.4/a_2, 0.6/a_2\}\rangle\\
C_2 &= \langle\{1/b_2, 1/b_3\}, \{0.6/a_1, 0.4/a_2, 0.2/a_3\}\rangle\\
C_3 &= \langle\{1/b_1, 1/b_2, 1/b_3\}, \{0.6/a_1, 0.2/a_2, 0.2/a_3\}\rangle\\
C_4 &= \langle\{0.75/b_2, 1/b_3\}, \{0.6/a_1, 0.6/a_2, 0.2/a_3\}\rangle
\end{aligned}$$

Fig. 1 Hasse diagram of $(\mathcal{M}_*, \preceq)$ and its concepts of Example 21

- $\|f_2 \Leftarrow f_1\|_{\mathcal{F}}$ is the degree in which the fuzzy implication $f_2 \Leftarrow f_1$ is valid in $\mathcal{F} \subseteq L_1^A$, defined as:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{F}} = \bigwedge_{f\in\mathcal{F}} \|f_2 \Leftarrow f_1\|_f$$

- $\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}$ is the degree in which the fuzzy implication $f_2 \Leftarrow f_1$ is valid in the multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$, defined as:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*} = \|f_2 \Leftarrow f_1\|_{Int(\mathcal{M}_*)}$$

Note that, the definition of validity of an attribute implication in a multi-adjoint concept lattice considers an external adjoint triple, which can be different from the ones given in the multi-adjoint frame.

An example of the computation of the degree of validity of an attribute implication in a given multi-adjoint framework and its semantic interpretation is presented below.

Example 21 Let $([0,1]_{10}, [0,1]_4, [0,1]_5, \leq, (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{L}, \nwarrow_{L}))$ be a multi-adjoint frame and $(A_{*_A}, B_{*_B}, R, \sigma)$ be a context where the set of attributes is $A = \{a_1, a_2, a_3\}$, the set of objects is $B = \{b_1, b_2, b_3\}$, $*_A$ is a family of identity truth-stressing $\swarrow$-hedges, $*_B$ is a family of globalization truth-stressing $\nwarrow$-hedges, and the relation $R\colon A \times B \to [0,1]_5$ and the mapping σ are given in Table 1.

The corresponding multi-adjoint concept lattice related to this framework and the context $(A_{*_A}, B_{*_B}, R, \sigma)$ is composed of 5 concepts, as it is shown in Fig. 1.

Now, we will consider $(\&_{DL}, \swarrow^{DL}, \nwarrow_{L})$ as an extra adjoint triple with respect to $([0,1]_{10}, \leq)$, * as the identity truth-stressing, the attribute implications displayed in Table 2 and we will compute their validity in the multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$.

Table 2 Attribute implications considered in Example 21

$f_2 \Leftarrow f_1$	Name
$\{0.8/a_3\} \Leftarrow \{1/a_1, 1/a_2\}$	I_1
$\{1/a_2\} \Leftarrow \{1/a_1, 0.2/a_2, 0.2/a_3\}$	I_2

Applying Definition 20, we have that:

$$\begin{aligned}\|I_1\|_{Int(\mathcal{M}_*)} &= \bigwedge_{f \in Int(\mathcal{M}_*)} \|I_1\|_f \\ &= \bigwedge_{f \in Int(\mathcal{M}_*)} \{S^1(f_2, f) \swarrow^{\mathrm{DL}} *(S^1(f_1, f))\} \\ &= \inf\{1, 1, 1, 1, 1\} = 1\end{aligned}$$

Then, due to the degree of validity equal to 1, we can ensure with total certainty that any object totally satisfying attributes a_1 and a_2 will also have attribute a_3 with a high truth-degree.

On the other hand, following the same procedure, we obtain that:

$$\|I_2\|_{Int(\mathcal{M}_*)} = \inf\{1, 0.4, 0.4, 0.2, 0.6\} = 0.2$$

In this case, since its validity degree is closer to 0, we could discard this attribute implication when extracting relevant information. □

Finally, we present different technical equivalences satisfied by the degree of inclusion S^1 and the degree of validity of an attribute implication, which will be useful later.

Lemma 22 *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that the conjunctor & is associative and $x \& \top_1 = x$, for all $x \in L_1$, and $*\colon L_1 \to L_1$ a truth-stressing $\nwarrow$-hedge. Given three subsets of attributes $f_1, f_2, f_3 \in L_1^A$ and $c \in L_1$, we have that:*

(1) $S^1(f_2, f_3) \nwarrow c = S^1(f_2 \& c, f_3)$
(2) $\|f_2 \Leftarrow f_1\|_{f_3} \nwarrow c = \|f_2 \& c \Leftarrow f_1\|_{f_3}$
(3) $c \preceq_1 \|f_2 \Leftarrow f_1\|_{f_3}$ *if and only if* $\|f_2 \& c \Leftarrow f_1\|_{f_3} = \top_1$

where $f_2 \& c$ denotes a fuzzy subset of attributes, for all $c \in L_1$, that is, $f_2 \& c\colon A \to L_1$ defined as $(f_2 \& c)(a) = f_2(a) \& c$, for all $a \in A$.

Proof (1) Given $f_2, f_3 \in L_1^A$ and $c \in L_1$, the next chain of equivalences is obtained:

$$\begin{aligned}S^1(f_2, f_3) \nwarrow c &= \left(\bigwedge_{a \in A} (f_3(a) \nwarrow f_2(a))\right) \nwarrow c && \text{(By Definition 15)} \\ &= \bigwedge_{a \in A} ((f_3(a) \nwarrow f_2(a)) \nwarrow c) && \text{(By Proposition 2(3))} \\ &= \bigwedge_{a \in A} (f_3(a) \nwarrow (f_2(a) \& c)) && \text{(By Proposition 4(2))} \\ &= S^1(f_2 \& c, f_3) && \text{(By Definition 15)}\end{aligned}$$

(2) Given $f_1, f_2, f_3 \in L_1^A$ and $c \in L_1$. The identity that we want to prove is deduced from the following chain of equalities:

$$\begin{aligned}
\| f_2 \Leftarrow f_1 \|_{f_3} \nwarrow c &= (S^1(f_2, f_3) \swarrow *(S^1(f_1, f_3))) \nwarrow c && \text{(By Definition 20)}\\
&= (S^1(f_2, f_3) \nwarrow c) \swarrow *(S^1(f_1, f_3)) && \text{(By Proposition 4(1))}\\
&= S^1(f_2 \,\&\, c, f_3) \swarrow *(S^1(f_1, f_3)) && \text{(By Lemma 22(1))}\\
&= \| f_2 \,\&\, c \Leftarrow f_1 \|_{f_3} && \text{(By Definition 20)}
\end{aligned}$$

(3) Given $f_1, f_2, f_3 \in L_1^A$ and $c \in L_1$, taking account Proposition 3(2), we obtain that:

$$c \preceq_1 \| f_2 \Leftarrow f_1 \|_{f_3} \quad \text{if and only if} \quad \| f_2 \Leftarrow f_1 \|_{f_3} \nwarrow c = \top_1$$

Applying Lemma 22(2), we have that $\| f_2 \Leftarrow f_1 \|_{f_3} \nwarrow c = \| f_2 \,\& c \Leftarrow f_1 \|_{f_3}$. Hence, we can conclude that the following equivalence is satisfied:

$$c \preceq_1 \| f_2 \Leftarrow f_1 \|_{f_3} \quad \text{iff} \quad \| f_2 \,\&\, c \Leftarrow f_1 \|_{f_3} = \top_1$$

□

4 Attribute implications. Theories and models

This section addresses the study of the notions of theory and model, as well as the entailment between these structures, for attribute implications in the multi-adjoint concept lattice with truth-stressing hedges framework. Following the same philosophy that in the residuated framework, a generalization of the definitions, properties and results presented in Belohlávek and Vychodil (2016) on theories and models of attribute implications will be introduced to the multi-adjoint framework. Hence, from now on, an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ will be fixed. The first definition to be extended is the notion of theory of attribute implications.

Definition 23 Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$, $\mathfrak{I}_A$ be a set of attribute implications over A and $f_1, f_2 \in L_1^A$.

- A *theory* is a fuzzy subset of attribute implications over A, that is $\mathcal{T} \colon \mathfrak{I}_A \to L_1$, and we say that $\mathcal{T}(f_2 \Leftarrow f_1)$ denotes the degree in which an attribute implication $f_2 \Leftarrow f_1$ belongs to $\mathcal{T}$.
- A *model* of $\mathcal{T}$ is a fuzzy subset of attributes $M \in L_1^A$ satisfying that $\mathcal{T}(f_2 \Leftarrow f_1) \preceq_1 \| f_2 \Leftarrow f_1 \|_M$, for all $f_1, f_2 \in L_1^A$. The set of all models of $\mathcal{T}$, denoted as $Mod(\mathcal{T})$, is defined as:

$$Mod(\mathcal{T}) = \{M \in L_1^A \mid \mathcal{T}(f_2 \Leftarrow f_1) \preceq_1 \| f_2 \Leftarrow f_1 \|_M, \text{ for all } f_1, f_2 \in L_1^A\}$$

- A *crisp theory* is a subset of attribute implications over A.
- A *model* of a crisp theory $\mathcal{T}_{\text{cr}}$ is a fuzzy subset of attributes $M \in L_1^A$ satisfying that $\| f_2 \Leftarrow f_1 \|_M = \top_1$, for all $f_1, f_2 \in L_1^A$ such that $f_2 \Leftarrow f_1 \in \mathcal{T}_{\text{cr}}$. The set of all models of $\mathcal{T}_{\text{cr}}$, denoted as $Mod(\mathcal{T}_{\text{cr}})$, is defined as:

$$Mod(\mathcal{T}_{\text{cr}}) = \{M \in L_1^A \mid \| f_2 \Leftarrow f_1 \|_M = \top_1, \text{ for all } f_2 \Leftarrow f_1 \in \mathcal{T}_{\text{cr}}\}$$

Next, we will present an example that computes the set of all models of a given theory.

Table 3 Initial attribute implications considered in Example 24

f_1	f_2	$f_2 \Leftarrow f_1$	Name
$\{1/a_1, 0.5/a_2\}$	$\{0.5/a_1\}$	$\{0.5/a_1\} \Leftarrow \{1/a_1, 0.5/a_2\}$	I_1
$\{1/a_2\}$	$\{1/a_1\}$	$\{1/a_1\} \Leftarrow \{1/a_2\}$	I_2
$\{0.5/a_1, 1/a_2\}$	$\{0.5/a_2\}$	$\{0.5/a_2\} \Leftarrow \{0.5/a_1, 1/a_2\}$	I_3
$\{0.5/a_2\}$	$\{0.5/a_1, 0.5/a_2\}$	$\{0.5/a_1, 0.5/a_2\} \Leftarrow \{0.5/a_2\}$	I_4

Table 4 Membership degrees of attributes implications to $\mathcal{T}$ given in Example 24 and their validity

	$\mathcal{T}(f_2 \Leftarrow f_1)$	$\|f_2 \Leftarrow f_1\|_{M_i}$ with $i \in \{1, 2, 3, 7, 8, 9\}$	$\|f_2 \Leftarrow f_1\|_{M_4}$	$\|f_2 \Leftarrow f_1\|_{M_5}$	$\|f_2 \Leftarrow f_1\|_{M_6}$
I_1	0.5	1	1	1	1
I_2	1	1	1	0	0.5
I_3	0.5	1	1	1	1
I_4	1	1	0	0	1

Example 24 Let $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$ be the adjoint triple defined from the discretization of the product t-norm on $([0, 1]_2, \leq)$, where $[0, 1]_2 = \{0, 0.5, 1\}$ and $*\colon [0, 1]_2 \to [0, 1]_2$ the globalization truth-stressing $\nwarrow$-hedge. Given the set of attributes $A = \{a_1, a_2\}$, we consider the four attribute implications displayed in Table 3, and the whole set of fuzzy subsets of attributes with respect to $[0, 1]_2^A$, that is, $M_i : \{a_1, a_2\} \to [0, 1]_2$ with $i \in \{1, \ldots, 9\}$, which are defined as:

$$
\begin{aligned}
M_1 &= \{\} \\
M_2 &= \{0.5/a_1\} \\
M_3 &= \{1/a_1\} \\
M_4 &= \{0.5/a_2\} \\
M_5 &= \{1/a_2\} \\
M_6 &= \{0.5/a_1, 1/a_2\} \\
M_7 &= \{1/a_1, 0.5/a_2\} \\
M_8 &= \{0.5/a_1, 0.5/a_2\} \\
M_9 &= \{1/a_1, 1/a_2\}
\end{aligned}
$$

Table 4 collects the values $\mathcal{T}(I_i)$, for all $i \in \{1, 2, 3, 4\}$ and the validity of the implications in the mappings $M_i : \{a_1, a_2\} \to [0, 1]_2$ with $i \in \{1, \ldots, 9\}$, from which we can see what of these mappings are models of $\mathcal{T}$.

We conclude that set of models is $Mod(\mathcal{T}) = \{M_1, M_2, M_3, M_7, M_8, M_9\}$, since $\mathcal{T}(f_2 \Leftarrow f_1) \leq \|f_2 \Leftarrow f_1\|_{M_i}$, for all $f_1, f_2 \in [0, 1]_2^A$, for all M_i with $i \in \{1, 2, 3, 7, 8, 9\}$. □

The following definition allows us to establish a semantic entailment between an attribute implication and a theory, in order to know whether an implication is deduced or derived from others.

Definition 25 Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$, $\mathcal{T}$ be a theory and $f_1, f_2 \in L_1^A$. The degree in which $f_2 \Leftarrow f_1$ *semantically follows* from $\mathcal{T}$, denoted as $\| f_2 \Leftarrow f_1 \|_{\mathcal{T}}$, is defined as:

$$\| f_2 \Leftarrow f_1 \|_{\mathcal{T}} = \bigwedge_{M \in Mod(\mathcal{T})} \| f_2 \Leftarrow f_1 \|_M$$

Moreover, in the fuzzy framework, practical considerations often lead to work with fully true attribute implications, that is, attribute implications whose degree of validity is equal to 1. This class of attribute implications plays a important role in the residuated case, as it was remarked in Belohlávek and Vychodil (2016), due to the degree in which an attribute implication semantically follows from a theory $\mathcal{T}$ can be reduced in terms of them. As a consequence of Lemma 22, a similar result can be obtained in the multi-adjoint framework, as the following theorem shows.

Theorem 26 *If the conjunctor & of the adjoint triple* $(\&, \swarrow, \nwarrow)$ *is associative and* $x \& \top_1 = x$*, for all* $x \in L_1$*. Given a theory* $\mathcal{T}$ *and two subsets of attributes* $f_1, f_2 \in L_1^A$*, we have that:*

$$\| f_2 \Leftarrow f_1 \|_{\mathcal{T}} = \bigvee \{ c \in L_1 \mid \| f_2 \,\&\, c \Leftarrow f_1 \|_{\mathcal{T}} = \top_1 \}$$

Proof Given $f_1, f_2, f_3 \in L_1^A$ and $c \in L_1$, we have that:

$$\begin{aligned}
\| f_2 \Leftarrow f_1 \|_{\mathcal{T}} &\overset{(i)}{=} \bigwedge_{M \in Mod(\mathcal{T})} \| f_2 \Leftarrow f_1 \|_M \\
&\overset{(ii)}{=} \bigvee \{ c \in L_1 \mid c \preceq_1 \| f_2 \Leftarrow f_1 \|_M, \text{ for all } M \in Mod(\mathcal{T}) \} \\
&\overset{(iii)}{=} \bigvee \{ c \in L_1 \mid \| f_2 \,\&\, c \Leftarrow f_1 \|_M = \top_1, \text{ for all } M \in Mod(\mathcal{T}) \} \\
&\overset{(iv)}{=} \bigvee \{ c \in L_1 \mid \bigwedge_{M \in Mod(\mathcal{T})} \| f_2 \,\&\, c \Leftarrow f_1 \|_M = \top_1 \} \\
&\overset{(v)}{=} \bigvee \{ c \in L_1 \mid \| f_2 \,\&\, c \Leftarrow f_1 \|_{\mathcal{T}} = \top_1 \}
\end{aligned}$$

where (i) and (v) are obtained by Definition 25, (ii) is satisfied by the definition of infimum of a set as the greatest lower bound of the set, (iii) follows from Lemma 22(3), and (iv) holds by the infimum property. □

The following theorem provides an interesting result that allows us to express any given theory as a crisp theory, and therefore, with the same models.

Theorem 27 *Given a theory* $\mathcal{T}$*, two subsets of attributes* $f_1, f_2 \in L_1^A$ *and the crisp theory* $\mathcal{T}_{cr}$ *defined as:*

$$\mathcal{T}_{cr} = \{ f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \Leftarrow f_1 \mid f_1, f_2 \in L_1^A \text{ and } f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \neq f_{\perp_1} \}$$

where $f_{\perp_1} \colon A \to L_1$ *is the fuzzy subset defined as* $f_{\perp_1}(a) = \perp_1$*, for all* $a \in A$*. If the conjunctor & of the adjoint triple* $(\&, \swarrow, \nwarrow)$ *is associative and* $x \& \top_1 = x$*, for all* $x \in L_1$*, then we have:*

(1) $Mod(\mathcal{T}) = Mod(\mathcal{T}_{cr})$
(2) $\| f_2 \Leftarrow f_1 \|_{\mathcal{T}} = \| f_2 \Leftarrow f_1 \|_{\mathcal{T}_{cr}}$

Proof Given a theory $\mathcal{T}$, by Definition 23, we have that:

$$Mod(\mathcal{T}) = \{M \in L_1^A \mid \mathcal{T}(f_2 \Leftarrow f_1) \preceq_1 \|f_2 \Leftarrow f_1\|_M,\ \text{for all } f_1, f_2 \in L_1^A\} \quad (6)$$
$$Mod(\mathcal{T}_{\rm cr}) = \{M \in L_1^A \mid \|f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \Leftarrow f_1\|_M = \top_1,\ \text{for all } f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}\} \quad (7)$$

The equality $Mod(\mathcal{T}) = Mod(\mathcal{T}_{\rm cr})$ is deduced from the following chain of equivalences:

$$\begin{aligned} M \in Mod(\mathcal{T}) &\text{ if and only if } \mathcal{T}(f_2 \Leftarrow f_1) \preceq_1 \|f_2 \Leftarrow f_1\|_M \quad \text{(By Equation (6))}\\ &\text{ if and only if } \|f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \Leftarrow f_1\|_M = \top_1 \quad \text{(By Lemma 22(3))}\\ &\text{ if and only if } M \in Mod(\mathcal{T}_{\rm cr}) \quad \text{(By Equation (7))} \end{aligned}$$

As a consequence, applying Definition 25, we obtain that:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{T}} = \bigwedge_{M \in Mod(\mathcal{T})} \|f_2 \Leftarrow f_1\|_M = \bigwedge_{M \in Mod(\mathcal{T}_{\rm cr})} \|f_2 \Leftarrow f_1\|_M = \|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}}$$

and hence, we conclude that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}} = \|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}}$. □

As a direct consequence of the previous theorem, a similar result to the one presented in Theorem 26 can be obtained by means of crisp theories.

Corollary 28 *Given a theory $\mathcal{T}$, two subsets of attributes $f_1, f_2 \in L_1^A$ and the crisp theory $\mathcal{T}_{cr}$ defined as:*

$$\mathcal{T}_{cr} = \{f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \Leftarrow f_1 \mid f_1, f_2 \in L_1^A \text{ and } f_2 \,\&\, \mathcal{T}(f_2 \Leftarrow f_1) \neq f_{\perp_1}\}$$

where $f_{\perp_1}\colon A \to L_1$ is the fuzzy subset defined as $f_{\perp_1}(a) = \perp_1$, for all $a \in A$. If the conjunctor & of the adjoint triple $(\&, \swarrow, \nwarrow)$ is associative and $x\&\top_1 = x$, for all $x \in L_1$, then we have:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{T}} = \bigvee\{c \in L_1 \mid \|f_2 \,\&\, c \Leftarrow f_1\|_{\mathcal{T}_{cr}} = \top_1\}$$

Proof The identity follows directly from Theorems 26 and 27. □

Now, we will focus on two key properties for the attribute implications theories: completeness and non-redundancy. These properties are crucial for defining basis of attribute implications, which will be studied in detail later on.

Definition 29 A theory $\mathcal{T}$ is called *complete* if for every attribute implication $f_2 \Leftarrow f_1$ is satisfied that:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{T}} = \|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}$$

The completeness of a theory can be studied in terms of its models, as the following result shows.

Theorem 30 *Given a theory $\mathcal{T}$, if $Mod(\mathcal{T}) = Int(\mathcal{M}_*)$ then $\mathcal{T}$ is complete.*

R	a_1	a_2
b_1	0.5	1
b_2	0	0.5

σ	a_1	a_2
b_1	$\&_{DP}$	$\&_{DP}$
b_2	$\&_{DL}$	$\&_{DL}$

$Int(C_0) = \{0.5/a_2\}$
$Int(C_1) = \{0.5/a_1, 1/a_2\}$
$Int(C_2) = \{1/a_1, 1/a_2\}$

Fig. 2 Relation R, mapping σ, and intents of Example 31

Proof Let $\mathcal{T}$ be a theory such that $Mod(\mathcal{T}) = Int(\mathcal{M}_*)$. Taking into account Definition 29, we demonstrate that $\mathcal{T}$ is complete, from the following chain of equalities, for all $f_1, f_2 \in L_1^A$:

$$\begin{aligned}\|f_2 \Leftarrow f_1\|_{\mathcal{T}} &\overset{(i)}{=} \bigwedge_{M \in Mod(\mathcal{T})} \|f_2 \Leftarrow f_1\|_M \\ &\overset{(ii)}{=} \bigwedge_{M \in Int(\mathcal{M}_*)} \|f_2 \Leftarrow f_1\|_M \\ &\overset{(iii)}{=} \|f_2 \Leftarrow f_1\|_{Int(\mathcal{M}_*)} \\ &\overset{(iv)}{=} \|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}\end{aligned}$$

where (i) is obtained by Definition 25, (ii) is satisfied because $Mod(\mathcal{T}) = Int(\mathcal{M}_*)$, (iii) and (iv) hold by Definition 20. □

It is important to mention that, in contrast to the residuated case [Belohlávek and Vychodil (2016), Theorem 5.3], the above condition is sufficient but not necessary, i.e, the fact that a theory $\mathcal{T}$ is complete does not necessarily imply that the equality $Mod(\mathcal{T}) = Int(\mathcal{M}_*)$ holds, as the following example shows.

Example 31 Let $([0, 1]_2, \leq, (\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}}), (\&_{\mathrm{DL}}, \swarrow^{\mathrm{DL}}, \nwarrow_{\mathrm{DL}}))$ be a multi-adjoint frame where $(\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}})$ and $(\&_{\mathrm{DL}}, \swarrow^{\mathrm{DL}}, \nwarrow_{\mathrm{DL}})$ are the adjoint triples obtained from the discretization of the product and Łukasiewicz t-norms on $[0, 1]_2$, respectively, $(A_{*_A}, B_{*_B}, R, \sigma)$ be a context where the set of attributes is $A = \{a_1, a_2\}$, the set of objects is $B = \{b_1, b_2\}$, the relation $R: A \times B \to [0, 1]_2$ and the mapping σ are given in Fig. 2 (right side), $*_A$ is a family of identity truth-stressing $\swarrow$-hedges and $*_B$ is a family of identity truth-stressing $\nwarrow$-hedges.

The multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ related to this framework and the context is composed of 3 concepts, whose intents are displayed in Fig. 2 (left side).

Now, we consider an arbitrary complete theory $\mathcal{T}$ and we will see that the equality $Mod(\mathcal{T}) = Int(\mathcal{M}_*)$ is not satisfied in the considered framework. Given $M = \{1/a_2\} \notin Int(\mathcal{M}_*)$, we will show that $M \in Mod(\mathcal{T})$, in order to conclude that $Mod(\mathcal{T}) \nsubseteq Int(\mathcal{M}_*)$, and hence, $Mod(\mathcal{T}) \neq Int(\mathcal{M}_*)$ in this case. For this purpose, according to Definition 23, we need to check that the inequality $\mathcal{T}(f_2 \Leftarrow f_1) \leq \|f_2 \Leftarrow f_1\|_M$ holds, for all $f_1, f_2 \in [0, 1]_2^A$. By Definition 20, considering $(\&_{\mathrm{DP}}, \swarrow^{\mathrm{DP}}, \nwarrow_{\mathrm{DP}})$ as a extra triple with respect to $([0, 1]_2, \leq)$ and $*$ as the identity truth stressing hedge, this is equivalent to prove that $\mathcal{T}(f_2 \Leftarrow f_1) \leq S^1(f_2, M) \swarrow^{\mathrm{DP}} S^1(f_1, M)$. We will differentiate two possible cases regarding to f_1 and f_2:

- If $f_2 \leq f_1$, then the inequality $\mathcal{T}(f_2 \Leftarrow f_1) \leq S^1(f_2, M) \swarrow S^1(f_1, M)$ is always satisfied. This fact follows from the following chain of consequences:

$$
\begin{aligned}
f_2 \leq f_1 \text{ then } & (M \nwarrow_{\mathrm{DP}} f_1) \leq (M \nwarrow_{\mathrm{DP}} f_2) \quad \text{(By Proposition 2(1))} \\
\text{then } & \bigwedge_{a \in A} (M(a) \nwarrow_{\mathrm{DP}} f_1(a)) \leq \bigwedge_{a \in A} (M(a) \nwarrow_{\mathrm{DP}} f_2(a)) \quad \text{(By the infimum property)} \\
\text{then } & S^1(f_1, M) \leq S^1(f_2, M) \quad \text{(By Definition 15)} \\
\text{then } & S^1(f_2, M) \swarrow^{\mathrm{DP}} S^1(f_1, M) = 1 \quad \text{(By Proposition 3(1))} \\
\text{then } & \mathcal{T}(f_2 \Leftarrow f_1) \leq S^1(f_2, M) \swarrow^{\mathrm{DP}} S^1(f_1, M) \quad \text{(Due to } \mathcal{T}(f_2 \Leftarrow f_1) \in [0, 1]_2)
\end{aligned}
$$

- If $f_1 < f_2$ or f_1 and f_2 are incomparable, then we need to compute the corresponding degrees of validity presented in Table 5. Notice that, the value $\|f_2 \Leftarrow f_1\|_{\mathcal{T}}$ corresponds to the degree of validity $\|f_2 \Leftarrow f_1\|_{\mathcal{M}_*}$, since $\mathcal{T}$ is complete. Considering the computed data presented in Table 5, we can see that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}} \leq \|f_2 \Leftarrow f_1\|_M$ holds. Moreover, due to the inequality $\mathcal{T}(f_2 \Leftarrow f_1) \leq \|f_2 \Leftarrow f_1\|_{\mathcal{T}}$ always holds, we obtain that $\mathcal{T}(f_2 \Leftarrow f_1) \leq \|f_2 \Leftarrow f_1\|_M$ is also satisfied in this case.

Then, we have shown that $\mathcal{T}(f_2 \Leftarrow f_1) \leq \|f_2 \Leftarrow f_1\|_M$ for all $f_1, f_2 \in [0, 1]_2^A$. According to Definition 23, this fact implies that $M \in Mod(\mathcal{T})$. Due to $M \notin Int(\mathcal{M}_*)$, we can conclude that $Mod(\mathcal{T}) \nsubseteq Int(\mathcal{M}_*)$, and hence, $Mod(\mathcal{T}) \neq Int(\mathcal{M}_*)$. □

As a consequence of the above example, we have that the completeness condition of a theory $\mathcal{T}$ does not guarantee that $Mod(\mathcal{T}) \subseteq Int(\mathcal{M}_*)$ is fulfilled. However, the following theorem shows that the other inclusion $Int(\mathcal{M}_*) \subseteq Mod(\mathcal{T})$ always holds.

Theorem 32 *Given a theory $\mathcal{T}$, if $\mathcal{T}$ is complete then $Int(\mathcal{M}_*) \subseteq Mod(\mathcal{T})$.*

Proof Let $M \in Int(\mathcal{M}_*)$. In order to conclude that $M \in Mod(\mathcal{T})$, applying Definition 23, we need to prove that $\mathcal{T}(f_2 \Leftarrow f_1) \preceq_1 \|f_2 \Leftarrow f_1\|_M$ is satisfied, for all $f_1, f_2 \in L_1^A$. This fact is deduced from the following chain of inequalities:

$$
\begin{aligned}
\mathcal{T}(f_2 \Leftarrow f_1) & \overset{(i)}{\preceq_1} \bigwedge_{M' \in Mod(\mathcal{T})} \|f_2 \Leftarrow f_1\|_{M'} \\
& \overset{(ii)}{=} \|f_2 \Leftarrow f_1\|_{\mathcal{T}} \\
& \overset{(iii)}{=} \|f_2 \Leftarrow f_1\|_{Int(\mathcal{M}_*)} \\
& \overset{(iv)}{=} \bigwedge_{M'' \in Int(\mathcal{M}_*)} \|f_2 \Leftarrow f_1\|_{M''} \\
& \overset{(v)}{\preceq_1} \|f_2 \Leftarrow f_1\|_M
\end{aligned}
$$

where (i) is deduced from Definition 23 and the infimum property, (ii) is satisfied by Definition 25, (iii) is obtained from Definition 29, (iv) holds by Definition 20, and (v) follows from the infimum property and due to $M \in Int(\mathcal{M}_*)$. □

After presenting the notion of completeness for a theory of attribute implications, and discussing sufficient conditions that ensure it, we will focus on studying the non-redundancy property.

From now on, in order to introduce formally the definition of non-redundancy, we will consider $\mathcal{T}$ as a crisp theory. This fact can be assumed without loss of generality, due to the results established in Theorem 27.

Table 5 Computed degrees of validity of Example 31

f_1	f_2	$\|f_2 \Leftarrow f_1\|_{\mathcal{T}}$	$S^1(f_1, M)$	$S^1(f_2, M)$	$\|f_2 \Leftarrow f_1\|_M$
$\{0.5/a_2\}$	$\{1/a_1, 0.5/a_2\}$	0	1	0	0
$\{0.5/a_2\}$	$\{1/a_1, 1/a_2\}$	0	1	0	0
$\{0.5/a_2\}$	$\{0.5/a_1, 0.5/a_2\}$	0	1	0	0
$\{0.5/a_2\}$	$\{0.5/a_1, 1/a_2\}$	0	1	0	0
$\{0.5/a_2\}$	$\{1/a_1\}$	0	1	0	0
$\{0.5/a_2\}$	$\{0.5/a_1\}$	0	1	0	0
$\{1/a_2\}$	$\{1/a_1, 1/a_2\}$	0	1	0	0
$\{1/a_2\}$	$\{0.5/a_1, 1/a_2\}$	0	1	0	0
$\{1/a_2\}$	$\{1/a_1, 0.5/a_2\}$	0	1	0	0
$\{1/a_2\}$	$\{1/a_1\}$	0	1	0	0
$\{1/a_2\}$	$\{0.5/a_1\}$	0	1	0	0
$\{1/a_2\}$	$\{0.5/a_1, 0.5/a_2\}$	0	1	0	0
$\{0.5/a_1\}$	$\{1/a_1, 0.5/a_2\}$	0.5	0	0	1
$\{0.5/a_1\}$	$\{1/a_1, 1/a_2\}$	0.5	0	0	1
$\{0.5/a_1\}$	$\{0.5/a_1 .5/a_2\}$	1	0	0	1
$\{0.5/a_1\}$	$\{0.5/a_1, 1/a_2\}$	1	0	0	1
$\{0.5/a_1\}$	$\{0.5/a_2\}$	1	0	1	1
$\{0.5/a_1\}$	$\{1/a_2\}$	1	0	1	1
$\{0.5/a_1, 0.5/a_2\}$	$\{1/a_1, 1/a_2\}$	0.5	0	0	1
$\{0.5/a_1, 0.5/a_2\}$	$\{1/a_1\}$	0.5	0	0	1
$\{0.5/a_1, 0.5/a_2\}$	$\{1/a_2\}$	1	0	1	1
$\{0.5/a_1, 1/a_2\}$	$\{1/a_1, 1/a_2\}$	0.5	0	0	1
$\{0.5/a_1, 1/a_2\}$	$\{1/a_1\}$	0.5	0	0	1
$\{0.5/a_1, 1/a_2\}$	$\{1/a_1, 0.5/a_2\}$	0.5	0	0	1
$\{1/a_1\}$	$\{1/a_1, 0.5/a_2\}$	1	0	0	1
$\{1/a_1\}$	$\{1/a_1, 1/a_2\}$	1	0	0	1
$\{1/a_1\}$	$\{1/a_2\}$	1	0	1	1
$\{1/a_1\}$	$\{0.5/a_2\}$	1	0	1	1
$\{1/a_1\}$	$\{0.5/a_1, 1/a_2\}$	1	0	0	1
$\{1/a_1\}$	$\{0.5/a_1, 0.5/a_2\}$	1	0	0	1
$\{1/a_1, 0.5/a_2\}$	$\{1/a_1, 1/a_2\}$	1	0	0	1
$\{1/a_1, 0.5/a_2\}$	$\{1/a_2\}$	1	0	1	1
$\{1/a_1, 0.5/a_2\}$	$\{0.5/a_1, 1/a_2\}$	1	0	0	1

Definition 33 A crisp theory $\mathcal{T}_{\text{cr}}$ is called non-redundant if for every implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\text{cr}}$ it is satisfied that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\}} \neq \top_1$.

It is convenient to pay special attention to the notation $\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\}$, which means that the implication $f_2 \Leftarrow f_1$ is removed from the set $\mathcal{T}_{\text{cr}}$.

The above definition can be characterized by the following lemma.

Lemma 34 *Given a theory $\mathcal{T}_{cr}$, the following conditions are equivalent:*

(1) $\mathcal{T}_{cr}$ is a non-redundant set of implications.

(2) $Mod(\mathcal{T}_{cr}) \subset Mod(\mathcal{T}_{cr} \setminus \{f_2 \Leftarrow f_1\})$, *for every implication* $f_2 \Leftarrow f_1 \in \mathcal{T}_{cr}$.

(3) For every implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{cr}$ *there exists* $f_4 \Leftarrow f_3$ *such that* $\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{cr}\setminus\{f_2\Leftarrow f_1\}} \prec_1 \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{cr}}$.

Proof First of all, we will prove that (1) implies (2). Since $Mod(\mathcal{T}_{\rm cr}) \subseteq Mod(\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\})$ it only remains to prove that $Mod(\mathcal{T}_{\rm cr}) \neq Mod(\mathcal{T}_{\rm cr} \setminus \{f_2 \Leftarrow f_1\})$. For this purpose, we will suppose the equality $Mod(\mathcal{T}_{\rm cr}) = Mod(\mathcal{T}_{\rm cr} \setminus \{f_2 \Leftarrow f_1\})$ holds and demonstrate that it leads to a contradiction. Indeed, if $Mod(\mathcal{T}_{\rm cr}) = Mod(\mathcal{T}_{\rm cr} \setminus \{f_2 \Leftarrow f_1\})$, then the following chain of equivalences is satisfied for all $f_1, f_2 \in L_1^A$:

$$\begin{aligned}
\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\}} &\overset{(i)}{=} \bigwedge_{M\in Mod(\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\})} \|f_2 \Leftarrow f_1\|_M \\
&\overset{(ii)}{=} \|f_2 \Leftarrow f_1\|_{Mod(\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\})} \\
&\overset{(iii)}{=} \|f_2 \Leftarrow f_1\|_{Mod(\mathcal{T}_{\rm cr})} \\
&\overset{(iv)}{=} \bigwedge_{M\in Mod(\mathcal{T}_{\rm cr})} \|f_2 \Leftarrow f_1\|_M \\
&\overset{(v)}{=} \|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}}
\end{aligned}$$

where (i) and (v) holds by Definition 25, (ii) and (iv) are satisfied by Definition 20, and (iii) is obtained due to $Mod(\mathcal{T}_{\rm cr}) = Mod(\mathcal{T}_{\rm cr} \setminus \{f_2 \Leftarrow f_1\})$.

In particular, if $f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}$, we have that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}} = \top_1$, by Definition 23. As a consequence, we obtain that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\}} = \top_1$, which contradicts the initial hypothesis about the non-redundancy of the set $\mathcal{T}_{\rm cr}$.

Secondly, we will see that (2) implies (3). Given a attribute implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}$, we can assume that there exists $M \in Mod(\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\})$ such that $M \notin Mod(\mathcal{T}_{\rm cr})$, which implies in particular that $\|f_2 \Leftarrow f_1\|_M \prec_1 \top_1$. Therefore, if we suppose that $(f_4 \Leftarrow f_3) = (f_2 \Leftarrow f_1)$, we obtain the desired inequality:

$$\begin{aligned}
\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\}} &\overset{(i)}{=} \|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\}} \\
&\overset{(ii)}{=} \bigwedge_{M'\in Mod(\mathcal{T}_{\rm cr}\setminus\{f_2\Leftarrow f_1\})} \|f_2 \Leftarrow f_1\|_{M'} \\
&\overset{(iii)}{\preceq_1} \|f_2 \Leftarrow f_1\|_M \\
&\overset{(iv)}{\prec_1} \top_1 \\
&\overset{(v)}{=} \|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\rm cr}} \\
&\overset{(vi)}{=} \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}}
\end{aligned}$$

where (i) and (vi) are obtained because we have considered that $(f_4 \Leftarrow f_3) = (f_2 \Leftarrow f_1)$, (ii) holds by Definition 25, (iii) follows from the infimum property and since $M \in Mod(\mathcal{T}_{\rm cr}\setminus \{f_2 \Leftarrow f_1\})$, (iv) is obtained due to the previous strict inequality previously obtained because $M \notin Mod(\mathcal{T}_{\rm cr})$, and (v) is satisfied because $f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}$.

Finally, we will prove that (3) implies (1). For this purpose, we will assume that the theory $\mathcal{T}_{\rm cr}$ is redundant, and we will show that this assumption leads to a contradiction, thereby proving the non-redundancy of $\mathcal{T}_{\rm cr}$. Then, there exists an implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}$ such

Table 6 Attribute implications of the theory $\mathcal{T}_{\text{cr}}$ given in Example 35

$f_2 \Leftarrow f_1$
$\{0.25/a_1\} \Leftarrow \{1/a_1, 0.5/a_2\}$
$\{1/a_1\} \Leftarrow \{1/a_2\}$
$\{0.5/a_2\} \Leftarrow \{0.5/a_1, 1/a_2\}$
$\{0.25/a_1, 0.25/a_2\} \Leftarrow \{0.5/a_2\}$

Table 7 Computed degrees of validity of the attribute implications of $\mathcal{T}_{\text{cr}}$ given in Example 35

$\|f_2 \Leftarrow f_1\|_{M_i}$ with $i \in \{1,2,3,7,8,9\}$	$\|f_2 \Leftarrow f_1\|_{M_4}$	$\|f_2 \Leftarrow f_1\|_{M_5}$	$\|f_2 \Leftarrow f_1\|_{M_6}$
1	1	1	1
1	1	0	0.5
1	1	1	1
1	0	0	1

that $\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\}} = \top_1$, which, by Definition 25, is equivalent to:

$$\bigwedge_{M\in Mod(\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\})} \|f_2 \Leftarrow f_1\|_M = \top_1$$

Then, given $M \in Mod(\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\})$ we have that $\|f_2 \Leftarrow f_1\|_M = \top_1$. In addition, taking into account that $\|f' \Leftarrow f\|_M = \top_1$ is also verified, for every attribute implication $f' \Leftarrow f \in \mathcal{T}_{\text{cr}}\setminus\{f_2 \Leftarrow f_1\}$, we can deduce that $M \in Mod(\mathcal{T}_{\text{cr}})$. Consequently, we obtain that $Mod(\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\}) \subseteq Mod(\mathcal{T}_{\text{cr}})$. Now, due the inclusion $Mod(\mathcal{T}_{\text{cr}}) \subseteq Mod(\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\})$ holds, we can deduce that $Mod(\mathcal{T}_{\text{cr}}) = Mod(\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\})$. As a result, considering Definition 25, we have that the equality $\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\}} = \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\text{cr}}}$ holds for all $f_3, f_4 \in L_1^A$, which contradicts the condition presented in (3) that we have assumed by hypothesis. Therefore, we conclude that $\mathcal{T}_{\text{cr}}$ has to be a non-redundant theory. □

This section finishes with an example of a redundant theory of attribute implications.

Example 35 Let $(\&_{\text{DP}}, \swarrow^{\text{DP}}, \nwarrow_{\text{DP}})$ be the adjoint triple defined from the discretization of the product t-norm on $([0,1]_2, \leq)$, where $[0,1]_2 = \{0, 0.5, 1\}$ and $*\colon [0,1]_2 \to [0,1]_2$ is the globalization truth-stressing $\nwarrow$-hedge. Given the set of attributes $A = \{a_1, a_2\}$, we consider a crisp theory $\mathcal{T}_{\text{cr}}$ composed of the four attribute implications displayed in Table 6.

Now, we will prove that the theory $\mathcal{T}_{\text{cr}}$ is redundant. For this purpose, we will consider the attribute implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\text{cr}}$, with $f_1 = \{1/a_1, 0.5/a_2\}$ and $f_2 = \{0.25/a_1\}$, and we will compute the degree of validity $\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\}}$, in order to show that the condition presented in Definition 33 is not satisfied. Applying Definition 25, we have that:

$$\|f_2 \Leftarrow f_1\|_{\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\}} = \bigwedge_{M\in Mod(\mathcal{T}_{\text{cr}}\setminus\{f_2\Leftarrow f_1\})} \|f_2 \Leftarrow f_1\|_M$$

In this case, taking into account the data presented in Table 7 and Definition 23, we can deduce that:

$$Mod(\mathcal{T}_{\text{cr}} \setminus \{f_2 \Leftarrow f_1\}) = \{M_1, M_2, M_3, M_7, M_8, M_9\}$$

Then, considering the degrees of validity of the attribute implication $f_2 \Leftarrow f_1$ in each of the above models, given in Table 7, we obtain that:

$$\| f_2 \Leftarrow f_1 \|_{\mathcal{T}_{\mathrm{cr}} \setminus \{f_2 \Leftarrow f_1\}} = \inf\{1, 1, 1, 1, 1, 1\} = 1$$

Consequently, attending to Definition 33, we can conclude that $\mathcal{T}_{\mathrm{cr}}$ is a redundant theory. □

5 Bases of fuzzy attribute implications

The main objective of this section is to present an extension of the notion of a basis of attribute implications to the multi-adjoint framework, establishing the corresponding results for this purpose. In particular, we focus on the most usual basis considered in FCA, such as in Belohlávek and Vychodil (2016), Ganter and Wille (1999), which is composed of the implications given by pseudo-intents. This basis is usually called Duquenne-Guigues-basis, although the bases proposed by Guigues and Duquenne were slightly different, see Guigues and Duquenne (1986), Dubois et al. (2025) for more detail.

From a complete theory we can define the notion of a basis of attribute implications.

Definition 36 Given a theory $\mathcal{T}_{\mathrm{cr}}$ is called a basis of $(\mathcal{M}_*, \preceq)$ if it is complete and no proper subset of $\mathcal{T}_{\mathrm{cr}}$ is complete.

In other words, a basis refers to a set of attribute implications that must satisfy two primary conditions: completeness and non-redundancy. Completeness ensures that all other implications can be derived from this set, providing comprehensive information. Non-redundancy means that the implications within the bases cannot be deduced from each other. Therefore, a basis of attribute implications is just a complete and non-redundant theory, as the following result shows.

Theorem 37 *A theory $\mathcal{T}_{cr}$ is a basis of $(\mathcal{M}_*, \preceq)$ if and only if the following two conditions are satisfied:*

(1) $\mathcal{T}_{cr}$ is complete.
(2) $\mathcal{T}_{cr}$ is non-redundant.

Proof Firstly, we will prove that if $\mathcal{T}_{\mathrm{cr}}$ satisfies conditions (1) and (2) then it is a basis of $(\mathcal{M}_*, \preceq)$. For this purpose, as we suppose that $\mathcal{T}_{\mathrm{cr}}$ is complete, considering Definition 36, we only need to prove that no proper subset of $\mathcal{T}_{\mathrm{cr}}$ is complete. Indeed, due to the set $\mathcal{T}_{\mathrm{cr}}$ is also non-redundant, applying Lemma 34, we have that for every implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\mathrm{cr}}$ there exists an implication $f_4 \Leftarrow f_3$ such that $\| f_4 \Leftarrow f_3 \|_{\mathcal{T}_{\mathrm{cr}} \setminus \{f_2 \Leftarrow f_1\}} \prec_1 \| f_4 \Leftarrow f_3 \|_{\mathcal{T}_{\mathrm{cr}}}$. Moreover, taking into account that the set $\mathcal{T}_{\mathrm{cr}}$ is complete in $(\mathcal{M}_*, \preceq)$, applying Definition 29, we have that $\| f_4 \Leftarrow f_3 \|_{\mathcal{T}_{\mathrm{cr}}} = \| f_4 \Leftarrow f_3 \|_{\mathcal{M}_*}$, which implies that $\| f_4 \Leftarrow f_3 \|_{\mathcal{T}_{\mathrm{cr}} \setminus \{f_2 \Leftarrow f_1\}} \neq | f_4 \Leftarrow f_3 \|_{\mathcal{M}_*}$, for all $f_3, f_4 \in L_1^A$. As a result, we have that the proper subset $\mathcal{T}_{\mathrm{cr}} \setminus \{f_2 \Leftarrow f_1\}$ is not complete, for all $f_2 \Leftarrow f_1 \in \mathcal{T}_{\mathrm{cr}}$. Therefore, according to Definition 36, we can conclude that $\mathcal{T}_{\mathrm{cr}}$ is a basis of $(\mathcal{M}_*, \preceq)$.

Secondly, we will prove that if $\mathcal{T}_{\mathrm{cr}}$ is a basis, then conditions (1) and (2) are satisfied. Let $\mathcal{T}_{\mathrm{cr}}$ be a basis of $(\mathcal{M}_*, \preceq)$, by Definition 36, this is equivalent to $\mathcal{T}_{\mathrm{cr}}$ is complete and no proper subset of $\mathcal{T}_{\mathrm{cr}}$ is complete. Then, condition (1) is verified. Now, we will prove condition (2) is also satisfied. Given $f_2 \Leftarrow f_1 \in \mathcal{T}_{\mathrm{cr}}$, since no proper subset of $\mathcal{T}_{\mathrm{cr}}$ is complete in $(\mathcal{M}_*, \preceq)$, the subset $\mathcal{T}_{\mathrm{cr}} \setminus \{f_2 \Leftarrow f_1\}$ cannot be complete. As a consequence, considering Definition 29, there exists an implication $f_4 \Leftarrow f_3$ such that:

$$\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\}} \neq \|f_4 \Leftarrow f_3\|_{\mathcal{M}_*} = \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}}$$

i.e, $\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\}} \neq \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}}$. Moreover, since $Mod(\mathcal{T}_{\rm cr}) \subseteq Mod(\mathcal{T}_{\rm cr} \setminus \{f_2 \Leftarrow f_1\})$ holds, applying Definition 25 and the infimum property, we have that:

$$\begin{aligned}
\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\}} &= \bigwedge_{M \in Mod(\mathcal{T}_{\rm cr}\setminus\{f_4 \Leftarrow f_3\})} \|f_4 \Leftarrow f_3\|_M \\
&\prec_1 \bigwedge_{M \in Mod(\mathcal{T}_{\rm cr})} \|f_4 \Leftarrow f_3\|_M \\
&= \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}}
\end{aligned}$$

Hence, we have proved that for every attribute implication $f_2 \Leftarrow f_1 \in \mathcal{T}_{\rm cr}$ there exists $f_4 \Leftarrow f_3$ such that $\|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}\setminus\{f_2 \Leftarrow f_1\}} \prec_1 \|f_4 \Leftarrow f_3\|_{\mathcal{T}_{\rm cr}}$, which, by Lemma 34, implies that the set $\mathcal{T}_{\rm cr}$ is non-redundant. □

As we mentioned at the beginning of the section, we are interested in defining the Duquenne-Guigues basis in the multi-adjoint framework with truth-stressing hedges. This basis depends strongly on the notion of pseudo-intents (Ganter 2010; Ganter and Wille 1999). The definition of pseudo-intents presented in Ganter and Wille (1999) demands that the composition of concept-forming operators be a closure operator (Davey and Priestley 2002). Specifically, it is required that the property $f \preceq_1 f^{\downarrow^*\uparrow_*}$ is satisfied for all $f \in L_1^A$. In the multi-adjoint framework with truth-stressing hedges, by Proposition 12(2), we have that this property is not verified in general and hence, ${}^{\downarrow^*\uparrow_*}$ is not a closure operator in general. However, we can guarantee that ${}^{\downarrow^*\uparrow_*}$ is a closure operator when the identity is only considered in the family $*_A$. This fact is shown in the following proposition.

Proposition 38 *If the conjunctor & of the adjoint triple* $(\&, \swarrow, \nwarrow)$ *satisfies* $x \,\&\, \top_1 = x$ *and* $\top_1 \,\&\, y = y$, *for all* $x, y \in L_1$. *If the truth-stressing* $\swarrow$*-hedges in* $*_A$ *are only the identity then* ${}^{\downarrow\uparrow_*}$ *is a closure operator, that is, the following properties are satisfied, for all* $f, f_1, f_2 \in L_1^A$:

(1) $f \preceq_1 f^{\downarrow\uparrow_*}$
(2) If $f_1 \preceq_1 f_2$ *then* $f_1^{\downarrow\uparrow_*} \preceq_1 f_2^{\downarrow\uparrow_*}$
(3) $f^{\downarrow\uparrow_*} = (f^{\downarrow\uparrow_*})^{\downarrow\uparrow_*}$

Proof The proof follows directly from Proposition 12. □

Therefore, although in this case the concept-forming operators does not form a Galois connection, the composition ${}^{\downarrow\uparrow_*}$ provides a closure operator, which will also be fundamental to compute an implicational base.

In order to introduce formally the notion of pseudointents, from now on, we will consider that:

$$\begin{aligned}
*_A &= \{*_a \colon L_1 \to L_1 \mid *_a \text{ is the identity mapping, for all } a \in A\} \\
*_B &= \{*_b \colon L_2 \to L_2 \mid *_b \text{ is a truth-stressing } \nwarrow\text{-hedge, for all } b \in B\}
\end{aligned}$$

We will write (A, B_*, R, σ) instead of $(A_{*_A}, B_{*_B}, R, \sigma)$ to simplify the notation.

Definition 39 A system $\mathcal{P} \subseteq L_1^A$ is said to be a *system of pseudointents* if, given $P \in L_1^A$, we have that:

$$P \in \mathcal{P} \text{ if and only if } P \neq P^{\downarrow\uparrow_*} \text{ and } \|Q^{\downarrow\uparrow_*} \Leftarrow Q\|_P = \top_1, \text{ for all } Q \in \mathcal{P}, Q \neq P$$

Next, we present an example of systems of pseudo-intents for a given multi-adjoint framework and contexts.

Example 40 Consider the multi-adjoint frame and the context of Example 31, the external adjoint triple $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$ on $[0,1]_2$, the globalization truth-stressing $\nwarrow$-hedge $*\colon [0,1]_2 \to [0,1]_2$, and the whole set of not empty fuzzy subsets of attributes with respect to $[0,1]_2^A$ such that are not intents, that is, $P_i\colon \{a_1, a_2\} \to [0,1]_2$ with $i \in \{0, \ldots, 5\}$, which are defined as:

$$\begin{aligned} P_0 &= \{\} \\ P_1 &= \{0.5/a_1\} \\ P_2 &= \{1/a_1\} \\ P_3 &= \{1/a_2\} \\ P_4 &= \{1/a_1, 0.5/a_2\} \\ P_5 &= \{0.5/a_1, 0.5/a_2\} \end{aligned}$$

Specifically, the closure of each of the above fuzzy subsets of attributes is given by:

$$P_0^{\downarrow\uparrow} = \{0.5/a_2\}$$
$$P_1^{\downarrow\uparrow} = P_3^{\downarrow\uparrow} = P_5^{\downarrow\uparrow} = \{0.5/a_1, 1/a_2\}$$
$$P_2^{\downarrow\uparrow} = P_4^{\downarrow\uparrow} = \{1/a_1, 1/a_2\}$$

Due to P_0 satisfies all the attribute implications, it must be in every system of pseudo-intents. Hence, by definition and the results in Table 8, P_1 and P_2 cannot be in a system of pseudo-intents. According to Definition 39, the only system of pseudo-intents is $\mathcal{P} = \{P_0, P_3, P_5\}$. Clearly, we have that $P_3 \neq P_3^{\downarrow\uparrow}$ and $P_5 \neq P_5^{\downarrow\uparrow}$. Moreover, taking into account the data presented in Table 8, we obtain that:

$$\|P_3^{\downarrow\uparrow} \Leftarrow P_3\|_{P_5} = \|P_5^{\downarrow\uparrow} \Leftarrow P_5\|_{P_3} = 1$$

Furthermore, no exits P_i, with $i \in \{1, 2, 4\}$, such that

$$\|P_0^{\downarrow\uparrow} \Leftarrow P_0\|_{P_i} = \|P_3^{\downarrow\uparrow} \Leftarrow P_3\|_{P_i} = \|P_5^{\downarrow\uparrow} \Leftarrow P_5\|_{P_i} = 1$$

For example, $\mathcal{Q} = \{P_0, P_3, P_4\}$ is not a system of pseudo-intents because although $P_3 \neq P_3^{\downarrow\uparrow}$ and $P_4 \neq P_4^{\downarrow\uparrow}$, and

$$\|P_3^{\downarrow\uparrow} \Leftarrow P_3\|_{P_4} = \|P_4^{\downarrow\uparrow} \Leftarrow P_4\|_{P_3} = 1$$

we have that: $\|P_0^{\downarrow\uparrow} \Leftarrow P_0\|_{P_5} = \|P_3^{\downarrow\uparrow} \Leftarrow P_3\|_{P_5} = \|P_4^{\downarrow\uparrow} \Leftarrow P_4\|_{P_5} = 1$, which implies by Definition 39 that P_5 must be included in $\mathcal{Q}$, but in this case we obtain $\|P_5^{\downarrow\uparrow} \Leftarrow P_5\|_{P_4} \neq 1$. Thus, $\mathcal{Q} = \{P_0, P_3, P_4\}$ cannot be a system of pseudo-intents. □

The importance of the notion of a system of pseudo-intents derives in that it has a direct application to extract bases of attribute implications, being key to define the attribute implications of a base extending the crisp Duquenne-Guigues basis to the fuzzy setting. Before introducing this result, a technical lemma is presented.

Lemma 41 *If the conjunctor* & *of the adjoint triple* $(\&, \swarrow, \nwarrow)$ *is associative,* $x \,\&\, \top_1 = x$ *and* $\top_1 \,\&\, y = y$*, for all* $x, y \in L_1$*, and* $\swarrow$ *is a forcing-implication. Given a multi-adjoint concept lattice* $(\mathcal{M}_*, \preceq)$*, then the theory defined as* $\mathcal{T}_{\mathcal{P}} = \{P^{\downarrow\uparrow_*} \Leftarrow P \mid P \in \mathcal{P}\}$ *verifies that* $Mod(\mathcal{T}_{\mathcal{P}}) = Int(\mathcal{M}_*)$.

Table 8 Computed degrees of validity of Example 40

f_1	f_2	$\|f_2 \Leftarrow f_1\|_{P_0}$	$\|f_2 \Leftarrow f_1\|_{P_1}$	$\|f_2 \Leftarrow f_1\|_{P_2}$	$\|f_2 \Leftarrow f_1\|_{P_3}$	$\|f_2 \Leftarrow f_1\|_{P_4}$	$\|f_2 \Leftarrow f_1\|_{P_5}$
P_0	$P_0^{\downarrow\uparrow}$	0	0	0	1	1	1
P_1	$P_1^{\downarrow\uparrow}$	1	0	0	1	0.5	0.5
P_2	$P_2^{\downarrow\uparrow}$	1	1	0	1	0.5	1
P_3	$P_3^{\downarrow\uparrow}$	1	1	1	0	1	1
P_4	$P_4^{\downarrow\uparrow}$	1	1	1	1	0.5	1
P_5	$P_5^{\downarrow\uparrow}$	1	1	1	1	0.5	0.5

Proof First of all, we assume that $Mod(\mathcal{T}_\mathcal{P}) \nsubseteq Int(\mathcal{M}_*)$ which will lead us to a contradiction. As $Mod(\mathcal{T}_\mathcal{P}) \nsubseteq Int(\mathcal{M}_*)$, there exists $M \in Mod(\mathcal{T}_\mathcal{P})$ such that $M \notin Int(\mathcal{T}_\mathcal{P})$. Then:

- As $M \in Mod(\mathcal{T}_\mathcal{P})$, we have that $\|Q^{\downarrow\uparrow_*} \Leftarrow Q\|_M = \top_1$, for all $Q \in \mathcal{P}$. By Definition 39, this fact implies that $M \in \mathcal{P}$, that is, M is also a pseudo-intent. As a consequence, we have that $M^{\downarrow\uparrow_*} \Leftarrow M \in \mathcal{T}_\mathcal{P}$.
- As $M \notin Int(\mathcal{M}_*)$, we have that $M \neq M^{\downarrow\uparrow_*}$. Taking into account that the inequality $M \preceq_1 M^{\downarrow\uparrow_*}$ is fulfilled by Proposition 12(2), we can deduce that $M^{\downarrow\uparrow_*} \npreceq_1 M$. As a result, applying Proposition 19, we have that $S^1(M^{\downarrow\uparrow_*}, M) \neq \top_1$. From this fact and taking into account that $z \swarrow \top_1 = z$ for all $z \in L_1$, by Proposition 3(3), we obtain for the attribute implication $M^{\downarrow\uparrow_*} \Leftarrow M \in \mathcal{T}_\mathcal{P}$ that:

$$\begin{aligned}\|M^{\downarrow\uparrow_*} \Leftarrow M\|_M &= S^1(M^{\downarrow\uparrow_*}, M) \swarrow *(S^1(M, M))\\ &= S^1(M^{\downarrow\uparrow_*}, M) \swarrow \top_1\\ &= S^1(M^{\downarrow\uparrow_*}, M)\\ &\neq \top_1\end{aligned}$$

This chain of equalities contradicts that $M \in Mod(\mathcal{T}_\mathcal{P})$. Hence, $Mod(\mathcal{T}_\mathcal{P}) \subseteq Int(\mathcal{M}_*)$.

On the other hand, we will prove that the inclusion $Int(\mathcal{M}_*) \subseteq Mod(\mathcal{T}_\mathcal{P})$ is also satisfied. Given $M \in Int(\mathcal{M}_*)$, we need to deduce that $\|P^{\downarrow\uparrow_*} \Leftarrow P\|_M = \top_1$, for all $P \in \mathcal{P}$. By Definition 20, this is equivalent to prove that $S^1(P^{\downarrow\uparrow_*}, M) \swarrow *(S^1(P, M)) = \top_1$. For this purpose, by Proposition 3(1), we only need to see that $*(S^1(P, M)) \preceq_1 S^1(P^{\downarrow\uparrow_*}, M)$. This fact follows from the following chain of inequalities:

$$\begin{aligned}*(S^1(P, M)) &\overset{(i)}{\preceq}_1 *(S^2(M^{\downarrow}, P^{\downarrow}))\\ &\overset{(ii)}{\preceq}_1 S^2(*(M^{\downarrow}), *(P^{\downarrow}))\\ &\overset{(iii)}{\preceq}_1 S^1((P^{\downarrow})^{\uparrow_*}, (M^{\downarrow})^{\uparrow_*})\\ &\overset{(iv)}{=} S^1(P^{\downarrow\uparrow_*}, M)\end{aligned}$$

where (i) holds by Lemma 16(1), (ii) by Proposition 18(1), (iii) by Corollary 17(2), and the idempotency condition in Definition 7, and (iv) due to $M \in Int(\mathcal{M}_*)$.

Thus, we have proven that the equality $Mod(\mathcal{T}_\mathcal{P}) = Int(\mathcal{M}_*)$ holds. □

Next, we prove that every system of pseudo-intents provides a base of the given context.

Theorem 42 *If the conjunctor & of the adjoint triple* $(\&, \swarrow, \nwarrow)$ *is associative,* $x \,\&\, \top_1 = x$ *and* $\top_1 \,\&\, y = y$*, for all* $x, y \in L_1$*, and* $\swarrow$ *is a forcing-implication. Given a multi-adjoint concept lattice* $(\mathcal{M}_*, \preceq)$*, if* $\mathcal{P}$ *is a system of pseudo-intents of* $(\mathcal{M}_*, \preceq)$*, then the theory defined as:*

$$\mathcal{T}_{\mathcal{P}} = \{P^{\downarrow\uparrow_*} \Leftarrow P \mid P \in \mathcal{P}\}$$

is a basis of $(\mathcal{M}_*, \preceq)$.

Proof In order to prove that $\mathcal{T}_{\mathcal{P}}$ is a basis of $(\mathcal{M}_*, \preceq)$, we need to demonstrate that $\mathcal{T}_{\mathcal{P}}$ is a complete and no proper subset of $\mathcal{T}_{\mathcal{P}}$ is complete.

Completeness of $\mathcal{T}_{\mathcal{P}}$. Taking into account Lemma 41 and Theorem 30, we conclude that $\mathcal{T}_{\mathcal{P}}$ is complete.

Non redundancy of $\mathcal{T}_{\mathcal{P}}$. Now, we will prove that no proper subset of $\mathcal{T}_{\mathcal{P}}$ is complete. Given $\mathcal{T}_{\mathcal{P}}' \subset \mathcal{T}_{\mathcal{P}}$, by the definition of the theory $\mathcal{T}_{\mathcal{P}}$, we can ensure that there exists a pseudo-intent $P \in \mathcal{P}$ such that $P^{\downarrow\uparrow_*} \Leftarrow P \notin \mathcal{T}_{\mathcal{P}}'$. For this attribute implication we have that:

$$\begin{aligned}
\|P^{\downarrow\uparrow_*} \Leftarrow P\|_{\mathcal{T}_{\mathcal{P}}'} &\overset{(i)}{=} \bigwedge_{M \in Mod(\mathcal{T}_{\mathcal{P}}')} \|P^{\downarrow\uparrow_*} \Leftarrow P\|_M \\
&\overset{(ii)}{\preceq_1} \|P^{\downarrow\uparrow_*} \Leftarrow P\|_P \\
&\overset{(iii)}{\neq} \top_1 \\
&\overset{(iv)}{=} \|P^{\downarrow\uparrow_*} \Leftarrow P\|_{\mathcal{T}_{\mathcal{P}}} \\
&\overset{(v)}{=} \|P^{\downarrow\uparrow_*} \Leftarrow P\|_{Int(\mathcal{M}_*)}
\end{aligned}$$

where (i) is obtained by Definition 25, (ii) due to $P \in Mod(\mathcal{T}_{\mathcal{P}}')$ and the infimum property, (iii) is satisfied since $\|P^{\downarrow\uparrow_*} \Leftarrow P\|_P = S^1(P^{\downarrow\uparrow_*}, P) \neq \top_1$, (iv) holds since $P^{\downarrow\uparrow_*} \Leftarrow P \in \mathcal{T}_{\mathcal{P}}$, and (v) because of $\mathcal{T}_{\mathcal{P}}$ is complete.

Then, we obtain that $\|P^{\downarrow\uparrow_*} \Leftarrow P\|_{\mathcal{T}_{\mathcal{P}}'} \neq \|P^{\downarrow\uparrow_*} \Leftarrow P\|_{Int(\mathcal{M}_*)}$, which, by Definition 29, implies that $\mathcal{T}_{\mathcal{P}}'$ is not complete.

Therefore, we have shown that $\mathcal{T}_{\mathcal{P}}$ is a complete and no proper subset of it is complete, then according to Definition 36, we can conclude that $\mathcal{T}_{\mathcal{P}}$ is a basis of $(\mathcal{M}_*, \preceq)$. □

Notice that, the requirements in Theorem 42 on the operators & and $\swarrow$ only refers on the "external" adjoint triple and the adjoint triples in the multi-adjoint frame can be as general as necessary.

The following example illustrates how a specific basis of attribute implications can be constructed from a given context, based on the results presented in Theorem 42.

Example 43 This example computes a basis of attribute implications from the system of pseudo-intents obtained in Example 40. To do this, we need to check the hypothesis required in Theorem 42.

Specifically, we must check that the conjuctor $\&_{\mathrm{DP}}$ of the considered extra adjoint triple is associative, it satisfies the boundary conditions, and that the associated implication $\swarrow^{\mathrm{DP}}$ is a forcing implication.

(i) **Boundary conditions.** From the data in Table 9, we observe that the boundary conditions are fulfilled, i.e., the following equalities

$$x \,\&_{\mathrm{DP}}\, 1 = x \quad \text{and} \quad 1 \,\&_{\mathrm{DP}}\, y = y$$

Table 9 Definition of $\&_{DP}$ and $\swarrow^{DP}$ on $[0,1]_2$ of Example 43

$\&_{DP}$	0	0.5	1
0	0	0	0
0.5	0	0.5	0.5
1	0	0.5	1
$\swarrow^{DP}$	0	0.5	1
0	1	0	0
0.5	1	1	0.5
1	1	1	1

are satisfied for all $x, y \in [0,1]_2$.

(ii) **Associativity of** $\&_{DP}$. We have to check that the equality

$$x \&_{DP}(y \&_{DP} z) = (x \&_{DP} y) \&_{DP} z$$

holds for all $x, y, z \in [0,1]_2$. Based on the values computed in Table 9, we can differentiate the following cases:

- If any of x, y, or z is equal to 0, then both sides of the equality are 0, hence the identity is satisfied.
- If any of x, y, or z is equal to 1, then the boundary conditions apply with this element, and the identity also holds.
- If $x = y = z = 0.5$, then we have:

$$0.5 \&_{DP}(0.5 \&_{DP} 0.5) = 0.5 \&_{DP} 0.5 = (0.5 \&_{DP} 0.5) \&_{DP} 0.5$$

which confirms the associativity in this case as well.

(iii) **Forcing implication.** To verify that the implication $\swarrow^{DP}$ is a forcing implication, we refer to Definition 6, which requires that the implication be order-preserving in the left argument, order-reversing in the right argument, and satisfy the following equivalence:

$$z \swarrow^{DP} y = 1 \quad \text{if and only if} \quad y \leq z \quad \text{, for all} \quad y, z \in [0,1]_2$$

The monotonicity conditions are satisfied due to $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$ is an adjoint triple (see Proposition 2) and the previous equivalence can be directly checked from the computed values in Table 9.

Therefore, the hypothesis required in Theorem 42 are satisfied, and consequently, applying such theorem, for the system of pseudo-intents $\mathcal{P}$, we can ensure that the theory defined as $\mathcal{T}_{\mathcal{P}} = \{P^{\downarrow\uparrow_*} \Leftarrow P \mid P \in \mathcal{P}\}$ is a basis of $(\mathcal{M}_*, \preceq)$. Specifically, we obtain that:

$$\mathcal{T}_{\mathcal{P}} = \{P_0^{\downarrow\uparrow} \Leftarrow P_0, P_3^{\downarrow\uparrow} \Leftarrow P_3, P_5^{\downarrow\uparrow} \Leftarrow P_5\}$$

Taking into account the closure of each fuzzy set of attributes P_i, with $i \in \{0, 3, 5\}$, that is:

$$P_0^{\downarrow\uparrow} = \{0.5/a_2\}$$
$$P_3^{\downarrow\uparrow} = P_5^{\downarrow\uparrow} = \{0.5/a_1, 1/a_2\}$$

R	a_1	a_2
b_1	0	0

σ	a_1	a_2
b_1	$\&_{DP}$	$\&_{DG}$

Fig. 3 Relation R and mapping σ of Example 44

we have that the resulting basis is composed of the following attribute implications:

$$\mathcal{T}_{\mathcal{P}} = \{\{0.5/a_2\} \Leftarrow \{\}, \{0.5/a_1, 1/a_2\} \Leftarrow \{1/a_2\}, \\ \{0.5/a_1, 1/a_2\} \Leftarrow \{0.5/a_1, 0.5/a_2\}\}$$

Usually, the bases are depicted removing the redundant information in the consequent to remark the most significant information, that is,

$$\mathcal{T}_{\mathcal{P}} = \{\{0.5/a_2\} \Leftarrow \{\}, \{0.5/a_1\} \Leftarrow \{1/a_2\}, \\ \{1/a_2\} \Leftarrow \{0.5/a_1, 0.5/a_2\}\}$$

□

Next, a trivial example is introduced to show that the truth-stressing hedge is fundamental in order to obtain the set of pseudo-intents.

Example 44 Let $([0,1]_2, \leq, (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}))$ be a multi-adjoint frame where $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$ and $(\&_{DG}, \swarrow^{DG}, \nwarrow_{DG})$ are the adjoint triples obtained from the discretization of the product and G del t-norms on $[0,1]_2$, respectively, $(A_{*_A}, B_{*_B}, R, \sigma)$ be a context where the set of attributes is $A = \{a_1, a_2\}$, the set of objects is $B = \{b_1\}$, the relation $R: A \times B \to [0,1]_2$ and the mapping σ are given in Fig. 3, $*_A$ is a family of identity truth-stressing $\swarrow$-hedges and $*_B$ is a family of identity truth-stressing $\nwarrow$-hedges.

The multi-adjoint concept lattice $(\mathcal{M}_*, \preceq)$ related to this framework and the context is composed of 2 concepts, whose intents are:

$$Int(C_0) = \{\}$$
$$Int(C_1) = \{1/a_1, 1/a_2\}$$

Now, we will compute the systems of pseudo-intents associated with the given context, considering two different types of truth-stressing hedges, and we will compare the obtained results. Specifically, we will use the identity and globalization truth-stressing $\nwarrow$-hedges. To do this, we will consider the external adjoint triple $(\&_{DP}, \swarrow^{DP}, \nwarrow_{DP})$, whose operators were defined in Table 9, and the whole set of non-empty fuzzy subsets of attributes with respect to $[0,1]_2^A$ which are not intents. Such fuzzy subsets of attributes $P_i: \{a_1, a_2\} \to [0,1]_2$, with $i \in \{1, \ldots, 5\}$, are defined as:

$$P_1 = \{0.5/a_1\}$$
$$P_2 = \{1/a_1\}$$
$$P_3 = \{0.5/a_2\}$$
$$P_4 = \{1/a_2\}$$
$$P_5 = \{0.5/a_1, 1/a_2\}$$
$$P_6 = \{1/a_1, 0.5/a_2\}$$
$$P_7 = \{0.5/a_1, 0.5/a_2\}$$

Table 10 Computed degrees of validity using the identity truth-stressing hedge in Example 44

f_1	f_2	$\|f_2 \Leftarrow f_1\|_{P_1}$	$\|f_2 \Leftarrow f_1\|_{P_2}$	$\|f_2 \Leftarrow f_1\|_{P_3}$	$\|f_2 \Leftarrow f_1\|_{P_4}$	$\|f_2 \Leftarrow f_1\|_{P_5}$	$\|f_2 \Leftarrow f_1\|_{P_6}$	$\|f_2 \Leftarrow f_1\|_{P_7}$
P_1	$P_1^{\downarrow\uparrow}$	0	0	1	1	0.5	0.5	0.5
P_2	$P_2^{\downarrow\uparrow}$	0	0	1	1	1	0.5	1
P_3	$P_3^{\downarrow\uparrow}$	1	1	0	0	0.5	0.5	0.5
P_4	$P_4^{\downarrow\uparrow}$	1	1	0	0	0.5	1	1
P_5	$P_5^{\downarrow\uparrow}$	1	1	1	1	0.5	1	1
P_6	$P_6^{\downarrow\uparrow}$	1	1	1	1	1	0.5	1
P_7	$P_7^{\downarrow\uparrow}$	1	1	1	1	0.5	0.5	0.5

Table 11 Computed degrees of validity using the globalization truth-stressing hedge in Example 44

f_1	f_2	$\|f_2 \Leftarrow f_1\|_{P_1}$	$\|f_2 \Leftarrow f_1\|_{P_2}$	$\|f_2 \Leftarrow f_1\|_{P_3}$	$\|f_2 \Leftarrow f_1\|_{P_4}$	$\|f_2 \Leftarrow f_1\|_{P_5}$	$\|f_2 \Leftarrow f_1\|_{P_6}$	$\|f_2 \Leftarrow f_1\|_{P_7}$
P_1	$P_1^{\downarrow\uparrow}$	0	0	1	1	0.5	0.5	0.5
P_2	$P_2^{\downarrow\uparrow}$	1	0	1	1	1	0.5	1
P_3	$P_3^{\downarrow\uparrow}$	1	1	0	0	0.5	0.5	0.5
P_4	$P_4^{\downarrow\uparrow}$	1	1	1	0	0.5	1	1
P_5	$P_5^{\downarrow\uparrow}$	1	1	1	1	0.5	1	1
P_6	$P_6^{\downarrow\uparrow}$	1	1	1	1	1	0.5	1
P_7	$P_7^{\downarrow\uparrow}$	1	1	1	1	0.5	0.5	0.5

By Definition 39, we obtain the following systems of pseudo-intents corresponding to the identity and globalization truth-stressing hedges, by using the degrees of validity given in Tables 10 and 11.

- Systems of pseudo-intents ($\mathcal{P}^I$) using the identity truth-stressing hedge:

$$\mathcal{P}_1^I = \{P_2, P_4, P_7\} \qquad \mathcal{P}_2^I = \{P_1, P_4\} \qquad \mathcal{P}_3^I = \{P_1, P_3\} \qquad \mathcal{P}_4^I = \{P_2, P_3\}$$

- Systems of pseudo-intents ($\mathcal{P}^G$) using the globalization truth-stressing hedge:

$$\mathcal{P}^G = \{P_1, P_3\}$$

Therefore, applying Theorem 42, the above systems of pseudo-intents give rise to different bases of attribute implications. □

As a consequence, the selection of the truth-stressing hedge is a new parameter that can be considered to improve the modelling of the given knowledge system. In the future, we will study the relationship among the selected truth-stressing hedge and the possible systems of pseudo-intents.

6 Conclusions and future work

This paper have presented a theoretical development of the main notions and results on theories of attribute implications, models and the entailment relations between these structures in the multi-adjoint concept lattice framework enriched with truth-stressing hedges, in order to introduce the notion of basis of attribute implications in this context.

Specifically, truth-stressing hedges have been considered in the definition of the concept-forming operators of the multi-adjoint concept lattice framework. We have shown that, in this case, the composition of these operators do not form a closure operator in general, which is fundamental for computing a basis of attribute implications. However, we have proven that, if we consider the identity truth-stressing hedge for each attribute of the given context, these operators indeed form a closure operator. This fact has enabled us to introduce the notion of pseudo-intent, which is directly related to the construction of implicational bases.

Furthermore, we have explored two properties of attribute implication theories, completeness and non-redundancy, which play a significant role in defining a basis of attribute implications. We have discussed the necessary and sufficient conditions to guarantee these properties. Based on these considerations, we have extended the concept of a basis of attribute implications to the multi-adjoint framework, characterizing it as a complete and non-redundant theory. In addition, we have established the conditions under which the Duquenne-Guigues basis can be extended to this framework.

As a future work, we will focus on the development of efficient mechanisms for the extraction of valid attribute implications, which will contribute to optimizing the construction of the bases of attributes implications proposed in this paper. Additionally, we will study complementary procedures to those introduced in this paper for the construction of implicational bases, providing comparative analysis and carrying out empirical validations, together with detailed examples. These objectives will not only allow us to advance the theoretical framework, but also enhance its impact and improve the practical applicability of the flexible multi-adjoint attribute implications, particularly in real-world scenarios, such as those presented for digital forensic and renewable energies in Aragón et al. (2022), Cornejo et al. (2022), Cornejo et al. (2024).

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Declarations

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