ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 26 (2026) #P1.10
https://doi.org/10.26493/1855-3974.3410.9bc
(Also available at http://amc-journal.eu)

Mobile mutual-visibility sets in graphs*

Magda Dettlaff

Faculty of Mathematics, Physics and Informatics, University of Gdańsk, Poland

Magdalena Lemańska

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Poland

Juan A. Rodríguez-Velázquez †

Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Spain

Ismael G. Yero ‡

Departamento de Matemáticas, Universidad de Cádiz, Algeciras Campus, Spain

Received 12 June 2024, accepted 22 December 2024, published online 18 December 2025

Abstract

Given a connected graph G, the mutual-visibility number of G is the cardinality of a largest set S such that for every pair of vertices $x, y \in S$ there exists a shortest x, y-path whose interior vertices are not contained in S. Assume that a robot is assigned to each vertex of the set S. At each stage, one robot can move to a neighbouring vertex. Then S is a mobile mutual-visibility set of G if there exists a sequence of moves of the robots such that all the vertices of G are visited while maintaining the mutual-visibility property at all times. The mobile mutual-visibility number of G, denoted $\mathrm{Mob}_\mu(G)$, is the cardinality of a largest mobile mutual-visibility set of G. In this paper we introduce the concept of the mobile mutual-visibility number of a graph. We begin with some basic properties of the mobile mutual-visibility number of G and its relationship with the mutual-visibility number of G. We give exact values of $\mathrm{Mob}_\mu(G)$ for particular classes of graphs, i.e. cycles, wheels, complete bipartite graphs, and block graphs (in particular trees). Moreover,

*We first want to sincerely thank the reviewers for their careful revisions of our manuscript. In particular, we want to thank that reviewer who found some gaps in our arguments from Section 4 (among other remarks), and pointing out some counterexamples for the results of this section in our first version. We have now included such counterexamples in our exposition.

†J. A. Rodríguez-Velázquez has been partially supported by the SGR Grant 2021 00115 and by Project HERMES funded by the European Union NextGenerationEU/PRTR via INCIBE.

‡Corresponding author. I. G. Yero has been partially supported by the Spanish Ministry of Science and Innovation through the grant PID2023-146643NB-I00. Moreover, this investigation was initiated while this author (I.G. Yero) was making a temporary stay at "Universitat Rovira i Virgili" supported by the program "Ayudas para la recualificación del sistema universitario español para 2021-2023, en el marco del Real Decreto 289/2021, de 20 de abril de 2021".

we present bounds for the lexicographic product of two graphs and show characterizations of the graphs achieving the limit values of some of these bounds. As a consequence of this study, we deduce that the decision problem concerning finding the mobile mutual-visibility number is NP-hard. Finally, we focus our attention on the mobile mutual-visibility number of line graphs of complete graphs, prism graphs and strong grids of two paths.

Keywords: Mobile mutual-visibility set, mutual-visibility number, total mutual-visibility.

Math. Subj. Class. (2020): 05C12, 05C76

1 Introduction

The research on mutual-visibility in graphs was introduced in 2022 by Di Stefano [14]. This seminal work was motivated by a computer science model of mobile entities in a network requiring some "mutual visibility" properties while communicating between them, so that there will be a "private channel" of communication; this private channel is a shortest path between the two entities that does not contain any other entity that is involved in the model. After this first work on the topic, a series of subsequent contributions have rapidly appeared. Among them, we find for instance [1–3, 5–11, 20, 21, 24].

One of the main antecedents of this concept (see [13]) is the one considering a set of robots placed at some locations in a network, and then by making some movements, the robots attain a configuration in which the locations occupied by the robots have some mutual visibility properties in the previously mentioned sense. Once the robots occupy this configuration, they are not required to continue moving. In our setting, we consider the continuation of this model in the sense that, after reaching a configuration with the mutual visibility property, we require the robots to continue moving, by keeping the property at all times, until all the nodes of the network have been visited at least once.

Consider a set of vertices $S \subseteq V(G)$ of a graph G. Two vertices $x, y \in V(G)$ are *S-visible* if there exists a shortest x,y-path whose interior vertices are not in S. The set $S \subseteq V(G)$ is a *mutual-visibility set* of G if every two distinct vertices $x, y \in S$ are S-visible. The *mutual-visibility number* of G is the cardinality of a largest mutual-visibility set of G, and is denoted by $\mu(G)$, see [14]. A mutual-visibility set of cardinality $\mu(G)$ is called a μ-set of G.

From now on, assume that one robot is assigned to each of the vertices from a mutual-visibility set $S \subseteq V(G)$ of a graph G. Then, the robots move through the graph from one vertex to another adjacent vertex. A move of a robot is *legal* if the robot moves from $x \in S$ to an adjacent unoccupied vertex $y \notin S$ such that the new set $(S \setminus \{x\}) \cup \{y\}$ is a mutual-visibility set of G as well. In this sense, we say that the set S is a *mobile mutual-visibility set* of G if there exists a sequence of legal moves beginning at the vertices of S, such that that each vertex of G is visited at least once by at least one robot. Now, the *mobile mutual-visibility number* $\mathrm{Mob}_\mu(G)$ of G represents the largest cardinality among all possible mobile mutual-visibility sets of G. A mobile mutual-visibility set of cardinality $\mathrm{Mob}_\mu(G)$ will be called a *mobile μ-set* of G for short. By a robot x we mean a robot located at a vertex x. Moreover, a move from a robot x to an adjacent vertex y shall be written as $x \rightsquigarrow y$.

E-mail addresses: magda.dettlaff@ug.edu.pl (Magda Dettlaff), magdalena.lemanska@pg.edu.pl (Magdalena Lemańska), juanalberto.rodriguez@urv.cat (Juan A. Rodríguez-Velázquez), ismael.gonzalez@uca.es (Ismael G. Yero)

Note now that any two vertices of a non-trivial graph G form a mobile mutual-visibility set of G since any two vertices of G form a mutual-visibility set. Also, any mobile mutual-visibility set is of course a mutual-visibility set. Thus, the following straightforward bounds are clear for any non-trivial graph G.

$$2 \leq \mathrm{Mob}_\mu(G) \leq \mu(G). \tag{1.1}$$

Now, in order to get a better understanding of the central concept of this work, we consider the non-trivial example of the hypercube Q_3. It is already known from [7] that $\mu(Q_3) = 5$. An example of a μ-set of Q_3 appears in the drawing of Figure 1 (a). The subsequent drawings of Figure 1 (b)–(e) show a sequence of legal moves that allow every vertex of Q_3 to be visited at least once, which allows us to conclude that $\mathrm{Mob}_\mu(Q_3) = \mu(Q_3) = 5$.

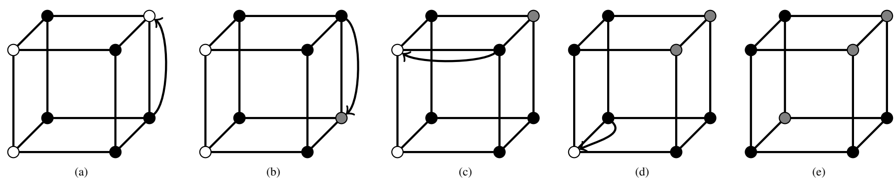

Figure 1: The moves of a mobile μ-set in Q_3 from (a) to (e): bold vertices represent the robots of the mobile μ-set; white vertices are not visited yet; gray vertices have already been visited by a robot from the mobile μ-set.

The article is organized as follows. We first study the relationship between the mobile mutual-visibility number and the mobile general position number of a graph, which is a closely related concept defined in [16]. We continue with a few basic properties of the mobile mutual-visibility number of G and its interrelation with the mutual-visibility number $\mu(G)$. In particular, we show that there exists a graph with $\mathrm{Mob}_\mu(G) = a$ and $\mu(G) = b$ for any pair of integers a and b with $2 \leq a \leq b$. We also give exact values of $\mathrm{Mob}_\mu(G)$ for particular classes of graphs, i.e. cycles, wheels, complete bipartite graphs and block graphs. We prove that trees are the only graphs G with $\mathrm{Mob}_\mu(G) = 2$. Moreover, we prove bounds for the lexicographic product of two graphs and give characterizations of the graphs achieving the limit values of some of these bounds. As a consequence of this study of the lexicographic product, we deduce that the decision problem concerning finding the mobile mutual-visibility number of graphs is NP-hard. Finally, we focus our attention on considering mobile mutual-visibility sets for the line graph of complete graphs, and also for prism graphs and for the strong grid of two paths.

We note that throughout our exposition all graphs considered are connected, undirected, and without loops and multiple edges. A *universal vertex* is a vertex of a graph that is adjacent to all other vertices of the graph. As usual, we use the notation K_n, C_n, P_n, and N_n for complete graphs, cycle graphs, path graphs, and empty graphs of order n, respectively. Given a vertex $v \in V(G)$ of a graph G, the vertex-deletion subgraph $G - v$ is the subgraph of G induced the set $V(G) \setminus \{v\}$. The order of a graph G will be denoted by $n(G)$. Moreover, in order to simplify the writing, we shall assume that $[n] = \{1, \ldots, n\}$ for any integer $n \geq 1$. Other notation and terminology will be introduced when required.

2 Relationships with mobile general position

A parallel situation to the mobile mutual-visibility parameter is related to the general position problem. The *general position number* of a graph G is the cardinality of a largest set of vertices (called a general position set) of G such that it contains no three collinear vertices, where lines are taken as geodesics (shortest paths) in the graph. This concept is in fact known from the '90s (see [18]), although it was recently and independently rediscovered in [4] and [22]. It is now a very well-studied parameter, and a large number of remarkable contributions on it can be found in the literature, for example [15, 17, 19, 25] to name just a few of the most recent ones.

One of the works on general position considered the idea of "moving" the vertices of general position sets in order to traverse all the vertices of the graphs, see [16]. General position sets and mobile general position numbers (denoted as $\mathrm{Mob}_{\mathrm{gp}}(G)$) are defined analogously to mutual-visibility sets and mobile mutual-visibility numbers. It is clear that, if a set of vertices of a graph G is a mobile general position set, then it is also a mobile mutual-visibility set, which implies that

$$\mathrm{Mob}_{\mathrm{gp}}(G) \leq \mathrm{Mob}_{\mu}(G). \tag{2.1}$$

for all connected graphs G. There are several graphs for which equality occurs in the relationship above. For instance, it is known from [16] that $\mathrm{Mob}_{\mathrm{gp}}(C_n) = 3$ for any cycle C_n with $n \neq 4, 6$, while $\mathrm{Mob}_{\mathrm{gp}}(C_4) = \mathrm{Mob}_{\mathrm{gp}}(C_6) = 2$. We next see that $\mathrm{Mob}_{\mu}(C_n) = 3$ for every $n \geq 3$.

Proposition 2.1. *For any integer $n \geq 3$, $\mathrm{Mob}_{\mu}(C_n) = 3$.*

Proof. If $n \in \{4, 6\}$, then it is straightforward to check the conclusion. Assume that $n \geq 3$ and $n \neq 4, 6$. By (1.1) and (2.1) we have $\mathrm{Mob}_{\mathrm{gp}}(C_n) \leq \mathrm{Mob}_{\mu}(C_n) \leq \mu(C_n)$. Since $\mu(C_n) = 3$ (see [14]) and $\mathrm{Mob}_{\mathrm{gp}}(C_n) = 3$ (see [16]), we deduce the desired equality. □

Other examples with equality in the relationship (2.1) are the block graphs, whose mobile mutual-visibility number will be further computed (Corollary 3.3) to be equal to the clique number of the block graph, which is indeed the value for their mobile general position number as well, as proved in [16].

In contrast to the results above, there are also several graph classes where the equality in (2.1) does not occur. Let us consider a graph G obtained from two complete graphs K_p and K_s, $1 \leq p \leq s$ and a complete graph K_2 with vertex set $\{u_1, u_2\}$ by joining u_i with every vertex from $V(K_p) \cup V(K_s)$. It is readily observed that $\mu(G) = p + s + 1$ and that $V(G) \setminus \{u_1\}$ is a mutual-visibility set of G. Also, $u_1 \rightsquigarrow u_2$ is a legal move after which each vertex of G is visited. Hence, $\mathrm{Mob}_{\mu}(G) = p + s + 1$. On the other hand, notice that any general position set S of G such that $S \cap V(K_p) \neq \emptyset$ and $S \cap V(K_s) \neq \emptyset$ satisfies $S \cap \{u_1, u_2\} = \emptyset$. It implies that $V(K_s) \cup \{u_1\}$ is a maximum mobile general position set of G, hence $\mathrm{Mob}_{\mathrm{gp}}(G) = s + 1$. Finally, $\mathrm{Mob}_{\mu}(G) - \mathrm{Mob}_{\mathrm{gp}}(G) = p$.

3 Basic results

We begin this section by considering the inequality (1.1). Since we can have equality in $\mu(G) \leq n(G)$, it is a natural question to consider when $\mathrm{Mob}_{\mu}(G) = n(G)$ holds. This case is straightforward, as the whole vertex set of a graph G is a mobile μ-set of G if and only if there is no pair of non-adjacent vertices in G.

Remark 3.1. Let G be a graph of order n. Then $\mathrm{Mob}_\mu(G) = n$ if and only if G is a complete graph.

We next center our attention into the other issues related to the inequality (1.1). That is, first characterizing all graphs for which $\mathrm{Mob}_\mu(G) = 2$, giving some examples of graphs G such that $\mathrm{Mob}_\mu(G) = \mu(G)$, and the realization of all possible values for $\mathrm{Mob}_\mu(G)$ and $\mu(G)$ inside the inequality. To this end, we first focus on considering graphs having cut vertices (recall that for a connected graph G, a vertex v is called a *cut vertex* if $G - v$ is disconnected).

Proposition 3.2. *If v is a cut vertex of a graph G, where $C_1, \ldots, C_k$ are the components of $G - v$ and $G_i = G[V(C_i) \cup \{v\}]$ with $i \in [k]$, then*

$$\max\{\mathrm{Mob}_\mu(G_i) \,:\, i \in [k]\} \le \mathrm{Mob}_\mu(G) \le \max\{\mu(G_i) \,:\, i \in [k]\}.$$

Proof. Let $S \subseteq V(G)$ be a mobile μ-set of G such that $v \in S$. Since $\mathrm{Mob}_\mu(G) = |S| \ge 2$, there exists $i \in [k]$ such that $S \cap V(C_i) \neq \emptyset$. Now, if there exist two vertices $x \in S \cap V(C_i)$ and $y \in S \cap V(C_j)$, with $i \neq j$, then they are not S-visible (v lies in every shortest x, y-path), which is a contradiction. Hence $S \subseteq V(G_i)$, which implies that S is a mutual-visibility set of G_i. Thus, the upper bound follows. On the other hand, if one considers a mobile μ-set of any G_i, it is clear that it is also a mobile mutual-visibility set of G, and so, we conclude that $\mathrm{Mob}_\mu(G) \ge \max\{\mathrm{Mob}_\mu(G_i) \,:\, i \in [k]\}$. □

To see that the upper bound above is tight, consider for instance the graph G obtained from a complete bipartite graph $K_{2,3}$ by adding a pendant vertex to one vertex of the partition set of cardinality two. In such case, the graphs G_1 and G_2 are the complete bipartite graph $K_{2,3}$ and the path P_2. It can be noticed that $\mathrm{Mob}_\mu(G) = 4 = \max\{\mu(K_{2,3}, \mu(P_2)\}$. In addition, notice that $\mathrm{Mob}_\mu(K_{2,3}) = 3$ and $\mathrm{Mob}_\mu(P_2) = 2$, showing that the lower bound is not reached in this case. Situations where this lower bound is tight are later shown for the case of block graphs.

A graph G is a *block graph* if every block (maximal 2-connected component) of G is a clique. We use $\omega(G)$ to represent the *clique number* of G, i.e. the cardinality of the maximum clique of G. By the fact that all the vertices of a clique form a mobile mutual-visibility set of the clique itself, a recursive argument on the cut vertices of block graphs leads to the following conclusion by using Proposition 3.2.

Corollary 3.3. *If G is a block graph, then $\mathrm{Mob}_\mu(G) = \omega(G)$. In particular, if G is a tree, then $\mathrm{Mob}_\mu(G) = 2$.*

Proof. If G is a complete graph, then the result follows from Proposition 3.2. Hence, we may assume that G has at least two blocks. In this sense, by using an induction procedure on the number of cut vertices of G, we deduce the desired equality. □

We next show that indeed the trees are the only graphs G satisfying $\mathrm{Mob}_\mu(G) = 2$. Notice that this, in addition, represents the characterization of the graph class attaining the lower bound in (1.1).

Theorem 3.4. *Let G be a connected graph. Then $\mathrm{Mob}_\mu(G) = 2$ if and only G is a tree.*

Proof. If T is a tree, then the result holds by Corollary 3.3. Assume that G is not a tree, and let C be a shortest cycle in G. We shall show that there exists a mobile mutual-visibility set of G of cardinality three. Since C is an isometric cycle, Proposition 2.1 implies that three mutually visible robots R_1, R_2, R_3 can traverse C and visit every vertex of the cycle. Let x be any vertex of $V(G) \setminus V(C)$ and P be a shortest path from x to C. Let x_0 be the vertex of $V(P) \cap V(C)$ and x_1 the neighbour of x_0 on P. Temporarily label the vertices of C so that C is the cycle $v_0v_1 \cdots v_{k-1}v_0$, where $v_0 = x_0$. Move the three robots around C until R_1 is at x_0 and suppose that R_2 and R_3 are at v_i and v_j respectively, where $j > i$. We can choose the distance $j - i$ to be the maximum of the distances between the robots. Perform the move $x_0 \rightsquigarrow x_1$, then move R_2 and R_3 alternately according to $v_i \rightsquigarrow v_{i-1}$, $v_j \rightsquigarrow v_{j+1}$, $v_{i-1} \rightsquigarrow v_{i-2}$, $v_{j+1} \rightsquigarrow v_{j+2}, \ldots$, and so on, until R_2 and R_3 are stationed at the neighbours v_1 and v_{k-1} of x_0. Then move robot R_1 along the rest of the path P until it reaches x. It can be seen that at each stage the robots are mutually visible. These steps can now be reversed and a robot sent to any other not yet visited vertex of $V(G) \setminus V(C)$. Therefore, $\mathrm{Mob}_\mu(G) \geq 3$. □

Now, in connection with the bounds from (1.1), one may think about the realization of $\mathrm{Mob}_\mu(G)$ with respect to $\mu(G)$. This can be seen as follows.

Remark 3.5. For any pair of integers a and b with $2 \leq a \leq b$, there exists a graph G with $\mathrm{Mob}_\mu(G) = a$ and $\mu(G) = b$.

Proof. If $a = b$, then we simply consider the complete graph K_a. Hence, we may assume $2 \leq a < b$ (this means $b \geq 3$). We construct a graph $G_{a,b}$ as follows. Let us use the representation $b = p \cdot (a - 1) + q$ where $q < a - 1$. We begin with p disjoint cliques of cardinality $a - 1$, and one other clique of cardinality q. Next, to obtain $G_{a,b}$, we add an extra vertex x, and join by an edge all the vertices from all the mentioned cliques to the vertex x.

It can be noted that, all the vertices of $G_{a,b}$ but the vertex x form a mutual-visibility set of $G_{a,b}$, and so, $\mu(G_{a,b}) \geq p \cdot (a-1) + q = b$. Moreover, since x is a cut vertex and $b \geq 3$, it holds that x does not belong to any μ-set of $G_{a,b}$. Therefore, $\mu(G_{a,b}) = b$.

Now, the equality $\mathrm{Mob}_\mu(G_{a,b}) = a$ is obtained by using Proposition 3.2. □

Based on the results above concerning graphs having cut vertices, we now center our attention on computing the value of $\mathrm{Mob}_\mu(G)$ for some basic families of graphs having no cut vertices. To this end, we need to recall that the mutual-visibility number of complete bipartite graphs is known from [14, Table 2]. We also recall that a *wheel* W_n is a graph obtained from a cycle C_n by adding a vertex v and joining it by an edge to every vertex of C_n.

Proposition 3.6.

(i) *For any integers* $r \geq s \geq 1$, $\mathrm{Mob}_\mu(K_{r,s}) = \begin{cases} 2, & \text{if } s = 1, \\ r + s - 2, & \text{if } 2 \leq s \leq 3, \\ r + s - 3, & \text{if } s \geq 4. \end{cases}$

(ii) *For any wheel graph* W_n, $\mathrm{Mob}_\mu(W_n) = \begin{cases} 4; & \text{if } n = 3, 4, 5, 6, \\ 3; & \text{if } n \geq 7. \end{cases}$

Proof. (i) If $s = 1$, then $K_{r,s}$ is a star, and so, Corollary 3.3 leads to the equality. Assume now $s \geq 2$.

From (1.1) and the results from [14, Table 2] we know that $\mathrm{Mob}_\mu(K_{r,s}) \leq r + s - 2$. On the other hand, it can be readily seen that $\mathrm{Mob}_\mu(K_{r,s}) \geq r + s - 3$ by just considering a set of vertices of cardinality $r + s - 3$ which does not contain two vertices from one bipartition set and one from the other one. Such a set is clearly a mutual-visibility set and one can make a sequence of legal moves to visit all the vertices of $K_{r,s}$, thus the conclusion.

Let $V(K_{r,s}) = U \cup V$ such that $|U| = r$ and $|V| = s$ are the bipartition sets of $K_{r,s}$. If $\mathrm{Mob}_\mu(K_{r,s}) = r + s - 2$, then there are always two vertices without a robot. If both U and V have one robot, then in the following move one of U and V will have two empty vertices. Hence, the robots in the partite set with two unoccupied vertices will not be mutually visible unless it has at most three vertices. Thus, $s \leq 3$, in which case it is easily seen that we have equality in $\mathrm{Mob}_\mu(K_{r,s}) = r+s-2$. On the other hand, for $s \geq 4$ we have $\mathrm{Mob}_\mu(K_{r,s}) = r + s - 3$.

(ii) Let $C = [n]$ be the vertices of the outer cycle in W_n and let w be the central vertex. The cases $n = 3, 4, 5, 6$ are straightforward. Hence, assume $n \geq 7$. It can be easily observed that $S = \{w, 1, 2\}$ is a mutual-visibility set of W_n. Moreover, the moves $w \rightsquigarrow i \rightsquigarrow w$ are legal for any $i \neq 1, 2$. Therefore, $\mathrm{Mob}_\mu(W_n) \geq 3$.

Suppose next that $\mathrm{Mob}_\mu(W_n) \geq 4$ and let S' be a mobile μ-set of W_n. Regardless of any move that the vertices of S' would do, at some point, the vertex w needs to be visited, and there will be at least 3 vertices on the cycle. Since $n \geq 7$, there will be two robots on the cycle such that all shortest paths between them pass either through the robot at w or through the third robot on the cycle. Thus, they are not S'-visible, which is not possible. Therefore, $\mathrm{Mob}_\mu(W_n) \leq 3$ and the equality follows. □

4 Lexicographic products with an application

Given two graphs G and H, the *join graph* $G + H$ is the graph obtained from G and H by adding all the possible edges between vertices of G and vertices of H. Clearly, if G and H are complete graphs, then $G + H$ is also complete and so by Remark 3.1, we next consider only join graphs $G + H$ in which at least one of them is not complete.

Theorem 4.1. *Let G and H be two graphs not both complete with $n(G) \geq 2$ and $n(H) \geq 2$. Then*

$$n(G) + n(H) - 3 \leq \mathrm{Mob}_\mu(G + H) \leq n(G) + n(H) - 1.$$

Proof. Since either G or H is not complete, the join $G + H$ is also not complete. Thus, by Remark 3.1, the upper bound follows. To prove the lower bound, notice that any set of vertices of $G + H$ of cardinality $n(G) + n(H) - 3$, such that it contains either $n(G) - 2$ vertices of G and $n(H) - 1$ vertices of H, or $n(G) - 1$ vertices of G and $n(H) - 2$ vertices of H, is a mutual-visibility set of $G + H$. Consequently, robots placed on the vertices of any such set can make legal moves to visit the three remaining vertices of $G + H$, and so it is a mobile mutual-visibility set, which leads to the desired bound. □

The *lexicographic product graph* $G \circ H$ of two graphs G and H is obtained as follows. The set of vertices of $G \circ H$ is $V(G \circ H) = V(G) \times V(H)$. Also, the set of edges of $G \circ H$ is given as follows. There is an edge between (x, y) and (x', y') in $G \circ H$ whenever there is an edge between x and x' in G; or $x = x'$ and there is an edge between y and y' in H. Clearly, $K_2 \circ G \cong G + G$ for any graph G. If $u \in V(G)$, then the subgraph of

$G \circ H$ induced by $\{u\} \times V(H)$ is called an H-*layer*, and is denoted by H_u. Similarly, the G-layers G_v are defined for vertices of $v \in V(H)$. Notice that $G \circ H$ is connected whenever G is connected. Thus, from now on in this section, we assume G is a connected graph in the lexicographic product $G \circ H$.

We first note the following fact for the case of lexicographic product graphs. If $g \in V(G)$ and $h, h' \in V(H)$ (with $h \neq h'$) are arbitrary vertices of G and H, respectively, then the set $S = (V(G) \times (V(H) \setminus \{h\})) \setminus \{(g, h')\}$ is a mutual-visibility set of $G \circ H$. This means that any robot placed at any neighbour (g', h') of (g, h') can make the legal move $(g', h') \rightsquigarrow (g, h')$, and the set S can achieve any configuration in which there are two vertices not in S from one copy of H. Now, to visit any vertex (g'', h) by a legal move, we just need to have the robots placed at a set S' such that $(g'', h') \notin S'$. Such S' can clearly be reached by the arguments above. This means that the robots placed at S can legally move to visit all the vertices of $G \circ H$. Consequently, it is deduced that S is a mobile mutual-visibility set of $G \circ H$. Thus, for any non-trivial graphs G and H,

$$\mathrm{Mob}_\mu(G \circ H) \geq n(G)(n(H) - 1) - 1. \tag{4.1}$$

On the other hand, since the join of two copies of a graph H can be seen as the lexicographic product $K_2 \circ H$ and join graphs were studied above, we next consider those lexicographic product graphs $G \circ H$ in which G is not K_2.

In order to present our results, we need to introduce the following notion and terminology, which were first described in [11]. A set of vertices S of a graph G is called a *total mutual-visibility set* if every two vertices of $V(G)$ are S-visible (notice that the visibility property must be satisfied for every pair of vertices of the graph). The cardinality of a largest total mutual-visibility set of G is the *total mutual-visibility number*, and is denoted by $\mu_t(G)$. Some further contributions on this parameter were given in [9,24]. Moreover, the total mutual-visibility number for the lexicographic product of graphs was studied in [21]. There was proved the following result.

Theorem 4.2 ([21])**.** *If G is a connected graph and has no universal vertices, then for any graph H,*

$$\mu_t(G \circ H) = n(G)(n(H) - 1) + \mu_t(G).$$

We shall also need the following structural lemma concerning the mobile mutual-visibility sets of $G \circ H$.

Lemma 4.3. *Let G be a connected non-trivial graph. If H is a graph with $n(G) < n(H) + 1$, then there exists a mobile μ-set S of $G \circ H$ such that $S \cap V(H_u) \neq \emptyset$ for every $u \in V(G)$.*

Proof. Suppose to the contrary that for every mobile μ-set S of $G \circ H$ there exists a vertex $x \in V(G)$ such that $S \cap V(H_x) = \emptyset$. This means that $|S| \leq n(H)(n(G) - 1)$. On the other hand, by (4.1), we know that $|S| \geq n(G)(n(H) - 1) - 1$. Thus, it must happen $n(G)(n(H) - 1) - 1 \leq n(H)(n(G) - 1)$, which means that $n(G) \geq n(H) + 1$, and this is not possible due to the assumption. Therefore, there exists a mobile μ-set S of $G \circ H$ such that $S \cap V(H_u) \neq \emptyset$ for every $u \in V(G)$. □

Notice that, if the assumption $n(G) < n(H) + 1$ is not include in the lemma above, then there could exist graphs G and H (satisfying the remaining conditions of the lemma) for which a mobile μ-set S of $G \circ H$ exists such that $S \cap V(H_u) = \emptyset$ for some $u \in V(G)$. To see this, we can consider the graphs $C_5 \circ N_2$ (N_2 is the empty graph on two vertices) or $C_5 \circ K_2$ as examples. In the former case, this graph has the mobile mutual visibility number equal to 6, but there is no mobile μ-set with a robot in each copy of N_2. A similar situation occurs in the latter case.

Theorem 4.4. *Let G be a connected non-trivial graph and let H be any graph of order at least two. Then*

$$\mathrm{Mob}_\mu(G \circ H) \geq n(G)(n(H) - 2) + 2\mu_t(G).$$

Moreover, if $n(G) < n(H) + 1$, then

$$\mathrm{Mob}_\mu(G \circ H) \leq n(G)(n(H) - 1) + \mu_t(G).$$

In addition, if $n(G) < n(H)+1$, H is not an empty graph and G has no universal vertices, then

$$\mathrm{Mob}_\mu(G \circ H) = \mu_t(G \circ H) = n(G)(n(H) - 1) + \mu_t(G).$$

Proof. To prove the lower bound, let S be a μ_t-set of G and let v, w be two vertices of H. We claim that the set $S' = (S \times V(H)) \cup ((V(G) \setminus S) \times (V(H) \setminus \{v, w\}))$ is a mobile mutual-visibility set of $G \circ H$. It can be readily observed that S' is a mutual-visibility set in $G \circ H$, since S is a μ_t-set of G, and the fact that every H-layer H_u such that $u \notin S$ has two vertices not in S'. Thus, any two vertices of S' are S'-visible by using shortest paths with those vertices (u, v) or (u, w). Moreover, by the same reasons, any vertex (x, v) or (x, w) with $x \in S$ can develop a sequence of legal moves and visit all the vertices of $V(G \circ H) \setminus S'$. Thus, S' is a mobile mutual-visibility set and the lower bound follows.

Now, to prove the upper bound, let X be a mobile μ-set of $G \circ H$ satisfying that $X \cap V(H_u) \neq \emptyset$ for every $u \in V(G)$, which can be assumed based on Lemma 4.3, since $n(G) < n(H) + 1$ by assumption. Let $X_G = \{u \in V(G) : |X \cap V(H_u))| = n(H)\}$ (namely, the vertices of G whose corresponding H-layers are whole included in X). We claim that such X_G is a total mutual-visibility set of G. Suppose for the contrary that X_G is not such. Hence, there are at least two vertices $x, y \in V(G)$ which are not X_G-visible. But then, this means that there are two vertices $v, v' \in V(H)$ such that the two vertices $(x, v), (y, v') \in X$ (the existence of these two vertices in X is confirmed by the Lemma 4.3) are not X-visible in $G \circ H$, which is not possible. Thus, X_G is a total mutual-visibility set of G as claimed, and so, the following holds.

$$\begin{aligned}
\mathrm{Mob}_\mu(G \circ H) = |X| &= \sum_{x \in X_G} |X \cap V(H_x)| + \sum_{x \notin X_G} |X \cap V(H_x)| \\
&\leq n(H)|X_G| + (n(H) - 1)(n(G) - |X_G|) \\
&= n(G)(n(H) - 1) + |X_G| \\
&\leq n(G)(n(H) - 1) + \mu_t(G).
\end{aligned}$$

From now on, assume H is not an empty graph. Let A be a μ_t-set of G and let w be a vertex of H of degree at least one. We claim that the set $B = (A \times V(H)) \cup ((V(G) \setminus A) \times (V(H) \setminus \{w\}))$ is a mobile mutual-visibility set of $G \circ H$. To see this, we first observe that B is a total mutual-visibility set as shown in [21] (proof of Theorem 4.2), and so, also a mutual-visibility set of $G \circ H$.

Let w' be a neighbor of w in H and let $u \in V(G) \setminus A$ (note that $(u, w') \in B$ while $(u, w) \notin B$). Due to the structure of the lexicographic product, we observe that the set $B' = B \setminus \{(u, w')\} \cup \{(u, w)\}$ is also a total mutual-visibility set of $G \circ H$. This means that $(u, w') \rightsquigarrow (u, w)$ is a legal move that keeps the mutual visibility property. In this sense, we readily see that there exists a sequence of legal move that allows us to visit all the vertices of $G \circ H$. Therefore, B is a mobile mutual-visibility set and so,

$$\mathrm{Mob}_\mu(G \circ H) \geq \mu_t(G \circ H) = n(G)(n(H) - 1) + \mu_t(G). \tag{4.2}$$

Therefore, the general upper bound of the theorem together with (4.2), lead to the equality $\mathrm{Mob}_\mu(G \circ H) = \mu_t(G \circ H) = n(G)(n(H)-1) + \mu_t(G)$, which completes the proof. □

We remark that the condition $n(G) < n(H) + 1$ in the theorem above is only required for the general upper bound of the theorem. We have also noted that there are several other cases in which this upper bound is true although the condition $n(G) < n(H) + 1$ would not be satisfied. However, we have not been able to identify all the necessary conditions required by G and/or H so that this upper bound is satisfied. Some of the following results show this situation.

We next focus our attention on the special case of lexicographic products whose second factor is an empty graph N_s, for some $s \geq 2$, and shall prove that there are some of such lexicographic products that attain either the lower, or the upper, or none of the bounds given in Theorem 4.4. We recall that if the first factor G is K_2, then $G \circ N_s$ is a complete bipartite graph $K_{s,s}$, which were studied previously. Thus, from now on we consider the first factor has order at least 3.

Proposition 4.5. *For any integer $r \geq 3$ and $s \geq 2$,*

$$\mathrm{Mob}_\mu(P_r \circ N_s) = rs - r + 1.$$

Proof. Let $P_r = u_1 u_2 \cdots u_r$ taken in the usual way for the adjacency of vertices. Let $v, w \in V(N_s)$ be two fixed vertices. We claim that the set $S = (\{u_1, u_r\} \times V(N_s)) \cup (\{u_2, \ldots, u_{r-1}\} \times (V(N_s) \setminus \{v\})) \setminus \{(u_2, w)\}$ is a mobile mutual-visibility set for $P_r \circ N_s$. Since each copy of N_s corresponding to all the vertices in $\{u_2, \ldots, u_{r-1}\}$ is not whole included in S, it follows that any two vertices of S are S-visible. Thus, S is a mutual-visibility set.

If $r = 3$, then for any $y \in V(N_s)$ the sequence $(u_1, y) \rightsquigarrow (u_2, v) \rightsquigarrow (u_1, y) \rightsquigarrow (u_2, w)$ is a sequence of legal moves, since in every step the condition for the corresponding set of vertices occupied by a robot is a mutual-visibility set. This is based on the fact that in the copy of N_s corresponding to u_2, there are two vertices not included in S in the first setting. As a consequence, S is a mobile mutual-visibility set as claimed, and so $\mathrm{Mob}_\mu(P_r \circ N_s) \geq r(s-2) + 4$.

Assume now that $r \geq 4$. By using similar arguments as above, for the vertex $w \in V(N_s)$, the sequence $(u_3, w) \rightsquigarrow (u_2, v) \rightsquigarrow (u_3, w) \rightsquigarrow (u_2, w)$ is a sequence of legal moves (note that we involve here the copies of N_s corresponding to u_2, u_3). At this moment, the new set of vertices occupied by the robots is $S = (\{u_1, u_r\} \times V(N_s)) \cup (\{u_2, \ldots, u_{r-1}\} \times (V(N_s) \setminus \{v\}) \setminus \{(u_3, w)\}$. Next we repeat the process, but involving the copies of N_s corresponding to u_3, u_4, instead of u_2, u_3 (if this is possible, based on the value of r). At the final step, when considering the copies of N_s corresponding to u_{r-2}, u_{r-1}, we need to make an extra move which is $(u_{r-2}, w) \rightsquigarrow (u_{r-1}, v)$. At each step, the condition of being a mutual-visibility set remains, and we can visit all the vertices of $P_r \circ N_s$. Thus, S is mobile mutual-visibility set, and therefore, $\mathrm{Mob}_\mu(P_r \circ N_s) \geq r(s-2) + 4 + r - 3$.

On the other hand, for a set of vertices in $P_r \circ N_s$ to be a mutual-visibility set, we can have at most all but one vertex from each copy of N_s corresponding to the vertices in $\{u_2, \ldots, u_{r-1}\}$ together with all the vertices in the copies corresponding to u_1, u_r. However, if we consider a set having the maximum possible of such set, then we cannot perform any legal move to visit all the vertices of $P_r \circ N_s$. Hence, at least one vertex needs to be excluded from such set. Thus, this means precisely that $\mathrm{Mob}_\mu(P_r \circ N_s) \leq r(s-2)+4+r-3$, which gives the equality. □

Notice that in the result above, if $r = 3$, then we have $\mathrm{Mob}_\mu(P_r \circ N_s) = r(s-2)+4 = n(P_r)(n(N_s) - 2) + 2\mu_t(P_r)$, since $\mu_t(P_r) = 2$, which means that the lower bound of Theorem 4.4 is achieved. Moreover, from the cases $r \geq 4$, we see that none of the (lower or upper) bounds are achieved. It remains to prove that the upper can also be achieved when the second factor of the product is an empty graph. To this end, we need the concept of *mobile total mutual-visibility sets* analogously defined to the one of mobile mutual-visibility sets.

In this sense, we say that a graph G is a *total-mobile graph* if it has mobile total mutual-visibility set of cardinality $\mu_t(G) > 0$.

An example of such graphs is for instance the direct product of two complete graphs K_r and K_s with $r, s \geq 5$. It was proved in [12], that $\mu_t(K_r \times K_s) = rs - 4$ and the set $V(K_r \times K_s) \setminus \{(1,1),(2,2),(3,3),(4,4)\}$ (where the vertices of K_n is taken as $[n]$) is a maximum total mutual-visibility set of such graph. It can be noted that the sequence $(5,5) \rightsquigarrow (1,1) \rightsquigarrow (2,2) \rightsquigarrow (3,3) \rightsquigarrow (4,4)$ is a sequence of legal moves that keep the property of the corresponding sets being a total mutual-visibility set. Thus, $K_r \times K_s$ is a total-mobile graph.

Theorem 4.6. *Let G be a graph without universal vertices and let $s \geq 2$ be an integer such that $n(G) < s + 1$. Then $\mathrm{Mob}_\mu(G \circ N_s) = n(G)(s-1) + \mu_t(G)$ if and only if G is a total-mobile graph.*

Proof. Assume first that G is a total-mobile graph, and let A be a mobile total mutual-visibility set of G and let $w \in V(N_s)$. We claim that the set $S = (A \times V(N_s)) \cup ((V(G) \setminus A) \times (V(N_s) \setminus \{w\}))$ is a mobile mutual-visibility set of $G \circ N_s$.

First, since A is a total mutual-visibility set of G, we have that S is a mutual-visibility set of $G \circ N_s$ (indeed such set is a total mutual-visibility set as proved in [21]). Now, since $\mu_t(G) > 0$, we consider a sequence Q of legal moves in G which keeps the total mobility properties. At each step of this process, the corresponding set of vertices A' is a total mutual-visibility set, and so, by using the result from [21] (as mentioned), we deduce that such each $S' = (A' \times V(N_s)) \cup ((V(G) \setminus A') \times (V(N_s) \setminus \{w\}))$ is also a mutual-visibility set of $G \circ N_s$. Thus, by using the sequence Q in G we can define a sequence Q' in $G \circ N_s$ (just by taking the sequence Q in the copy of G corresponding to the vertex $w \in V(N_s)$) that is a sequence of legal moves in $G \circ N_s$, and we can visit all the vertices of $G \circ N_s$. Therefore, S is a mobile mutual-visibility set of $G \circ N_s$ as claimed, and so, $\mathrm{Mob}_\mu(G \circ N_s) \geq n(G)(s-1) + \mu_t(G)$. The equality then follows from Theorem 4.4, since $n(G) < s - 1$ is an assumption of the result.

On the other hand, assume that $\mathrm{Mob}_\mu(G \circ N_s) = n(G)(s-1) + \mu_t(G)$, with $\mu_t(G) > 0$, and let S be a mobile mutual-visibility set of $G \circ N_s$. We need first to show that the projection $S_G = \{u \in V(G) : (u,v) \in S \text{ for every } v \in V(N_s)\}$ is a total mutual-visibility set of G. If this is not the case, then there are two vertices $x, y \in V(G)$ that are not

S_G-visible. But then, any two vertices $(x, z), (y, z) \in V(G \circ N_s)$ are not S-visible, which is not possible.

Next, we need to show also that $|S \cap (\{u\} \times V(N_s))| \geq s - 1$ for every $u \in V(G)$. Suppose that this is not the case. Hence, since $\mathrm{Mob}_\mu(G \circ N_s) = |S| = n(G)(s-1) + \mu_t(G)$, it must happen that $|S_G| > \mu_t(G)$, or equivalently, S_G is not a total mutual-visibility set of G, which is a contradiction. Therefore, S_G is a μ_t-set of G. It remains to see that S_G is a mobile total mutual-visibility set of G.

Consider now a sequence R of legal moves in $G \circ N_s$. For any move $(x, y) \rightsquigarrow (x', y')$ from R, we define a move $x \rightsquigarrow x'$ in G. At each step of the moves in R, the corresponding set is a mutual-visibility set in $G \circ N_s$, and its corresponding projection over G is a total mutual-visibility set of G, by using the same arguments as above. This means that the sequence that is defined in G (from the one in $G \circ N_s$) is a sequence of legal moves in G that keeps the total mutual-visibility properties. Thus, the set S_G is a mobile total mutual-visibility set of G, and G is a total mobile graph, which completes the proof. □

An interesting consequence of Theorem 4.4 relates to the decision problem concerning finding the mobile mutual-visibility number of graphs, which is next stated. In connection with this, notice that it is not exactly on whether this problem belongs to the NP class, since checking that a mobile mutual-visibility set is indeed such, depends on finding some strategy for moving the robots placed at vertices of the initial mutual-visibility set so that all the vertices of the graphs are visited at least once by at least one robot.

MOBILE MUTUAL-VISIBILITY PROBLEM
Input: A connected graph $G = (V, E)$ and $k \leq n(G)$.
Question: Is $\mathrm{Mob}_\mu(G) \geq k$?

Theorem 4.7. *The MOBILE MUTUAL-VISIBILITY PROBLEM is NP-hard.*

Proof. We consider any connected graph G without universal vertices, and a non-empty graph H satisfying that $n(G) < n(H) + 1$. From Theorem 4.4, we know that $\mathrm{Mob}_\mu(G \circ H) = n(G)(n(H) - 1) + \mu_t(G)$. Since the decision problem concerning finding $\mu_t(G)$ is NP-complete, as shown in [9], and we have a polynomial reduction from this problem to our Mobile Mutual-Visibility Problem, we hence complete our proof. □

5 Line graphs of complete graphs

If G is a graph, then its *line graph*, which is denoted $L(G)$, is the graph with vertex set $V(L(G)) = \{e_{uv} : uv \in E(G)\}$. Two vertices $e_{uv}, e_{u'v'} \in V(L(G))$ are adjacent whenever the two edges $uv, u'v'$ are incident in the original graph G. For a given set of edges $F \subseteq E(G)$, the set of vertices $e_{uv} \in V(L(G))$ such that $uv \in F$ is written as S_F. In addition, by G_F we denote the edge-induced subgraph of G based on F, i.e., the edges in F together with any vertices that are their endpoints.

The mutual-visibility number of $L(K_n)$ for any $n \geq 3$ was studied in [12]. Notice that $L(K_n)$ is a $(2n-4)$-regular graph of order $\frac{n(n-1)}{2}$ and diameter two. This last fact means that, while considering the mutual-visibility of two vertices of $L(K_n)$, we only need to care about vertices that are at distance two, or equivalently, of edges that are not incident in the original K_n. We next illustrate this using the graphs $L(K_4)$ and $L(K_5)$. It was proved in [12] that $\mu(L(K_4)) = 5$ and that $\mu(L(K_5)) = 8$. For the case of K_4, any set of five edges becomes a μ-set of $L(K_4)$. Thus, in this case it is easy to see that $\mathrm{Mob}_\mu(L(K_4)) = \mu(L(K_4)) = 5$. For the case of K_5, any set of 8 edges excluding two independent edges in K_5 becomes a μ-set of $L(K_5)$ (see Figure 2 (a) for an example). We now readily observe that some black edges can move to the red ones so that the mutual-visibility property remains all the time among the black edges. Thus, we conclude that $\mathrm{Mob}_\mu(L(K_5)) = \mu(L(K_5)) = 8$.

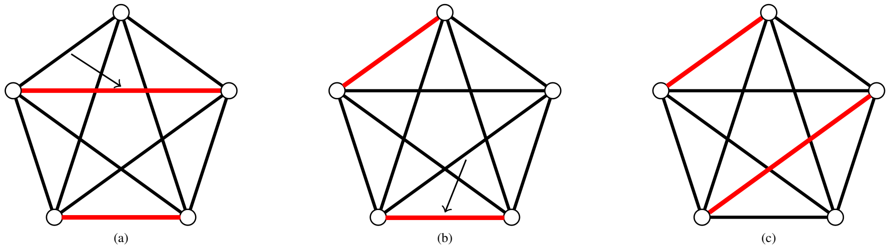

Figure 2: The graph K_5 where a μ-set is represented by the black edges in each of the configurations, and the corresponding moves to achieve one configuration from the other one.

In the examples above, it happens that $\mathrm{Mob}_\mu(L(K_n)) = \mu(L(K_n))$ if $n \leq 5$ (if $n = 3$, then it is trivial to see this fact). We next show that these are the only cases in which this situation occurs. To see this, we first need the following results from [12], where $T(n, 3)$ represents the Turán graph, which is a complete 3-partite graph of order n in which the sizes of the 3 parts are as equal as possible.

Theorem 5.1 ([12]). *Let $n \geq 3$ be an integer and $F \subseteq E(K_n)$. Then $S_F \subseteq V(L(K_n))$ is a μ-set of $L(K_n)$ if and only if $(K_n)_F \cong T(n, 3)$.*

Another point of view in the result above, and based on the definition of Turán graphs, is that a set $S_F \subseteq V(L(K_n))$ is a μ-set of $L(K_n)$ if and only if the subgraph of K_n generated by the edges in F together with the vertices in the edges of F, does not have a complete graph K_4 as an induced subgraph. With the tool that gives us Theorem 5.1, we are able to show the following.

Theorem 5.2. *If $n \geq 6$ is an integer, then $\mathrm{Mob}_\mu(L(K_n)) < \mu(L(K_n))$.*

Proof. Let $S_F \subseteq V(L(K_n))$ be a μ-set of $L(K_n)$ for some $F \subset E(K_n)$. By Theorem 5.1, $(K_n)_F \cong T(n, 3)$, which means $(K_n)_F$ represents a complete 3-partite graph of the largest size (number of edges) and order n in which the cardinalities of the 3 parts are as equal as possible. Let A, B, C be the partite sets of $(K_n)_F$. Note that for any edge $xy \in F$, it holds that x, y belong to different partite sets of $(K_n)_F$. Assume $x \in A$ and $y \in B$.

Now, in order for a vertex $e_{x'y'} \notin S_F$ to be visited by a robot placed at the vertex $e_{xy} \in S_F$ (notice that either $x = x'$, or $y = y'$; assume $x = x'$), we need to move the robot e_{xy} to $e_{x'y'}$ and the new set $S_{F'} = S_F \setminus \{e_{xy}\} \cup \{e_{x'y'}\}$ must be a μ-set of $L(K_n)$ as well. However, by Theorem 5.1, $(K_n)_{F'}$ must be isomorphic to $T(n, 3)$ too, that is, $(K_n)_{F'}$ must be a complete 3-partite graph of order n in which the cardinalities of the 3 parts are as equal as possible, as already mentioned. But, this is not possible since $n \geq 6$, which means the cardinalities of A, B and C are at least two, and so, there are vertices

$b \in B \setminus \{y\}$ and $c \in C$ such that all the edges between the four vertices $x = x', y', b, c$ (note that $y' \in A$) belong to F', and they induce a K_4, which is a subgraph of $(K_n)_{F'}$, a contradiction.

As a consequence of this, we deduce that S_F cannot be a mobile mutual-visibility set of $L(K_n)$, and the bound follows. □

6 Prism graphs

The *prism* of a graph G is obtained as the Cartesian product of G and a complete graph K_2. The Cartesian product of two graphs G and H is the graph $G \square H$ with vertex set $V(G \square H) = V(G) \times V(H)$, where two vertices (x, y) and (x', y') are adjacent if and only if either $x = x'$ and $yy' \in E(H)$, or $xx' \in E(G)$ and $y = y'$. We first observe the following bound.

Theorem 6.1. *For any graph G, $\mathrm{Mob}_\mu(G) + 1 \leq \mathrm{Mob}_\mu(G \square K_2) \leq 2\mu(G)$, and the bounds are tight.*

Proof. Let x, y be the two vertices of K_2. Let S be a mobile μ-set of G and let $w \in V(G)$ be any vertex. Let $S' = \{(u, x) : u \in S\} \cup \{(w, y)\}$. Note that any vertex (u, x) can be visited by legal moves using only the vertices of $S \times \{x\}$. Also, the choice of w does not influence the visibility of vertices from S', which means that (w, y) can visit all the vertices of $V(G) \times \{y\}$ by using legal moves. Thus, S' is a mobile mutual-visibility set, and the lower bound follows. To see the tightness, consider the complete graph $G = K_n$, $n \geq 2$ (notice that $\mu(K_n) = n$). In this case, if $\mathrm{Mob}_\mu(K_n \square K_2) \geq n + 2$ and S is a mobile μ-set, then there will be two vertices $u, v \in V(K_n)$ such that $(u, x), (u, y), (v, x), (v, y) \in S$. Thus, we readily observe that (for instance) $(u, x), (v, y)$ are not S-visible, which is a contradiction. Thus, $\mathrm{Mob}_\mu(K_n \square K_2) \leq n + 1$, and the equality follows.

The upper bound follows directly, since each copy of G cannot have more than $\mu(G)$ vertices in any mobile μ-set. To see that this upper bound is also tight we consider G as a cycle graph C_n with $n \geq 7$ (recall that $\mu(C_n) = 3$). Let $S = \{i, j, k\}$ (with $i < j < k$) be a μ-set of C_n such that the distance between the vertices in S is as equidistant as possible, and let $S' = S \times \{x, y\}$ where $V(K_2) = \{x, y\}$. First observe that S' is a mutual-visibility set of $C_n \square K_2$. Since $n \geq 7$, there are at least two vertices of S such that the distance between them is at least three. Without loss of generality i and j are such vertices. Now, we make the following moves $(i, x) \rightsquigarrow (i + 1, x)$ and $(i, y) \rightsquigarrow (i + 1, y)$, and we indeed notice that these are legal moves since we obtain a new mutual-visibility set of $C_n \square K_2$. By repeating this process with k instead of i and later with j instead of k, we always obtain a mutual-visibility set of $C_n \square K_2$ (operations with i, j, k are done modulo n). A cyclic repetition of these procedures will allow us to visit all of the vertices of $C_n \square K_2$ by using legal moves. Thus, S' is a mobile mutual-visibility set of $C_n \square K_2$ of cardinality $2\mu(C_n)$, which gives the equality in the upper bound. □

Next, some other examples showing the tightness of the bounds of Theorem 6.1 are given.

Proposition 6.2. *For any path P_n,* $\mathrm{Mob}_\mu(P_n \square K_2) = \begin{cases} 3, & \text{if } 2 \leq n \leq 3, \\ 4, & \text{if } n \geq 4. \end{cases}$

Proof. If $n = 2$, then $P_n \square K_2 = C_4$, and we have the conclusion from Proposition 2.1. If $n = 3$, then by some simple calculations we get the conclusion. Hence, assume $n \geq 4$, and consider the set S of vertices formed by all the vertices of degree 2 in $P_n \Box K_2$. Let $V(P_n) = [n]$ and $V(K_2) = [2]$. Hence, we readily observe that the sequences defined at next are sequences of legal moves.

- $(1, i) \rightsquigarrow (2, i) \rightsquigarrow \cdots \rightsquigarrow (n-2, i) \rightsquigarrow (n-3, i) \rightsquigarrow \cdots \rightsquigarrow (1, i)$ with $i \in [2]$,
- $(n, i) \rightsquigarrow (n-1, i)$ with $i \in [2]$.

Thus, S is a mobile mutual-visibility set of $P_n \Box K_2$, and so, $\mathrm{Mob}_\mu(P_n \Box K_2) \geq 4$.

On the other hand, if $\mathrm{Mob}_\mu(P_n \Box K_2) \geq 5$, then there will be three vertices from a mobile mutual-visibility set S' in one of the copies of the path P_n, and so, two of them will not be S'-visible, which is not possible. Therefore $\mathrm{Mob}_\mu(P_n \Box K_2) = 4$. □

Notice that there are graphs G for which the bounds of Theorem 6.1 are not achieved, taking into account that for instance, $\mu(K_{1,n}) = n$ and $\mathrm{Mob}_\mu(K_{1,n}) = 2$.

Proposition 6.3. *For any integer* $n \geq 3$, $\mathrm{Mob}_\mu(K_{1,n} \Box K_2) = n + 1$.

Proof. Let $V(K_{1,n}) = [n] \cup \{0\}$ where 0 is the central vertex, and let $V(K_2) = [2]$. Let $S = ([n] \times \{1\}) \cup \{(0, 2)\}$. Hence, we note that the sequences defined below are sequences of legal moves.

- $(0, 2) \rightsquigarrow (n, 2)$,
- $(i, 1) \rightsquigarrow (i, 2)$ for every $i \in [n-1]$,
- $(n, 1) \rightsquigarrow (0, 1)$.

Thus, S is a mobile mutual-visibility set of $K_{1,n} \Box K_2$ and so, $\mathrm{Mob}_\mu(K_{1,n} \Box K_2) \geq n + 1$.

Suppose for contradiction that $\mathrm{Mob}_\mu(K_{1,n} \Box K_2) > n + 1$ and let S' be a mobile μ-set. This means that there is at least one $i \in \{0\} \cup [n]$ such that both $(i, 1), (i, 2) \in S'$. Consider now a moment in which one robot visits a central vertex $(0, j)$ for some $j \in [2]$, say for instance $(0, 1)$. Let S'' be the set of vertices occupied by the robots at this moment. If there are two vertices $(j, 1), (j', 1) \in S''$, then they are not S''-visible since the unique $(j, 1), (j', 1)$-shortest path in $K_{1,n} \Box K_2$ passes through $(0, 1)$, a contradiction. Thus, there can be at most one vertex $(j, 1) \in S''$ for some $j \in [n]$. We have the following situations.

Case 1: Assume that $(j, 1) \in S''$ with $j \in [n]$. If $(j, 2) \in S''$, then $(j, 1)$ is not S''-visible with every other vertex $x \in S'' \setminus \{(j, 1), (0, 1), (j, 2)\}$, which is not possible (notice that there is at least one other vertex different from $(j, 1), (0, 1), (j, 2)$ since $n \geq 3$ and we have assumed that $|S''| = \mathrm{Mob}_\mu(K_{1,n} \Box K_2) > n + 1 \geq 4$). Thus, it holds that $(j, 2) \notin S''$. If $(0, 2) \in S''$, then again $(j, 1)$ is not S''-visible with every other vertex $x \in S'' \setminus \{(j, 1), (0, 1), (0, 2)\}$, and this is also not possible. Consequently, we deduce that there is no $i \in \{0\} \cup [n]$ such that both $(i, 1), (i, 2) \in S'$, which contradicts our assumption.

Case 2: There is no vertex $(j, 1) \in S''$ for every $j \in [n]$. This means that the integer $i \in \{0\} \cup [n]$ such that both $(i, 1), (i, 2) \in S'$ (assumed at the beginning) is precisely $i = 0$. Moreover, for every $i \in [n]$ it holds that $(i, 2) \in S''$ since we have supposed that $\mathrm{Mob}_\mu(K_{1,n} \Box K_2) > n + 1$. But then, each two distinct vertices $(j, 2), (j', 2) \in S''$ are not S''-visible, and this is a contradiction. Again, we deduce that there cannot be $i \in \{0\} \cup [n]$ such that both $(i, 1), (i, 2) \in S'$, a final contradiction with our assumption. □

7 The strong grid

We consider in this section the strong product of two paths. To this end, we first recall that the *strong product* of two graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$. Two vertices $(g,h),(g',h')$ are adjacent in $G \boxtimes H$ if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$, or $gg' \in E(G)$ and $hh' \in E(H)$.

First notice that, if H is isomorphic to K_2, then $G \boxtimes H$ is isomorphic to $G \circ H$. Thus, from Theorem 4.4 we deduce the following result.

Corollary 7.1. *For any connected graph G, $\text{Mob}_\mu(G \boxtimes K_2) = n(G) + \mu_t(G)$. In particular, for any $r \geq 2$, $\text{Mob}_\mu(P_r \boxtimes P_2) = r + 2$.*

We shall consider in this section the strong product of two paths P_r and P_s. In view of the corollary above, we must assume $r, s \geq 3$. In order to simplify the writing, from now on assume $V(P_n) = [n]$ and that $V(P_r \boxtimes P_s) = [r] \times [s]$. For a given vertex $(i,j) \in V(P_r \boxtimes P_s)$, we denote by $D^+(i,j)$ the largest set of vertices of the form $\{(i',j') : i - i' = j - j'\}$ and call it the *increasing diagonal* through (i,j). Similarly, the *decreasing diagonal* through (i,j) is the largest convex set $D^-(i,j) = \{(i',j') : i - i' = j' - j\}$. Also, by the *border* of $P_r \boxtimes P_s$, we mean the set of vertices $B = (\{1,r\} \times [s]) \cup ([r] \times \{1,s\})$. See Figure 3 for clarifying examples.

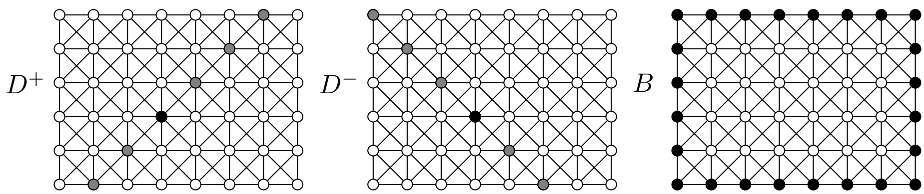

Figure 3: The strong grid $P_8 \boxtimes P_6$ with the sets $D^+(4,3)$, $D^-(4,3)$ (illustrated in gray), and the border B respectively (illustrated in black). Note that the vertex $(4,3)$ belongs to both diagonals, although we have drawn it in bold.

The mutual-visibility number of the strong grid graphs was studied in [11], where the following result was proved.

Theorem 7.2 ([11])**.** *For any integers $r, s \geq 3$, $\mu(P_r \boxtimes P_s) = 2r + 2s - 4$.*

To prove the result above, the authors of [11] have shown that the border of $P_r \boxtimes P_s$, i.e., $B = (\{1,r\} \times [s]) \cup ([r] \times \{1,s\})$ is a μ-set of $P_r \boxtimes P_s$. Moreover, in the proof of this result it is implicitly shown that this set is in fact the unique μ-set of $P_r \boxtimes P_s$.

Remark 7.3. If $r, s \geq 3$, then the graph $P_r \boxtimes P_s$ has a unique μ-set, and this set is the border of $P_r \boxtimes P_s$.

We are now prepared to present the main result of this section.

Theorem 7.4. *For any integers $r \geq 4$ and $s \geq 3$,*

$$2r + 2s - 6 \leq \text{Mob}_\mu(P_r \boxtimes P_s) \leq 2r + 2s - 5.$$

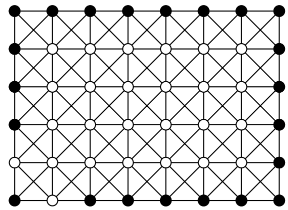

Figure 4: The strong grid $P_8 \boxtimes P_6$ with the set S in bold.

Proof. We claim that the set $S = (\{1, r\} \times [s]) \cup ([r] \times \{1, s\}) \setminus \{(2, 1), (1, 2)\} = B \setminus \{(2, 1), (1, 2)\}$ is a mobile mutual-visibility set of $P_r \boxtimes P_s$. See Figure 4 for an example.

Clearly, since S is a proper subset of the border of $P_r \boxtimes P_s$, proved in [11] to be a μ-set of $P_r \boxtimes P_s$, we deduce that S is a mutual-visibility set of $P_r \boxtimes P_s$.

To prove that S is a mobile mutual-visibility set of $P_r \boxtimes P_s$, we need to develop a sequence of legal moves that preserve the mutual-visibility property. To this end, the first idea is that the robot placed at $(1, 1)$ will move through the vertices of the increasing diagonal $D^+(1, 1)$, but to do so, we need to make some other legal moves with some of the other vertices of S.

We must observe that "circular" moves from the robots in S made along the border B in $P_r \boxtimes P_s$ keep the mutual-visibility properties. For instance a sequence like $(3, 1) \rightsquigarrow (2, 1)$, $(4, 1) \rightsquigarrow (3, 1), \ldots, (r, 1) \rightsquigarrow (r - 1, 1)$, $(r, 2) \rightsquigarrow (r, 1)$, etc. is such a circular sequence of moves.

Now, in order that we make the move $(1, 1) \rightsquigarrow (2, 2)$, we need first to make the circular move $(3, 1) \rightsquigarrow (2, 1)$ or $(1, 3) \rightsquigarrow (1, 2)$. Assume, we make $(3, 1) \rightsquigarrow (2, 1)$. Observe that this means the following. To be able to make the move $(1, 1) \rightsquigarrow (2, 2)$, we need to have one of the ends of the decreasing diagonal $D^-(2, 2)$ not occupied by a robot. The new set of vertices occupied by the robots is readily seen to be a mutual-visibility set. Next, we need to make the move $(2, 2) \rightsquigarrow (3, 3)$, and to do so, we need to have one of the ends ($(5, 1)$ or $(1, 5)$) of the decreasing diagonal $D^-(3, 3)$ not occupied by a robot. However, in this case we cannot make the move $(4, 1) \rightsquigarrow (3, 1)$ because then the vertices $(3, 1)$ and $(1, 3)$ would not be visible. Thus, we need to do the circular moves $(1, 3) \rightsquigarrow (1, 2)$, $(1, 4) \rightsquigarrow (1, 3)$ and $(1, 5) \rightsquigarrow (1, 4)$, and then we can make $(2, 2) \rightsquigarrow (3, 3)$. See Figure 5 for these moves.

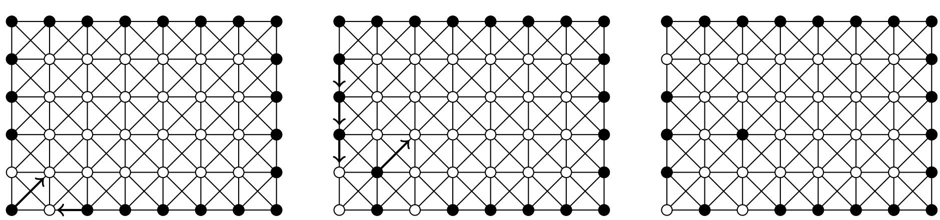

Figure 5: The strong grid $P_8 \boxtimes P_6$. (left hand side): the original set S in bold and arrows showing the moves $(3, 1) \rightsquigarrow (2, 1)$ and $(1, 1) \rightsquigarrow (2, 2)$; (center): the set S' once the two moves have been made and arrows showing the moves $(1, 3) \rightsquigarrow (1, 2)$, $(1, 4) \rightsquigarrow (1, 3)$, $(1, 5) \rightsquigarrow (1, 4)$, and $(2, 2) \rightsquigarrow (3, 3)$; (right hand side): the set S'' once the last moves have been made.

The next move through the increasing diagonal $D^+(1,1)$ must be $(3,3) \rightsquigarrow (4,4)$. To make this, we need one of the ends of the decreasing diagonal $D^-(4,4)$ not occupied by a robot. Now, we cannot make circular moves from the side we used in the previous step. Instead, we need to make the circular moves from the other direction. That is, $(4,1) \rightsquigarrow (3,1)$, $(5,1) \rightsquigarrow (4,1)$, $(6,1) \rightsquigarrow (5,1)$, and $(7,1) \rightsquigarrow (6,1)$. After this, we can make the move $(3,3) \rightsquigarrow (4,4)$. Notice that at each of these moves, the mutual-visibility properties are preserved.

For a next round of moves, that is, $(4,4) \rightsquigarrow (5,5)$, we will need to make circular moves from the other side than the one used. This process can be repeated until the robot that is moving in the increasing diagonal arrives to a vertex (i,i) in which the next vertex $(i+1,i+1)$ is occupied by a robot. At this point, all the vertices of the increasing diagonal $D^+(1,1)$ have been visited by a sequence of legal moves keeping the mutual-visibility properties. Then, we turn back all the moves made (in strictly opposite order) until we achieve the original configuration of the robots placed at the set S.

Now, to visit the vertices of any other increasing diagonal $D^+(i,j)$ where either $i=1$ or $j=1$, we first make some circular moves so that (i,j) is occupied by a robot and the two neighbors of (i,j) from the border B are not occupied by a robot. For instance, if we want to visit the vertices of the increasing sequence $D^+(4,1)$, we need the circular moves $(1,1) \rightsquigarrow (1,2)$, $(3,1) \rightsquigarrow (2,1)$, $(2,1) \rightsquigarrow (1,1)$, $(4,1) \rightsquigarrow (3,1)$, $(3,1) \rightsquigarrow (2,1)$, and $(5,1) \rightsquigarrow (4,1)$.

At this configuration, the robot placed at $(4,1)$ will visit all the vertices from the increasing sequence $D^+(4,1)$, but at each step, a sequence of circular moves need to be done so that the mutual-visibility properties will remain. As an example, the first move that the robot $(4,1)$ needs to do is $(4,1) \rightsquigarrow (5,2)$. In this sense, one need to have one of the ends of the decreasing diagonal $D^-(5,2)$ not occupied by a robot. Thus, we can make the move $(6,1) \rightsquigarrow (5,1)$, see Figure 6 for these configurations and moves.

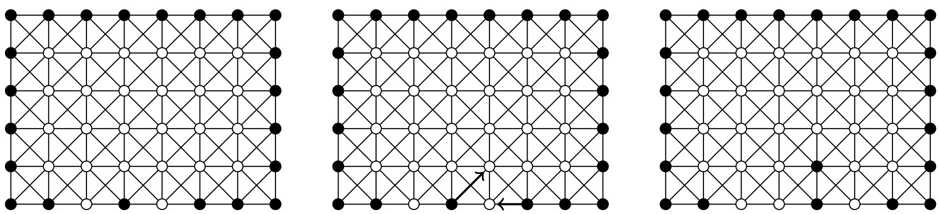

Figure 6: The strong grid $P_8 \boxtimes P_6$. (left hand side): the modified set S in bold whether we want to visit the vertices of the increasing diagonal $D^+(4,1)$; (center): the set S and arrows showing the moves $(6,1) \rightsquigarrow (5,1)$ and $(4,1) \rightsquigarrow (5,2)$; (right hand side): the set S' once these last moves have been made.

In this sense, we observe that all the vertices of any increasing diagonal $D^+(i,j)$ where either $i=1$ or $j=1$ can be visited by a robot beginning with the original set $S = (\{1,r\} \times [s]) \cup ([r] \times \{1,s\}) \setminus \{(2,1),(1,2)\} = B \setminus \{(2,1),(1,2)\}$. Therefore, S is a mobile mutual-visibility set of $P_r \boxtimes P_s$ as claimed, and so, $\mathrm{Mob}_\mu(P_r \boxtimes P_s) \geq 2r+2s-6$.

On the other hand, from (1.1) and Theorem 7.2, we deduce that $\mathrm{Mob}_\mu(P_r \boxtimes P_s) \leq 2r+2s-4$. Now, from Remark 7.3 we know that $P_r \boxtimes P_s$ has a unique μ-set given by the border B of $P_r \boxtimes P_s$. This means that no vertex from B can make a legal move. Thus, $\mathrm{Mob}_\mu(P_r \boxtimes P_s) \leq 2r+2s-5$, which is the upper bound. □

The general position number of the strong grid $P_r \boxtimes P_s$ was studied in [23], where it was proved to be equal to 4. Thus, clearly $\mathrm{Mob_{gp}}(P_r \boxtimes P_s) \leq 4$. Based on the result above, we see that the strong grids are another non-trivial example of graphs for which the difference $\mathrm{Mob}_\mu(G) - \mathrm{Mob_{gp}}(G)$ is arbitrarily large.

8 Open problems

We close with some open problems that we have yet to settle.

(P1) An example given in the introductory section suggests studying the mobile mutual-visibility number of the hypercubes. Is it the case that $\mathrm{Mob}_\mu(Q_d) = \mu(Q_d)$ for any $d \geq 3$?

(P2) Finding the characterizations of the classes of graphs G such that $\mathrm{Mob}_\mu(G) = \mathrm{Mob_{gp}}(G)$ or $\mathrm{Mob}_\mu(G) = \mu(G)$.

(P3) In Section 4 we have studied the lexicographic product of graphs G and H whether the condition $n(G) < n(H) + 1$ is satisfied. In particular, we have proved that $\mathrm{Mob}_\mu(G \circ H) \leq n(G)(n(H) - 1) + \mu_t(G)$ follows in this situation. It hence remain to consider bounding (or computing) the value of $\mathrm{Mob}_\mu(G \circ H)$ in general, or at least finding the required properties of G and/or H in order that such upper bound would be fulfilled.

(P4) We have proved in Section 4 that the decision problem concerning finding the mobile mutual-visibility number of graphs is NP-hard, since it is not clear on whenever deciding that a given set of vertices is a mobile mutual-visibility set or not belongs to the NP class. In this sense, the following question is of interest. Which is the complexity of checking that a given set of vertices of a graph is a mobile mutual-visibility set or not?

(P5) By Theorem 5.2 we know that if $n \geq 6$ is an integer, then $\mathrm{Mob}_\mu(L(K_n)) < \mu(L(K_n))$. A fundamental question is now how large the difference $\mu(L(K_n)) - \mathrm{Mob}_\mu(L(K_n))$ can be.

(P6) In Theorem 7.4 we have proved that $\mathrm{Mob}_\mu(P_r \boxtimes P_s)$ can only take two possible values. Are both of them possible or it is always one of them? In relation with this family of graphs, it would also be of interest to know $\mathrm{Mob}_\mu(G)$ whether G is a grid (the Cartesian version).

ORCID iDs

Magda Dettlaff https://orcid.org/0000-0002-7296-1893
Magdalena Lemańska https://orcid.org/0000-0002-0924-9924
Juan A. Rodríguez-Velázquez https://orcid.org/0000-0002-9082-7647
Ismael G. Yero https://orcid.org/0000-0002-1619-1572

References

[1] B. Brešar and I. G. Yero, Lower (total) mutual-visibility number in graphs, *Appl. Math. Comput.* **465** (2024), Paper No. 128411, 11 pp., doi:10.1016/j.amc.2023.128411, `https://doi.org/10.1016/j.amc.2023.128411`.

[2] C. Bujtás, S. Klavžar and J. Tian, Total mutual-visibility in hamming graphs, *Opuscula Math.* **45** (2025), 63–78, doi:10.7494/OpMath.2025.45.1.63, https://doi.org/10.7494/OpMath.2025.45.1.63.

[3] M. Cera López, P. García-Vázquez, J. C. Valenzuela-Tripodoro and I. G. Yero, The k-distance mutual-visibility problem in graphs, *Bull. Malays. Math. Sci. Soc.* **48** (2025), Paper No. 25.

[4] S. U. Chandran and G. J. Parthasarathy, The geodesic irredundant sets in graphs, *Int. J. Math. Comb.* **4** (2016), 135.

[5] S. Cicerone, A. Di Fonso, G. Di Stefano and A. Navarra, The geodesic mutual visibility problem for oblivious robots: the case of trees, in: *Proceedings of the 24th International Conference on Distributed Computing and Networking*, pp. 150–159, 2023.

[6] S. Cicerone, A. Di Fonso, G. Di Stefano and A. Navarra, Time-optimal geodesic mutual visibility of robots on grids within minimum area, in: *Stabilization, Safety, and Security of Distributed Systems*, Springer, Cham, volume 14310 of *Lecture Notes in Comput. Sci.*, pp. 385–399, 2023, doi:10.1007/978-3-031-44274-2_29, https://doi.org/10.1007/978-3-031-44274-2_29.

[7] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra and F. Piselli, Mutual visibility in hypercube-like graphs, in: *Structural Information and Communication Complexity*, Springer, Cham, volume 14662 of *Lecture Notes in Comput. Sci.*, pp. 192–207, 2024, doi:10.1007/978-3-031-60603-8_11, https://doi.org/10.1007/978-3-031-60603-8_11.

[8] S. Cicerone and G. Di Stefano, Mutual-visibility in distance-hereditary graphs: a linear-time algorithm, *Procedia Comput. Sci.* **223** (2023), 104–111.

[9] S. Cicerone, G. Di Stefano, L. Drožđek, J. Hedžet, S. Klavžar and I. G. Yero, Variety of mutual-visibility problems in graphs, *Theoret. Comput. Sci.* **974** (2023), Paper No. 114096, 13 pp., doi:10.1016/j.tcs.2023.114096, https://doi.org/10.1016/j.tcs.2023.114096.

[10] S. Cicerone, G. Di Stefano and S. Klavžar, On the mutual visibility in Cartesian products and triangle-free graphs, *Appl. Math. Comput.* **438** (2023), Paper No. 127619, 9 pp., doi:10.1016/j.amc.2022.127619, https://doi.org/10.1016/j.amc.2022.127619.

[11] S. Cicerone, G. Di Stefano, S. Klavžar and I. G. Yero, Mutual-visibility in strong products of graphs via total mutual-visibility, *Discrete Appl. Math.* **358** (2024), 136–146, doi:10.1016/j.dam.2024.06.038, https://doi.org/10.1016/j.dam.2024.06.038.

[12] S. Cicerone, G. Di Stefano, S. Klavžar and I. G. Yero, Mutual-visibility problems on graphs of diameter two, *European J. Comb.* **120** (2024), Paper No. 103995, 16, doi:10.1016/j.ejc.2024.103995, https://doi.org/10.1016/j.ejc.2024.103995.

[13] G. A. Di Luna, P. Flocchini, S. Gan Chaudhuri, F. Poloni, N. Santoro and G. Viglietta, Mutual visibility by luminous robots without collisions, *Inform. and Comput.* **254** (2017), 392–418, doi:10.1016/j.ic.2016.09.005, https://doi.org/10.1016/j.ic.2016.09.005.

[14] G. Di Stefano, Mutual visibility in graphs, *Appl. Math. Comput.* **419** (2022), Paper No. 126850, 13 pp., doi:10.1016/j.amc.2021.126850, https://doi.org/10.1016/j.amc.2021.126850.

[15] V. Iršič, S. Klavžar, G. Rus and J. Tuite, General position polynomials, *Results Math.* **79** (2024), Paper No. 110, 16 pp., doi:10.1007/s00025-024-02133-3, https://doi.org/10.1007/s00025-024-02133-3.

[16] S. Klavžar, A. Krishnakumar, J. Tuite and I. G. Yero, Traversing a graph in general position, *Bull. Aust. Math. Soc.* **108** (2023), 353–365, doi:10.1017/s0004972723000102, https://doi.org/10.1017/s0004972723000102.

[17] S. Klavžar and G. Rus, The general position number of integer lattices, *Appl. Math. Comput.* **390** (2021), Paper No. 125664, 4 pp., doi:10.1016/j.amc.2020.125664, https://doi.org/10.1016/j.amc.2020.125664.

[18] J. Körner, On the extremal combinatorics of the Hamming space, *J. Comb. Theory Ser. A* **71** (1995), 112–126, doi:10.1016/0097-3165(95)90019-5, https://doi.org/10.1016/0097-3165(95)90019-5.

[19] D. Korže and A. Vesel, General position sets in two families of Cartesian product graphs, *Mediterr. J. Math.* **20** (2023), Paper No. 203, 12 pp., doi:10.1007/s00009-023-02416-z, https://doi.org/10.1007/s00009-023-02416-z.

[20] D. Korže and A. Vesel, Mutual-visibility sets in Cartesian products of paths and cycles, *Results Math.* **79** (2024), Paper No. 116, 20 pp., doi:10.1007/s00025-024-02139-x, https://doi.org/10.1007/s00025-024-02139-x.

[21] D. Kuziak and J. A. Rodríguez-Velázquez, Total mutual-visibility in graphs with emphasis on lexicographic and Cartesian products, *Bull. Malays. Math. Sci. Soc.* **46** (2023), Paper No. 197, 17 pp., doi:10.1007/s40840-023-01590-3, https://doi.org/10.1007/s40840-023-01590-3.

[22] P. Manuel and S. Klavžar, A general position problem in graph theory, *Bull. Aust. Math. Soc.* **98** (2018), 177–187, doi:10.1017/S0004972718000473, https://doi.org/10.1017/S0004972718000473.

[23] P. Manuel and S. Klavžar, The graph theory general position problem on some interconnection networks, *Fund. Inform.* **163** (2018), 339–350, doi:10.3233/fi-2018-1748, https://doi.org/10.3233/fi-2018-1748.

[24] J. Tian and S. Klavžar, Graphs with total mutual-visibility number zero and total mutual-visibility in Cartesian products, *Discuss. Math. Graph Theory* **44** (2024), 1277–1291.

[25] J. Tian and K. Xu, The general position number of Cartesian products involving a factor with small diameter, *Appl. Math. Comput.* **403** (2021), Paper No. 126206, 10 pp., doi:10.1016/j.amc.2021.126206, https://doi.org/10.1016/j.amc.2021.126206.