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Handling incomplete information in formal concept analysis - a possibilistic approach

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Abstract

It is very common to find databases with missing information. Therefore, it is important to develop formal tools. This paper focuses on the contribution of Formal Concept Analysis to this fundamental goal. For this purpose, five forms of attribute implications are extracted from incomplete contexts, which are analyzed from two different approaches. The first approach follows the traditional recipe taking into account the well-known characterizations about the validity of attribute implications. The second approach goes further by considering possibility theory. Specifically, possibility and necessity measures are defined in order to establish the plausibility and certainty of relevant statements pertaining to an incomplete context.

Keywords Possibility Theory · Formal Concept Analysis · Attribute Implications · Incomplete Information

Mathematics Subject Classification 03E72 · 03B52 · 03G10 · 06B05 · 06A15

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1 Introduction

Formal Concept Analysis (FCA) was introduced by Wille (1982) as a mathematical tool to extract information from relational datasets, which are represented as formal contexts composed of a set of objects, a set of attributes and a binary relation. In this framework, an important question is how to extract valid attribute implications (Cordero et al. 2013; Dubois et al. 2021; Ganter et al. 2005; Guigues and Duquenne 1986), expressing dependence among subsets of attributes, from formal contexts.

In order to analyze more complex data, non-binary relations are also considered, leading to many-valued contexts (Ganter and Wille 1999). An interesting particular case of them is the three-valued context understood as an incomplete context (Burmeister and Holzer 2000, 2005; Holzer 2004a, b), also known as partial formal contexts (Ganter and Meschke 2009). These contexts are meaningful to analyze incomplete data, since it is possible to express that an object and an attribute are known to be related, known to be unrelated or that their relationship is unknown. To this end, the use of completions of a partial context (Holzer 2004a) facilitates this kind of studies, since they decompose incomplete contexts into potential scenarios obtained by removing the lack of knowledge.

Independently, Possibility Theory (Zadeh 1978) was proposed by Zadeh as an extension of fuzzy restrictions (Zadeh 1975) given in Fuzzy Set Theory (Zadeh 1965). It was later extended to model incomplete information in terms of graded plausibility and certainty of events, by Dubois and Prade (1988).

This paper presents a novel point of view of incomplete information processing applying possibility theory to the FCA framework introducing new advantages in the area. The idea is to consider an incomplete context as a set of complete ones, one of which is the right one. We then can define the possibility and necessity measures of sets of complete contexts to study them with respect to a given incomplete context. Then, we will analyze the validity of attribute implications in incomplete contexts in terms of their validity in the completions of the given incomplete context. To this end, we will take advantage of possibility theory, determining the possibility and necessity of the implications under consideration. Therefore, we will focus on evaluating the possibility and necessity of attribute implications taking into account all possible scenarios where these implications are valid. We have also analyzed the relations with respect to other approaches, such as the ones given in Burmeister and Holzer (2000); Demin et al. (2012); Kriegel (2017); Obiedkov (2002); Pérez-Gámez et al. (2022).

The outline followed in the paper is shown below. Section 2 recalls preliminary notions about different kinds of contexts and derivation operators depending on the level of knowledge present in them. In Sect. 3 completions are introduced to decompose incomplete contexts into potential complete contexts. Then, Sect. 4 presents some basic notions of possibility theory and the extension of this framework to the FCA setting. Next, Sect. 5 analyzes the validity of attribute implications in incomplete contexts in terms of derivation operators and possibility theory. Section 6 collects previous works focused on the analysis of incomplete information in FCA and compares them with this proposal. Finally, different conclusions and lines of future work are exposed in Sect. 7.

2 Representation and extraction of knowledge in FCA

Relational datasets are represented as formal contexts in FCA. This fact is essential to analyze these datasets in mathematical terms, and as a result, to extract knowledge about the given relation by means of derivation operators.

Definition 1 A *formal context* is a tuple $FC = (X, A, R)$ where X and A are non-empty sets of objects and attributes, respectively, and R is a binary relation between them. In order to express, by means of relation R, the fact that an object $x \in X$ is known to possess an attribute $a \in A$, we write $(x, a) \in R$. Otherwise, we write $(x, a) \notin R$.

The *derivation operators over* FC are the mappings ${}^{\uparrow} : \mathcal{P}(X) \to \mathcal{P}(A)$ and ${}^{\downarrow} : \mathcal{P}(A) \to \mathcal{P}(X)$ defined, for every $Y \subseteq X$ and $B \subseteq A$, as:

$$Y^{\uparrow} = \{a \in A \mid (x, a) \in R, \text{ for all } x \in Y\}$$
$$B^{\downarrow} = \{x \in X \mid (x, a) \in R, \text{ for all } a \in B\}$$

Derivation operators associate each subset of objects (attributes, respectively) with the attributes (objects, respectively) related to them. Therefore, the use of these operators allows us to study in detail the formal context under consideration. Moreover, they satisfy some important properties, in particular they form an antitone Galois connection, which ensures that the partial order in the sets $\mathcal{P}(X)$ and $\mathcal{P}(A)$ is preserved, and their computation can be decomposed into simpler ones.

Proposition 2 (Ganter and Wille (1999)) *Let* $FC = (X, A, R)$ *be a formal context.*

- $({}^{\uparrow}, {}^{\downarrow})$ *is an antitone Galois connection, that is, the following equivalence is satisfied, for every* $Y \subseteq X$ *and* $B \subseteq A$*:*

$$B \subseteq Y^{\uparrow} \text{ if and only if } Y \subseteq B^{\downarrow}$$

- *The following equalities hold:*

$$Y^{\uparrow} = \left(\bigcup_{x \in Y} x\right)^{\uparrow} = \bigcap_{x \in Y} x^{\uparrow}, \text{ for all } Y \subseteq X$$
$$B^{\downarrow} = \left(\bigcup_{a \in B} a\right)^{\downarrow} = \bigcap_{a \in B} a^{\downarrow}, \text{ for all } B \subseteq A$$

where $x^{\uparrow} = \{a \in A \mid (x, a) \in R\}$ *and* $a^{\downarrow} = \{x \in X \mid (x, a) \in R\}$.

The following example illustrates a formal context and its features by using Definition 1 and Proposition 2.

Example 3 Let $FC = (X, A, R)$ be a formal context where $X = \{x_1, x_2, x_3\}$ is the set of objects, $A = \{a_1, a_2, a_3\}$ is the set of attributes and R is the relation between objects and attributes displayed in Table 1.

A cross in cell (x, a) of the relation indicates that $(x, a) \in R$. According to Table 1 and Definition 1, we have that the object x_1 is related to the attributes a_1 and a_2, the object x_2 is related to the attributes a_2 and a_3, and the object x_3 is related to the attributes a_1 and a_3, or equivalently, $(x_1, a_1), (x_1, a_2), (x_2, a_2), (x_2, a_3), (x_3, a_1), (x_3, a_3) \in R$. Considering the blank cells in Table 1, we obtain that $(x_1, a_3), (x_2, a_1), (x_3, a_2) \notin R$.

Now, some features of this context are described using Proposition 2. With this purpose, we will apply the derivation operators $^{\uparrow}$ and $^{\downarrow}$ over the sets X and A respectively, obtaining that:

$$X^{\uparrow} = \{x_1, x_2, x_3\}^{\uparrow} = x_1^{\uparrow} \cap x_2^{\uparrow} \cap x_3^{\uparrow} = \{a_1, a_2\} \cap \{a_2, a_3\} \cap \{a_1, a_3\} = \varnothing$$

Table 1 Formal context FC given in Example 3

R	a_1	a_2	a_3
x_1	×	×	
x_2		×	×
x_3	×		×

$$A^{\downarrow} = \{a_1, a_2, a_3\}^{\downarrow} = a_1^{\downarrow} \cap a_2^{\downarrow} \cap a_3^{\downarrow} = \{x_1, x_3\} \cap \{x_1, x_2\} \cap \{x_2, x_3\} = \varnothing$$

From these computations, we can ensure that no attribute in FC is related to all the objects of X. In a similar way, we conclude that no object in FC is related to all the attributes of A. □

It is convenient to mention that different interpretations about the meaning of the blank cells in formal contexts can be given Burmeister and Holzer (2005). Specifically, given a formal context (X, A, R), an object $x \in X$ and an attribute $a \in A$, the statement $(x, a) \notin R$ could mean that:

- the object x does not possess attribute a,
- or that it is unknown whether or not the object x possesses attribute a,
- or it could even have any other meaning (such as: attribute a is meaningless for x,...)

These interpretations gave rise to different approaches for representing and analyzing the knowledge, such as the *ontic* or *mixed approach* (Chacón-Gómez et al. 2023; Dubois et al. 2021; Hu and Wang 2024; Ren et al. 2023; Rodríguez-Jiménez et al. 2014) which follows the philosophy of the aforementioned first item, and the *epistemic approach* (Burmeister and Holzer 2005; Dubois et al. 2021; Holzer 2004b; Obiedkov 2002), which follows the second one. To the best of our knowledge, the third item has not yet been addressed. In this paper, we consider both approaches simultaneously in order to carry out more exhaustive studies. With this purpose, we will denote each interpretation in a different way. Henceforth, given an object $x \in X$ and an attribute $a \in A$, in order to express that:

- it is known that the object x is related to the attribute a by relation R, we write $R(x, a) = +$.
- it is known that the object x does not have the attribute a by relation R, we write $R(x, a) = -$.
- it is not known whether or not the object x is related to the attribute a by relation R, we write $R(x, a) = ?$.

This notation highlights the presence of incomplete information, interpreting the relation as a mapping $R\colon X \times A \to \{+, -, ?\}$. Different notions of context arise from the previous approaches and the fixed notation, which are collected below.

Definition 4 Let X and A be non-empty sets of objects and attributes, respectively.

- $CC = (X, A, \{+, -\}, R_{CC})$ is a *complete context* where R_{CC} is a two-valued relation between objects and attributes $R_{CC}\colon X \times A \to \{+, -\}$.
- $SC = (X, A, \{+, ?\}, R_{SC})$ is a *standard context* where R_{SC} is a two-valued relation between objects and attributes $R_{SC}\colon X \times A \to \{+, ?\}$.
- $IC = (X, A, \{+, -, ?\}, R_{IC})$ is an *incomplete context* where R_{IC} is a three-valued relation between objects and attributes $R_{IC}\colon X \times A \to \{+, -, ?\}$.

Notice that we take an epistemic view on data modelled by contexts: complete contexts express complete knowledge of objects while standard contexts express positive or incomplete information. Incomplete contexts, containing positive, negative information and unknown features encompass the two previous cases.

Next, we introduce derivation operators over complete contexts, standard contexts and incomplete contexts. Taking into account the point that $(x, a) \in R$ and $R(x, a) = +$ have the same meaning, derivation operators over complete contexts will be denoted as $({}^{\uparrow_c}, {}^{\downarrow^c})$. Then, Definition 1 is adapted to complete contexts in the following way.

Definition 5 Let $CC = (X, A, \{+, -\}, R_{CC})$ be a complete context. The *derivation operators over* CC are the mappings ${}^{\uparrow_c} : \mathcal{P}(X) \to \mathcal{P}(A)$ and ${}^{\downarrow^c} : \mathcal{P}(A) \to \mathcal{P}(X)$ defined, for every $Y \subseteq X$ and $B \subseteq A$, as:

$$\begin{aligned} Y^{\uparrow_c} &= \{a \in A \mid R_{CC}(x, a) = +, \text{ for all } x \in Y\} \\ B^{\downarrow^c} &= \{x \in X \mid R_{CC}(x, a) = +, \text{ for all } a \in B\} \end{aligned} \tag{1}$$

This definition is similar to Definition 1 for standard contexts.[1] Furthermore, it is easy to check that Proposition 2 is satisfied by derivation operators over complete and standard contexts.

Finally, and taking into account the three-valued relations defined in incomplete contexts, two different pairs of derivation operators arise (Holzer 2004a). The first one focuses on the known relationships, as in Definition 5, while the second one also considers the unknown relationships.

Definition 6 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context. The pairs of *derivation operators over* IC, denoted as $({}^{\uparrow\Box}, {}^{\downarrow^\Box})$ and $({}^{\uparrow\Diamond}, {}^{\downarrow^\Diamond})$, are given by the mappings ${}^{\uparrow\Box}, {}^{\uparrow\Diamond} : \mathcal{P}(X) \to \mathcal{P}(A)$ and ${}^{\downarrow^\Box}, {}^{\downarrow^\Diamond} : \mathcal{P}(A) \to \mathcal{P}(X)$ defined, for every $Y \subseteq X$ and $B \subseteq A$, as:

$$\begin{aligned} Y^{\uparrow\Box} &= \{a \in A \mid R_{IC}(x, a) = +, \text{ for all } x \in Y\} \\ B^{\downarrow^\Box} &= \{x \in X \mid R_{IC}(x, a) = +, \text{ for all } a \in B\} \\ Y^{\uparrow\Diamond} &= \{a \in A \mid R_{IC}(x, a) \neq -, \text{ for all } x \in Y\} \\ B^{\downarrow^\Diamond} &= \{x \in X \mid R_{IC}(x, a) \neq -, \text{ for all } a \in B\} \end{aligned}$$

As usual, $Y^{\uparrow\Box}$ is the set of attributes certainly possessed by all objects in Y, while $Y^{\uparrow\Diamond}$ excludes those attributes that are known not to be possessed by at least one object in Y. In other words, $Y^{\uparrow\Diamond}$ is the set of attributes that all objects in Y possibly possess. $B^{\downarrow^\Box}$ and $B^{\downarrow^\Diamond}$ have a similar meaning, exchanging objects and attributes.

Derivation operators over incomplete contexts also form antitone Galois connections, as it is shown in Holzer (2004a). As a consequence, the derivation of subsets of objects and attributes can be decomposed in terms of the derivation of their elements.

Proposition 7 (Holzer (2004a)) *Let* $IC = (X, A, \{+, -, ?\}, R_{IC})$ *be an incomplete context.*

- *The pairs* $({}^{\uparrow\Box}, {}^{\downarrow^\Box})$ *and* $({}^{\uparrow\Diamond}, {}^{\downarrow^\Diamond})$ *are antitone Galois connections.*

[1] But, as a result, we do not handle the positive and the negative in a symmetric way in complete contexts.

Table 2 Complete context

R_{CC}	a_1	a_2	a_3
x_1	+	+	−
x_2	−	+	+
x_3	+	−	+

Table 3 Standard context

R_{SC}	a_1	a_2	a_3
x_1	+	+	?
x_2	?	+	+
x_3	+	?	+

Table 4 Incomplete context

R_{IC}	a_1	a_2	a_3
x_1	+	+	−
x_2	?	+	+
x_3	+	−	+

- *The following properties hold, for each $Y \subseteq X$ and $B \subseteq A$:*

$$Y^{\uparrow\Box} = \left(\bigcup_{x\in Y} x\right)^{\uparrow\Box} = \bigcap_{x\in Y} x^{\uparrow\Box} \quad Y^{\uparrow\Diamond} = \left(\bigcup_{x\in Y} x\right)^{\uparrow\Diamond} = \bigcap_{x\in Y} x^{\uparrow\Diamond}$$
$$B^{\downarrow^\Box} = \left(\bigcup_{a\in B} a\right)^{\downarrow^\Box} = \bigcap_{a\in B} a^{\downarrow^\Box} \quad B^{\downarrow^\Diamond} = \left(\bigcup_{a\in B} a\right)^{\downarrow^\Diamond} = \bigcap_{a\in B} a^{\downarrow^\Diamond}$$

The next example deals with the formal context given in Example 3 from the mixed and epistemic approach. With this purpose, we present a complete, a standard and an incomplete context obtained from the formal one interpreting the blank cells in a different way.

Example 8 Coming back to the environment of Example 3, we consider three different relations R_{CC}, R_{SC} and R_{IC}, presented in Tables 2, 3 and 4, respectively. These relations correspond to a complete context $CC = (X, A, \{+, -\}, R_{CC})$, a standard context $SC = (X, A, \{+, ?\}, R_{SC})$ and an incomplete context $IC = (X, A, \{+, -, ?\}, R_{IC})$, respectively.

On the one hand, the complete context CC assumes that the objects x_1, x_2 and x_3 are not related to the attributes a_3, a_1 and a_2, respectively. On the other hand, the standard context SC interprets the relationships between objects x_1, x_2 and x_3, and attributes a_3, a_1 and a_2, respectively, as unknown. Finally, the incomplete context IC discerns two different cases. An unknown relationship is considered for the object x_2 and the attribute a_1, whereas for the objects x_1 and x_3, and the attributes a_3 and a_2, it is understood that they are not related.

Since the derivation operators over complete and standard contexts only take into account known positive relationships, the same conclusions given in Example 3 can be extracted. In incomplete contexts, a more detailed extraction of knowledge can be carried out by means of the derivation operators $({}^{\uparrow\Box}, {}^{\downarrow^\Box})$ and $({}^{\uparrow\Diamond}, {}^{\downarrow^\Diamond})$. For instance:

$$X^{\uparrow\Box} = \{x_1, x_2, x_3\}^{\uparrow\Box} = x_1^{\uparrow\Box} \cap x_2^{\uparrow\Box} \cap x_3^{\uparrow\Box} = \{a_1, a_2\} \cap \{a_2, a_3\} \cap \{a_1 \cap a_3\} = \varnothing$$

$$X^{\uparrow\Diamond} = \{x_1, x_2, x_3\}^{\uparrow\Diamond} = x_1^{\uparrow\Diamond} \cap x_2^{\uparrow\Diamond} \cap x_3^{\uparrow\Diamond} = \{a_1, a_2\} \cap \{a_1, a_2, a_3\} \cap \{a_1, a_3\} = \{a_1\}$$

As a result, although there is no attribute related to all the objects of IC, it may be that attribute a_1 is related to them. Then, it is important to take this information into account in order to analyze the given incomplete context appropriately. Moreover:

$$A^{\downarrow\Box} = \{a_1, a_2, a_3\}^{\downarrow\Box} = a_1^{\downarrow\Box} \cap a_2^{\downarrow\Box} \cap a_3^{\downarrow\Box} = \{x_1, x_3\} \cap \{x_1, x_2\} \cap \{x_2, x_3\} = \varnothing$$
$$A^{\downarrow\Diamond} = \{a_1, a_2, a_3\}^{\downarrow\Diamond} = a_1^{\downarrow\Diamond} \cap a_2^{\downarrow\Diamond} \cap a_3^{\downarrow\Diamond} = \{x_1, x_2, x_3\} \cap \{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\}$$

Therefore, no object is related to all the attributes according to the known information of IC. However, object x_2 may be related to all the attributes because the relationship between x_2 and a_1 is unknown. □

Remark 9 A similar approach can be introduced considering the dual context (interchanging $+$ by $-$) and operators dual to the ones given in Definition 6. For example, the following derivation operators can be defined, for every $Y \subseteq X$ and $B \subseteq A$, as:

$$Y^{\uparrow\Diamond_\delta} = \{a \in A \mid R_{IC}(x, a) \neq + \text{ for all } x \in Y\}$$
$$B^{\downarrow\Diamond_\delta} = \{x \in X \mid R_{IC}(x, a) \neq + \text{ for all } a \in B\}$$

which are dual to ${}^{\uparrow\Diamond}$, ${}^{\downarrow\Diamond}$. Thus, due to duality, it is only necessary to study the pairs in Definition 6.

The introduction of the derivation operators $({}^{\uparrow\Diamond}, {}^{\downarrow\Diamond})$ helps to draw more conclusions about relational data sets by considering unknown relationships, which leads to different cases where the given objects and attributes may or may not be related. These scenarios are depicted by means of completions, which are introduced in the following section.

3 Completions

The unknown relationships present in incomplete contexts make it difficult to handle the information contained in them. The notion of completion arises to address this issue by replacing each ? with $+$ or $-$. As a consequence, completions present possible scenarios in agreement with the given incomplete context by removing the uncertainty and preserving the known information.

Definition 10 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context. A *completion of* IC is a complete context obtained from IC, denoted as $CO = (X, A, \{+, -\}, R_{CO})$, where the relationship R_{CO} between objects and attributes is defined, for each $(x, a) \in X \times A$, as follows:

- If $R_{IC}(x, a) = +$, then $R_{CO}(x, a) = +$.
- If $R_{IC}(x, a) = -$, then $R_{CO}(x, a) = -$.
- If $R_{IC}(x, a) = ?$, then either $R_{CO}(x, a) = +$ or $R_{CO}(x, a) = -$.

We denote by $\text{Comp}(IC)$ the set of all completions of the incomplete context IC.

Completions of standard contexts can also be considered, since incomplete contexts are more general than standard contexts. Notice that, if there exist n pairs of objects and attributes with unknown relationship, then 2^n different completions can be obtained. It is possible to define different partial orders among complete contexts, depending on the considered

Table 5 Completion 1 (CO_1) extracted from Table 3

R_{CO_1}	a_1	a_2	a_3
x_1	+	+	+
x_2	+	+	+
x_3	+	+	+

Table 6 Completion 2 (CO_2) extracted from Table 3

R_{CO_2}	a_1	a_2	a_3
x_1	+	+	−
x_2	+	+	+
x_3	+	+	+

Table 7 Completion 3 (CO_3) extracted from Table 3

R_{CO_3}	a_1	a_2	a_3
x_1	+	+	+
x_2	−	+	+
x_3	+	+	+

relationship between $+$ and $-$. This paper is based on the order of truthfulness, that is, $- < +$, which leads us to introduce the following partial order in the set of completions.

Definition 11 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context and two completions $CO_1 = (X, A, R_{CO_1})$, $CO_2 = (X, A, R_{CO_2})$ in Comp(IC). The relation $\preccurlyeq$ is defined in Comp(IC) as follows:

$$CO_1 \preccurlyeq CO_2 \text{ if and only if } \quad R_{CO_2}(x, a) = + \text{ for all } (x, a) \in X \times A \\ \text{such that } R_{CO_1}(x, a) = +$$

It is important to highlight the greatest and least completions according to Definition 11. Each of them is obtained by replacing all the relationships ? by $+$, called upper completion, or by $-$, called lower completion.

Definition 12 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context.

- The *upper completion* $CO_u = (X, A, R_{CO_u})$ *of* IC is the completion of IC with R_{CO_u} satisfying that $R_{CO_u}(x, a) = +$, for all $(x, a) \in X \times A$ such that $R_{IC}(x, a) = ?$.
- The *lower completion* $CO_l = (X, A, R_{CO_l})$ *of* IC is the completion of IC with R_{CO_l} satisfying that $R_{CO_l}(x, a) = -$, for all $(x, a) \in X \times A$ such that $R_{IC}(x, a) = ?$.

The upper and lower completions will play an important role in the study of the validity of attribute implications in incomplete contexts, which will be carried out in Sect. 5. Now, we show all the completions of the standard context presented in Example 8.

Example 13 Coming back to Example 8, it is immediate to check that there are three unknown relationships in the standard context represented in Table 3, corresponding to $R_{SC}(x_1, a_3) = R_{SC}(x_2, a_1) = R_{SC}(x_3, a_2) = ?$. Therefore, as we have commented above, we obtain $2^3 = 8$ completions of SC, which are represented in Tables 5, 6, 7, 8, 9, 10, 11 and 12.

Clearly, one of these completions matches with the real information, which has not been accurately identified in the original dataset due to the three existing unknown relationships.

Table 8 Completion 4 (CO_4) extracted from Table 3

R_{CO_4}	a_1	a_2	a_3
x_1	+	+	+
x_2	+	+	+
x_3	+	−	+

Table 9 Completion 5 (CO_5) extracted from Table 3

R_{CO_1}	a_1	a_2	a_3
x_1	+	+	−
x_2	−	+	+
x_3	+	+	+

Table 10 Completion 6 (CO_6) extracted from Table 3

R_{CO_2}	a_1	a_2	a_3
x_1	+	+	−
x_2	+	+	+
x_3	+	−	+

Table 11 Completion 7 (CO_7) extracted from Table 3

R_{CO_3}	a_1	a_2	a_3
x_1	+	+	+
x_2	−	+	+
x_3	+	−	+

Moreover, by Definition 12, we obtain that CO_1 and CO_8, represented in Tables 5 and 12 are the upper and lower completions of SC, respectively. □

Henceforth, given a completion CO, we will denote as $B^{\downarrow^c}_{CO}$ the derivation of the subset of attributes B over CO. Derivation operators over incomplete contexts and completions address the management of incomplete information in FCA from different perspectives, but they are closely related. In particular, the following result shows that the derivation operator $\downarrow^{\Box}$ is related to the lower completion and $\downarrow^{\Diamond}$ to the upper one.

Proposition 14 *Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context. Then, for each $B \subseteq A$, the following equalities hold:*

$$B^{\downarrow^{\Box}} = B^{\downarrow^c}_{CO_l}$$
$$B^{\downarrow^{\Diamond}} = B^{\downarrow^c}_{CO_u}$$

Table 12 Completion 8 (CO_8) extracted from Table 3

R_{CO_4}	a_1	a_2	a_3
x_1	+	+	−
x_2	−	+	+
x_3	+	−	+

Proof We prove that $B^{\downarrow^{\square}} = B^{\downarrow^{c}}_{CO_l}$ holds. Given $x \in B^{\downarrow^{\square}}$, by Definition 6, we have that $R_{IC}(x,a) = +$, for all $a \in B$. Taking into account the fact that $CO_l \in \text{Comp}(IC)$, by Definition 10, we deduce that $R_{CO_l}(x,a) = +$, for all $a \in B$. Hence $x \in B^{\downarrow^{c}}_{CO_l}$, and as a consequence, $B^{\downarrow^{\square}} \subseteq B^{\downarrow^{c}}_{CO_l}$. Conversely, if $x \in B^{\downarrow^{c}}_{CO_l}$, then $R_{CO_l}(x,a) = +$, for all $a \in B$. Thus, as CO_l is the lower completion of IC, by Definition 12, we obtain that $R_{IC}(x,a) = +$, for all $a \in B$. Therefore, $x \in B^{\downarrow^{\square}}$ and $B^{\downarrow^{c}}_{CO_l} \subseteq B^{\downarrow^{\square}}$. In conclusion, $B^{\downarrow^{\square}} = B^{\downarrow^{c}}_{CO_l}$, for all $B \subseteq A$.

The proof of the equality $B^{\downarrow^{\diamond}} = B^{\downarrow^{c}}_{CO_u}$ is completely analogous. □

Now, we introduce a technical result about the monotonicity of the derivation operators given in Definition 6.

Lemma 15 *Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context. For each $CO \in$ Comp(IC) and $B \subseteq A$, the following chain of inclusions holds:*

$$B^{\downarrow^{\square}} \subseteq B^{\downarrow^{c}}_{CO} \subseteq B^{\downarrow^{\diamond}}$$

Proof For each $CO \in \text{Comp}(IC)$ and $B \subseteq A$, we have that:

$$B^{\downarrow^{\square}} \overset{(a)}{=} B^{\downarrow^{c}}_{CO_l} \overset{(b)}{\subseteq} B^{\downarrow^{c}}_{CO} \overset{(c)}{\subseteq} B^{\downarrow^{c}}_{CO_u} \overset{(d)}{=} B^{\downarrow^{\diamond}}$$

where (a) and (d) are deduced applying Proposition 14, and (b) and (c) are obtained due to $CO_l \preccurlyeq CO \preccurlyeq CO_u$ by Definition 11. □

This section has recalled the notion of completion in order to display each possible scenario in agreement with a given incomplete context. Completions will be of great importance in the rest of the work, in which we will relate FCA and possibility theory for the handling of incomplete information, since as suggested above, an incomplete context corresponds to a set of possible completions.

4 The possibility/necessity approach to incomplete information

In this section, we will address the problem of dealing with incomplete information in FCA from the perspective of possibility theory (Denœux et al. 2020; Zadeh 1978). This fact will allow us to compute plausibility and certainty that statements pertaining to an incomplete context are true or false. First of all, we recall possibility and necessity measures of possibility theory.

4.1 Possibility theory

A universe $\mathcal{U}$ is basic in possibility theory to represent states of affairs using a given language. A mapping $\pi : \mathcal{U} \to [0, 1]$ for which there exists an element $u \in \mathcal{U}$ such that $\pi(u) = 1$, is called a *normalized possibility distribution*. It represents an epistemic state, i.e., incomplete knowledge. By convention, $\pi(u) = 0$ means that the state of affairs u is ruled out, while $\pi(u) = 1$ means that u is fully plausible. Complete knowledge is when $\pi(u) = 1$ for some $u \in \mathcal{U}$ and $\pi(u') = 0$ for all $u' \neq u \in \mathcal{U}$.

From now on, given a universe $\mathcal{U}$, suppose a normalized possibility distribution π is fixed. For each subset of the universe, called event, two important mappings are defined to compute the possibility and the necessity of its occurrence, respectively (Dubois and Prade 1988), in agreement with the available knowledge described by π.

Definition 16 The *possibility and necessity measures* $\Pi, N : \mathcal{P}(\mathcal{U}) \to [0, 1]$ are defined, for each event $E \subseteq \mathcal{U}$, as follows:

$$\Pi(E) = \max\{\pi(s) \mid s \in E\}$$
$$N(E) = 1 - \max\{\pi(s) \mid s \notin E\}.$$

Notice that possibility and necessity measures are closely related, since $N(E) = 1 - \Pi(E^c)$, where E^c is the complementary set of E. Moreover, by convention $\Pi(\varnothing) = 0$.

The cases in which the previous measures take values 0 and 1 are interpreted as follows:

- $\Pi(E) = 0$ means that the event E is totally impossible, that is, it cannot occur because it is incompatible with the epistemic state π.
- $\Pi(E) = 1$ means that the event E is totally possible, but it may fail to occur.
- $N(E) = 0$ means that the event E is not necessary (= certain) at all, but it may occur (because its opposite has possibility 1).
- $N(E) = 1$ means that the event E is totally necessary, that is, it occurs with total certainty (because its opposite is impossible).

As a result, several properties can be deduced (Dubois and Prade 2015) for possibility and necessity measures. The next proposition recalls the most relevant ones for this work.

Proposition 17 (Dubois and Prade (2015)) *Let $\mathcal{U}$ be a universe, $E \subseteq \mathcal{U}$ and $\Pi, N : \mathcal{P}(\mathcal{U}) \to [0, 1]$ be the possibility and necessity measures. The following properties hold:*

1. $\Pi(\mathcal{U}) = N(\mathcal{U}) = 1$.
2. $N(\varnothing) = 0$.
3. $N(E) > 0$ *implies* $\Pi(E) = 1$.
4. $N(E) \leq \Pi(E)$.
5. *Let $C(\pi) = \{s \in \mathcal{U} \mid \pi(s) = 1\}$, and $S(\pi) = \{s \in \mathcal{U} \mid \pi(s) > 0\}$ where $\pi : \mathcal{U} \to [0, 1]$ is a normalized possibility distribution. Given an event $F \subseteq \mathcal{U}$, we have that:*

$$\Pi(F) = 1 \text{ if and only if } F \cap C(\pi) \neq \varnothing$$

$$N(F) = 1 \text{ if and only if } S(\pi) \subseteq F$$

Moreover, if $\pi : \mathcal{U} \to \{0, 1\}$ then $C(\pi) = S(\pi) = E$ and we have that:

$$N(F) = 1 \text{ if and only if } E \subseteq F; \quad \Pi(F) = 1 \text{ if and only if } E \cap F \neq \emptyset.$$

This result shows that the whole universe $\mathcal{U}$ is totally necessary (hence possible), while the empty set is totally impossible. The third item is equivalent to $\min(N(F), N(F^c)) = 0$, forbidding the possibility of believing a statement and its negation at the same time. This last item is particularly remarkable, since it characterizes totally possible and necessary events depending on their relations to the possibility distribution that collects all the more or less possible elements.

Note that $C(\pi)$ is the set of most plausible elements, and $\Pi(F) = 1$ means that F contains a most plausible element; $S(\pi)$ is the set of non impossible elements, and $N(F) = 1$ means that F is certain since no element outside F is possible.

In the binary case, which is the case in this paper, the computation of possibility and necessity measures is greatly simplified.

4.2 Incomplete contexts and associated possibility and necessity measures

This section extends the possibility and necessity measures of Definition 16 to the FCA framework. The main goal is to model incomplete information in this framework, represented by a set of possible complete contexts, taking into account the information present in the original dataset. With this purpose, we will define these measures over incomplete contexts, since they allow to manage more complex data, which will be fundamental to carry out meaningful studies.

To begin with, we define the possibility degree of a single complete context with respect to an incomplete context. With this purpose, we set $\mathcal{U}$ as the universe of complete contexts with $n \in \mathbb{N}$ objects and $m \in \mathbb{N}$ attributes, and we also consider an incomplete context with the same number of objects and attributes. For simplicity, we will refer to contexts with $n \in \mathbb{N}$ objects and $m \in \mathbb{N}$ attributes as *$n \times m$- contexts*.

Definition 18 Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts of the form $CC = (X, A, \{+, -\}, R_{CC})$. Given an $n \times m$-incomplete context $IC = (X, A, \{+, -, ?\}, R_{IC})$, the relationship between R_{IC} and R_{CC} is given by the mapping $f\colon \{+, -, ?\} \times \{+, -\} \to \{0, 1\}$, defined for each $(r_{IC}, r_{CC}) \in \{+, -, ?\} \times \{+, -\}$ as:

$$f(r_{IC}, r_{CC}) = \begin{cases} 1 & \text{if } r_{IC} = r_{CC} \text{ or } r_{IC} = ? \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The *possibility degree of CC with respect to IC is given by the mapping* $\pi_{IC}\colon \mathcal{U} \to \{0, 1\}$ defined as:

$$\pi_{IC}(CC) = \min\{f(R_{IC}(x, a), R_{CC}(x, a)) \mid (x, a) \in X \times A\} \tag{3}$$

Note that, Equation (2) implies that $f(?, +) = f(?, -) = 1$, which is consistent to the fact that the symbol ? represents total ignorance. Based on this remark, and taking into account Definition 10, we deduce that the possibility of a complete context is 1 if and only if it is a completion of the given incomplete context.

Proposition 19 *Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts of the form $CC = (X, A, \{+, -\}, R_{CC})$. Given an $n \times m$-incomplete context $IC = (X, A, \{+, -, ?\}, R_{IC})$, the binary possibility distribution induced by IC over $\mathcal{U}$ is given as follows:*

$$\pi_{IC}(CC) = \begin{cases} 1 \text{ if } CC \in Comp(IC) \\ 0 \text{ otherwise} \end{cases}$$

Proof Applying Definition 10, if $CC \in \text{Comp}(IC)$ then we obtain that:

$$(R_{IC}(x, a), R_{CC}(x, a)) \in \{(+, +), (-, -), (?, +), (?, -)\}$$

for each $(x, a) \in X \times A$. By Equation (2), we have that:

$$f(R_{IC}(x, a), R_{CC}(x, a)) = 1$$

for all $(x, a) \in X \times A$. According to the possibility distribution of CC with respect to IC given by (3), we conclude that $\pi_{IC}(CC) = 1$.

On the other hand, if $CC \notin \text{Comp}(IC)$, then there exists $(x, a) \in X \times A$ such that either $R_{CC}(x, a) = +$ and $R_{IC}(x, a) = -$ or $R_{CC}(x, a) = -$ and $R_{IC}(x, a) = +$. In any case, by Equation (2), $f(R_{IC}(x, a), R_{CC}(x, a)) = 0$. As a result, considering once again the possibility distribution of CC with respect to IC, given by (3), we conclude that $\pi_{IC}(CC) = 0$. □

Therefore, Proposition 19 reduces the computation of the possibility of a complete context to checking whether it is a completion of the given incomplete context. As a result, we deduce that only the information provided by completions is reliable with respect to the given incomplete context. Otherwise, complete contexts that are not completions of the given incomplete context cannot provide accurate information for that context. In addition, we also deduce that π_{IC} is a normalized binary possibility distribution, since $\text{Comp}(IC) \neq \varnothing$ for any incomplete context IC.

Now, we define the possibility and necessity measures of a set of complete contexts (event of the universe $\mathcal{U}$) with respect to an incomplete context. This definition, based on Definitions 16 and 18, is presented next.

Definition 20 Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts and IC be an $n \times m$-incomplete context. The possibility and necessity measures with respect to IC, are given by the mappings $\Pi_{IC}, N_{IC} : \mathcal{P}(\mathcal{U}) \to \{0, 1\}$ defined, for each subset $\mathcal{C} \subseteq \mathcal{U}$ of complete contexts, as follows:

$$\Pi_{IC}(\mathcal{C}) = \max\{\pi_{IC}(CC) \mid CC \in \mathcal{C}\}$$
$$N_{IC}(\mathcal{C}) = 1 - \max\{\pi_{IC}(CC) \mid CC \notin \mathcal{C}\}$$

Since Π_{IC} is a possibility measure, by convention, $\Pi_{IC}(\varnothing) = 0$. The following result, which is a particular case of Proposition 17, collects some important properties satisfied by the possibility and necessity measures of Definition 20. Its importance lies in that it allows to compute the possibility and necessity of a set of complete contexts $\mathcal{C}$ with respect to an incomplete context IC in terms of the relationship between $\mathcal{C}$ and $\text{Comp}(IC)$.

Corollary 21 *Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts, $\mathcal{C} \subseteq \mathcal{U}$ and IC be an $n \times m$-incomplete context. Then:*

1. $\Pi_{IC}(\mathcal{U}) = N_{IC}(\mathcal{U}) = 1$.
2. $N_{IC}(\varnothing) = 0$.
3. $N_{IC}(\mathcal{C}) \leq \Pi_{IC}(\mathcal{C})$.
4.

$$\Pi_{IC}(\mathcal{C}) = \begin{cases} 1 \text{ if } \mathcal{C} \cap Comp(IC) \neq \varnothing \\ 0 \text{ otherwise} \end{cases} \quad N_{IC}(\mathcal{C}) = \begin{cases} 1 \text{ if } Comp(IC) \subseteq \mathcal{C} \\ 0 \text{ otherwise} \end{cases}$$

Proof The proof of each item is immediately deduced from Proposition 17, taking into account that Π_{IC} and N_{IC} are possibility and necessity measures, and $\text{Comp}(IC) = \{CC \in \mathcal{U} \mid \pi_{IC}(CC) = 1\}$, by Proposition 19. □

In the characterization of the possibility measure, the existence of at least one completion of IC in $\mathcal{C}$ makes this set totally possible with respect to IC, since at least one reliable scenario of IC is under consideration in $\mathcal{C}$. Otherwise, it is totally impossible for the set $\mathcal{C}$ to accurately describe the information present in IC because no feasible scenario of IC is considered in $\mathcal{C}$. With respect to the necessity measure, the set $\mathcal{C}$ must contain all the completions $\text{Comp}(IC)$ in order to ensure an appropriate depiction of IC. Otherwise, at least one potential scenario of IC is not taken into account in $\mathcal{C}$, then this set is totally unnecessary to illustrate the unknown relationships in IC.

The possibility and necessity measures over complete contexts are useful for the analysis of a collection of scenarios of interest to the user with respect to some incomplete information. The following example illustrates the application of Corollary 21 to such studies.

Table 13 Complete context CC_1

R_{CC_1}	a_1	a_2	a_3
x_1	+	+	+
x_2	+	+	+
x_3	+	+	+

Table 14 Complete context CC_2

R_{CC_2}	a_1	a_2	a_3
x_1	+	+	−
x_2	+	+	+
x_3	+	+	+

Table 15 Complete context CC_3

R_{CC_3}	a_1	a_2	a_3
x_1	−	+	+
x_2	+	+	−
x_3	−	−	+

Table 16 Complete context CC_4

R_{CC_4}	a_1	a_2	a_3
x_1	+	+	−
x_2	−	−	+
x_3	+	+	−

Example 22 The purpose of this example is to analyze complete contexts in agreement with the standard context SC given in Table 3, in which specific objects and attributes are related. Thus, given the universe $\mathcal{U}$ of 3×3-contexts, we consider the set of complete contexts $\mathcal{C} = \{CC_1, CC_2, CC_3, CC_4\} \subseteq \mathcal{U}$, with $CC_i = (X, A, \{+, -\}, R_{CC_i})$ and $i \in \{1, 2, 3, 4\}$. Relationships R_{CC_i} are represented in Tables 13, 14, 15 and 16.

Note that, $R_{SC}(x_1, a_2) = +$ and $R_{CC_i}(x_1, a_2) = +$, for all $i \in \{1, 2, 3, 4\}$, which is the only relationship common to all the complete contexts. In addition, it is easy to check that the complete contexts CC_1 and CC_2 correspond to completions CO_1 and CO_2 of SC represented in Tables 5 and 6, respectively. Thus, by Proposition 19 we deduce that $\pi_{SC}(CC_1) = \pi_{SC}(CC_2) = 1$. On the other hand, $CC_3, CC_4 \notin \text{Comp}(SC)$ by Definition 10, since for example:

$$R_{CC_3}(x_1, a_1) = - \text{ and } R_{SC}(x_1, a_1) = +$$
$$R_{CC_4}(x_2, a_2) = - \text{ and } R_{SC}(x_2, a_2) = +$$

As a result, by Proposition 19 we obtain that $\pi_{SC}(CC_3) = \pi_{SC}(CC_4) = 0$. Finally, since $CC_1 \in \mathcal{C} \cap \text{Comp}(SC)$ and $\text{Comp}(SC) \nsubseteq \mathcal{C}$, because the completion CO_3 among others, represented in Table 7, is not considered in $\mathcal{C}$, by Corollary 21 we conclude that:

$$\Pi_{SC}(\mathcal{C}) = 1$$
$$N_{SC}(\mathcal{C}) = 0$$

Therefore, the information provided by the set $\mathcal{C}$ is possible with respect to SC, but also uncertain, since not all completions of SC have been considered in $\mathcal{C}$. As a result, the complete contexts considered in $\mathcal{C}$ can provide precise information about scenarios in which x_1 and a_2 are related, but we cannot guarantee its complete accuracy. □

Finally, notice that incomplete contexts can be seen as the set of all their completions, since one of them describes the real complete context in a completely accurate way. As a consequence, by Definition 20 and Corollary 21 is deduced that the possibility and necessity of the set of completions $Comp(IC)$ of any incomplete context IC is 1.

Corollary 23 *Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts and IC be an $n \times m$-incomplete context. Then:*

$$\Pi_{IC}(Comp(IC)) = N_{IC}(Comp(IC)) = 1$$

Proof By Definition 20, we have that:

$$\begin{aligned}\Pi_{IC}(\mathrm{Comp}(IC)) &= \max\{\pi_{IC}(CC) \mid CC \in \mathrm{Comp}(IC)\}\\ N_{IC}(\mathrm{Comp}(IC)) &= 1 - \max\{\pi_{IC}(CC) \mid CC \notin \mathrm{Comp}(IC)\}\end{aligned}$$

Then, applying Corollary 21(4) we conclude that:

$$\Pi_{IC}(\mathrm{Comp}(IC)) = N_{IC}(\mathrm{Comp}(IC)) = 1$$

□

Remark 24 Incomplete contexts are special subsets of complete contexts. Indeed the set of completions of an incomplete context is a partially ordered family of complete contexts (in the sense of the truth ordering $- < +$) which has a single greatest (replacing ? by $+$ in the incomplete context) and a single least element (replacing ? by $-$ in the incomplete context). A set $\mathcal{C}$ of complete contexts induces the following incomplete context:

$$R_{IC_{\mathcal{C}}}(x,a) = \begin{cases} + & \text{if } R_{CC}(x,a) = +, \text{ for all } CC \in \mathcal{C}\\ - & \text{if } R_{CC}(x,a) = -, \text{ for all } CC \in \mathcal{C}\\ ? & \text{otherwise}\end{cases}$$

Often, we shall have that $\mathcal{C} \subseteq Comp(IC_{\mathcal{C}})$ rather than an equality. So a subset of contexts cannot always be faithfully represented by the incomplete context $IC_{\mathcal{C}}$.

This section has applied the all-or-nothing case of possibility theory to the FCA framework by defining Boolean possibility and necessity measures of sets of complete contexts with respect to an incomplete context. This fact allows us to manage incomplete information in relational datasets, extracting sure and possible conclusions from them. With a similar philosophy, we shall study the possibility and necessity of attribute implications induced by an incomplete context in the next section.

5 Attribute implications in incomplete contexts

This section is devoted to the study of attribute implications in incomplete contexts from two different perspectives. The first one presents well-known characterizations (Ait-Yakoub et al. 2016, 2017; Holzer 2004a) about their validity in a more detailed way. The second one focuses on the definition and study of the possibility and necessity measures of attribute implications with respect to an incomplete context, taking into account Definition 20.

First of all, we recall the notion of attribute implication in formal contexts. Specifically, we pay attention to those attribute implications that are valid, since they express a total dependency of a subset of attributes with respect to another one.

Definition 25 Let $FC = (X, A, R)$ be a formal context and $D, E \subseteq A$.

- An *attribute implication* is denoted as $D \Rightarrow E$, where the set D is the *antecedent* of the implication and the set E is the *consequent*, such that $D \cap E = \emptyset$.
- $D \Rightarrow E$ is a *valid attribute implication in* FC if $D^{\downarrow} \subseteq E^{\downarrow}$.

Then, a valid attribute implication $D \Rightarrow E$ in the context FC means that all the objects of this context satisfying all the attributes in the antecedent D also satisfy all the attributes in the consequent E. Note that we assume any attribute implication $D \Rightarrow E$ is such that $D \cap E = \varnothing$, since otherwise part of the information given by that implication is redundant.

Notice also that, the notion of attribute implication can be defined in all the contexts presented in Definition 4 as in formal contexts, since they depend only on the set of attributes under consideration. Furthermore, the validity of attribute implications in complete contexts is determined as in formal contexts, but considering the derivation operator given in Equation (1). In this paper, we focus on validity in incomplete contexts.

5.1 Validity of attribute implications in incomplete contexts

The unknown information present in incomplete contexts makes it difficult to determine the validity of attribute implications in those contexts. Taking into account that completions remove the lack of information from incomplete contexts, some papers (Ait-Yakoub et al. 2016, 2017; Holzer 2004a) focused on them to study the corresponding validity. The next definition collects the characterization given in Ait-Yakoub et al. (2016, 2017) for possible and certain attribute implications in terms of the upper and lower completions. Moreover, we include three new types of implications, which are impossible, conservative and optimistic attribute implications.

Definition 26 Let IC be an incomplete context, CO_l and CO_u be the lower and upper completions of IC, and $D \Rightarrow E$ be an attribute implication. It is said that:

- $D \Rightarrow E$ is a *possibly valid (or a possible) attribute implication in* IC if $D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_u}$.
- $D \Rightarrow E$ is a *certainly valid (or a certain) attribute implication in* IC if $D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l}$.
- $D \Rightarrow E$ is a *certainly non-valid (or an impossible) attribute implication in* IC if $D^{\downarrow^c}_{CO_l} \not\subseteq E^{\downarrow^c}_{CO_u}$.
- $D \Rightarrow E$ is a *conservatively valid (or a conservative) attribute implication in* IC if $D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_l}$.
- $D \Rightarrow E$ is an *optimistically valid (or an optimistic) attribute implication in* IC if $D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_u}$.

As per the definition, an attribute implication is possible (resp. certain) in an incomplete context IC if the objects related to the attributes of the antecedent in the lower (resp. upper) completion of IC are also related to the attributes of the consequent in the upper (resp. lower) completion of IC.

Notice that if a complete context CC is considered instead of an incomplete context IC, we obtain that $CO_l = CO_u = CC$ because all the information is already known. In this case, we deduce that any attribute implication is impossible or certain, since certain, possible, conservative and optimistic attribute implications coincide. Furthermore, this group of implications are valid attribute implications in formal contexts interpreted as complete contexts. Analogously, any attribute implication in a standard context is possible, since there is no negative information. Among these possible implications there are also conservative attribute implications, which corresponds to valid implications of standard formal contexts. Indeed, the conservative attribute implications coincide with the standard definition of attribute implication in standard contexts.

The validity of attribute implications in incomplete contexts is related to their validity in the completions. In this way, we provide the following characterization.

Proposition 27 *Let IC be an incomplete context, CO_l and CO_u be the lower and upper completions of IC, and $D \Rightarrow E$ be an attribute implication.*

- *$D \Rightarrow E$ is a possible attribute implication in IC if and only if it is valid in some completion of IC.*
- *$D \Rightarrow E$ is a certain attribute implication in IC if and only if it is valid in all the completions of IC.*
- *$D \Rightarrow E$ is an impossible attribute implication in IC if and only if it is not valid in any completion of IC.*
- *$D \Rightarrow E$ is a conservative attribute implication in IC if and only if it is valid in CO_l.*
- *$D \Rightarrow E$ is an optimistic attribute implication in IC if and only if it is valid in CO_u.*

Proof We will only prove the characterizations for possible and certain attribute implications. The statement about impossible implications is proven by Definition 26 and by the characterization of possible implications. Likewise, the results about conservative and optimistic attribute implications are immediately deduced from Definition 26.

- First of all, we suppose that $D \Rightarrow E$ is a possible attribute implication in IC. Then, by Definition 26, it is $D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_u}$. Now, we consider a completion $CO \in \text{Comp}(IC)$ satisfying, for each $x \in X$, that:

$$R_{CO}(x,a) = - \text{ for each } \quad a \in D \quad \text{such that } R_{IC}(x,a) = ? \tag{4}$$

$$R_{CO}(x,a) = + \text{ for each } \quad a \in E \quad \text{such that } R_{IC}(x,a) = ? \tag{5}$$

  We begin by proving that $D^{\downarrow^c}_{CO_l} = D^{\downarrow^c}_{CO}$. The inclusion $D^{\downarrow^c}_{CO_l} \subseteq D^{\downarrow^c}_{CO}$ is trivially obtained because $CO_l \preccurlyeq CO$. Conversely, let $x \in D^{\downarrow^c}_{CO}$. Therefore, $R_{CO}(x,a) = +$ for all $a \in D$. Taking into account that $CO \in \text{Comp}(IC)$ and Equation (4), it is obtained that $R_{IC}(x,a) = +$, for all $a \in D$. As a consequence, by Definition 12, we obtain that $R_{CO_l}(x,a) = +$, for all $a \in D$. Thus, $x \in D^{\downarrow^c}_{CO_l}$ and $D^{\downarrow^c}_{CO} \subseteq D^{\downarrow^c}_{CO_l}$. In conclusion, $D^{\downarrow^c}_{CO_l} = D^{\downarrow^c}_{CO}$.
  Now, we prove that $E^{\downarrow^c}_{CO_u} = E^{\downarrow^c}_{CO}$. Let $x \in E^{\downarrow^c}_{CO_u}$. Then, we deduce that $R_{CO_u}(x,a) = +$, for all $a \in E$. Hence, since $CO_u \in \text{Comp}(IC)$ by Definition 10 we obtain that $R_{IC}(x,a) \neq -$, for all $a \in E$. Then, if there exists $a \in E$ such that $R_{CO}(x,a) = -$, and taking into account that $CO \in \text{Comp}(IC)$, it is deduced that $R_{IC}(x,a) = ?$, which contradicts Equation (5). Consequently, $R_{CO}(x,a) = +$ for all $a \in E$. Therefore, $x \in E^{\downarrow^c}_{CO}$ and $E^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO}$. Reciprocally, taking into account that $CO \preccurlyeq CO_u$, we

deduce that $E^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO_u}$. In conclusion, $E^{\downarrow^c}_{CO_u} = E^{\downarrow^c}_{CO}$.
Since by hypothesis $D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_u}$, and we have shown that $D^{\downarrow^c}_{CO_l} = D^{\downarrow^c}_{CO}$ and $E^{\downarrow^c}_{CO_u} = E^{\downarrow^c}_{CO}$, we conclude that $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$. Thus, we conclude that $D \Rightarrow E$ is valid in CO.
Next, we suppose that $D \Rightarrow E$ is valid in some completion of IC. Then, there exists a completion $CO \in \text{Comp}(IC)$ such that $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$. Moreover, since $CO_l \preccurlyeq CO \preccurlyeq CO_u$ we deduce that $D^{\downarrow^c}_{CO_l} \subseteq D^{\downarrow^c}_{CO}$ and $E^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO_u}$. Therefore, we obtain that:

$$D^{\downarrow^c}_{CO_l} \subseteq D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO_u}$$

In conclusion, $D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_u}$, and by Definition 26 $D \Rightarrow E$ is a possible attribute implication in IC.

- For the second item, we begin by supposing that $D \Rightarrow E$ is a certain attribute implication in IC. Then, by Definition 26, $D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l}$. Since $CO_l \preccurlyeq CO \preccurlyeq CO_u$ for each $CO \in \text{Comp}(IC)$, it is easy to obtain the following chain of inclusions:

$$D^{\downarrow^c}_{CO} \subseteq D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO}$$

As a result, $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$ for each $CO \in \text{Comp}(IC)$, and $D \Rightarrow E$ is valid in all the completions of IC.
Finally, we suppose that $D \Rightarrow E$ is valid in all the completions of IC, so that $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$ for all $CO \in \text{Comp}(IC)$. We will prove that $D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l}$ by reductio ad absurdum. If $D^{\downarrow^c}_{CO_u} \nsubseteq E^{\downarrow^c}_{CO_l}$, then there exists $x \in D^{\downarrow^c}_{CO_u}$ such that $x \notin E^{\downarrow^c}_{CO_l}$. In this way, we deduce that $R_{CO_u}(x, a) = +$ for all $a \in D$, and that there exists $b \in E$ such that $R_{CO_l}(x, b) = -$. Then, we consider a completion $CO \in \text{Comp}(IC)$ satisfying that $R_{CO}(x, a) = +$ for all $a \in D$ and that $R_{CO}(x, b) = -$. It is easy to check that $x \in D^{\downarrow^c}_{CO}$ and $x \notin E^{\downarrow^c}_{CO}$. Consequently, $D^{\downarrow^c}_{CO} \nsubseteq E^{\downarrow^c}_{CO}$, leading to a contradiction. In conclusion, $D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l}$.

□

On the other hand, in Holzer (2004a) the validity of attribute implications in incomplete contexts was determined without the need to compute any completion, but it is given in terms of the derivation operators of Definition 6. Notice that, this result is supported by Definition 26 and Proposition 14. This characterization is recalled next, including the new results about impossible, conservative and optimistic attribute implications.

Proposition 28 *Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be an incomplete context and $D \Rightarrow E$ be an attribute implication. Then:*

- *$D \Rightarrow E$ is possible in IC if and only if $D^{\downarrow^\Box} \subseteq E^{\downarrow^\Diamond}$.*
- *$D \Rightarrow E$ is certain in IC if and only if $D^{\downarrow^\Diamond} \subseteq E^{\downarrow^\Box}$.*
- *$D \Rightarrow E$ is impossible in IC if and only if $D^{\downarrow^\Box} \nsubseteq E^{\downarrow^\Diamond}$.*
- *$D \Rightarrow E$ is conservative in IC if and only if $D^{\downarrow^\Box} \subseteq E^{\downarrow^\Box}$.*
- *$D \Rightarrow E$ is optimistic in IC if and only if $D^{\downarrow^\Diamond} \subseteq E^{\downarrow^\Diamond}$.*

Proof All the items can be immediately proven by Definition 26, taking into account that $B^{\downarrow^\Box} = B^{\downarrow^c}_{CO_l}$ and $B^{\downarrow^\Diamond} = B^{\downarrow^c}_{CO_u}$ for all $B \subseteq A$ by Proposition 14. □

Therefore, an attribute implication is possible in an incomplete context if and only if the objects certainly related to the attributes of the antecedent are possibly related to the attributes of the consequent. The meaning of the rest of implications is interpreted in an analogous way.

Note that if $D^{\downarrow^{\Diamond}} = \emptyset$ the implication $D \Rightarrow E$ will have no support on the incomplete context, and no support in any of its completions. As a consequence, it will be certainly valid, although trivially so. Moreover if IC is a standard (incomplete) context, its upper completion contains only $+$, so that no attribute implication (other than those with empty support) will be certainly valid, and all attribute implications will be possible.

In the following example, we study different attribute implications to determine their validity in the incomplete context of Example 8, taking advantage of Proposition 28.

Example 29 This example focuses on the incomplete context represented in Table 4, analyzing the different kinds of attribute implications introduced in Definition 26. By Definition 6, it is easy to check that:

$$a_1^{\downarrow^{\Box}} = \{x_1, x_3\} \tag{6}$$

$$a_1^{\downarrow^{\Diamond}} = \{x_1, x_2, x_3\} \tag{7}$$

$$a_2^{\downarrow^{\Box}} = a_2^{\downarrow^{\Diamond}} = \{a_1, a_2\}^{\downarrow^{\Diamond}} = \{x_1, x_2\} \tag{8}$$

$$a_3^{\downarrow^{\Box}} = a_3^{\downarrow^{\Diamond}} = \{a_1, a_3\}^{\downarrow^{\Diamond}} = \{x_2, x_3\} \tag{9}$$

$$\{a_1, a_2\}^{\downarrow^{\Box}} = \{x_1\} \tag{10}$$

$$\{a_1, a_3\}^{\downarrow^{\Box}} = \{x_3\} \tag{11}$$

$$\{a_2, a_3\}^{\downarrow^{\Box}} = \{a_2, a_3\}^{\downarrow^{\Diamond}} = \{x_2\} \tag{12}$$

We begin by studying the attribute implication $a_1 \Rightarrow a_3$. By Equations (6) and (9) we deduce that:

$$a_1^{\downarrow^{\Box}} = \{x_1, x_3\} \nsubseteq \{x_2, x_3\} = a_3^{\downarrow^{\Diamond}}$$

As a result, by Proposition 28, $a_1 \Rightarrow a_3$ is an impossible attribute implication in IC. Hence, this implication is not valid in any possible scenario of IC. Furthermore, it is easy to check that it is neither possible nor certain nor conservative nor optimistic.

Now, we focus on the attribute implication $a_2 \Rightarrow a_1$. Taking into account Equations (6), (7) and (8), we obtain that:

$$a_2^{\downarrow^{\Diamond}} = a_2^{\downarrow^{\Box}} = \{x_1, x_2\} \subseteq \{x_1, x_2, x_3\} = a_1^{\downarrow^{\Diamond}} \tag{13}$$

$$a_2^{\downarrow^{\Box}} = a_2^{\downarrow^{\Diamond}} = \{x_1, x_2\} \nsubseteq \{x_1, x_3\} = a_1^{\downarrow^{\Box}} \tag{14}$$

Consequently, by Equation (13) we conclude that the attribute implication $a_2 \Rightarrow a_1$ is possible and optimistic in IC. Moreover, it is neither certain nor conservative in IC because of Equation (14). Therefore, this implication is valid in some scenarios of IC, but not in all of them. In particular, taking into account that the completions of IC are represented in Tables 10 and 12, it is easy to deduce that the attribute implication $a_2 \Rightarrow a_1$ is valid in the upper completion given in Table 10, but it is not valid in the lower completion, corresponding to Table 12. As a consequence, obtaining more precise information affects the validity of attribute implications, that is, if we can replace a symbol ? with $+$ or $-$, we can change the validity of an implication. For instance, if we certify that the object x_2 and the attribute a_1 are

Table 17 Incomplete context IC

R_{IC}	a_1	a_2	a_3	a_4
x_1	$+$	$+$	?	?
x_2	?	$-$	$+$	$-$
x_3	$-$	$-$	$-$	$+$

not related, then the attribute implication $a_2 \Rightarrow a_1$ is not valid in the new context. Otherwise, this attribute implication becomes valid.

Finally, notice that there are neither certain nor conservative attribute implications in IC. Indeed, from Equations (6) to (12) it can be checked that:

$$D^{\downarrow^{\Box}} \not\subseteq E^{\downarrow^{\Box}}$$

for all $D, E \subseteq A$ with $D \cap E = \varnothing$ and $D, E \neq \varnothing$. Consequently, no conservative attribute implication is extracted from IC. Taking into account that $D^{\downarrow^{\Box}} \subseteq D^{\downarrow^{\Diamond}}$, by Lemma 15, we conclude that there are no certain implications neither. □

Example 29 has illustrated the study of attribute implications in incomplete contexts applying Proposition 28. Thus, it has been possible to analyze attribute implications in detail despite the lack of information that characterizes incomplete contexts.

The different types of validity introduced in Definition 26 allow to analyze in detail dependencies between attributes, extracting significant conclusions of the given context. It is important to remark that these kinds of validity are closely related due to Proposition 27.

Remark 30 Given $IC = (X, A, \{+, -, ?\}, R_{IC})$ an incomplete context and $D \Rightarrow E$ an attribute implication, by applying Proposition 27, we can ensure that:

- If $D \Rightarrow E$ is a certain attribute implication in IC, then it is a conservative and optimistic attribute implication in IC.
- If $D \Rightarrow E$ is a conservative or optimistic attribute implication in IC, then it is a possible attribute implication in IC.
- If $D \Rightarrow E$ is a certain, conservative or optimistic attribute implication in IC, then it is not an impossible attribute implication in IC.

The next example shows that there are no more relationships among the different kinds of attribute implications given in this paper. With this purpose, we will show that the reciprocal of the items in Remark 30 are not true in general.

Example 31 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be the incomplete context whose set of objects is $X = \{x_1, x_2, x_3\}$, the set of attributes is $A = \{a_1, a_2, a_3, a_4\}$ and the relation R_{IC} between both sets is given in Table 17.

We will illustrate that the attribute implications introduced in Definition 26 are not completely equivalent by taking into account the information given in Table 17. First of all, we consider the attribute implication $a_1 \Rightarrow a_2$. By Definition 6 we obtain that:

$$a_1^{\downarrow^{\Box}} = \{x_1\}$$
$$a_1^{\downarrow^{\Diamond}} = \{x_1, x_2\}$$
$$a_2^{\downarrow^{\Box}} = a_2^{\downarrow^{\Diamond}} = \{x_1\}$$

Then, we have that:

$$a_1^{\downarrow^{\Box}} = \{x_1\} \subseteq \{x_1\} = a_2^{\downarrow^{\Box}} \tag{15}$$

$$a_1^{\downarrow^{\Diamond}} = \{x_1, x_2\} \nsubseteq \{x_1\} = a_2^{\downarrow^{\Box}} \tag{16}$$

$$a_1^{\downarrow^{\Box}} = \{x_1\} \subseteq \{x_1\} = a_2^{\downarrow^{\Diamond}} \tag{17}$$

$$a_1^{\downarrow^{\Diamond}} = \{x_1, x_2\} \nsubseteq \{x_1\} = a_2^{\downarrow^{\Diamond}} \tag{18}$$

As a result, by Equations (15) and (16) and Proposition 28 we conclude that $a_1 \Rightarrow a_2$ is a conservative attribute implication in IC, but it is not certain. In a similar way, from Equations (17) and (18) we deduce that it is possible, but not optimistic. Furthermore, we have also proven that it is conservative, but not optimistic.

Now, we focus on the attribute implication $a_3 \Rightarrow a_1$. By Definition 6:

$$a_3^{\downarrow^{\Diamond}} = \{x_1, x_2\}$$

$$a_3^{\downarrow^{\Box}} = \{x_2\}$$

Consequently, we obtain that:

$$a_3^{\downarrow^{\Diamond}} = \{x_1, x_2\} \subseteq \{x_1, x_2\} = a_1^{\downarrow^{\Diamond}} \tag{19}$$

$$a_3^{\downarrow^{\Diamond}} = \{x_1, x_2\} \nsubseteq \{x_1\} = a_1^{\downarrow^{\Box}} \tag{20}$$

Therefore, by Equation (19) the attribute implication $a_3 \Rightarrow a_1$ is optimistic in IC, and by Equation (20) it is not certain.

From the previous computations, we can also extract more relationships between attribute implications. Indeed, by considering the attribute implication $a_2 \Rightarrow a_3$, we have that:

$$a_2^{\downarrow^{\Box}} = \{x_1\} \subseteq \{x_1, x_2\} = a_3^{\downarrow^{\Diamond}} \tag{21}$$

$$a_2^{\downarrow^{\Diamond}} = \{x_1\} \subseteq \{x_1, x_2\} = a_3^{\downarrow^{\Diamond}} \tag{22}$$

$$a_2^{\downarrow^{\Box}} = \{x_1\} \nsubseteq \{x_2\} = a_3^{\downarrow^{\Box}} \tag{23}$$

Then, we deduce that $a_2 \Rightarrow a_3$ is possible and optimistic in IC by Equations (21) and (22) respectively, but it is not conservative due to Equation (23).

Next, we study the attribute implication $\{a_2, a_3\} \Rightarrow a_4$. Applying Definition 6, we obtain that:

$$\{a_2, a_3\}^{\downarrow^{\Box}} = \varnothing$$

$$\{a_2, a_3\}^{\downarrow^{\Diamond}} = \{x_1\}$$

$$a_4^{\downarrow^{\Box}} = \{x_3\}$$

$$a_4^{\downarrow^{\Diamond}} = \{x_1, x_3\}$$

Hence, we have that:

$$\{a_2, a_3\}^{\downarrow^{\Box}} = \varnothing \subseteq \{x_3\} = a_4^{\downarrow^{\Box}} \tag{24}$$

$$\{a_2, a_3\}^{\downarrow^{\Diamond}} = \{x_1\} \subseteq \{x_1, x_3\} = a_4^{\downarrow^{\Diamond}} \tag{25}$$

$$\{a_2, a_3\}^{\downarrow^{\Diamond}} = \{x_1\} \nsubseteq \{x_3\} = a_4^{\downarrow^{\Box}} \tag{26}$$

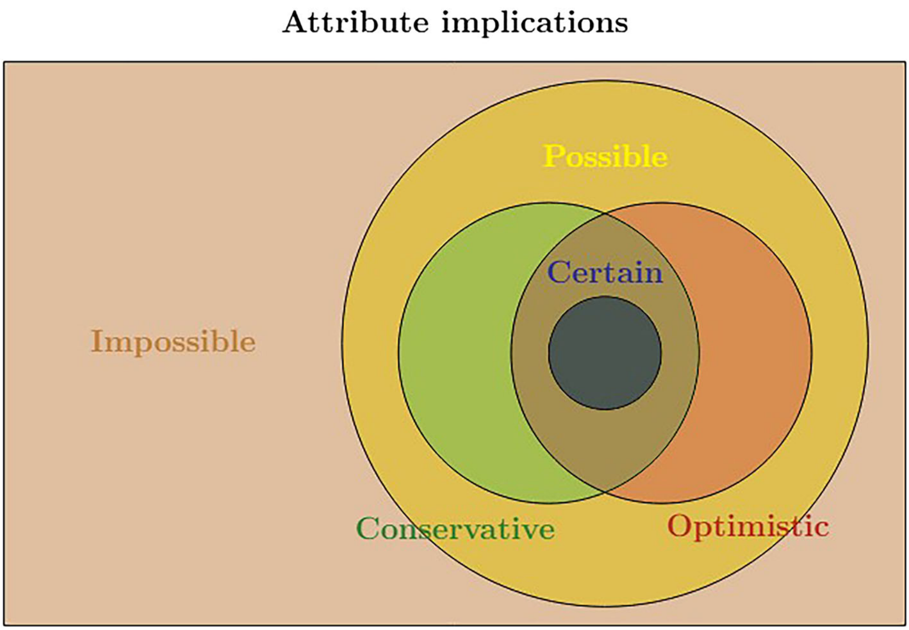

Fig. 1 Relationship between attribute implications in incomplete contexts

Attending to Equations (24) and (25), $\{a_2, a_3\} \Rightarrow a_4$ is a conservative and optimistic attribute implication in IC, while by Equation (26) we conclude that it is not certain in IC.

Finally, notice that none of the previous attribute implications is impossible, since all of them are possible by Remark 30. An example of impossible attribute implication is $a_3 \Rightarrow a_4$, since:

$$a_3^{\downarrow^{\Box}} = \{x_2\} \nsubseteq \{x_1, x_3\} = a_4^{\downarrow^{\Diamond}}$$

It is easy to check that this attribute implication is neither possible nor certain nor conservative nor optimistic. □

Remark 30 and Example 31 have illustrated all the existing relationships among the attribute implications given in Definition 26. As a summary, we represent these links in the Venn diagram of Fig. 1.

Notice that, we lose the richness of the notions and results presented in this article when we restrict ourselves to particular environments. For example, if a standard context is considered, then any pair of subsets of attributes constitutes a possible attribute implication. In a similar way, any implication considered in a complete context is impossible or certain due to all the information is assumed to be known. Hence, in these non-proper contexts different cases in Fig. 1 collapse/coincide or are empty. Anyway, we obtain the same information given from the original approaches, such as in standard contexts we obtain conservative implications.

In the next section, we will continue to study attribute implications in incomplete contexts incorporating possibility theory. With this purpose, we will follow the same philosophy as in Sect. 4.2 by introducing the possibility and necessity measures of attribute implications with respect to an incomplete context. This fact will allow us to analyze the possibility and necessity of attribute implications and to extract relevant conclusions from them.

5.2 Possibility and necessity measures of attribute implications

This section presents the possibility and necessity of attribute implications with respect to an incomplete context. These values will be obtained from Definition 20, considering once again $\mathcal{U}$ as the universe of $n \times m$-complete contexts. The corresponding event needed to compute the previous values is the set of completions where the given implication is valid, which is defined next.

Definition 32 Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts, IC be an $n \times m$-incomplete context and $\mathcal{I}_{IC}$ be the set of attribute implications in IC. The *mapping of valid completions* with respect to IC will be denoted as $\mathfrak{C}_{IC} : \mathcal{I}_{IC} \to \mathcal{P}(\mathcal{U})$ and is defined for each $D \Rightarrow E \in \mathcal{I}_{IC}$ as follows:

$$\mathfrak{C}_{IC}(D \Rightarrow E) = \{CO \in \text{Comp}(IC) \mid D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}\}$$

Then, the possibility and necessity of $D \Rightarrow E$ are given by the possibility and necessity of the set of completions $\mathfrak{C}_{IC}(D \Rightarrow E)$.

Definition 33 Let $\mathcal{U}$ be the universe of $n \times m$-complete contexts, IC be an $n \times m$-incomplete context and $\mathcal{I}_{IC}$ be the set of attribute implications in IC. Given $D \Rightarrow E \in \mathcal{I}_{IC}$, the *possibility and necessity of $D \Rightarrow E$ with respect to IC* are the values:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \Pi_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E))$$
$$\mathbf{N}_{IC}(D \Rightarrow E) = N_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E))$$

The following result specifies the detail of these definitions.

Proposition 34 *In the setting of Definition 33, we have that:*

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \max\{\pi_{IC}(CO) \mid CO \in \mathit{Comp}(IC) \text{ and } D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}\}$$
$$\mathbf{N}_{IC}(D \Rightarrow E) = 1 - \max\{\pi_{IC}(CO) \mid CO \in \mathit{Comp}(IC) \text{ and } D^{\downarrow^c}_{CO} \nsubseteq E^{\downarrow^c}_{CO}\}$$

Proof By definition we obtain that

$$\begin{aligned}\Pi_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E)) &= \max\{\pi_{IC}(CO) \mid CO \in \mathfrak{C}_{IC}(D \Rightarrow E)\}\\ &= \max\{\pi_{IC}(CO) \mid CO \in \text{Comp}(IC) \text{ and } D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}\}\\ N_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E)) &= 1 - \max\{\pi_{IC}(CO) \mid CO \notin \mathfrak{C}_{IC}(D \Rightarrow E)\}\\ &= 1 - \max\{\pi_{IC}(CO) \mid CO \in \text{Comp}(IC) \text{ and } D^{\downarrow^c}_{CO} \nsubseteq E^{\downarrow^c}_{CO}\}\end{aligned}$$

where the last inequality holds because:

$$(\mathfrak{C}_{IC}(D \Rightarrow E))^c = (\text{Comp}(IC))^c \cup \{CO \in \text{Comp}(IC) \mid D^{\downarrow^c}_{CO} \nsubseteq E^{\downarrow^c}_{CO}\}$$

Nevertheless, for the necessity of $D \Rightarrow E$, the set $\{CO \in \text{Comp}(IC) \mid D^{\downarrow^c}_{CO} \nsubseteq E^{\downarrow^c}_{CO}\}$ has only been considered, since by Proposition 19, $\pi_{IC}(CC) = 0$ for all $CC \in (\text{Comp}(IC))^c$. Then, these contexts are irrelevant for the computation of the necessity of attribute implications. □

Notice also that, the possibility and necessity of attribute implications can be obtained without computing the possibility and necessity measures of the event $\mathfrak{C}_{IC}(D \Rightarrow E)$, since all its elements are completions, and their possibility is 1, by Proposition 19. Therefore, it is only necessary to compute that event.

Proposition 35 *Let IC be an incomplete context and $D \Rightarrow E$ be an attribute implication. Then:*

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \begin{cases} 1 \text{ if } \mathfrak{C}_{IC}(D \Rightarrow E) \neq \varnothing \\ 0 \text{ otherwise} \end{cases}$$

$$\mathbf{N}_{IC}(D \Rightarrow E) = \begin{cases} 1 \text{ if } \mathfrak{C}_{IC}(D \Rightarrow E) = Comp(IC) \\ 0 \text{ otherwise} \end{cases}$$

$$\mathbf{N}_{IC}(D \Rightarrow E) \leq \mathbf{\Pi}_{IC}(D \Rightarrow E)$$

Proof First of all, we focus on $\mathbf{\Pi}_{IC}(D \Rightarrow E)$. Notice that:

$$\mathfrak{C}_{IC}(D \Rightarrow E) = \{CO \in \text{Comp}(IC) \mid D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}\} \subseteq \text{Comp}(IC)$$

As a result, if $\mathfrak{C}_{IC}(D \Rightarrow E) \neq \varnothing$ then $\mathfrak{C}_{IC}(D \Rightarrow E) \cap \text{Comp}(IC) = \mathfrak{C}_{IC}(D \Rightarrow E) \neq \varnothing$. Then, by Corollary 21(4), we deduce that

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \Pi_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E)) = 1$$

Otherwise, if $\mathfrak{C}_{IC}(D \Rightarrow E) = \varnothing$, then:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \Pi_{IC}(\varnothing) = 0$$

Now, we analyze the necessity of $D \Rightarrow E$. If $\mathfrak{C}_{IC}(D \Rightarrow E) = \text{Comp}(IC)$, in particular, we have that $\text{Comp}(IC) \subseteq \mathfrak{C}_{IC}(D \Rightarrow E)$. Therefore, by Corollary 21(4):

$$\mathbf{N}_{IC}(D \Rightarrow E) = N_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E)) = 1$$

Analogously, if $\mathfrak{C}_{IC}(D \Rightarrow E) \neq \text{Comp}(IC)$, and taking into account that $\mathfrak{C}_{IC}(D \Rightarrow E) \subseteq \text{Comp}(IC)$, we deduce that $\text{Comp}(IC) \nsubseteq \mathfrak{C}_{IC}(D \Rightarrow E)$. Thus, by Corollary 21(4):

$$\mathbf{N}_{IC}(D \Rightarrow E) = N_{IC}(\mathfrak{C}_{IC}(D \Rightarrow E)) = 0$$

Finally, from the previous equalities it is easy to deduce that $\mathbf{N}_{IC}(D \Rightarrow E) \leq \mathbf{\Pi}_{IC}(D \Rightarrow E)$. □

Notice that, the previous cases of the possibility and necessity of attribute implications are totally related to their kind of validity in IC. In particular, $\mathfrak{C}_{IC}(D \Rightarrow E) \neq \varnothing$ means that $D \Rightarrow E$ is valid in at least one completion of IC, whereas $\mathfrak{C}_{IC}(D \Rightarrow E) = \text{Comp}(IC)$ expresses that $D \Rightarrow E$ is valid in all the completions of IC. Consequently, and taking into account Proposition 28, the following characterization is deduced.

Proposition 36 *Let IC be an incomplete context and $D \Rightarrow E$ be an attribute implication. Then:*

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1 \textit{ if and only if } D^{\downarrow^{\square}} \subseteq E^{\downarrow^{\diamond}} \textit{ if and only if } D^{\downarrow^c}_{CO_l} \subseteq E^{\downarrow^c}_{CO_u}$$

$$\mathbf{N}_{IC}(D \Rightarrow E) = 1 \textit{ if and only if } D^{\downarrow^{\diamond}} \subseteq E^{\downarrow^{\square}} \textit{ if and only if } D^{\downarrow^c}_{CO_u} \subseteq E^{\downarrow^c}_{CO_l}$$

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = 0 \textit{ if and only if } D^{\downarrow^{\diamond}} \nsubseteq E^{\downarrow^{\square}} \textit{ if and only if } D^{\downarrow^c}_{CO_u} \nsubseteq E^{\downarrow^c}_{CO_l}$$

Proof We will only prove the first equivalence for the two first cases. The second equivalence is immediately obtained by Proposition 14, and the last statement is immediately deduced from the first one and Lemma 15. We begin by supposing that $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$. By Proposition 35, we deduce that $\mathfrak{C}_{IC}(D \Rightarrow E) \neq \varnothing$, that is, there exists $CO \in \text{Comp}(IC)$

such that $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$. As a result, by Definition 26, $D \Rightarrow E$ is a possible attribute implication in IC. Applying Proposition 28, we conclude that $D^{\downarrow^\Box} \subseteq E^{\downarrow^\Diamond}$. The counterpart can be proven analogously.

Now, we suppose that $\mathbf{N}_{IC}(D \Rightarrow E) = 1$. Then, by Proposition 35, $\mathfrak{C}_{IC}(D \Rightarrow E) = \text{Comp}(IC)$ and $D^{\downarrow^c}_{CO} \subseteq E^{\downarrow^c}_{CO}$, for all $CO \in \text{Comp}(IC)$. Consequently, by Definition 26, $D \Rightarrow E$ is a certain attribute implication in IC, and by Proposition 28, we deduce that $D^{\downarrow^\Diamond} \subseteq E^{\downarrow^\Box}$. The counterpart is proven following a similar reasoning. □

Then, the possibility of an attribute implication with respect to an incomplete context is 1 if and only if it is possibly valid in this context. A similar conclusion is deduced from the necessity measure regarding certainly valid attribute implications. In this way, Proposition 36 allows the direct computation of the possibility and necessity measures of attribute implications by the derivation operators of Definition 6. Therefore, it is not necessary to compute the set $\mathfrak{C}_{IC}(D \Rightarrow E)$ explicitly, in order to determine possibly or necessarily valid attribute implications.

It is important to emphasize that Propositions 35 and 36 compute the possibility and necessity of attribute implications from different perspectives. On the one hand, Proposition 35 studies attribute implications from the standpoint of possibility theory, since the event $\mathfrak{C}_{IC}(D \Rightarrow E)$ is analyzed. On the other hand, Proposition 36 focuses on the FCA framework, because of the corresponding characterizations are given in terms of the derivation operators $\downarrow^\Box$, $\downarrow^\Diamond$ and $\downarrow^c$. Therefore, the derivation of both results has shown that possibility theory naturally applies to the study of incomplete FCA frameworks.

With respect to conservative and optimistic attribute implications, it is not possible to obtain a double equivalence between the possibility measure and the derivation operators. Instead, we have that every conservative or optimistic attribute implication has a possibility measure equal to one, because these implications are also possibly valid.

Proposition 37 *Let IC be an incomplete context and $D \Rightarrow E$ be an attribute implication. The following statement holds:*

$$\text{If } D^{\downarrow^\Box} \subseteq E^{\downarrow^\Box} \text{ (respectively } D^{\downarrow^\Diamond} \subseteq E^{\downarrow^\Diamond}\text{) then } \mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$$

Proof We only prove that $D^{\downarrow^\Box} \subseteq E^{\downarrow^\Box}$ implies $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$, since the other case can be proven analogously. By Lemma 15, we have that $E^{\downarrow^\Box} \subseteq E^{\downarrow^\Diamond}$. As a result, $D^{\downarrow^\Box} \subseteq E^{\downarrow^\Diamond}$, and by Proposition 36, we conclude that $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$. □

Now, we introduce other properties about Definition 33 supported by the decomposition of attribute implications. Consequently, the possibility and necessity of an attribute implication are obtained from the possibility and necessity of simpler implications, which facilitates the computation of these measures for the original attribute implication.

Proposition 38 *Let IC be an incomplete context and $D \Rightarrow E$ be an attribute implication with $D = \{d_1, \ldots, d_{m_1}\}$ and $E = \{e_1, \ldots, e_{m_2}\}$. Then, the following equalities and inequalities hold:*

$$\begin{aligned}
\mathbf{\Pi}_{IC}(D \Rightarrow E) &= \min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\}\\
\mathbf{\Pi}_{IC}(D \Rightarrow E) &\geq \max\{\mathbf{\Pi}_{IC}(d_i \Rightarrow E) \mid i \in \{1, \ldots, m_1\}\}\\
\mathbf{N}_{IC}(D \Rightarrow E) &= \min\{\mathbf{N}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\}\\
\mathbf{N}_{IC}(D \Rightarrow E) &\geq \max\{\mathbf{N}_{IC}(d_i \Rightarrow E) \mid i \in \{1, \ldots, m_1\}\}
\end{aligned}$$

Proof As $\mathbf{\Pi}_{IC}(D \Rightarrow E) \in \{0, 1\}$, in order to demonstrate the equality:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\}$$

we only need to prove the following statements:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1 \quad \text{implies} \quad \min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} = 1$$

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = 0 \quad \text{implies} \quad \min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} = 0$$

First of all, we suppose that $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$. Then, by Propositions 36 and 7, taking into account that $E = \{e_1, \ldots, e_{m_2}\}$, we deduce that:

$$D^{\downarrow^{\Box}} \subseteq E^{\downarrow^{\Diamond}} = \bigcap_{j=1}^{m_2} e_j^{\downarrow^{\Diamond}}$$

As a consequence, we obtain that $D^{\downarrow^{\Box}} \subseteq e_j^{\downarrow^{\Diamond}}$, for all $j \in \{1, \ldots, m_2\}$. Hence, by Proposition 36, we deduce that $\mathbf{\Pi}_{IC}(D \Rightarrow e_j) = 1$, for all $j \in \{1, \ldots, m_2\}$. In conclusion, $\min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} = 1$.

Now, we suppose that $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 0$. By Propositions 36 and 7, we deduce that:

$$D^{\downarrow^{\Box}} \not\subseteq E^{\downarrow^{\Diamond}} = \bigcap_{j=1}^{m_2} e_j^{\downarrow^{\Diamond}}$$

Therefore, there exists $k \in \{1, \ldots, m_2\}$ such that $D^{\downarrow^{\Box}} \not\subseteq e_k^{\downarrow^{\Diamond}}$. As a result, by Proposition 36, we obtain that $\mathbf{\Pi}_{IC}(D \Rightarrow e_k) = 0$, and consequently, $\min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} = 0$. Thus, we conclude that:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) = \min\{\mathbf{\Pi}_{IC}(D \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\}$$

Now, we prove that $\mathbf{\Pi}_{IC}(D \Rightarrow E) \geq \max\{\mathbf{\Pi}_{IC}(d_i \Rightarrow E) \mid i \in \{1, \ldots, m_1\}\}$. If the considered maximum is equal to 0, then the inequality trivially holds. If this maximum is equal to 1, then there exists $k \in \{1, \ldots, m_1\}$ such that $\mathbf{\Pi}_{IC}(d_k \Rightarrow E) = 1$. Then, by Proposition 36, we obtain that $d_k^{\downarrow^{\Box}} \subseteq E^{\downarrow^{\Diamond}}$. In addition, by Proposition 7:

$$D^{\downarrow^{\Box}} = \bigcap_{i=1}^{m_1} d_i^{\downarrow^{\Box}} \subseteq d_k^{\downarrow^{\Box}}$$

Consequently, $D^{\downarrow^{\Box}} \subseteq E^{\downarrow^{\Diamond}}$, and by Proposition 36 we conclude that $\mathbf{\Pi}_{IC}(D \Rightarrow E) = 1$. In conclusion:

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) \geq \max\{\mathbf{\Pi}_{IC}(d_i \Rightarrow E) \mid i \in \{1, \ldots, m_1\}\}$$

The proofs for the statements related to the necessity measure are completely analogous taking into account the characterization given in Proposition 36. □

Therefore, Proposition 38 is useful to split large computations into small ones, which are much easier to carry out. By considering both characterizations of the possibility and necessity simultaneously, the following result is obtained.

Table 18 Incomplete context IC

R_{IC}	a_1	a_2	a_3	a_4
x_1	+	+	+	?
x_2	+	−	−	−
x_3	−	+	−	−

Corollary 39 *Let IC be an incomplete context and $D \Rightarrow E$ be an attribute implication with $D = \{d_1, \ldots, d_{m_1}\}$ and $E = \{e_1, \ldots, e_{m_2}\}$. Then:*

$$\mathbf{\Pi}_{IC}(D \Rightarrow E) \geq \max\{\min\{\mathbf{\Pi}_{IC}(d_i \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} \mid i \in \{1, \ldots, m_1\}\}$$
$$\mathbf{N}_{IC}(D \Rightarrow E) \geq \max\{\min\{\mathbf{N}_{IC}(d_i \Rightarrow e_j) \mid j \in \{1, \ldots, m_2\}\} \mid i \in \{1, \ldots, m_1\}\}$$

Proof It is immediately obtained from Proposition 38. □

The following example illustrates the potential of the combination of possibility theory and FCA. In particular, different attribute implications with unknown information will be analyzed in terms of possibility and necessity measures by using Propositions 36 and 38.

Example 40 Let $IC = (X, A, \{+, -, ?\}, R_{IC})$ be the incomplete context with $X = \{x_1, x_2, x_3\}$ the set of objects, $A = \{a_1, a_2, a_3, a_4\}$ the set of attributes and R_{IC} the relation given in Table 18.

We will apply the different statements in Proposition 38 to obtain some possibility and necessity measures with simple computations, which are given by Proposition 36. First of all, we will study the attribute implications $a_1 \Rightarrow a_3$, $a_2 \Rightarrow a_3$ and $\{a_1, a_2\} \Rightarrow a_3$. By Definition 6, we obtain that:

$$\begin{aligned} a_1^{\downarrow^\Box} &= a_1^{\downarrow^\Diamond} = \{x_1, x_2\} \\ a_3^{\downarrow^\Box} &= a_3^{\downarrow^\Diamond} = \{x_1\} \\ a_2^{\downarrow^\Box} &= a_2^{\downarrow^\Diamond} = \{x_1, x_3\} \\ \{a_1, a_2\}^{\downarrow^\Box} &= \{a_1, a_2\}^{\downarrow^\Diamond} = \{x_1\} \\ a_4^{\downarrow^\Box} &= \varnothing \\ a_4^{\downarrow^\Diamond} &= \{x_1\} \end{aligned}$$

Since $a_1^{\downarrow^\Box} \not\subseteq a_3^{\downarrow^\Diamond}$, by Proposition 36 we deduce that $\mathbf{\Pi}_{IC}(a_1 \Rightarrow a_3) = 0$. Analogously, it can be obtained that $\mathbf{\Pi}_{IC}(a_2 \Rightarrow a_3) = 0$ and, since $\{a_1, a_2\}^{\downarrow^\Box} \subseteq a_3^{\downarrow^\Diamond}$, $\mathbf{\Pi}_{IC}(\{a_1, a_2\} \Rightarrow a_3) = 1$. Therefore, we conclude that:

$$\max\{\mathbf{\Pi}_{IC}(a_1 \Rightarrow a_3), \mathbf{\Pi}_{IC}(a_2 \Rightarrow a_3)\} = 0 < 1 = \mathbf{\Pi}_{IC}(\{a_1, a_2\} \Rightarrow a_3)$$

In this case, a strict inequality holds. Analogously, by Proposition 36 it can be checked that:

$$\max\{\mathbf{N}_{IC}(a_1 \Rightarrow a_3), \mathbf{N}_{IC}(a_2 \Rightarrow a_3)\} = 0 < 1 = \mathbf{N}_{IC}(\{a_1, a_2\} \Rightarrow a_3)$$

Thus, although the attribute implications $a_1 \Rightarrow a_3$ and $a_2 \Rightarrow a_3$ are impossible in IC, that is, they are not valid in any possible scenario of IC, the attribute implication $\{a_1, a_2\} \Rightarrow a_3$ is certain in IC, which implies it is valid in each possible scenario of IC.

On the other hand, we showed above that $\mathbf{\Pi}_{IC}(a_2 \Rightarrow a_3) = 0$. Thus, by Proposition 38, we conclude that:

$$\mathbf{\Pi}_{IC}(a_2 \Rightarrow \{a_1, a_3\}) = \mathbf{\Pi}_{IC}(a_2 \Rightarrow \{a_3, a_4\}) = \mathbf{\Pi}_{IC}(a_2 \Rightarrow \{a_1, a_3, a_4\}) = 0$$

As a result, if a given attribute implication is impossible in IC, any implication with the same antecedent and a greater consequent with respect to the inclusion order, will be also impossible. Analogously, $\mathbf{N}_{IC}(a_1 \Rightarrow a_3) = 0$, leading to:

$$\mathbf{N}_{IC}(a_1 \Rightarrow \{a_2, a_3\}) = \mathbf{N}_{IC}(a_1 \Rightarrow \{a_3, a_4\}) = \mathbf{N}_{IC}(a_1 \Rightarrow \{a_2, a_3, a_4\}) = 0$$

Similarly, since $\mathbf{\Pi}_{IC}(a_4 \Rightarrow a_3) = \mathbf{N}_{IC}(a_4 \Rightarrow a_3) = 1$, applying Proposition 38 we deduce that:

$$\mathbf{\Pi}_{IC}(\{a_1, a_4\} \Rightarrow a_3) = \mathbf{\Pi}_{IC}(\{a_2, a_4\} \Rightarrow a_3) = \mathbf{\Pi}_{IC}(\{a_1, a_2, a_4\} \Rightarrow a_3) = 1$$
$$\mathbf{N}_{IC}(\{a_1, a_4\} \Rightarrow a_3) = \mathbf{N}_{IC}(\{a_2, a_4\} \Rightarrow a_3) = \mathbf{N}_{IC}(\{a_1, a_2, a_4\} \Rightarrow a_3) = 1$$

Therefore, if an attribute implication is certain, and consequently possible, conservative and optimistic, in IC, then any implication with the same consequent and a greater antecedent will be also certain, possible, conservative and optimistic.

As a final remark, we have shown (e.g., in Example 29) that enriching an incomplete context with new information (replacing ? by + or −) directly affects the validity of attribute implications. □

6 Related works

The management of incomplete information in FCA has been discussed in several papers. Let us outline their main features and compare them with our present study.

6.1 Burmeister and Holzer's approach

Incomplete contexts modeled by the use of +, −, ? in the context matrix were proposed by Burmeister and Holzer (2000). They also present different ways of encoding incomplete contexts in the style of standard ones (using × like in Table 1) by augmenting the number of attributes (e.g., adding their negations or the fact that they are unknown). They also define certainty contexts (replacing + by ×) and possibility contexts (replacing +, ? by ×), as well as the certain and possible validity of attribute implications. In Holzer (2004a) some results are presented to determine if an attribute implication is possibly or certainly valid in terms of derivation operators. These results are also applied to analyze dependence among subsets of attributes in the procedure known as attribute exploration (Burmeister and Holzer 2000, 2005; Holzer 2004a, b). However, this line of study has not been addressed in this paper. In particular, we have studied different forms of validity of attribute implications in incomplete contexts (like optimistic and conservative) and we have shown that possibility theory (Dubois and Prade 1988) (rather than Kleene logic like in Burmeister and Holzer (2000)) can account for the notions of certain and possible attribute implications.

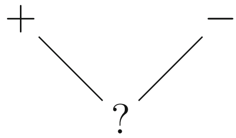

Fig. 2 Order defined in incomplete contexts given in Pérez-Gámez et al. (2023)

6.2 Obiedkov's approach

Incomplete contexts are also studied in Obiedkov (2002) in order to determine the validity of propositional formulas. To do that, a modal logic is defined, reinterpreting the symbol "?" present in incomplete contexts as "inapplicable" or "nonsense" instead of "unknown". Nevertheless, this has not been the view pursued in this paper, since we have focused on unknown relationships and attribute implications.

6.3 Pérez Gámez's approach

In Pérez-Gámez et al. (2023) a knowledge order is considered to deal with the unknown information contained in contexts, defining that $? \leq -$, $? \leq +$, and that $-$ and $+$ are incomparable. Thus, this order represents the existing knowledge about the relationship, and it is shown in Fig. 2.

In this paper, we have considered a truthfulness order to express the presence of each attribute in each object, so that $- < ? < +$. Hence, the meaning of the given relations, and consequently, of the extracted knowledge, is different in both papers. Moreover, (Pérez-Gámez et al. 2023) introduces a new logic for reasoning about implications, while in this paper, we have focused on studying their validity in incomplete contexts by means of derivation operators and possibility theory.

6.4 Demin and Kriegel's approach

The analysis of incomplete information in FCA has also been addressed from a probabilistic perspective (Demin et al. 2012; Kriegel 2017). Thus, several complete contexts are considered by assigning different probabilities to them, so that the validity of implications depends on the probabilities of the contexts in which they are valid. Despite some similarities between probabilistic and possibilistic approaches, the development presented in this paper is different from the one given in Demin et al. (2012); Kriegel (2017). We have focused on the possibilistic one because we wanted to analyze the potential and compulsory validity of attribute implications in incomplete contexts. This procedure has been carried out by using the necessity and possibility measures defined in possibility theory.

6.5 Negative information in complete and incomplete contexts

The approach presented in this paper focuses on incomplete contexts, that is, with contexts with positive, negative and unknown relationships.

In this paper we have focused on positive information to determine attribute implications and their type of validity. If we have an incomplete context $IC = (X, A, \{+, -, ?\}, R_{IC})$, such that for some $a \in A$, $R_{IC}(x, a) \neq +$ for all $x \in X$, we conclude that the object x and the attribute a are not related. This fact causes that all attribute implications with a in the consequent are impossible. However, it would be important to take advantage of negative information, expressing that a certain object does not possess a particular attribute. It looks natural to consider a symmetric handling of $+$ and $-$ in incomplete contexts.

One way to extract knowledge that accounts for both positive and negative relationships is to consider negative attributes on top of positive ones, thus augmenting the size of contexts. Namely, it is possible to consider the attribute "not a", usually denoted as $\neg a$, which has the opposite meaning of a. As a consequence, we obtain that $R_{IC}(x, \neg a) = +$, which becomes to obtain new attribute implications with $\neg a$ in the consequent possible, and so we can extract new relationships of dependence between subsets of attributes.

This kind of approach was introduced in Pérez-Gámez et al. (2022); Rodríguez-Jiménez et al. (2014, 2016a, b) allowing to define and analyze a wider variety of attribute implications.

7 Conclusions and future work

In this paper, we have extended the framework of Burmeister and Holzer for the handling of incomplete contexts and the extraction of attribute implications from them. We have shown that possibility theory can account for this approach, without resorting to Kripke semantics nor Kleene logic. Five forms of attribute implications have been laid bare and interesting properties have been analyzed.

In the future, we will involve negative attributes in our approach so as to restore the symmetry between positive and negative knowledge, and extract more general attribute implications where attributes and their negations can appear. However, it would be good to do so without augmenting the set of attributes with their negations. Indeed such enlarged contexts forget the interaction between an attribute and its negation. Doing so, we will be able to analyze in more detail incomplete contexts, extracting a significant greater amount of knowledge from them.

Besides, it would be of interest to take advantage of the fact that possibility theory is graded for modeling the fact that some completions have a higher plausibility than other ones (just as probabilities are attached to completions in Demin et al. (2012); Kriegel (2017)). More generally, graded possibility would enable us to deal with more general contexts where each pair (x, a) would be associated with a normalized possibility distribution over $\{+, -\}$ (encoding degrees of certainty that an object possesses or not an attribute), as already suggested in Ait-Yakoub et al. (2016).

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